

# Dynamics of a Bertrand duopoly with differentiated products and nonlinear costs: analysis, comparisons and new evidences

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## Nonlinear duopoly with price competition and horizontal product differentiation

**Bischi et al. (2010)** study **nonlinear dynamics in oligopolies**

- by considering the case of quantity-setting firms
- when information is incomplete (**Bischi et al., 1998, 2007**)

**Fanti et al. (2013)**

- consider the case of price-setting firms and incomplete information
- assume linear costs (constant returns to scale)

We extend the study of Fanti et al. (2013) by considering a nonlinear duopoly with price competition, horizontal differentiation and **quadratic production costs** (decreasing returns to scale)

## The economy is composed by firms and consumers

There exists:

- a **competitive sector** that produces the numeraire good  $k \geq 0$  (whose price is normalised to 1)
- a **duopolistic sector** where firm 1 and firm 2 produce (horizontally) differentiated products of variety 1 and variety 2, respectively
- $p_i \geq 0$  and  $q_i \geq 0$  are the **price and quantity of product of firm  $i$**  ( $i = 1, 2$ ), respectively

## Consumers

- Consumers' preferences towards  $q_1$  and  $q_2$  are captured by the utility function  $U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}(q_1^2 + q_2^2 + 2dq_1 q_2)$

- The direct demands are then given by

$$q_1 = \frac{a(1-d)-p_1+dp_2}{1-d^2} \quad \text{and} \quad q_2 = \frac{a(1-d)-p_2+dp_1}{1-d^2}$$

$a > 0$  is the extent of market demand of both goods and  $d \in (-1, 1)$  is the **deg. of horizontal product differentiation**

- If  $d = 0$  products of variety 1 and variety 2 are independent and each firm behaves as a **monopolist**

- If  $d > 0$  (resp.  $d < 0$ ) products are **substitutes** (resp. **complements**)

- If  $d \rightarrow 1$  (resp.  $d \rightarrow -1$ ) they are **perfect substitutes** (resp. **perfect complements**)

## Duopolistic firms

- The cost function can be written as  $c_i = wq_i^2$
- Firm  $i$  maximises profits  $\Pi_i = p_i q_i - wq_i^2$  with respect to  $p_i$
- Marginal profits of the  $i$ th firm are given by

$$\frac{\partial \Pi_i}{\partial p_i} = \frac{[a(1-d) + dp_j](1-d^2 + 2w) - 2(1-d^2 + w)p_i}{(1-d^2)^2}$$

$$i, j = 1, 2, \quad i \neq j$$

## Dynamic setting

- Both players have limited information
- Both firms follow an adjustment process based on local estimates of their own marginal profit in the current period:

$$p_{i,t+1} = p_{i,t} + \alpha p_{i,t} \frac{\partial \Pi_i(p_{i,t}, p_{j,t})}{\partial p_{i,t}}, \quad i = 1, 2, \quad t \in \mathbb{Z}_+$$

where  $\alpha > 0$  is a coefficient that captures the speed of adjustment of firm  $i$ 's price with respect to a marginal change in profits,  $\alpha p_{i,t}$  is the intensity of the reaction of player  $i$

**FINAL SYSTEM  $T_q$** 

$$x' = \left[ 1 + \frac{\alpha a}{1+d} + 2 \frac{\alpha a w}{(1-d)(1+d)^2} \right] x - 2 \left[ \frac{\alpha}{1-d^2} + \frac{\alpha w}{(1-d^2)^2} \right] x^2 + \left[ \frac{\alpha d}{1-d^2} + 2 \frac{\alpha d w}{(1-d^2)^2} \right] xy$$

$$y' = \left[ 1 + \frac{\alpha a}{1+d} + 2 \frac{\alpha a w}{(1-d)(1+d)^2} \right] y - 2 \left[ \frac{\alpha}{1-d^2} + \frac{\alpha w}{(1-d^2)^2} \right] y^2 + \left[ \frac{\alpha d}{1-d^2} + 2 \frac{\alpha d w}{(1-d^2)^2} \right] xy$$

where  $x' = p_{1,t+1}$ ,  $x = p_{1,t}$ ,  $y' = p_{2,t+1}$ , and  $y = p_{2,t}$ ,  
 $d \in (-1, 1)$ ,  $a > 0$ ,  $w \geq 0$ ,  $\alpha > 0$

**OUR AIM:** to analyze the dynamics of system  $T_q$  (quadratic costs) and to compare it with system  $T_l$  (linear costs)



## FEASIBLE SET

- A trajectory  $\psi_t = \{(x(t), y(t))\}_{t=0}^{\infty}$  is said *feasible* if  $(x(t), y(t)) \in R_+^2$  for all  $t$
- the set  $D \subseteq R_+^2$  whose points generate feasible trajectories is *feasible set*

### PROPOSITION

If  $d \rightarrow \pm 1$  then the set  $D - \{(0, 0)\}$  is empty

**PROPOSITION: steady states**

System  $T_q$  admits four fixed points for all parameter values:

$$1 \quad E_0 = (0, 0)$$

$$2 \quad E_{1q} = \left( 0, \frac{a(1-d)(1-d^2+2w)}{2(1-d^2+w)} \right)$$

$$3 \quad E_{2q} = \left( \frac{a(1-d)(1-d^2+2w)}{2(1-d^2+w)}, 0 \right)$$

$$4 \quad E_q^* = (x_q^*, x_q^*) = \left( \frac{a(1-d^2+2w)}{(1+d)(2-d)+2w}, \frac{a(1-d^2+2w)}{(1+d)(2-d)+2w} \right)$$

- $E_q^*$  is the unique interior Nash equilibrium of this model
- the equilibrium value of the quantity  $q_i^*$  is positive for all parameter values
- condition  $a > w$  must hold to ensure  $q_i^* > 0$

**PROPOSITION: comparison**

Let  $a > w$ . If  $a > \frac{17}{8}$  then  $x_q^* > x_l^*$ ; if  $a \leq 1$  then  $x_q^* \leq x_l^*$

The extent of market demand  $a$  determines **whether prices under decreasing returns to scale are higher w.r.t. constant returns to scale:**

- when market demand is large (resp. small), equilibrium prices under decreasing returns to scale are higher (resp. lower) than under constant returns to scale
- this because when costs are nonlinear and the extent of market demand is large firms operate close to their full production capacity and prices tend to be higher than when cost are linear

## Local stability

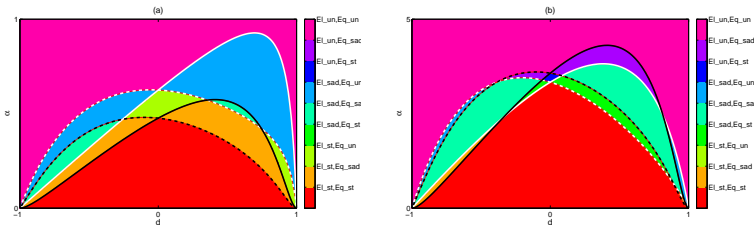
- $E_0$  is an unstable node
- $E_{1q}$  and  $E_{2q}$  can be both unstable nodes or saddle points
- The interior fixed point  $E_q^*$  may lose stability iff at least one eigenvalue crosses  $-1$  (as for the case with linear costs)
- Only the interior fixed point can be attractive

**PROPOSITION: comparison**

Let  $a > w$

- $\forall d \in (-1, 1), \exists \bar{\alpha}(d)$  such that if  $\alpha > \bar{\alpha}(d)$  then  $E_q^*$  and  $E_l^*$  are both locally unstable
- $\exists I(0)$  such that  $\forall d \in I(0), \exists \tilde{\alpha}(d)$  such that if  $\alpha < \tilde{\alpha}(d)$  then  $E_q^*$  and  $E_l^*$  are both locally stable
- $\exists I(0)$  such that  $\forall d \in I(0)$  then  $\lambda_{\parallel}(E_q^*) < \lambda_{\parallel}(E_l^*)$  and  $\lambda_{\perp}(E_q^*) < \lambda_{\perp}(E_l^*)$  iff  $a > \frac{1}{2}$

## Fixed points and local stability



**Figure:** Bifurcation curves on the parameter plane identifying regions at which different stability regimes occurs for  $w = 0.2$ . In panel (a)  $a = 3$  while in panel (b)  $a = 0.3$ . *st* means locally stable, *un* means locally unstable while *sad* means saddle.

- 1 The diagonal  $\Delta$  is an invariant set also for  $T_q$ , i.e.  $T_q(x, x) = (x', x')$ , as for  $T_l$
- 2 **THE RESTRICTION  $\phi_q$  OF THE SYSTEM TO  $\Delta$**  is topologically conjugate to the logistic map  $z' = \mu_q z(1 - z)$  with

$$\mu_q = 1 + \frac{\alpha a}{1 - d^2} \left( \frac{1 - d^2 + 2w}{1 + d} \right)$$

by the linear transformation

$$x = \frac{(1 - d^2)(1 + d) + \alpha a(1 - d^2 + 2w)}{\alpha((2 - d)(1 + d) + 2w)} z$$

- 3 Equal initial conditions imply equal dynamic behavior forever: trajectories embedded into  $\Delta$ , i.e. those characterized by  $x = y$  for all  $t$ , are called **synchronized trajectories** (Bischi and Gardini 2000 and Bischi et al. 1998)

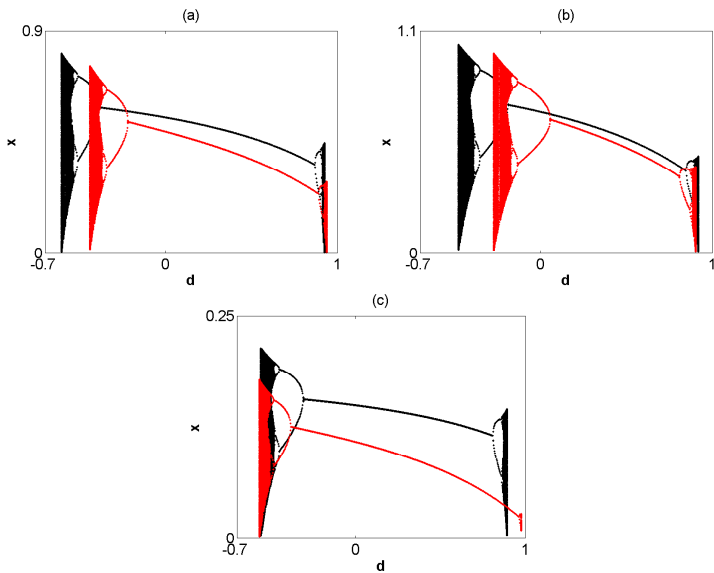
## PROPOSITION: comparison between $d$ -intervals corresponding to local stability

Assume  $a > w$

- 1 If  $\alpha a(1 + 2w) < 2$ , then  $x_q^*$  is a locally stable fixed point of  $\phi_q \forall d \in (d_{1q}, d_{2q})$ ,  $-1 < d_{1q} < 0 < d_{2q} < 1$
- 2 Let  $a \leq \frac{1}{2}$  and  $\alpha(a + w) < 2$  or  $a > \frac{1}{2}$  and  $\alpha a(1 + 2w) < 2$ . Then:
  - $x_q^*$  is a locally stable fixed point of  $\phi_q \forall d \in (d_{1q}, d_{2q})$
  - $x_l^*$  is a locally stable fixed point of  $\phi_l \forall d \in (d_{1l}, d_{2l})$
  - Let  $P = 2a(a - 1)(4 - \alpha) + \alpha w$  and  $Q = 4(2a - 1) - a\alpha$ . Then:
    - if  $P > 0$  and  $Q < 0$  then  $d_{1q} > d_{1l}$  and  $d_{2q} > d_{2l}$ ;
    - if  $P < 0$  and  $Q \neq 0$  then  $d_{1q} < d_{1l}$  and  $d_{2q} > d_{2l}$ ;
    - if  $P > 0$  and  $Q > 0$  the case is open



## Invariant sets and synchronized trajectories



**REMARK**

- 1 If  $a\alpha(1 + 2w) < 3$  then  $\phi_q$  admits an attractor  
 $\forall d \in (\bar{d}_{1q}, \bar{d}_{2q})$
- 2 The comparison between the intervals  $(\bar{d}_{1q}, \bar{d}_{2q})$  and  $(\bar{d}_{1l}, \bar{d}_{2l})$  (containing the  $d$ -values such that an attractor exists in the quadratic and linear cases) follows the same properties stated in Proposition

**PROPOSITION: existence of divergent trajectories**

- (i) A threshold value  $\bar{\alpha}_q$  does exist such that synchronized trajectories are divergent  $\forall \alpha > \bar{\alpha}_q$  given the other parameter values (the same result holds if  $a > \bar{a}$  or  $w > \bar{w}$ )
- (ii) Let  $\bar{\alpha}_l$  be the corresponding value of  $\bar{\alpha}_q$  in the linear costs model. Then:
  - if  $w = 0$  then  $\bar{\alpha}_q = \bar{\alpha}_l$
  - if  $w > 0$  then  $\bar{\alpha}_q \geq \bar{\alpha}_l$  iff  $1 + d - 2a \geq 0$

## SOME CONCLUSIONS:

- 1 the **convergence of synchronized trajectories** toward the Nash equilibrium is necessarily associated with **intermediate values of  $d$** , confirming the result obtained with linear costs
- 2 if  $x_q^*$  is locally stable for a given  $d$ -value then it loses stability via a **period doubling bifurcation** due to an **increase in the degree of substitutability** (resp. complementarity) between products, i.e.  $d$  moves to 1 (resp. to  $-1$ )
- 3 synchronized dynamics **increases in complexity** while moving from the case of products of independent varieties to **complementary or substitutable**, while, at the limit cases ( $d \rightarrow \pm 1$ ) no bounded dynamics occurs on  $\Delta$

- 1 RECALL that a feasible trajectory starting from  $(x(0), y(0))$ ,  $x(0) \neq y(0)$ , is said to **synchronize** if  $|x(t) - y(t)| \rightarrow 0$  as  $t \rightarrow +\infty$
- 2 If  $\alpha$  is not too high and parameter  $d$  assumes intermediate values, then  $E_q^*$  is locally stable and trajectories starting from a neighborhood of  $E_q^*$  synchronizes, i.e. if firms start from different initial conditions, they will behave in the same way in the long term
- 3 If  $E_q^*$  loses its local stability THEN:

## PROPOSITION

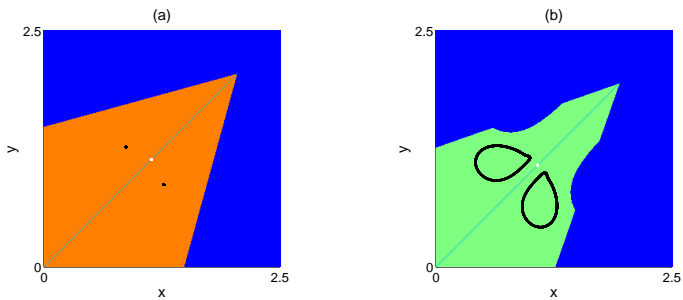
- (i) Let  $a\alpha(1 + 2w) < 2$ . Then  $\lambda_{\perp}(E_q^*) < \lambda_{\parallel}(E_q^*)$ ,  $\forall d \in I^+(0)$  and  $\lambda_{\perp}(E_q^*) > \lambda_{\parallel}(E_q^*)$ ,  $\forall d \in I^-(0)$
- (ii) Let  $\alpha(a + w) < 2$ . Then  $\lambda_{\perp}(E_l^*) < \lambda_{\parallel}(E_l^*)$ ,  $\forall d \in I^+(0)$  and  $\lambda_{\perp}(E_l^*) > \lambda_{\parallel}(E_l^*)$ ,  $\forall d \in I^-(0)$

Condition for  $E_q^*$  to be locally stable for  $d = 0$  is  $\alpha a(1 + 2w) < 2$

- If synchronized trajectories converge to  $x_q^*$  with independent products, then trajectories starting from initial conditions close to it with  $x(0) \neq y(0)$ :
  - synchronize in the long term as long as  $d \in I(0)$
  - do not synchronize (i.e. the fixed point loses firstly its transverse stability) as  **$d$  increases**

**Hence** If there exists a feasible initial condition  $(x(0), y(0))$  with  $x(0) \neq y(0)$ , then it necessarily converges to **ANOTHER BOUNDED ATTRACTOR  $B$**  existing out of the diagonal and coexisting with  $E_q^*$

## PRODUCTS ARE SUBSTITUTES



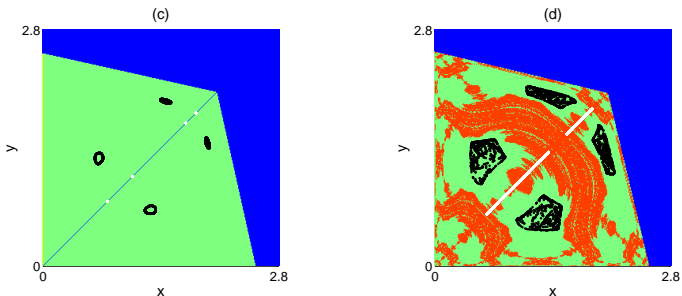
**Figure:** (a) A 2-period cycle (black points) coexists with the fixed point for  $\alpha = 0.4$ ,  $d = 0.4$ ,  $w = 0.5$  and  $a = 2$ . (b) If  $d = 0.5$  two cyclic attracting closed invariant curves have been created out of the diagonal

## PRODUCTS ARE COMPLEMENTS

*If  $E_q^*$  is locally stable for  $d = 0$  and  **$d$  decreases**, the Nash equilibrium loses firstly the stability along the diagonal, via a period doubling bifurcation which creates an attracting 2-cycle*

- Immediately after this first flip bifurcation, the 2-period cycle is locally stable and synchronization occurs
- if the degree of horizontal product differentiation still decreases, a sequence of flip bifurcation occurs on the diagonal, and attracting periodic cycles are created around the unstable Nash equilibrium

## PRODUCTS ARE COMPLEMENTS



**Figure:** (c) If  $d = -0.328$  a 4-period cycle attracts all synchronized trajectories, while a 4-piece quasi periodic attractor exists out of the diagonal and synchronization does not take place. (d) If  $d = -0.351$ , two coexisting complex attractors are owned and the basin structure is quite complex



## PRODUCTS ARE INDEPENDENT

Products of each variety are **INDEPENDENT** ( $d = 0$ )

In this case system becomes:

$$T_q(d = 0) : \begin{cases} x' = (1 + a\alpha + 2a\alpha w)x - 2\alpha(1 + w)x^2 \\ y' = (1 + a\alpha + 2a\alpha w)y - 2\alpha(1 + w)y^2 \end{cases} ,$$

which is a diagonal system having both equations conjugated to the logistic map

We proved that:

- 1 the eigenvalues associated to the Jacobian matrix are symmetric
- 2 any periodic point along the diagonal has always identical eigenvalues
- 3 any period doubling bifurcation along the diagonal, which is associated to the bifurcation cascade of the logistic map, is followed by a simultaneous period doubling bifurcation in the symmetric direction
- 4 the same holds in the case with linear costs

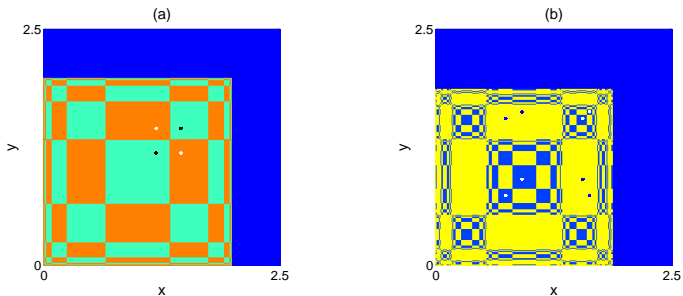
⇒ **COMPLEX PHENOMENON OF MULTISTABILITY**: many coexisting attracting cycles having their own basins of attraction (as in Bischi and Kopel 2003)

### PROPOSITION: comparison

Let  $d = 0$ ,  $a > w$  and  $\tilde{\alpha}_q := \frac{2}{a(1+2w)}$ ,  $\tilde{\alpha}_l := \frac{2}{a+w}$ . Then:

- (i) if  $a > \frac{1}{2}$  then  $E_q^*$  and  $E_l^*$  are both locally stable (resp. unstable)  $\forall \alpha < \tilde{\alpha}_q$  (resp.  $\forall \alpha > \tilde{\alpha}_l$ ); if  $\alpha \in (\tilde{\alpha}_q, \tilde{\alpha}_l)$  then  $E_q^*$  is locally unstable while  $E_l^*$  is locally stable;
- (ii) if  $a < \frac{1}{2}$  then  $E_q^*$  and  $E_l^*$  are both locally stable (resp. unstable)  $\forall \alpha < \tilde{\alpha}_l$  (resp.  $\forall \alpha > \tilde{\alpha}_q$ ); if  $\alpha \in (\tilde{\alpha}_l, \tilde{\alpha}_q)$  then  $E_q^*$  is locally stable while  $E_l^*$  is locally unstable

- 1 the primary period doubling bifurcation occurs earlier in the case of quadratic (resp. linear) costs if  $a > \frac{1}{2}$  (resp.  $a < \frac{1}{2}$ ), while it occurs simultaneously in the two cases if  $a = \frac{1}{2}$
- 2 also the second flip bifurcation occurs earlier in the quadratic costs (resp. linear costs) case if  $a > \frac{1}{2}$  (resp.  $a < \frac{1}{2}$ )
- 3 this result holds also for all the subsequent bifurcations







**Figure:** (a) If  $a = 2$ ,  $w = 0.5$  and  $\alpha = 0.51$ ,  $T_q(d = 0)$  admits two stable coexisting cycles of period 2, whose basins of attraction are represented by the green and orange regions respectively. (b) If  $\alpha = 0.62$ ,  $T_q(d = 0)$  admits two stable coexisting cycles of period 4, whose basins of attraction are represented by the yellow and blue regions respectively

## SOME CONCLUSIONS

- 1 Coexisting attractors can exist in either cases of substitutability and complementarity of products (positive and negative values of  $d$ , respectively)
- 2 In the case of substitutability the final outcome of the economy is predictable
- 3 In the case of complementarity, the structure of the basins of attraction may be more complex and the final outcome the economy may follow is unpredictable (sensitivity with respect to the final outcome)
- 4 Multistability occurs also in the case of independent products ( $d = 0$ )

**Thank you!**

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