A dynamic marketing model with best reply and inertia

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The Farris's Market Model

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The resulting profits are

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Setting $\beta_i = 1$, this leads to

$$x_i(t+1) = \sqrt{\frac{\sum_{j \neq i} a_j x_j^{(e)}(t+1)}{a_i}} - \sum_{j \neq i} a_j x_j^{(e)}(t+1)$$

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Assuming naïve expectations, $x_j^{(e)}(t+1) := x_j(t)$, Farris et al. derive the following "Best Response" dynamic model with adaptive adjustment:

$$x_i(t+1) = (1-\lambda_i)x_i(t) + \lambda_i \left(\sqrt{\frac{\sum_{j\neq i} a_j x_j(t)}{a_i}} - \sum_{j\neq i} a_j x_j(t) \right)$$

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Setting $N = 2$ and the new (rescaled) variables

$$x = a_1 a_2 x_1, \quad y = a_1 a_2 x_2$$

we have

$$Farris : \begin{cases} x' = (1-\lambda_1)x + \lambda_1 a_2 \left(\sqrt{y} - y \right) \\ y' = (1-\lambda_2)y + \lambda_2 a_1 \left(\sqrt{x} - x \right) \end{cases}$$

Similarities with the Puu' model

$$Puu \quad : \begin{cases} q_1' = (1-\lambda_1)\, q_1 + \lambda_1 \left(\sqrt{\dfrac{q_2}{c_1}} - q_2 \right) \\ q_2' = (1-\lambda_2)\, q_2 + \lambda_2 \left(\sqrt{\dfrac{q_1}{c_2}} - q_1 \right) \end{cases}$$

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We obtain the *Puu'* from the *Farris* setting $\boxed{a_2 = 1/a_1}$

Equilibria

The equilibria' abscissas follows from the fourth order algebric equation:

$$\eta\left[\frac{(1-a_1a_2)^2}{a_1a_2}\eta^3+2\left(1-a_1a_2\right)\eta^2+a_2\left(a_1+1\right)\eta-a_2\right]=0$$

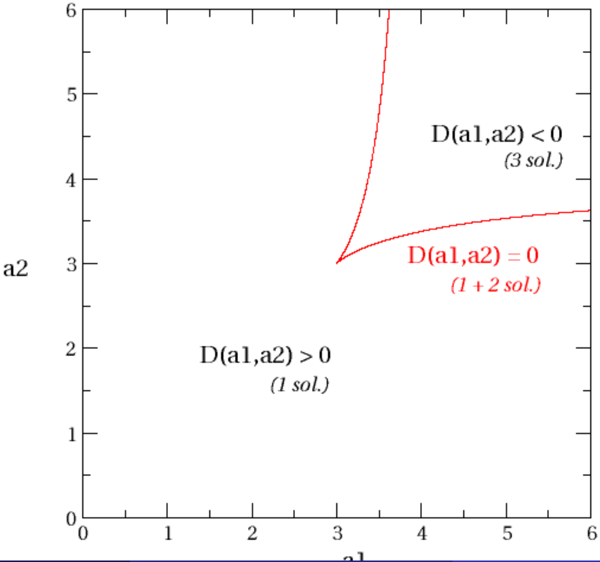
where $\eta := \sqrt{x}$. An analogous equation for $\zeta := \sqrt{y}$ holds.

From Cardano's formula, the number of real solutions is given provided the sign of the discriminant:

$$D(a_1,a_2):\begin{cases} >0 & \textit{1 real solution} \\ =0 & \textit{1 real solution and 2 coincident} \\ <0 & \textit{3 distinct real solutions}\end{cases}$$

Equilibria

Section of the function $D(a_1, a_2)$ with the plane $(a_1, a_2, 0)$



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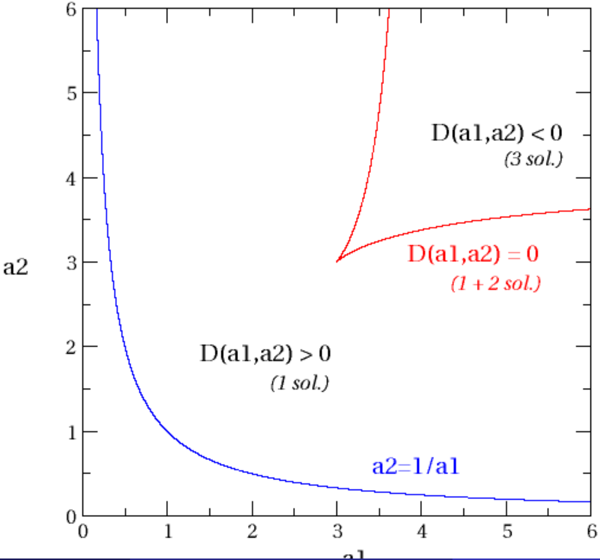


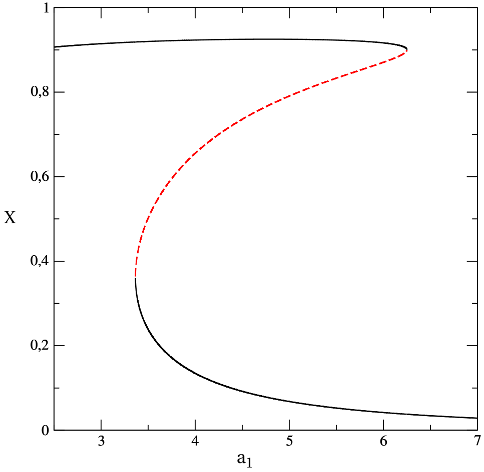
Proposition

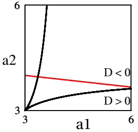
Besides $E_0 = (0,0)$ a non vanishing fixed point always exists in the region $S = (0,1) \times (0,1)$. If $a_1 a_2 \neq 1$ then two further distinct fixed points exist in the region S if the following inequality holds

$$D(a_1, a_2) = \frac{a_1^2 a_2^4}{108\,(1 - a_1 a_2)^6}\left[27 + a_1 a_2\,(4a_1 + 4a_2 - 18) - a_1^2 a_2^2\right] < 0$$

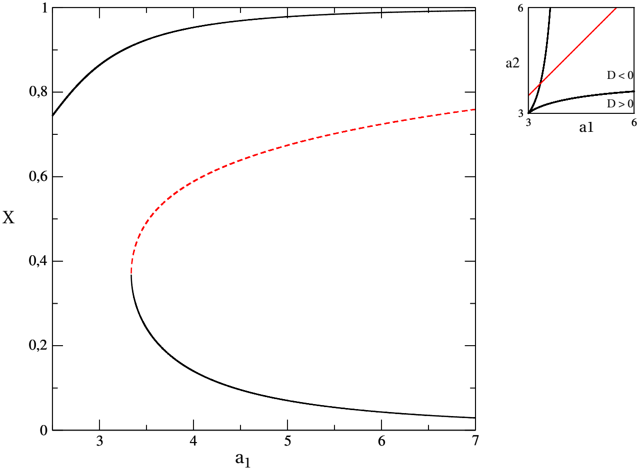
and if $D(a_1, a_2) = 0$ these two further fixed points are merging, i.e. there are two real coincident solutions of the cubic equation. In the particular case $a_1 a_2 = 1$ the unique fixed point $E = \left(\frac{1}{(a_1+1)^2}, \frac{1}{(a_2+1)^2}\right)$ is get.

Bifurcation path: $a_2 = -a_1/9 + 13/3$

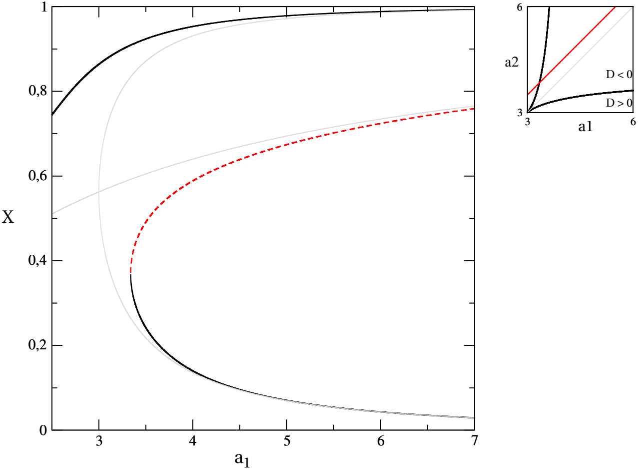




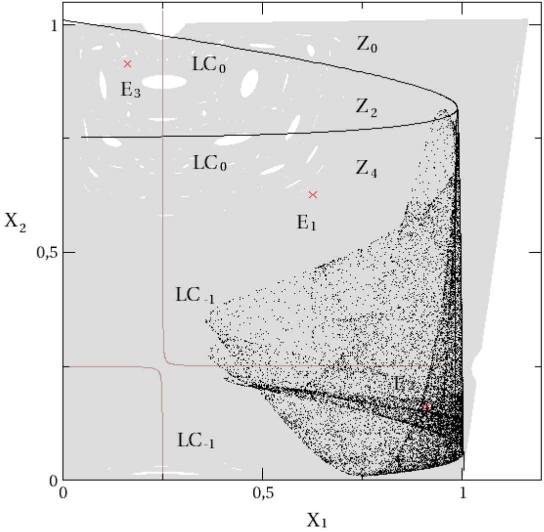
Bifurcation path: $a_2 = a_1 + 0.5$



Bifurcation path: $a_2 = a_1 + 0.5$, $a_2 = a_1$ (gray)



Typical scenario (after the *final bifurcation*)



The symmetric case: $a = a_1 = a_2$

Fixed poits are:

$$E_1 \quad = \left(\frac{a^2}{(1+a)^2}, \frac{a^2}{(1+a)^2} \right) \in \Delta = \{(x,y) \in \mathbb{R}^2 | x = y\}$$

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For $a \geqslant 3$ the two further fixed points in symmetric positions with respect to Δ:

$$E_2 = \frac{a^2}{2(a-1)^2(a+1)}\left(a - 1 + \sqrt{(a+1)(a-3)}, a - 1 - \sqrt{(a+1)(a-3)}\right)$$

$$E_3 = \frac{a^2}{2(a-1)^2(a+1)}\left(a - 1 - \sqrt{(a+1)(a-3)}, a - 1 + \sqrt{(a+1)(a-3)}\right)$$

The symmetric case: stability of $E_1 \in \Delta$

Local asymptotic stability for

$$a < a_p = 3$$

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Transverse instability (saddle) via pichfork, merging of two further fixed points E_2 and E_3, for

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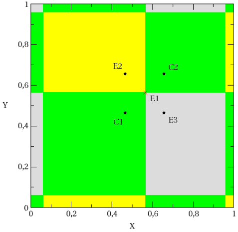
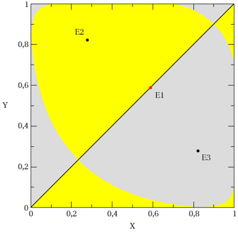
Unstability via flip for

$$a \geqslant a_f = 1 + 2\sqrt{1 - 2\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + \frac{4}{\lambda_1 \lambda_2}} \geqslant 3$$

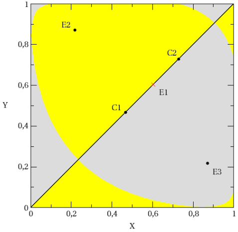
Fixed points and their basin of attraction



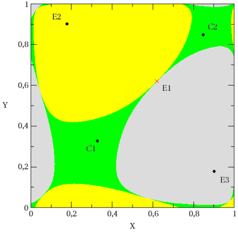
(a) Codim. 2 bifurcation



(b) Just after the pichfork



(c) Just after the flip



(d) Just after the subcritical flip

The symmetric case: stability of E_2 and E_3

Local asymptotic stability for

$$a < a_h = 1 + \sqrt[3]{2\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + 2\sqrt{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^2 - \frac{16}{27}}} + \sqrt[3]{2\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) - 2\sqrt{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^2 - \frac{16}{27}}}$$

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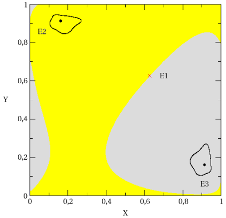
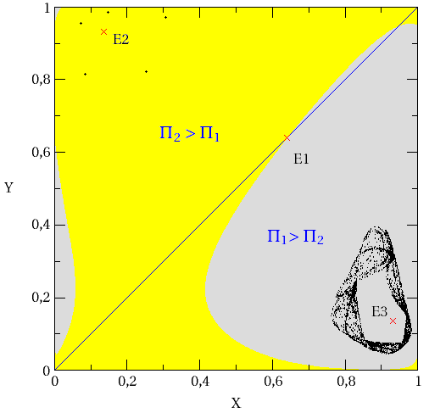
Unstability via Hopf for $a \geqslant a_h$



Profits

Computing the profits we get:

$$\Pi_1 > \Pi_2 \iff (x-y)(x+y-a^2) < 0$$



Conclusions: effect of eterogeneities

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  - Eterogeneities and initial conditions: asymptotic behaviour of the system

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