Heterogeneous population in binary choices with externalities

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Motivation and literature

Binary games with impulsive agents:

- Bischi, Gardini, Merlone (2009): *Discrete Dynamics in Nature and Society*
- Bischi, Gardini, Merlone (2009): *Journal of Dynamical Systems and Geometric Theories*

Extension to ternary games with impulsive agents and linear costs (Braess paradox):

- Dal Forno, Merlone (2013): *Mathematics and Computers in Simulation*
- Diback, Avrutin, Dal Forno: (*work in progress*)

Introduction of proportional agents (homogeneous population):

- Dal Forno, Merlone, Avrutin (2014): *Discrete Dynamics in Nature and Society*

Experiments and agent-based model (heterogeneity is necessary):

- Dal Forno, Merlone (2013): *Proceedings of the 2013 Winter Simulation Conference*

Our goal here:

When considering the heterogeneous population, which properties are inherited from the homogeneous populations, what is lost and what is gained?

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The formal model

The game:

- a repeated game
- a continuum of players chooses actions from $A = \{L, R\}$
- each player updates its choice at each time $t = 0, 1, 2, \ldots$
- the set of players is normalized to the interval $[0, 1]$

We introduce the following notation:

- $z_t^L \in [0, 1]$ denotes the fraction of players choosing action L at time t;
- $z_t^R \in [0, 1]$ denotes the fraction of players choosing action R at time t.

The formal model

Binary choices: when at any time t a fraction z_t^R of the population chooses action R, then a fraction $z_t^L = 1 - z_t^R$ chooses action L. The state of the system can be represented by

$$z = z^R \in [0, 1]$$

Cost functions are linear and depend on z:

- $L : [0, 1] \to \mathbb{R}$ is the cost associated to action L

$$L(z) = a_L + b_L z^L = a_L + b_L(1 - z)$$

- $R : [0, 1] \to \mathbb{R}$ is the cost associated to action R

$$R(z) = a_R + b_R z^R = a_R + b_R z$$

with $a_L, a_R, b_L, b_R > 0$.

The formal model

Agents are **cost minimizers** and **myopic**:

- If $L \succ R$ or, equivalently, $R\,(z_t) > L\,(z_t)$ then a fraction of the z_t agents who chose R switches to L
- If $L \prec R$ or, equivalently, $R\,(z_t) < L\,(z_t)$ then a fraction of the $(1 - z_t)$ agents who chose L switches to R

Consequently, we define the following intervals (or regions) where each strategy is dominant:

- $R_L = \{z^R \in [0,1] : L(z^R) < R(z^R)\}$
- $R_R = \{z^R \in [0,1] : L(z^R) > R(z^R)\}$

Dynamics with Homogeneous Population

The map with *impulsive* agents

Impulsive agents (X): the switching rate only depends on the sign of the difference between payoffs no matter how much they differ.

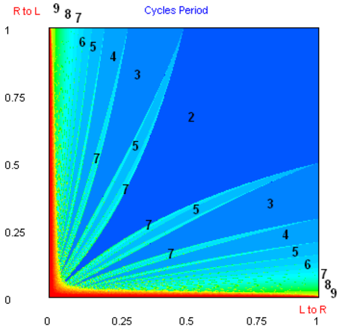
Taking into account that $x_t^L = 1 - x_t^R$, this is a 1D piecewise linear function with one discontinuity:

$$F_X : x_{t+1}^R = \begin{cases} (1-\delta_L)\, x_t^R & \text{if} \quad x_t^R \in R_L \\ (1-\delta_R)\, x_t^R + \delta_R & \text{if} \quad x_t^R \in R_R \end{cases}$$

where δ_R and δ_L model the fraction of players switching choice.

The map with *impulsive* agents

Proposition:
Given δ_L, δ_R, the map has only one attractor, a stable cycle of some period k, and any initial condition $x_0 \in [0,1]$ gives a trajectory converging to such k-cycle.



Bischi, Gardini, Merlone (2009a, 2009b)

The map with *proportional* agents

Proportional agents (Y): the switching rate depends not only on the sign of the difference between payoffs, but also on the relative difference between payoffs.

When $R \succ L$, the difference $L(y^R) - R(y^R)$ is normalized by the largest value of this difference, obtained in correspondence of $y^R = 0$:

$$L(0) - R(0) = a_L + b_L - a_R$$

When $R \prec L$, the difference $R(y^R) - L(y^R)$ is normalized by the largest value of this difference, obtained in correspondence of $y^R = 1$:

$$R(1) - L(1) = a_R + b_R - a_L$$

The map with *proportional* agents

Taking into account that $y_t^L = 1 - y_t^R$, this is a 1D piecewise smooth map:

$$F_Y : y_{t+1}^R = \begin{cases} (1 - \delta_L \rho_t^R) y_t^R & \text{if } y_t \in R_L \\ (1 - \delta_R \rho_t^L) y_t^R + \delta_R \rho_t^L & \text{if } y_t \in R_R \end{cases}$$

with

$$\rho_t^L = \frac{a_L + b_L - a_R - (b_L + b_R) y_t^R}{a_L + b_L - a_R}$$

and

$$\rho_t^R = \frac{a_R - a_L - b_L + (b_L + b_R) y_t^R}{a_R + b_R - a_L}$$

The map with *proportional* agents

Proposition:
Map F_y is continuous in $[0, 1]$ and (when feasible)

$$y^* = \frac{a_L - a_R + b_L}{b_L + b_R}$$

is the unique fixed point, which is globally stable – although it may be either locally stable or unstable.

Dal Forno, Merlone, Avrutin (2014)

Dynamics with Heterogeneous Population

Heterogeneous population

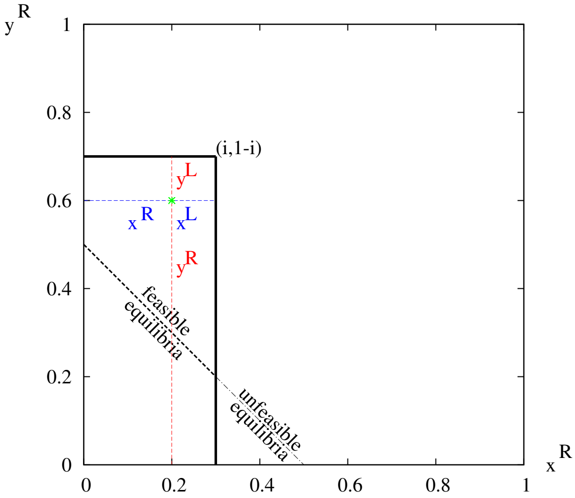
In the heterogeneous population agents compare the payoffs and react according to their own type, as:

$$(x^R_{t+1}, y^R_{t+1}) = (F_X(x^R_t), F_Y(y^R_t)) = \mathbf{F}(x^R_t, y^R_t)$$

with map $\mathbf{F} : [0, i] \times [0, 1-i] \to [0, i] \times [0, 1-i]$ defined as:

$$\mathbf{F}: \begin{cases} F_X : x^R_{t+1} = \begin{cases} (1-\delta_L)\, x^R_t \\ (1-\delta_R)\, x^R_t + \delta_R i \end{cases} & \text{if } x^R_t + y^R_t \in R_L \\ \\ F_Y : y^R_{t+1} = \begin{cases} (1-\delta_L \rho^R_t) y^R_t \\ (1-\delta_R \rho^L_t) y^R_t + \delta_R \rho^L_t (1-i) \end{cases} & \text{if } x^R_t + y^R_t \in R_R \end{cases}$$

A convenient graphical representation

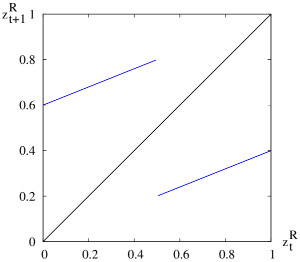


Example:

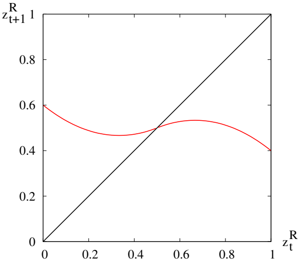
- $a_L = a_R = 27$
- $b_L = b_R = 24$
- $i = 0.3$

A convenient graphical representation

We represent the map using the aggregating variable $z^R := x^R + y^R$, which describes the state of the system.



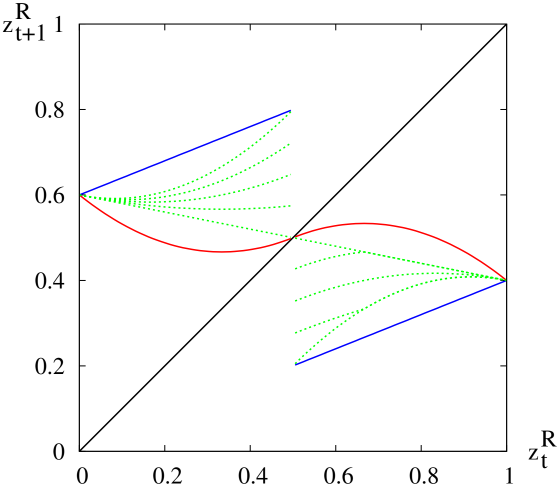
Homogenous impulsive



Homogenous proportional

- $a_L = 27$
- $a_R = 27$
- $b_L = 24$
- $b_L = 24$
- $\delta_L = 0.6$
- $\delta_R = 0.6$

A convenient graphical representation



- $a_L = 27$
- $a_R = 27$
- $b_L = 24$
- $b_L = 24$
- $\delta_L = 0.6$
- $\delta_R = 0.6$
- $i = 0.5$

Heterogeneous population: results

Proposition:
Assume $0 < z_t^R < 1$ and define

$$\alpha_t^L = \frac{x_t^L}{x^L + y_t^L} = \frac{i - x_t^R}{1 - z_t^R}, \qquad \alpha_t^R = \frac{x_t^R}{z_t^R}$$

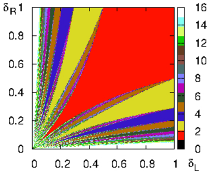
If $L\left(z_t^R\right) < R\left(z_t^R\right)$, then

$$z_{t+1}^R = \alpha^L \left[z_t^R + \delta_R \left(1 - z_t^R\right)\right] + \left(1 - \alpha^L\right) \left[z_t^R + \delta_R \rho_t^R \left(1 - z_t^R\right)\right];$$

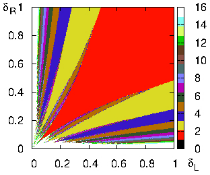
if $L\left(z_t^R\right) < R\left(z_t^R\right)$, then

$$z_{t+1}^R = \alpha^R \left[z_t^R - \delta_L \left(1 - z_t^R\right)\right] + \left(1 - \alpha^R\right) \left[z_t^R - \delta_L \rho_t^R \left(1 - z_t^R\right)\right].$$

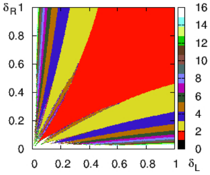
From impulsive to proportional behavior: what happens in between?



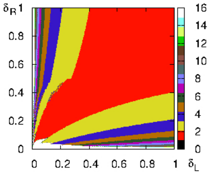
$i = 1.00$



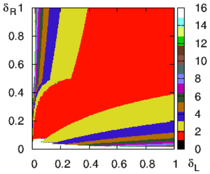
$i = 0.90$



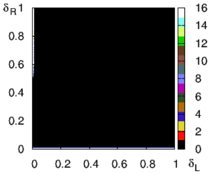
$i = 0.75$



$i = 0.25$



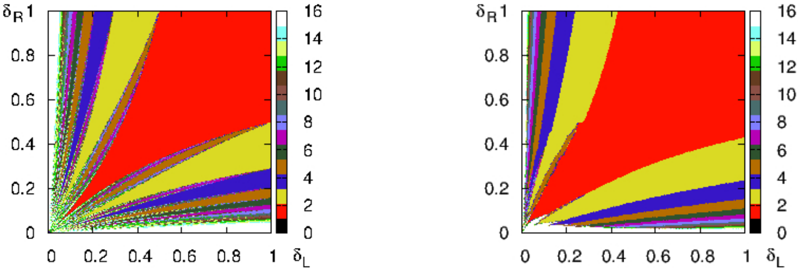
$i = 0.10$



$i = 0.00$

From impulsive to proportional behavior: what happens in between?

Consider $a_L = a_R = 27$, $b_L = b_R = 24$



$i = 1$ $\qquad$ $i = .5$

We still have cycles, but we have lost period adding structure and symmetry.

Period 2 cycles

To find 2-cycles we consider orbits belonging to R_L and R_R:

$$(\mathbf{x}, \mathbf{y}) = f_R\left(f_L\left(\mathbf{x}, \mathbf{y}\right)\right)$$

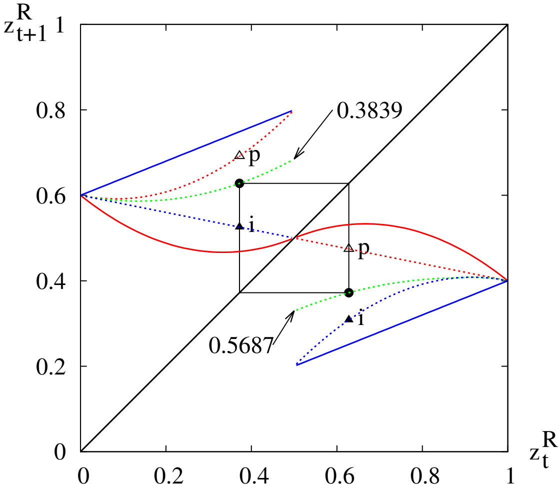
The orbit of a cycle of period two is given by the feasible solution $x^{R*} \in [0, i]$ and $y^{R*} \in [0, 1-i]$ of the system

$$\begin{cases} x^R = \frac{\delta_R}{\delta_L(1-\delta_R)+\delta_R} i \\ \gamma_0 \left(y^R\right)^4 + \gamma_1 \left(y^R\right)^3 + \gamma_2 \left(y^R\right)^2 + \gamma_3 y^R + \gamma_4 = 0 \end{cases}$$

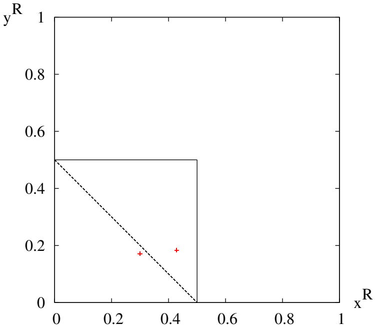
where $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4$ can be computed from x^R and the map parameters: $a_L, b_L, a_R, b_R, i, \delta_L, \delta_R$.

Period 2 cycles: an example

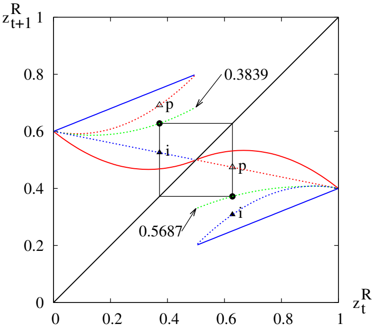
In general, in the two regions, the ratios of impulsive agents determine different α and the different heterogeneous population dynamics which are selected



Period 2 cycles: an example

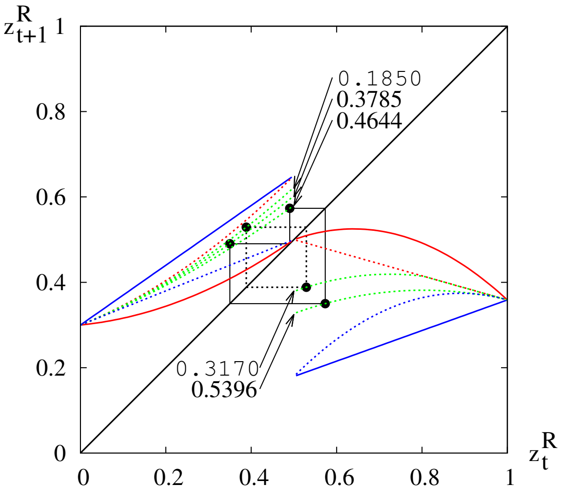


State space

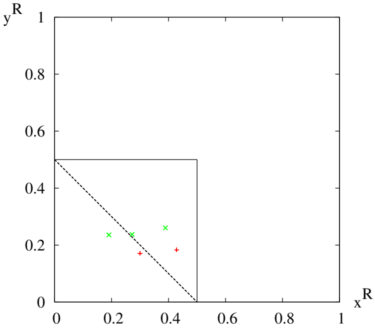


Map

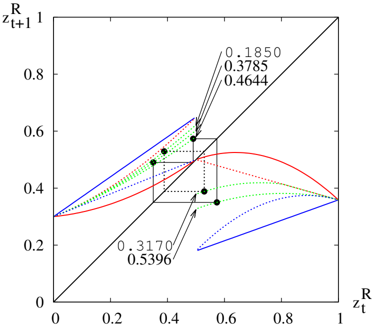
Coexistence: an example



Coexistence: an example



State space



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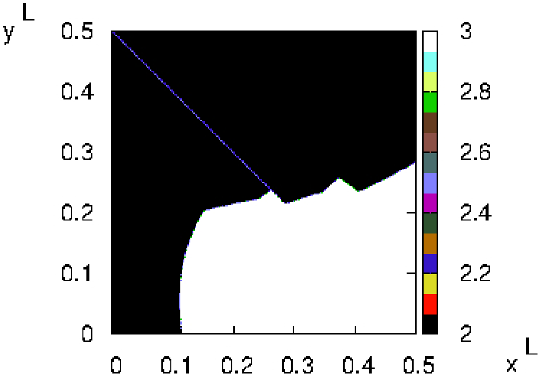

Depending on the initial condition we have either a 2- or a 3-cycle

Conclusion

Conclusion

With heterogeneous population the dynamics is much more complex

- the dynamics of the heterogeneous population is a convex linear combination of the homogeneous populations dynamics
- even a small percentage of impulsive agents qualitatively rules the dynamics

What is inherited?

- We still have cycles: overshooting is still persistent

What is lost?

- We lose period adding and symmetry: less elegant mathematical structure

What is gained?

- We have coexistence: the initial condition matters