A model of ecological dynamics

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Abstract

What we study is a model of a small open economy with two sectors (a traditional, dependent on environmental resources, and an industrial one), where the inter-sectoral labour mobility is regulated by the wage rate and, moreover, the workers' welfare is affected by how much each sector contributes to natural resources depletion. All these interactions produce a dynamics, which, in our model, takes place in a 3-dimensional box. In fact, if the carrying capacity of environmental resources lies in an interval determined by the total number of the workers, there exists exactly one equilibrium, which, for certain values of the parameters, turns out to be attractive (i.e. a sink). However, even in this case, there are trajectories leading, in a finite time, to the boundary of the box, i.e. to situations where the stock of environmental resources or the number of workers employed in the traditional sector are zero. Then the question arises about the dynamics when the unique equilibrium is a saddle endowed with a 2-dimensional stable manifold. In such a case we prove that the box can be separated in two regions, in one of which the trajectories leave in a finite time the box, while in the other they tend, in an infinite time, to a *boundary equilibrium*. Finally we illustrate new dynamical patterns which can take place when the system is modified so as not to allow the trajectories *assume negative values* of N (number of workers employed in the traditional sector).

Structural change and workers' welfare

- Suppose you have an open economy with two sectors (a traditional and an industrial one), inter-sectoral labor mobility (caused by the wage rate) and heterogeneous agents (workers and entrepreneurs).
- Then the question arises: how a structural change (SC) in labor employment affects the welfare of the workers?

Ecological dynamics

- Clearly the answer depends on whether the traditional , dependent on free-access environmental resources (E), or the industrial (I) sector more contributes to deterioration of natural resources.
- Therefore a dynamics takes place and can be studied.

The model

- Aggregate production functions:

$$Y_I = \left(\overline{N} - N\right)^{\alpha} K^{1-\alpha}$$

$$Y_E = \eta N E$$

0<α<1, β>0, while $N \in \left[0, \overline{N}\right]$ represents the labor force employed in the E-sector. Moreover let:

$$0 \le K \le \overline{K},\ 0 \le E \le \overline{E}$$

and let

$$w = \alpha\left(\overline{N} - N\right)^{\alpha-1} K^{1-\alpha}$$

be the wage rate in the industrial sector.

An adaptive dynamics

We assume that the inter-sectoral mobility produced by the wage rate gives place to the following three-dimensional dynamics

$$\dot{K} = s(1-\alpha)(\overline{N}-N)^{\alpha} K^{1-\alpha} - dK$$

$$\dot{N} = \gamma\left(\eta E - (\overline{N}-N)^{\alpha-1} K^{1-\alpha}\right)$$

$$\dot{E} = E\left(\overline{E} - E\right) - \delta\eta N E - \varepsilon (\overline{N}-N)^{\alpha} K^{1-\alpha}$$

After a suitable rescaling the system becomes

Rescaled system

$$\dot{K} = l\left[\left(\overline{N} - N\right)^{\alpha} K^{1-\alpha} - K\right]$$

$$\dot{N} = l\left(E - \frac{K^{1-\alpha}}{\left(\overline{N} - N\right)^{1-\alpha}} \right)$$

$$\dot{E} = E\left(\overline{E} - E - pN\right) - q\left(\overline{N} - N\right)^{\alpha} K^{1-\alpha}$$

where *l,p,q>0.* We analyse the system in the box

$$B = \left(0, \overline{K}\right] \times \left[0, \overline{N}\right) \times \left[0, \overline{E}\right] \text{ with } \overline{K} = \overline{N}$$

Local analysis

It is easily seen that a (unique) equilibrium exists in *B* if and only if

$$\overline{E} = (\lambda p + (1-\lambda)q)\overline{N} + 1, \; 0 < \lambda < 1$$

given by

$$P^* = ((1-\lambda)\overline{N}, \lambda\overline{N}, 1)$$

Then the local analysis yields that the equilibrium is a sink or a saddle (with one-dim. stable manifold) if *p>q*, whereas it is a source or a saddle with two-dim. stable manifold if *q>p*. All the four cases can occur.

A positively invariant region

Looking at the system clearly it emerges that there exist trajectories reaching, in a finite time, the boundary of the box, i.e. the sides $E=0$ or $N=0$.

On the other hand, if the unique equilibrium exists and is a sink, there is an open subset of the box constituted by trajectories remaining in B for any $t>0$ (say a positively invariant region).

So the question arises: what happens if the unique equilibrium is a saddle with two-dim. stable manifold? Does there still exist a positively invariant region?

Main Theorem

The following theorem provides a positive answer to the previous question.

Let

$$\overline{E} = (\lambda p + (1-\lambda)q)\overline{N} + 1,\ 0 < \lambda < 1$$

with *q>p>0*. Hence the unique equilibrium in the box is a source or a saddle with a two-dimensional manifold.

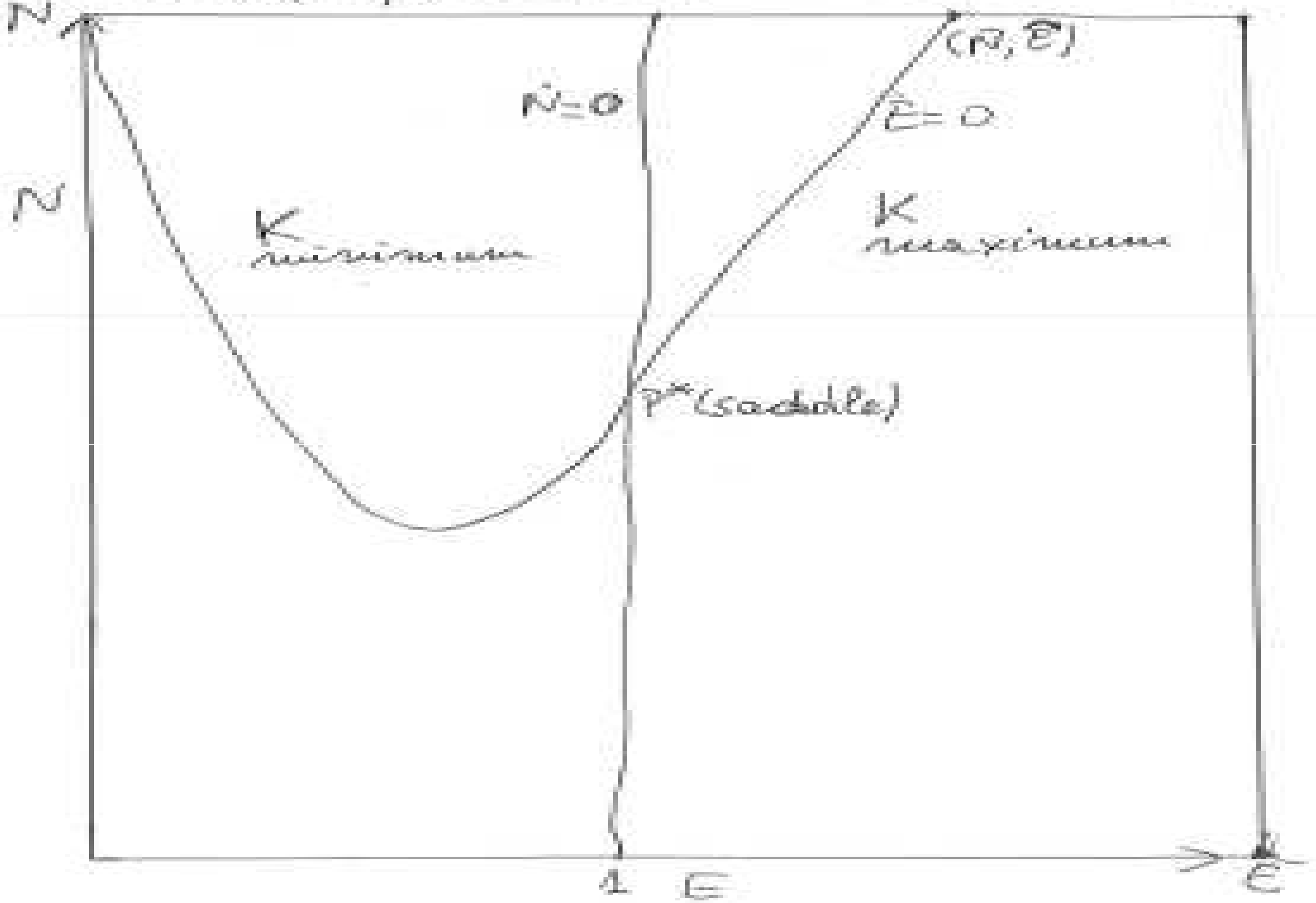
Then the following theorem holds

Theorem *All the trajectories starting from points of the segment*

$$\Sigma = \{K = \overline{N} - N, \lambda\overline{N} < N < \overline{N}, E = 1\}$$

converge to $\hat{P} = (0, \overline{N}, \hat{E}), \hat{E} = \overline{E} - p\overline{N}$ *as* $t \to +\infty$.

The case q > p. The plane is K + N = N̄



Consequences and Conjectures

The above Theorem implies that there exists a positively invariant region (containing a neighborhood of Σ) and characterizes the dynamics in such a region. Moreover it says that the equilibrium, saddle or source, belongs to the boundary of this region. Such facts support the following

Conjecture 1 *Suppose the unique equilibrium is a saddle endowed with a two-dimensional manifold. Then such a manifold separates the box in two sub-regions: in one of them the trajectories tend to the boundary point* $\hat{P}$ *as* $t \to +\infty$*, while in the other they leave the box in a finite time.*

A second conjecture, based on the same arguments, concerns the nature of a Hopf bifurcation through which the above saddle becomes a source.

Conjecture 2 *Whenever such a Hopf bifurcation occurs, it is supercritical, i.e. a cycle endowed with a two-dimensional stable manifold arises.*

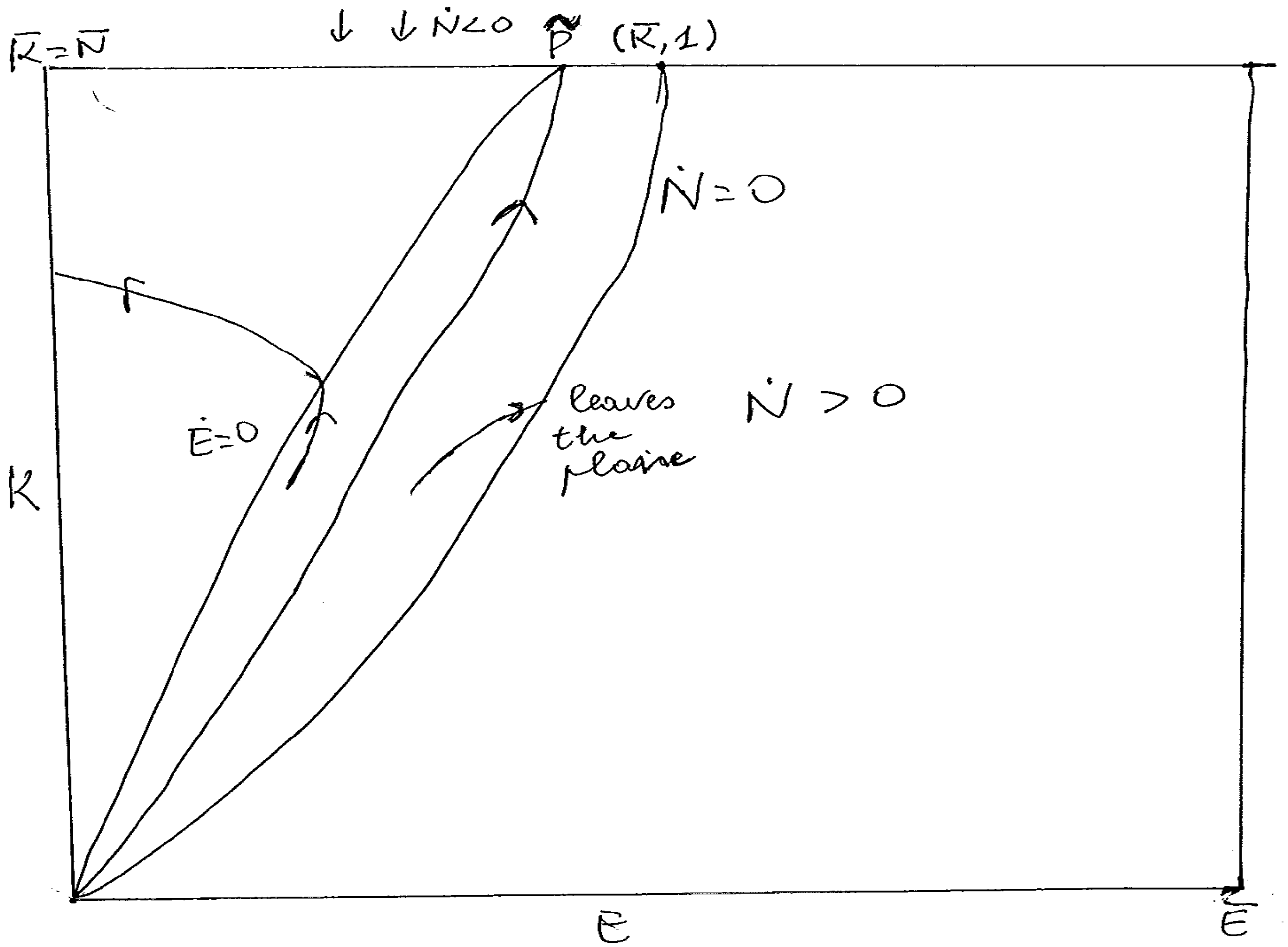
Dynamics on *N=0*

A classic modification of the above system is based on the assumption that a dynamics takes place also when no worker is employed in the traditional sector (i.e. *N=0*), in the sense that $\dot{N} < 0$ is replaced by $\dot{N} = 0$, while $\dot{N} > 0$ remains invaried. Usually such a modification is accomplished by introducing a discontinuity in the vector field. We prefer, instead, to reach the same goal by a C_∞ *(*except along a curve where the v.f. is lipschitzian) modification in an arbitrarily small neighborhood of *N=0.*

A saddle on the boundary

The reason is that, this way, new interesting dynamic patterns can emerge.

In particular assume *p>q* and an equilibrium *P** to lie in the interior of the box. Then it can be seen that the above smooth modification generates a saddle $\tilde{P} = (\overline{N}, 0, \tilde{E}), 0 < \tilde{E} < 1$, endowed with a two-dimensional stable manifold.



DYNAMICS ON N = 0

Other dynamic patterns

It is reasonable to conjecture that the stable manifold of the new *boundary saddle* separates the box in two regions, in one of which the trajectories remain in the box, while in the other one they leave the box through the plane *E=0* (in a finite time). However the dynamics in the positively invariant region can exhibit new patterns. For example, let

$$l = 0.7, p = 2.5, q = 1.5, \overline{N} = 2, \overline{E} = 5.3, \alpha = 0.9$$

Then numerical simulations suggest that the interior sink *P** has a *cylinder-shaped* basin, which, in its turn, is surrounded by an attracting cycle.

Conclusions

Hence, in such a case, three regimes would occur: in one the natural resources (*E*) run out in a finite time; in another one the trajectories converge to an equilibrium inside the box; finally, in the third one, large oscillations prevail and go on.

We conclude by observing that the present model is a limit case of one considered, but assuming an istantaneous adjustment of *N,* in an article by Antoci, Russu, Sordi and Ticci, where the production function in the traditional sector is a classic Cobb-Douglas, i.e. $Y_E = \eta N^{\beta} E^{\gamma}$, $0<\beta,\gamma<1$.

Do similar dynamic patterns take place in this more general context?