On the Risk Evaluation Method Based on the Market Model

Masaaki Kijima

Graduate School of Social Sciences, Tokyo Metropolitan University

September 18, 2014

This Talk is Mainly Based on

- Kijima, M. and Muromachi, Y. (2014), "On the risk evaluation method based on the market model," in this volume.

and a continuation of researches such as

1. Kijima, M., Suzuki, Y. and Tamba, Y. (2014), "Risk evaluation of mortgage-loan portfolios under low interest-rate environment," *Journal of Risk*, **16** (**5**), 3–37.
2. Kijima, M., Tanaka, K. and Wong, T. (2009), "A multi-quality model of interest rates," *Quantitative Finance*, **9**, (**2**), 133–145.
3. Kijima, M. and Muromachi, Y. (2000), "Evaluation of credit risk of a portfolio with stochastic interest rate and default processes," *Journal of Risk*, **3** (**1**), 5–36.

Plan of My Talk

1. Introduction: Motivation, Methodology
2. Interest-Rate Models: Literature review, This paper
3. Principal Component Analysis
4. The Change of Measure
   - General case
   - Possible models
5. Summary

Motivation

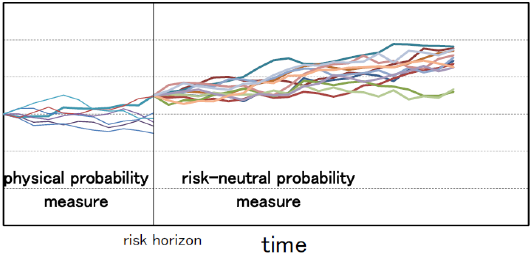
- Risk management has become more important (and been changed) than ever after the credit crunch.
- In particular,
  1. Low interest-rate environment is common all over the world.
  2. Multi-curve pricing approach has been adopted in the OTC market.
- It is important to develop new methodology for the risk evaluation purposes under these circumstances.
- To this end, we need to develop a risk evaluation model for interest-rate sensitive products which is
  1. consistent with empirical results
  2. within the no-arbitrage framework

General Methodology (1)

- Consider the risk evaluation of a portfolio at future time T.
  - T: risk horizon (one day, one-year, $\ldots$).
  - Risk Measures: VaR (Value at Risk), ES (Expected Shortfall),
- Kijima and Muromachi (2000) proposed a general framework for evaluating the financial risk.
- This paper also follows this approach: Namely,
- Risk factors (interest rates, hazard rates, stock prices, $\ldots$) are described by stochastic differential equations (SDEs).
- No-arbitrage prices are used for valuation at present and future.

General Methodology (2)

- Two probability measures are used.
  1. Physical measure $\mathbb{P}$ and pricing measure $\mathbb{Q}$
  2. $\mathbb{P}$ is used to generate scenarios up to T.
  3. $\mathbb{Q}$ is used to price assets at $[0, T]$
- Two measures $\mathbb{P}$ and $\mathbb{Q}$ are connected through the change of measure (or the market price of risk $\lambda(t)$).



Interest-Rate Models

1. Short rate models: Vasicek, CIR, Quadratic Gaussian, Affine
   - Easy to generate scenarios under $\mathbb{P}$
   - Easy to construct a model under the pricing measure $\mathbb{Q}$
   - Resulting term structures are poor to fit market yield curves
   - In particular, poor to explain empirical results for low interest rate environments $\Rightarrow$ Kijima et al. (2014)
   - Also, poor to explain empirical results for the multi-curve setting $\Rightarrow$ Need more work
2. Complicated models: Black's shadow rate, Vasicek with sticky boundary
   - Can explain S-shaped yield curves
   - Calibration is difficult

Interest-Rate Models: Term structure model

1. Market models: HJM, BGM
   - Suitable for the pricing of interest-rate derivatives
   - No attention to the empirical results of yield curve under $\mathbb{P}$
   - More work is needed for the multi-curve setting
2. Econometric models: PCA
   - Suitable for modeling the real world evolution of the term structure
   - Can provide point forecasts of yield-curve changes
   - Usually no attention to the arbitrage-free pricing
   - More work is needed for the multi-curve setting

Key Issues

- Under the physical measure $\mathbb{P}$, many empirical studies show that
  1. the movement of the yield curve can be well explained by Principal Component Analysis (PCA), and
  2. the dominant 3 factors are level, slope, and curvature.
     $\longrightarrow$ PCA should be a desirable starting block of the interest-rate risk evaluation model under $\mathbb{P}$.
- Under the pricing measure $\mathbb{Q}$,
  1. no-arbitrage framework is used for pricing.
  2. theoretical prices should be consistent with market prices.
     $\longrightarrow$ Market model may be a suitable model under $\mathbb{Q}$.

Purpose of This Study

In order to propose a better interest-rate risk evaluation model,

- We consider consistency of a market model under $\mathbb{Q}$ with the PCA model under $\mathbb{P}$.

Previous study: Norman (2009) considered the yield curve model under $\mathbb{P}$ consistent with the no-arbitrage framework.

1. Start from no-arbitrage market model under $\mathbb{Q}$ to construct an SDE under $\mathbb{P}$ by introducing the market price of risk $\boldsymbol{\lambda}(t)$.
2. Parameters of $\boldsymbol{\lambda}(t)$ are estimated by the idea of PCA.

In This Study

We take the reversed approach of Norman (2009), i.e.,

- We derive conditions for the no-arbitrage market models under $\mathbb{Q}$ to be consistent with the PCA model under $\mathbb{P}$. Namely,
  1. Start from an SDE consistent with PCA under $\mathbb{P}$.
  2. Introduce the market price of risk $\boldsymbol{\lambda}(t)$.
  3. Construct a market model under $\mathbb{Q}$.
  4. Impose the no-arbitrage condition on the SDE under $\mathbb{Q}$.
  5. And, we search some simple examples.

Notation

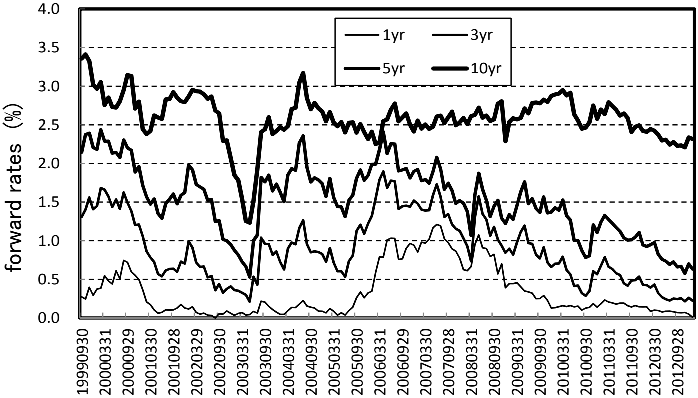
- t : time.
- $v(t,T)$: time-t price of the discount bond with maturity $T \geq t$
- $D_t(x) = v(t,t+x)$, $x > 0$: discount curve at t
- $R_t(x)$, $x \geq 0$: forward rates

$$R_t(x) = -\frac{\partial}{\partial x} \log D_t(x); \quad D_t(x) = \exp\left\{-\int_0^x R_t(u)\mathrm{d}u\right\}$$

- $L_t(x) = \log R_t(x)$: log forward rates
- $Y_t(x) = \sqrt{R_t(x)}$: square root of forward rates

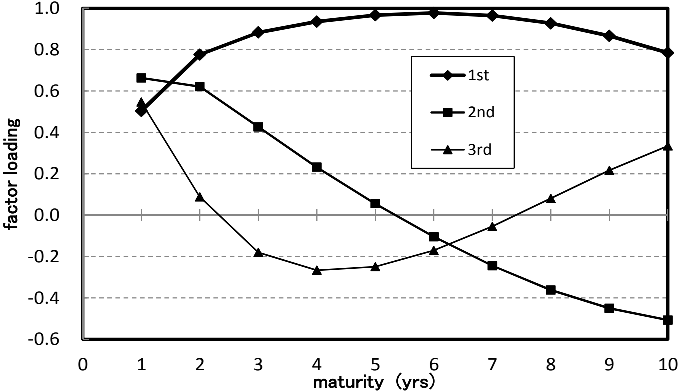
JGB Data

Forward Rates of JGBs (Japanese Government Bonds) at the end of months from September 1999 to January 2013.



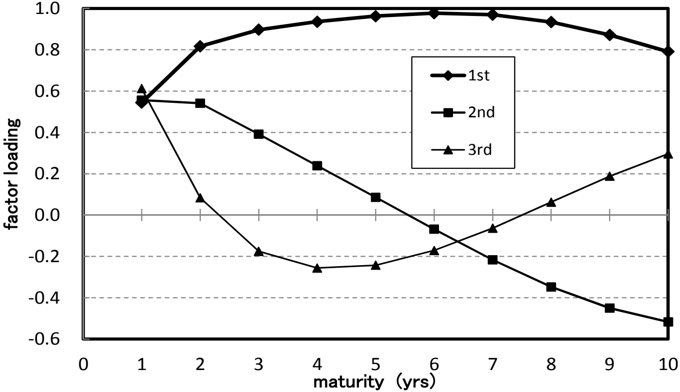
PCA for Forward Rates

Dominant three components of JGB forward rates movements. (Forward Rates: $\boldsymbol{R_t}(\boldsymbol{x})$)



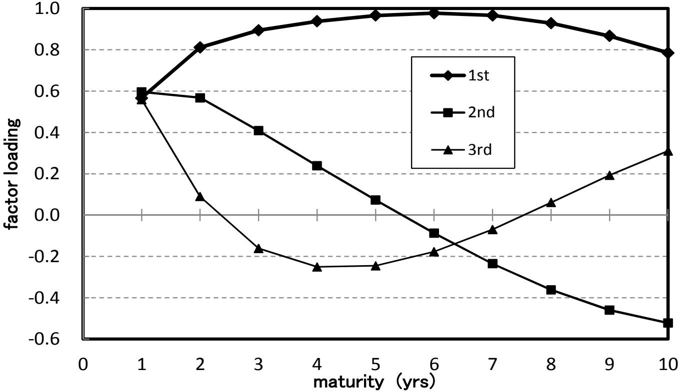
PCA for Log-Forward Rates

Dominant three components of JGB forward rates movements. (Log Forward Rates: $L_t(x) = \log R_t(x)$)



PCA for Square Root of Forward Rates

Dominant three components of JGB forward rates movements. (Square Root of Forward Rates: $Y_t(x) = \sqrt{R_t(x)}$)



Summary of PCA

Results are summarized as follow.

- Dominant three components are similar in all cases.
- The first component contributes about 84% of the movements, and the dominant three components contribute over 99%.
- The first component is not flat because the yield curve are typically S-shaped (the short rates are locked near zero) during the zero interest-rate policy (ZIRP).
- The second component is interpreted to be the slope, and the third is the curvature, as usual.

Yield-Curve Modeling

- Suppose that $\boldsymbol{R_t(x)}$ follows the SDE

$$\mathrm{d}\boldsymbol{R_t(x)} = \boldsymbol{\mu^r(t,x)}\mathrm{d}\boldsymbol{t} + \sum_{\boldsymbol{i}} \boldsymbol{\sigma_i^r(t,x)}\mathrm{d}\boldsymbol{w_{i,t}} \tag{1}$$

  where $\boldsymbol{w_{i,t}}$ are independent standard Brownian motions (SBMs) under $\mathbb{P}$.

- As in Brace et al. (1997), we assume

$$\mathrm{d}\boldsymbol{R_t(x)} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}\left[\left(\boldsymbol{R_t(x)} + \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\sigma^2(t,x)}\right)\mathrm{d}\boldsymbol{t} + \sum_{\boldsymbol{i}} \boldsymbol{\sigma_i(t,x)}\mathrm{d}\boldsymbol{w^*_{i,t}}\right] \tag{2}$$

  where $\boldsymbol{w^*_{i,t}}$ are independent SBMs under $\mathbb{Q}$, $\boldsymbol{\sigma_i(t,0) = 0}$, and $\boldsymbol{\sigma^2(t,x) = \sum_i \sigma_i^2(t,x)}$.

Why the Model (2) ?

- Note that, by definition, we have

$$\log v(t,T) = -\int_0^{T-t} R_t(u)\mathrm{d}u \tag{3}$$

- After some algebra using (2) and (3) , we obtain

$$\frac{\mathrm{d}v(t,T)}{v(t,T)} = R_t(0)\mathrm{d}t + \sum_i \sigma_i(t,T-t)\mathrm{d}w^*_{i,t} \tag{4}$$

  which means that the denominated discount bond price is a martingale under $\mathbb{Q}$.

- Hence, $\mathbb{Q}$ is a risk-neutral probability measure, and the model (2) is mandatory under $\mathbb{Q}$.

Market Price of Risk

Consider the model (1) under $\mathbb{P}$, and let $\boldsymbol{\lambda_i(t)}$ be the market price of risk associated with $\boldsymbol{w_{i,t}}$, that is,

$$\mathbf{d}\boldsymbol{w^*_{i,t}} = \mathbf{d}\boldsymbol{w_{i,t}} - \boldsymbol{\lambda_i(t)}\mathbf{d}\boldsymbol{t}$$

Then, the model (1) can be written as

$$\mathbf{d}\boldsymbol{R_t(x)} = \boldsymbol{\nu^r(t,x)}\mathbf{d}\boldsymbol{t} + \sum_{\boldsymbol{i}} \boldsymbol{\sigma^r_i(t,x)}\mathbf{d}\boldsymbol{w^*_{i,t}} \tag{5}$$

under $\mathbb{Q}$, where

$$\boldsymbol{\nu^r(t,x) = \mu^r(t,x)} + \sum_{\boldsymbol{i}} \boldsymbol{\sigma^r_i(t,x)\lambda_i(t)}$$

Main Result

Theorem

The model (1) under $\mathbb{P}$ can be consistent with the arbitrage-free model under $\mathbb{Q}$, if there exist $\boldsymbol{\lambda}_i(t)$ that satisfy the condition

$$\sum_i \boldsymbol{\lambda}_i(t)\boldsymbol{\sigma}_i^r(t,x) = \frac{\partial}{\partial x}R_t(x) - \mu^r(t,x) + \sum_i \boldsymbol{\sigma}_i(t,x)\boldsymbol{\sigma}_i^r(t,x)$$

for all $x > 0$, where

$$\boldsymbol{\sigma}_i(t,s) = \int_0^s \boldsymbol{\sigma}_i^r(t,u)\mathrm{d}u$$

Sketch of the Proof

From (3) and (5) and some algebra, we obtain

$$\begin{aligned}\frac{\mathrm{d}v(t,T)}{v(t,T)} &= \left(R_t(T-t) - \int_0^{T-t} \nu^r(t,u)\mathrm{d}u + \frac{1}{2}\sigma^2(t,T-t)\right)\mathrm{d}t \\ &\quad - \sum_i \sigma_i(t,T-t)\mathrm{d}w^*_{i,t} \qquad (6)\end{aligned}$$

The two equations, (6) and (4), coincide if the drift terms coincide, i.e.,

$$R_t(T-t) - \int_0^{T-t} \nu^r(t,u)\mathrm{d}u + \frac{1}{2}\sigma^2(t,T-t) = R_t(0)$$

Notice that $\lambda_i(t)$ can be stochastic, but can not depend on x (and T).
$\longleftarrow$ might be a strong constraint when constructing a consistent model

Plausible Models

As plausible specifications, we consider

- Forward Rate Case :

$$\mathrm{d}R_t(x) = a(m(x) - R_t(x))\mathrm{d}t + \sum_{i=1}^{3} F_i(x)\mathrm{d}w_{i,t} \tag{7}$$

  The solution is given as

$$R_t(x) = m(x) + (R_0(x) - m(x))\mathrm{e}^{-at} + \sum_{i=1}^{3} F_i(x)\int_0^t \mathrm{e}^{-a(t-s)}\mathrm{d}w_{i,s}$$

- Log Forward Rate Case :

$$\mathrm{d}L_t(x) = a^{\ell}(m^{\ell}(x) - L_t(x))\mathrm{d}t + \sum_{i=1}^{3} F_i^{\ell}(x)\mathrm{d}w_{i,t} \tag{8}$$

- Quadratic Gaussian Case : $R_t(x) = (y_t(x) + m^y(x))^2$ where

$$\mathrm{d}y_t(x) = -a^y y_t(x)\mathrm{d}t + \sum_{i=1}^{3} F_i^y(x)\mathrm{d}w_{i,t} \tag{9}$$

Forward Rate Model

Corollary

The PCA model (7) under $\mathbb{P}$ can be consistent with the arbitrage-free model (2), if there exist $\boldsymbol{\lambda_i(t)}$ that satisfy the condition

$$\begin{aligned}\sum_{i=1}^{3} \lambda_i(t) F_i(x_k) &= \frac{\partial}{\partial x}\left(m(x_k) + (R_0(x_k) - m(x_k))\mathrm{e}^{-at}\right) \\ &\quad + a(R_0(x_k) - m(x_k))\mathrm{e}^{-at} + \sum_{i=1}^{3} F_i(x_k) \int_0^{x_k} F_i(u)\mathrm{d}u \\ &\quad + \sum_{i=1}^{3} (f_i(x_k) + aF_i(x_k)) \int_0^t \mathrm{e}^{-(t-s)} \mathrm{d}w_{i,s}, \quad k = 1, 2, 3\end{aligned}$$

for all $x_1 < x_2 < x_3$ where $f_i(x) = \mathrm{d}F_i(x)/\mathrm{d}x$.

Single Factor Forward Rate Model

- We consider the single factor case.
- From the corollary, we have

$$\begin{aligned}\lambda_1(t) &= \frac{1}{F_1(x)}\left[\frac{\partial}{\partial x}\left(m(x)+(R_0(x)-m(x))\mathrm{e}^{-at}\right)\right.\\ &\qquad\left.+\ a(R_0(x)-m(x))\mathrm{e}^{-at}\right]\\ &\quad+\int_0^x F_1(u)\mathrm{d}u+\left(\frac{f_1(x)}{F_1(x)}+a\right)\int_0^t \mathrm{e}^{-a(t-s)}\mathrm{d}w_{1,s}\end{aligned}$$

- Given the volatility curve $F_1(x)$ and the initial forward-rate curve $R_0(x)$, the mean-reverting level $m(x)$ can be solved to satisfy the above equation.

$\Rightarrow$ At least, such a $\lambda_1(t)$ exists as the next slide shows.

Forward Rate Case: Single Factor, Cont.

Suppose that $F_1(x) = F_1(0)\mathrm{e}^{bx}$ for constants b, $b \neq 0, -a$, we find a solution

$$
\begin{aligned}
m(x) &= m(0) + \frac{\bar{\lambda}_1 F_1(0)}{b}\left(\mathrm{e}^{bx} - 1\right) - \frac{F_1^2(0)}{2b^2}\left(\mathrm{e}^{bx} - 1\right)^2 \\
R_0(x) &= m(x) + (R_0(0) - m(0))\mathrm{e}^{-ax} \\
&\quad + \frac{(\lambda_1(0) - \bar{\lambda}_1)F_1(0)}{a+b}\left(\mathrm{e}^{bx} - \mathrm{e}^{-ax}\right) \\
\lambda_1(t) &= \bar{\lambda}_1 + (\lambda_1(0) - \bar{\lambda}_1)\mathrm{e}^{-at} + (a+b)\int_0^t \mathrm{e}^{-a(t-s)}\mathrm{d}w_{1,s}
\end{aligned}
$$

(When $b = 0$ or $b = -a$, a more simple solution is obtained.)

Log Forward Rate Case

Similarly, consider the log forward rate case (8) with a single factor. Then, the constraint is given as

$$\begin{aligned}\lambda_1(t) \;=\; & \frac{1}{F_1^\ell(x)}\left[\frac{\partial}{\partial x}\left(m^\ell(x) + (L_0(x) - m^\ell(x))\mathrm{e}^{-a^\ell t}\right)\right. \\ & \left. + \; a^\ell (L_0(x) - m^\ell(x))\mathrm{e}^{-a^\ell t}\right] - \frac{1}{2}F_1^\ell(x) \\ & + \int_0^x F_1^\ell(u) R_t(u)\mathrm{d}u + \left(\frac{f_1^\ell(x)}{F_1^\ell(x)} + a^\ell\right)\int_0^t \mathrm{e}^{-a^\ell(t-s)}\mathrm{d}w_{1,s}\end{aligned}$$

So far, we cannot find any solutions yet (may be no solutions!)

Quadratic Gaussian Case

Similarly, consider the quadratic Gaussian case (9) with a single factor. Then, the constraint is given as

$$
\begin{aligned}
\lambda_1(t) \quad = \quad & \frac{1}{F_1^y(x)}\left[\frac{\partial}{\partial x}\left(y_0(x)\mathrm{e}^{-a^y t}+m^y(x)\right)+a^y y_0(x)\mathrm{e}^{-a^y t}\right] \\
& +2\mathrm{e}^{-a^y t}\int_0^x F_1^y(u)y_0(u)\mathrm{d}u+2\int_0^x F_1^y(u)m^y(u)\mathrm{d}u \\
& -\frac{1}{2}\left(\frac{y_0(x)\mathrm{e}^{-a^y t}+m^y(x)}{F_1^y(x)}+\int_0^t \mathrm{e}^{-a^y(t-s)}\mathrm{d}w_{1,s}\right)^{-1} \\
& +\left(\frac{f_1^y(x)}{F_1^y(x)}+2\int_0^x (F_1^y(u))^2\mathrm{d}u+a^y\right)\int_0^t \mathrm{e}^{-a^y(t-s)}\mathrm{d}w_{1,s}
\end{aligned}
$$

So far, we cannot find any solutions yet (may be no solutions!)

Summary

We present a risk evaluation model for interest-rate sensitive products within the no-arbitrage framework.

- Under the physical measure $\mathbb{P}$, a yield curve dynamics is modelled based on the results of the principal component analysis (PCA).
- Under the risk-neutral measure $\mathbb{Q}$, the market model is adopted for pricing interest-rate derivatives.
- We derive a sufficient condition for the two models to be consistent.
- We find a simple consistent model, although some market models often used in practice do not satisfy the condition.

Future Research

1. Develop an algorithm that can search consistent models in the multi-factor setting.
2. Econometric models better to fit the yield curve dynamics under the current environment.
3. Extend the model to the multi-curve setting.
4. Many others

Thank You for Your Attention