Evolutionary Cournot Models with Limited Market Knowledge

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Classical Cournot oligopoly model

Market with N firms $i = 1, ..., N$

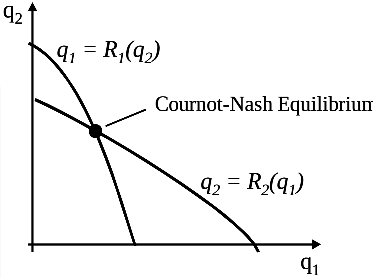
Inverse demand $p = f(Q)$, with $Q = \sum_{i=1}^{N} q_i$

Cost functions $C_i(q_i)$, $i = 1, ..., N$

$$q_i(t+1) = \arg\max_{q_i} [f(q_i + Q^e_{-i}(t+1))\, q_i - C_i(q_i)]$$

In simplest cases:

$q_i(t+1) = R_i\left(Q^e_{-i}(t+1)\right)$



Info set, rationality, computational ability

- Demand function $p = f(Q)$;
- Its own cost function $C_i(q_i)$
- Perfect Foresight
  $Q^e_{-i}(t+1) = Q_{-i}(t+1)$;
- Ability to solve the optimization problem

Best Reply (BR) with isoelastic demand and linear costs

$$q_i(t+1) = \arg\max_{q_i} \frac{q_i}{q_i + Q^e_{-i}(t+1)} - c_i q_i$$

naïve expectations: $Q^e_{-i}(t+1) = Q_{-i}(t)$

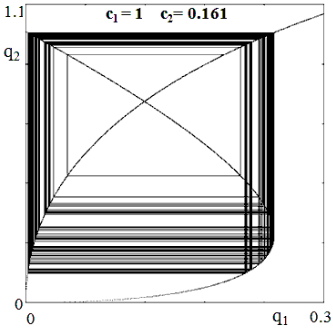
Duopoly

$$q_1(t+1) = R_1(q_2(t)) = \sqrt{\frac{q_2(t)}{c_1}} - q_2(t)$$

$$q_2(t+1) = R_2(q_1(t)) = \sqrt{\frac{q_1(t)}{c_2}} - q_1(t)$$

Proposition: Stability of NE under BR adjustment in the duopoly

The NE $(q_1^*; q_2^*) = \left(\frac{c_2}{(c_1+c_2)^2}; \frac{c_1}{(c_1+c_2)^2}\right)$ *is stable iff* $c_1/c_2 \in (3-2\sqrt{2}, 3+2\sqrt{2})$



Local Monopolistic Approximation (LMA)

Price competition:

Tuinstra, J. "A price adjustment process in a model of monopolistic competition", International game theory review, 6-3 (2004) pp. 417-442.

Quantity competition:

Bischi, G.I., A.K. Naimzada and L. Sbragia “Oligopoly Games with Local Monopolistic Approximation ” Journal of Economic Behavior and Organization, vol. 62 (2007) pp. 371-388.

Local Monopolistic Approximation (LMA)

Firms do not know the (global) demand function, at any time period they get a correct (local) estimate of

$$\frac{\partial f\left(q_i(t) + Q_{-i}\left(t\right)\right)}{\partial q_i} = f^{'}\left(Q\right)$$

by experimental computation of the effects of small quantity variations

$$\frac{f\left(q_i(t) + \Delta q_i + Q_{-i}(t)\right) - f\left(q_i(t) + Q_{-i}\left(t\right)\right)}{\Delta q_i}$$

or small price variations

$$\frac{\partial f\left(Q\right)}{\partial q_i} = \frac{df\left(Q\right)}{dQ} = \left[\frac{dQ(p)}{dp}\right]^{-1}$$

Each firm i uses this estimate to obtain

Conjectured demand function: Linear and monopolistic approximation

$$p^e(t+1) = p(t) + f'(Q)(q_i(t+1) - q_i(t)), \quad \text{where } p(t) = f(Q(t))$$

If producer i uses this LMA of expected price, FOC becomes

$$p(t) + 2f'(Q)q_i(t+1) - f'(Q)q_i(t)) - C_i'(q_i(t+1)) = 0$$

Theorem

The steady states of the optimization problem with LMA are the Cournot equilibria, located at the intersections of reaction curves.

$$f(Q) + 2f^{'}(Q)q_i(t+1) - f^{'}(Q)q_i(t)) - C_i^{'}(q_i(t+1)) = 0 \qquad i = 1, .., n$$

with $C_i = c_i q_i$ becomes *linear*, so an *explicit* dynamical system is get

LMA with linear cost

$$q_i(t+1) = \frac{1}{2} q_i(t) - \frac{f\left(Q(t)\right) - c_i}{2f'\left(Q(t)\right)} \qquad i = 1, ..., n$$

LMA duopoly with linear cost and Isoelastic demand p=f(Q)=1/Q

$$q_1(t+1) = \frac{1}{2}\left[2q_1(t) + q_2(t) - c_1\left(q_1(t) + q_2(t)\right)^2\right]$$
$$q_2(t+1) = \frac{1}{2}\left[q_1(t) + 2q_2(t) - c_2\left(q_1(t) + q_2(t)\right)^2\right]$$

Summing up, BR and LMA agents know their own current output, their own production cost $C_i(q_i)$ and observe the current selling price $p(t)$.

Information set/rationality for BR

(ia) knowledge of the demand function;

(ib) naïve expectations on the quantity of the rest of the industry;

(ic) ability to solve a non-linear optimization problem, which involves solving a non-linear equation.

Information set/rationality for LMA

(iA) local estimation of the demand function through $\frac{\partial f(Q)}{\partial q_i}$, the partial derivative of demand with respect to its own production

(iB) no expectations on other firms' future production when deciding its own production;

(iC) ability to solve a quadratic optimization problem, which involves solving a linear equation (when costs are linear or quadratic).

Proposition: stability of NE under LMA adjustment

The NE $(q_1^*; q_2^*) = \left(\frac{c_2}{(c_1+c_2)^2}; \frac{c_1}{(c_1+c_2)^2} \right)$ *is always asymptotically stable.*

- REMARK: With LMA the Nash equilibrium is always stable. In this case,

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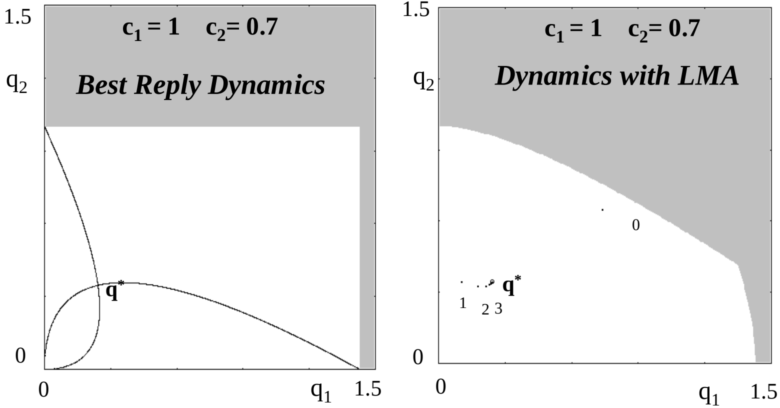
- REMARK: With LMA the Nash equilibrium is always stable. In this case,
- LESS INFORMATION IMPLIES MORE STABILITY.

Comparison of basins

For a set of parameters such that NE is stable under both kinds of adjustment, Bischi et al. 2007 numerically compare the basins of attraction of the NE when all agents use the same rule (BR or LMA)



What about profits?

- Evolutionary game with two behavioural rules H_k, $k = BR, LMA$ with information costs
- $N \geq 2$ firms. A fraction $r \in [0, 1]$ plays according to BR and the complementary fraction $(1 - r) N$ uses LMA.

Two steps:

1. A two-dim. discrete-time model of oligopoly with heterogeneous behavioural rules, exogenously given r. The stability of the NE is considered with respect to parameters N, r, inertia in adopting the computed output (i.e. anchoring attitude).
2. Then we endogenize $r = r(t)$ according to a *profit driven evolutionary pressure* given by an *exponential replicator (switching rule)* thus obtaining a 3-dim. discrete dynamical system, where the dynamic variable $r(t)$ expresses the time evolution of the fraction of population following one or the other behavioural rule.

Main research questions

- Does a more rational behavioral rule always perform better than a less rational one when information costs are negligible?
- What is the relationship between stability of the Nash equilibrium in the parameters space and stability of the Nash equilibrium in the state space when BR and LMA agents interact?
- What is the most likely long-run industry outcome when firms can switch between different (boundedly rational) behavioral rules but they fail to converge to a Nash equilibrium?
- Can behavioral heterogeneity arise endogenously as a result of market interaction among ex-ante identical agents who can select between different behavioral rules?

Non evolutionary oligopoly with heterogeneous firms

$x(t)$ output at time t of the representative firm of BR firms,
$y(t)$ output at time t of the representative firm of LMA firms.
$p = f(Q) = \frac{1}{Q}$, with $Q(t) = \sum_{i=1}^{N} q_i = N\left[rx(t) + (1-r)y(t)\right]$
Adjustment mechanisms of representative BR player with naïve expectations

$$x(t+1) = \max\left\{0, (1-\lambda)x(t) + \lambda\left[\sqrt{\frac{\overline{Q}_{-i}(t)}{c_x}} - \overline{Q}_{-i}(t)\right]\right\}$$

$\lambda \in [0,1]$ adaptive adjustment (inertia or anchoring), with
$\overline{Q}_{-i}(t) = (N-1)\left[rx(t) + (1-r)y(t)\right]$
Adjustment mechanisms of representative representative LMA player

$$y(t+1) = \max\left\{0, (1-\alpha)y(t) + \frac{\alpha}{2}\left[y(t) + \frac{c_y - f(Q(t))}{f'(Q(t))}\right]\right\}$$

where $\alpha \in [0,1]$ represents inertia. In the following $c_x = c_y = c > 0$

Cournot oligopoly with heterogeneous agents

Proposition: 2-dim local stability

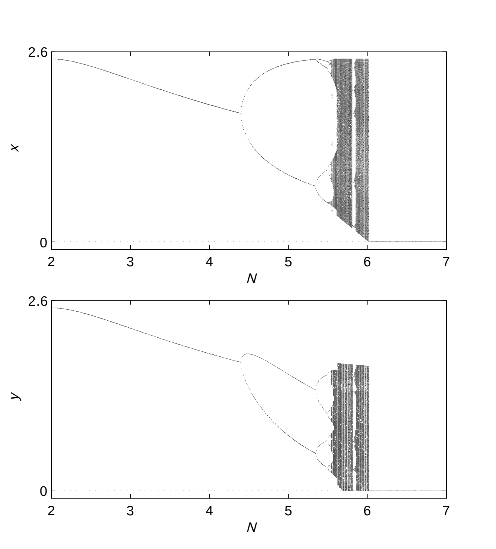
If $N \in \{2,3\}$ then NE is stable for all parameters' values;
For $N = 4$ stable for all $r \in [0,1]$ and $\lambda \in [0,1)$ or *$\lambda = 1$ with $r \in [0,1)$.*
If $5 \leq N < N_f$, then NE is a stable node, and it loses stability through a period doubling bifurcation *for N increasing beyond*

$$N_f = \frac{4\left[4+\alpha\left(1-2r\right)\right] - 2\lambda(1-r)(4+\alpha)}{\alpha(4+r(\lambda-4)-2\lambda)+4r\lambda}$$

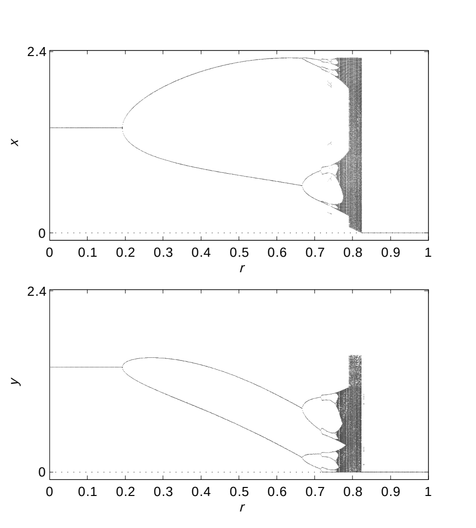
- If $\lambda \geq \alpha$ then $\frac{dN_f(r)}{dr} < 0$, i.e. an increasing fraction of BR players causes a reduction of the range of stability of the NE.
- N_f increases as α and λ decrease, i.e. more inertia enlarges the stability range of NE. If $\alpha = \lambda = 1$ (no inertia) $N_f(r) = \frac{2r+10}{r+2}$, so $N_f = 4$ for $r = 1$ and $N_f = 5$ for $r = 0$.
- If $\alpha \in (0,1)$ and $\lambda \in [0, \frac{4}{5})$ then NE is stable for any $r \in [0,1]$; if $\lambda \in \left[\frac{4}{5}, 1\right]$ then NE is stable for $r \in \left(0, \frac{8(1-\alpha)(2-\lambda)}{3[\alpha(\lambda-4)+4\lambda]}\right)$.

Some Global analysis

$\lambda = \alpha = 1$, $c = 0.1$ and $r = 0.5$



$\lambda = 0.9$, $\alpha = 0.7$, $c = 0.1$, $N = 6$



Increasing the fraction of BR agents brings more instability

The two cases of homogeneous behaviour

With $r = 0$

$$x(t+1) = f_0(x(t), y(t)) = \tag{1}$$

$$= \max\left\{0, (1-\lambda)x(t) + \lambda\left[\sqrt{\frac{(N-1)\,y(t)}{c}} - (N-1)\,y(t))\right]\right\} \tag{2}$$

$$y(t+1) = g_0(y(t))$$

$$= \max\left\{0, (1-\alpha)y(t) + \frac{\alpha}{2}\left[y(t) + Ny(t)(1-cNy(t))\right]\right\} \tag{3}$$

triangular or master/slave system.
$g_0(y)$ is a quadratic map, conjugate to the logistic map. Hence the positive value of y at which $g_0(y) = 0$, i.e.

$$y_0 = \frac{2+\alpha\,(N-1)}{\alpha c N^2} \tag{4}$$

gives the boundary of the basin: all trajectories starting with $y(0) > y_0$ go to the point $(0,0)$ in one step.

Analogously, if $r = 1$

$$x(t+1) = f_1(x(t)) = \tag{5}$$

$$= \max\left\{0, (1-\lambda)x(t) + \lambda\left[\sqrt{\frac{(N-1)\,x(t)}{c}} - (N-1)\,x(t))\right]\right\} \tag{6}$$

$$y(t+1) = g_1(x(t), y(t)) =$$

$$= \max\left\{0, (1-\alpha)y(t) + \frac{\alpha}{2}\left[y(t) + Nx(t)(1-cNx(t))\right]\right\} \tag{7}$$

and the positive value of x at which $f_1(x) = 0$:

$$x_1 = \frac{\lambda^2\,(N-1)}{c(\lambda N-1)^2} \tag{8}$$

gives the boundary of the basin.

Proposition: Basins comparison with pure strategies states

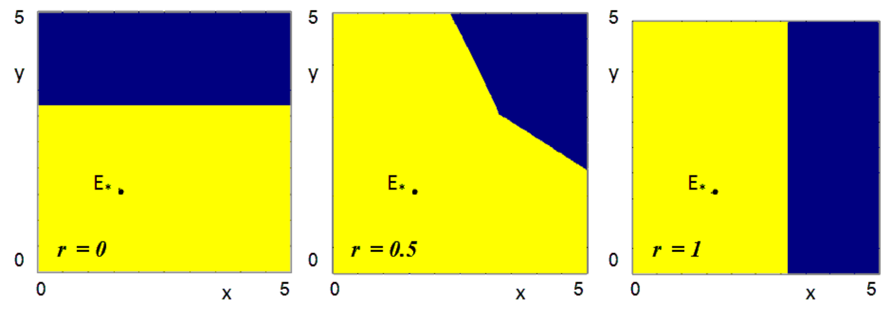
Consider the quantity dynamics system with homogenous marginal costs and $N < N_f$ so that the NE is asymptotically stable. If the following inequality holds

$$x_1 = \frac{\lambda^2 (N-1)}{(\lambda N - 1)^2} > \frac{2 + \alpha (N-1)}{\alpha N^2} = y_0 \quad (9)$$

then the basin of attraction of the NE is always larger in the case of all BR firms than in the case of all LMA firms.

- Remark: in particular, if inertia is neglected, i.e. with $\alpha = 1$ and $\lambda = 1$, $x_1 = \frac{1}{c(N-1)} > \frac{N+1}{cN^2} = y_0$. Hence *all BR* implies larger basin than *all LMA*.

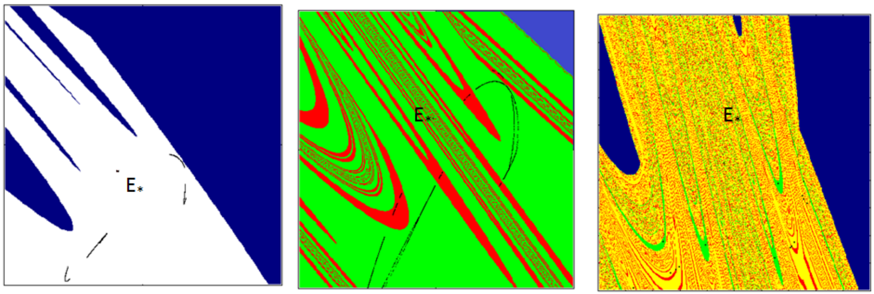
$\lambda = 0.7$, $\alpha = 0.5$, $c = 0.1$ and $N = 7$ basins at different r values



An increase of the fraction of BR firms on the one hand can reduce the range of stability of the Nash equilibrium in the *parameter space* (local stability), but on the other hand can increase the stability of the Nash equilibrium in the *state space* (basins of attraction).

Complex attractors, complex basins, coexisting attractors

More interesting scenarios are obtained after the Nash equilibrium is destabilized at $N = N_f$



Cyclic and complex attractors as well as intermingled structure of the basins of attraction

The Cournot game with evolution of behavioral rules

Let us endogenize the time evolution of the fraction $r = r(t)$ by an evolutionary mechanism based on observed profits:

$$\pi_{BR} = px - (c_x x + K_x) = \left(\frac{1}{Q} - c_x\right) x - K_x$$

$$\pi_{LMA} = py - (c_y y + K_y) = \left(\frac{1}{Q} - c_y\right) y - K_y$$

where K_i, $i = x, y$, represent the information costs of BR and LMA firms. We assume $K_x \geq K_y$.

The fraction $r(t)$ of agents choosing behavioural rule BR is updated according to the exponentially replicator equation

$$r(t+1) = r(t)\frac{e^{\beta \pi_{BR}(t)}}{r(t)e^{\beta \pi_{BR}(t)} + (1 - r(t))e^{\beta \pi_{LMA}(t)}}$$

$\beta > 0$ *intensity of choice*: $\beta = 0$ agents do not switch; $\beta = \infty$ implies $r(t) \to 1$ if $\pi_{BR}(t) > \pi_{LMA}(t)$ and $r(t) \to 0$ if $\pi_{BR}(t) < \pi_{LMA}(t)$.

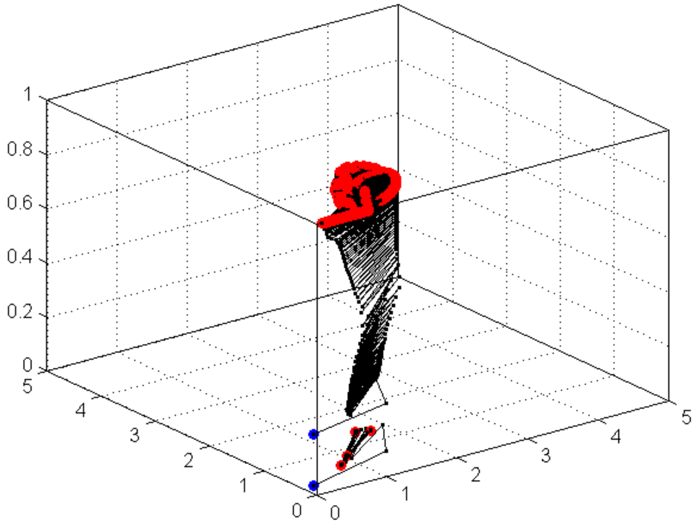
- Steady states: $r = 0$; $r = 1$; any $r^* \in (0, 1)$ such that $\pi_{BR} = \pi_{LMA}$.

Proposition: 3-dim local stability

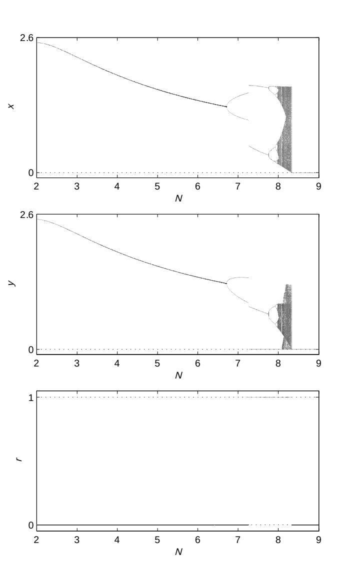
- *For the evolutionary model with* $c_x = c_y = c$,
- *if* $K_x = K_y$ *then a continuum of equilibrium points* E_* *exist along the segment* $E = \left(\frac{N-1}{cN^2}, \frac{N-1}{cN^2}, r\right)$, $r \in [0,1]$, *and these equilibria are marginally stable as far as the stability conditions for the Nash equilibrium are satisfied.*
- *If* $K_x > K_y$ *then only the two extremum points of the segment are equilibria,* $E_0 = \left(\frac{N-1}{cN^2}, \frac{N-1}{cN^2}, 0\right)$ *and* $E_1 = \left(\frac{N-1}{cN^2}, \frac{N-1}{cN^2}, 1\right)$, *characterized by all agents playing the same strategy. When* $N < N_f$, *the equilibrium* E_0 *(all LMA) is asymptotically stable, whereas* E_1 *(all BR) is unstable.*

Also for the 3D model, interesting scenarios are obtained when the NE is unstable

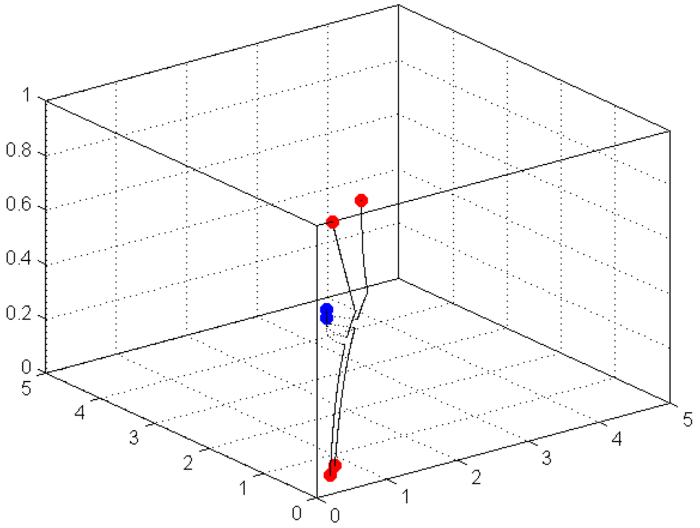
$\lambda = 0.6$, $\alpha = 0.7$, $c = 0.1$, $\delta = 0$, $\beta = 1$, $K_x = 0.01$, $K_y = 0$, $N = 8$, two different i.c.



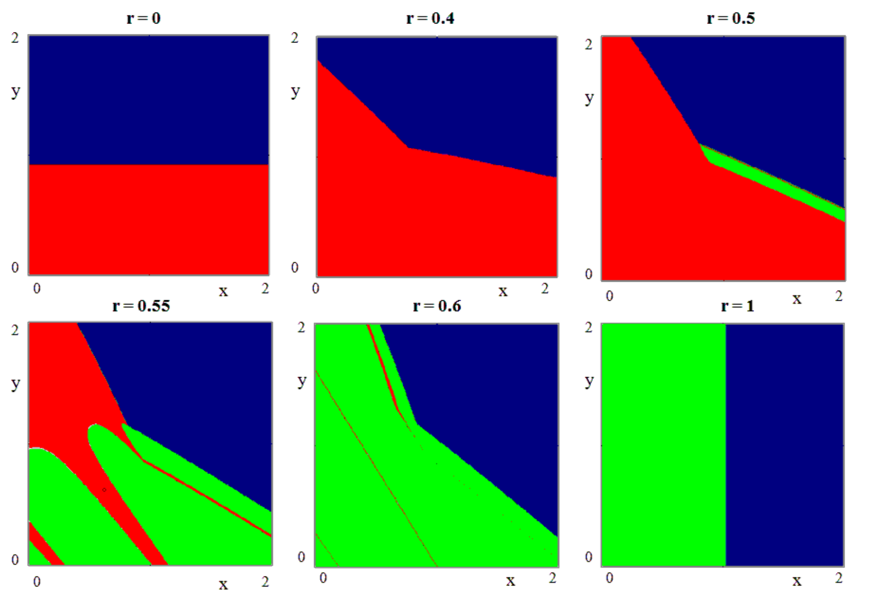
$\lambda = 0.6$, $\alpha = 0.7$, $c = 0.1$, $\delta = 0$, $\beta = 1$, $K_x = 0.01$, $K_y = 0$



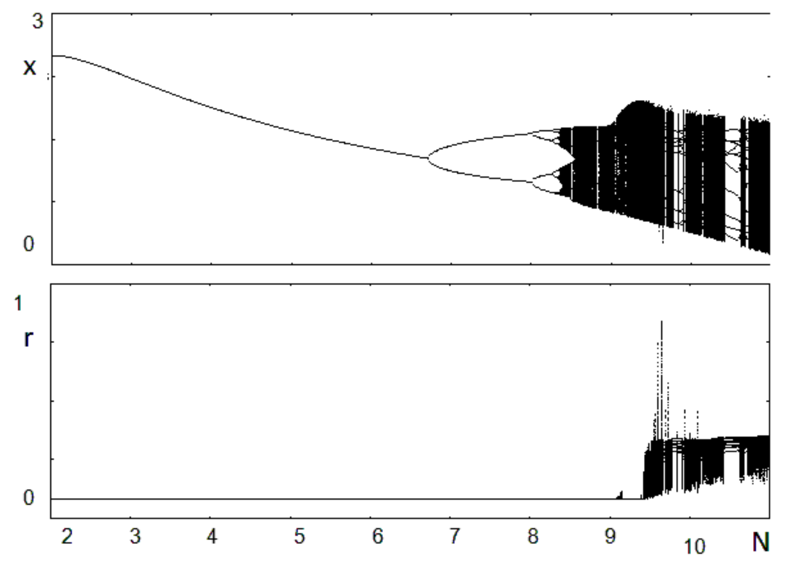
$\lambda = \alpha = 0.3$, $c = 0.1$, $\delta = 0$, $\beta = 1$, $K_x = 0.01$, $K_y = 0$, $N = 15$, two different i.c.



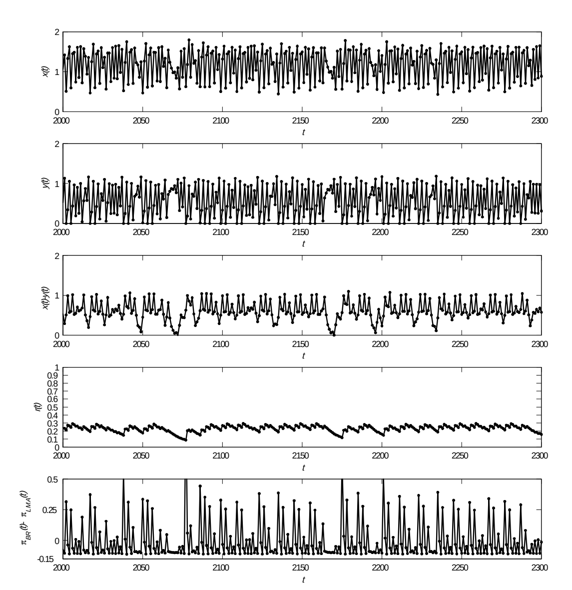
SECTIONS OF 3-D BASINS



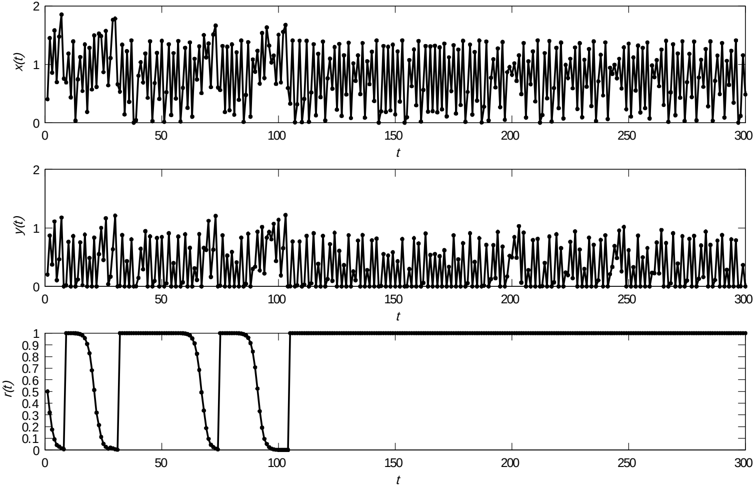
$\lambda = 0.5$, $\alpha = 0.7$, $c = 0.1$, $\delta = 0$, $\beta = 1$, $K_x = 0.1$, $K_y = 0$,i.C.$(x(0), y(0), r(0)) = (0.1, 0.2, 0.5)$



Same parameters and $N = 10$



Same parameters but info cost increased at $K_x = 0.8$



Some conclusions/1

- If the NE is stable and higher information costs are associated with BR, then LMA dominates, so all the firms play it in the long run.
- In the case of identical information costs, time evolutions can be obtained leading to coexistence in the long-run of both *BR* and *LMA* behavioral rules, i.e. even if the extra information to play the "more rational" *BR* strategy can be obtained for free, a fraction of firms prefer to use LMA in the long run. Strong path dependence.
- When the NE is unstable, and outputs exhibit oscillatory dynamics around it, if the oscillations remain inside a given neighborhood of NE then the presence of information costs force the firms to follow LMA, whereas oscillations of larger amplitude lead the firms to choose BR in the long run.

Some conclusions/2

- Situations are observed where two coexisting attractors exist in the invariant planes $r = 0$ and $r = 1$ where all players adopt LMA or BR respectively, each with its own basin of attraction, so that convergence to a pure behavioural rule strongly depends on the initial conditions. The motion on the invariant planes $r = 0$ and $r = 1$ are governed by two-dimensional triangular maps.
- Attractors in the interior of the phase space may exist, characterized by intermediate values of r, with dynamic properties which are much more complicated and worth of further investigations.
- What happens when the fitness of each strategy includes memory ?

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