Nonlinear Multiplier-Accelerator Model with Investment and Consumption Delays

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Outline

1. Introduction

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  - chaos theory in discrete-time models
- We reconsider the **lost roles of delays** for the emergence of persistent fluctuations in a continuous-time Goodwinian model

Multiplier-Accelerator Model

Goodwin, R. "The Nonlinear Accelerator and the Persistence of Business Cycle," *Econometrica*, 19, 1-17, 1951.

- Output adjustment process:

$$\varepsilon \dot{Y}(t) = \dot{K}(t) - (1-\alpha)Y(t)$$

$$\dot{K}(t) = I(t) = \varphi(\dot{Y}(t-\delta))$$

$$\text{with } \varphi(0) = 0,\ \varphi'(\dot{Y}) > 0,\ \varphi''(\dot{Y}) \neq 0$$

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$$\varepsilon \dot{Y}(t) = \varphi(\dot{Y}(t-\delta)) - (1-\alpha)Y(t)$$

- Approximated version: second order nonlinear differential equation

$$\varepsilon\delta \ddot{Y}(t) + [\varepsilon + (1-a)\delta]\dot{Y}(t) - \varphi(\dot{Y}(t)) + (1-\alpha)Y(t) = 0$$

Multiplier-Accelerator Model

Phillips, A., "Stabilization Policy in a Closed Economy," *Economic Journal*, 64, 832-842, 1954

- Two delay macro dynamic model

$$C(t) = \alpha Y(t-\eta),$$

$$I(t) = \varphi(\dot{Y}(t-\delta)),$$

$$Y(t) = \int_0^t \frac{1}{\varepsilon} e^{-\frac{t-\tau}{\varepsilon}} E(\tau) d\tau$$

where $E(\tau) = C(\tau) + I(\tau)$.

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- Differentiating the last equation with t yields a differential equation with two delays

$$\varepsilon \dot{Y}(t) = \varphi(\dot{Y}(t-\delta)) + Y(t) - \alpha Y(t-\eta) = 0$$

Multiplier-Accelerator Model

- Linearly approximated version:

$$\varepsilon \dot{Y}(t) + Y(t) - \nu \dot{Y}(t-\delta) - \alpha Y(t-\eta) = 0 \text{ with } \nu = \varphi'(0)$$

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- Simplification

$$\dot{Y}(t) + aY(t) - b\dot{Y}(t-\delta) - cY(t-\eta) = 0$$

$$a = \frac{1}{\varepsilon},\ b = \frac{\nu}{\varepsilon} \text{ and } c = \frac{\alpha}{\varepsilon}$$

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$$\lambda + a - b\lambda e^{-\delta\lambda} - ce^{-\eta\lambda} = 0.$$

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- **Assumption 1**. $\delta > \eta$

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- Characteristic equation

$$(\varepsilon - \nu)\lambda + (1-\alpha) = 0 \Longrightarrow \begin{cases} \varepsilon > \nu\text{: locally stable} \\ \varepsilon < \nu\text{: locally unstable} \end{cases}$$

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- **Assumption 2.** $\varepsilon > \nu$

Theorem

Given Assumption 2, the zero solution of one-delay differential equation

$$\lambda + a - b\lambda e^{-\delta\lambda} - ce^{-\eta\lambda} = 0$$

with $\delta = 0,\ \eta = 0$ *or* $\delta = \eta$ *is locally asymptotically stable for all* $\eta > 0,\ \delta > 0$ *or* $\delta = \eta > 0$.

$$(1-b)\lambda + a - ce^{-\eta\lambda} = 0 \text{ if } \delta = 0 \text{ and } \eta > 0,$$

$$\lambda + a - c - b\lambda e^{-\delta\lambda} = 0 \text{ if } \delta > 0 \text{ and } \eta = 0,$$

$$\lambda + a - (b\lambda + c)e^{-\delta\lambda} = 0 \text{ if } \delta = \eta > 0.$$

Two-Delay Model

- Characteristic equation with two delays

$$\lambda + a - b\lambda e^{-\delta\lambda} - ce^{-\eta\lambda} = 0$$

$$1 + a_1(\lambda)e^{-\delta\lambda} + a_2(\lambda)e^{-\eta\lambda} = 0$$

$$a_1(\lambda) = -\frac{b\lambda}{\lambda + a} \text{ and } a_2(\lambda) = -\frac{c}{\lambda + a}$$

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- Suppose that $\lambda = i\omega,\ \omega > 0$

$$a_1(i\omega) = -\frac{b\omega^2}{a^2 + \omega^2} - i\frac{ab\omega}{a^2 + \omega^2}$$

$$a_2(i\omega) = -\frac{ac}{a^2 + \omega^2} + i\frac{c\omega}{a^2 + \omega^2}$$

Two-Delay Model

- The absolute values

$$|a_1(i\omega)| = \sqrt{\left(\frac{b\omega^2}{a^2+\omega^2}\right)^2 + \left(\frac{ab\omega}{a^2+\omega^2}\right)^2} = \frac{b\omega}{\sqrt{a^2+\omega^2}}$$

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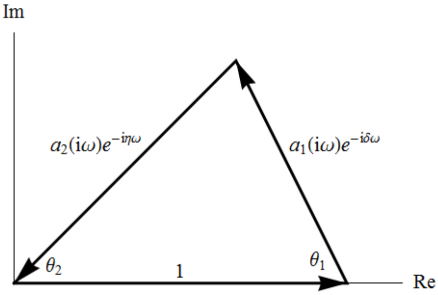
- The arguments

$$\arg\left[a_1(i\omega)\right] = \tan^{-1}\left(\frac{a}{\omega}\right) + \pi$$

$$\arg\left[a_2(i\omega)\right] = \pi - \tan^{-1}\left(\frac{\omega}{a}\right)$$

Two-Delay Model

- $1 + a_1(i\omega)e^{-\delta\lambda} + a_2(i\omega)e^{-\eta\lambda} = 0$



$$1 \leq |a_1(i\omega)| + |a_2(i\omega)|,$$

$$|a_1(i\omega)| \leq 1 + |a_2(i\omega)|,$$

$$|a_2(i\omega)| \leq 1 + |a_1(i\omega)|.$$

Two-Delay Model

- These three conditions to the following two conditions,

$$f(\omega) = (1 - b^2)\omega^2 - 2bc\omega + a^2 - c^2 \leq 0$$

and

$$g(\omega) = (1 - b^2)\omega^2 + 2bc\omega + a^2 - c^2 \geq 0$$

where $f(\omega)$ and $g(\omega)$ have the same discriminant,

$$D = 4[c^2 - a^2(1 - b^2)].$$

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- Let ω_1 and ω_2 be solutions of $f(\omega) = 0$ and let ω_3 and ω_4 be solutions of $g(\omega) = 0$.

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- Let ω_1 and ω_2 be solutions of $f(\omega) = 0$ and let ω_3 and ω_4 be solutions of $g(\omega) = 0$.
- It is confirmed that the two conditions, $f(\omega) \leq 0$ and $g(\omega) \geq 0$, are satisfied when ω is in interval $[\omega_3, \omega_4]$.

Two-Delay Model

- The internal angels, θ_1 and θ_2, of the triangle can be calculated by the law of cosine,

$$\theta_1(\omega) = \cos^{-1}\left(\frac{a^2 + (1+b^2)\omega^2 - c^2}{2b\omega\sqrt{a^2+\omega^2}}\right)$$

$$\theta_2(\omega) = \cos^{-1}\left(\frac{a^2 + (1-b^2)\omega^2 + c^2}{2c\sqrt{a^2+\omega^2}}\right)$$

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- Solving the following two equations for δ and η

$$\left\{\arg\left[a_1(i\omega)e^{-i\delta\omega}\right] + 2m\pi\right\} \pm \theta_1(\omega) = \pi$$

$$\left\{\arg\left[a_2(i\omega)e^{-i\eta\omega}\right] + 2n\pi\right\} \mp \theta_2(\omega) = \pi$$

Two-Delay Model

- Solutions are

$$\delta = \frac{1}{\omega}\left[\tan^{-1}\left(\frac{a}{\omega}\right) + \pi + (2m-1)\pi \pm \theta_1(\omega)\right]$$

and

$$\eta = \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{a}\right) + \pi + (2n-1)\pi \mp \theta_2(\omega)\right]$$

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- For any ω satisfying $f(\omega) \leq 0$ and $g(\omega) \geq 0$, we can find the pairs of (δ, η) constructing stability crossing curves for $\omega_3 \leq \omega \leq \omega_4$.

$$C_1(m,n) = \{\delta_1(\omega, m), \eta_1(\omega, n)\}$$

where

$$\delta_1(\omega, m) = \frac{1}{\omega}\left[\tan^{-1}\left(\frac{a}{\omega}\right) + 2m\pi + \theta_1(\omega)\right]$$

$$\eta_1(\omega, n) = \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{a}\right) + 2n\pi - \theta_2(\omega)\right]$$

Two-Delay Model

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$$C_2(m, n) = \{\delta_2(\omega, m), \eta_2(\omega, n)\}$$

where

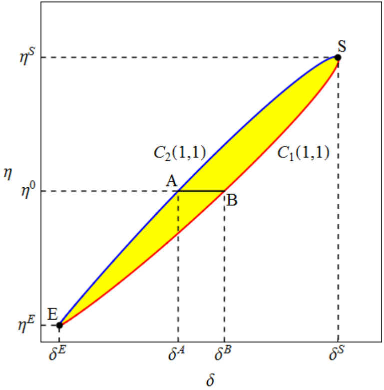
$$\delta_2(\omega, m) = \frac{1}{\omega}\left[\tan^{-1}\left(\frac{a}{\omega}\right) + 2m\pi - \theta_1(\omega)\right]$$

$$\eta_2(\omega, n) = \frac{1}{\omega}\left[-\tan^{-1}\left(\frac{\omega}{a}\right) + 2n\pi + \theta_2(\omega)\right]$$

with $m, n = 0, 1, 2, \ldots$ Notice that m and n are selected to be nonnegative integers so that $\delta > 0$ and $\eta > 0$.

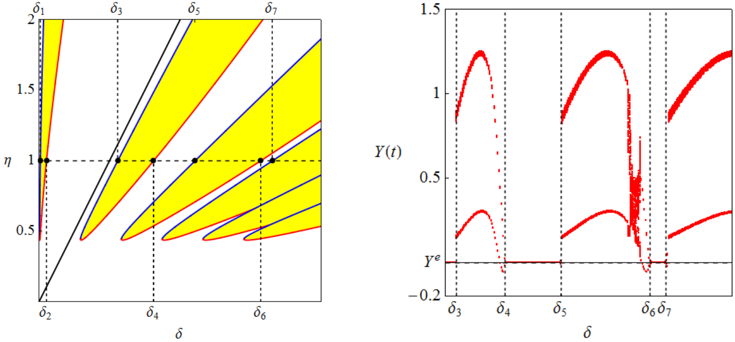
Two-Delay Model

- The stability crossing curve with
$$\varphi(x) = a_2 \left(\frac{a_1 + a_2}{a_1 e^{-x} + a_2} - 1 \right), \quad m = 1 \text{ and } n = 1$$



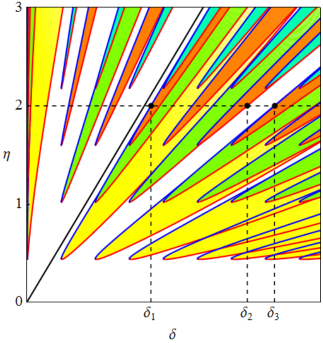
Two-Delay Model

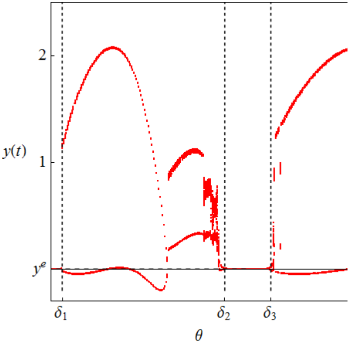
- Stability switches with $\eta = 1$, $m = 0, 1, 2, 3, 4, 5$ and $n = 1$



Two-Delay Model

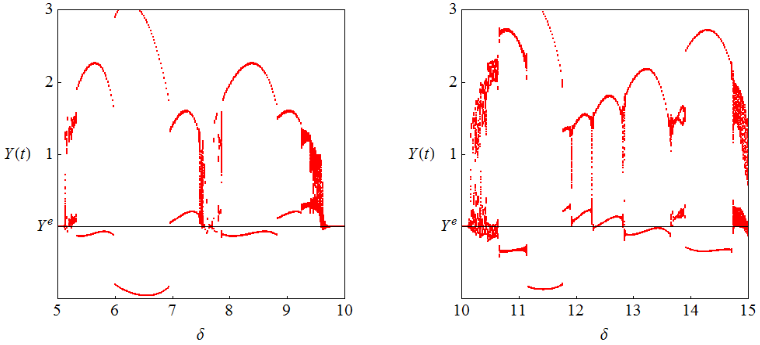
- Stability switches with $\eta = 2$, $m = 0, 1, ..., 8$ and $n = 1, 2, 3, 4$





Two-Delay Model

- Stability switching curves



$\eta = 5$ $\eta = 10$

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2. Asymptotical stability of the three different one-delay models is shown
3. In the two-delay case, the stability switching curves are obtained on which stability is lost.
4. stability loss and gain repeatedly occurs.