Globalization and Synchronization of Innovation Cycles

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Abstract: We propose and analyze a two-country model of endogenous innovation cycles. In autarky, innovation fluctuations in the two countries are *decoupled*. As the trade cost falls and intra-industry trade rises, they become *synchronized*. This is because globalization leads to the alignment of innovation incentives across firms based in different countries, as they operate in the increasingly global (hence common) market environment. Furthermore, synchronization occurs faster (i.e., with a smaller reduction in trade costs) when the country sizes are more unequal, and it is the larger country that dictates the tempo of global innovation cycles with the smaller country adjusting its rhythm to the rhythm of the larger country. These results suggest that adding endogenous sources of productivity fluctuations might help improve our understanding of why countries that trade more with each other have more synchronized business cycles.

Keywords: Endogenous innovation cycles and productivity co-movements; Globalization, Country size effect; Synchronized vs. Asynchronized cycles; Synchronization of coupled oscillators; Basins of attraction; Two-dimensional, piecewise smooth, noninvertible maps

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1. Introduction

How does globalization affect macroeconomic dynamics and their co-movements across countries? A vast majority of research addresses this question by assuming that productivity movements in each country are driven by some *exogenous* processes. As already demonstrated by innovation-based models of endogenous growth, however, globalization can change the growth rates of productivity. In this paper, we demonstrate that globalization can also change co-movements of productivity across countries. To this end, we propose and analyze a two-country model of *endogenous fluctuations* of innovation activities.[1]

The intuition we want to capture can be simply stated. Imagine that there are two structurally identical countries. In autarky, each of these countries experiences endogenous fluctuations of innovation, due to strategic complementarities in the timing of innovation among firms competing in their domestic market, which causes temporal clustering of innovation activities and hence aggregate fluctuations. Without trade, endogenous fluctuations in the two countries are obviously unrelated. As trade cost falls and firms based in the two countries start competing against each other, the innovators from both countries start responding to an increasingly global (hence common) market environment. This alignment of innovation incentives leads to a *synchronization* of innovation activities, and hence of productivity movements, across countries. To capture this intuition in a transparent manner, we consider a model that consists of two building blocks.

Our first building block is a model of endogenous fluctuations of innovations, originally proposed by Judd (1985). In this classic article, Judd developed dynamic extensions of the Dixit-Stiglitz monopolistic competitive model, in which innovators could pay a one-time fixed cost to introduce a new (horizontally differentiated) good. After showing that the equilibrium trajectory converges monotonically to a unique steady state under the assumption that innovators hold monopoly over their innovations indefinitely, he turned to the cases where they hold

[1] Empirically, countries that trade more with each other have more synchronized business cycles (references). The evidence is particularly strong among developed countries as well as among developing countries, while it is less so between developed and developing countries. Standard international RBC models have difficulty explaining this, and it is easy to see why. With exogenous productivity shocks driving business cycles in these models, more trade leads to more specialization, which means less synchronization, to the extent that the shocks have sector-specific components. Some attempts to resolve such "trade-co-movement puzzle" by appealing to vertical specialization across countries have met limited success, and several authors suggested that it would help to improve their performances if globalization would also synchronize co-movements of TFPs across countries (references). We hope that our model offers one such theoretical ingredient.

monopoly only for a limited time, so that each good is sold initially at the monopoly price and later at the competitive price. The assumption of temporary monopoly drastically changes the nature of dynamics and generates endogenous fluctuations. This is because, with free entry to innovation, the innovators need to recover their cost of innovation by earning enough profits during their monopoly. If they anticipate that more goods they would have to compete against during their monopoly are competitively priced, they have less incentive to innovate. This in turn suggests that past innovation discourages current innovators more than contemporaneous innovation, which leads to a temporal clustering of innovation, generating aggregate fluctuations of productivity.

Judd developed two versions of the model that formalizes this idea, of which we use the one, sketched by Judd (1985; Sec.4) and examined in greater detail by Deneckere-Judd (1992; DJ for short) for its analytical tractability. What makes it analytically tractable is the assumption that time is discrete and that the innovators hold their monopoly for just one period, the same period in which they introduce their goods. With this assumption, the entry game played by innovators in each period becomes effectively static (although their innovations will discourage future innovations). Furthermore, the state of the economy in each period (how saturated the market is from past innovations) is summarized by one variable (how many varieties of competitive goods the economy has inherited). As a result, for each initial condition, the equilibrium trajectory is unique, which can be obtained by iterating a one-dimensional (1D) map. This map turns out to be isomorphic to the *skew tent map*. That is, it is noninvertible and piecewise linear (PWL) with two branches. It depends on two parameters; σ (the elasticity of substitution between goods) and δ (the survival rate of the existing goods).[2] A higher σ increases the extent to which a past innovation, which is competitively sold, discourages innovators more than a contemporaneous innovation, which is monopolistically sold. A higher δ means more of the past innovations survive and carry over to discourage current innovations. For a sufficient high σ and/or a sufficiently high δ, strategic complementarities in the timing of innovation are strong enough to cause temporal clustering of innovation that makes the unique steady state unstable and the equilibrium trajectory fluctuate forever, starting from almost all initial

[2] In a model of horizontal innovation (or expanding variety), new goods are added to old goods without replacing them, so that the market could eventually become so saturated that innovations would stop permanently. One way to avoid this is to let the economy grow in size, exogenously as in Judd (1985) or endogenously as in Matsuyama (1999, 2001). Here, we assume instead, by following DJ (1992), that the existing goods are subject to idiosyncratic obsolescence shocks, so that only a constant fraction of them, δ, carries over to the next period.

conditions. For a moderately high σ and/or δ, the equilibrium trajectory asymptotically converges to a unique *period-2 cycle,* along which the economy alternates between the period of active innovation and the period of no innovation. For a much higher σ and/or δ, even the period-2 cycle is unstable, and the trajectory converges to a *chaotic attractor*. Since the equilibrium trajectory is unique, fluctuations are driven neither by multiplicity nor by self-fulfilling expectations. This feature of the model makes it useful as a building block to examine the effects of globalization on the nature of fluctuations across two countries.[3]

Our second building block is Helpman and Krugman (1985; Ch.10; HK for short), a model of international trade in horizontally differentiated (Dixit-Stiglitz) goods with *iceberg trade costs* between two structurally identical countries, which may differ only in size. This model has two key parameters; the distribution of country sizes and the degree of globalization, which is inversely related to the trade cost. In this model, the equilibrium number of firms based in each country is proportional to its size in autarky (when infinitely large trade costs). As trade costs fall, horizontally differentiated goods produced in the two countries mutually penetrate each other's home market (*Two-way flows of goods*), and the equilibrium distribution of firms become increasingly skewed toward the larger country *(Home Market Effect and its Magnification).*

By combining the DJ mechanism of endogenous fluctuations of innovations with the HK model of international trade, we show:

- The state space of our two-country model of the world economy is two-dimensional, i.e., how many varieties of competitive goods each country has inherited, which determines how saturated the two markets are from past innovations and represents the global market condition for the current innovators in the two countries.
- For each initial condition, the equilibrium trajectory is unique and obtained by iterating a two-dimensional (2D), piecewise smooth (PWS), noninvertible map, which has four parameters (the two coming from DJ and the two coming from HK).
- *In autarky,* with infinite trade costs, the dynamics of two countries are *decoupled* in the sense that the 2D-system can be decomposed into two independent 1D-systems, which are isomorphic to the original DJ model. Under the same parameter condition that ensures the

[3]It is worth pointing out that the discrete time specification is *not* responsible for causing fluctuations. Indeed, Judd (1985; Sec.3) showed that fluctuations must occur along any equilibrium trajectory for almost all initial conditions when the length of monopoly power is sufficiently *long* (but finite) *in continuous time.*

instability of the steady state in the DJ model, the dynamics of the two countries may converge to either *synchronized* or *asynchronized* fluctuations, depending on the initial conditions;

- *As trade costs fall*, and the goods produced in two countries mutually penetrate each other's home market, the dynamics become *synchronized* in the sense that the basin of attraction[4] for the synchronized cycle *expands and eventually covers a full measure of the state space*, and the basin of attraction for the asynchronized cycle *shrinks and eventually disappears*.[5] To put it differently, as trade costs fall, the innovation dynamics becomes more likely to converge to the synchronized 2-cycle, and for a sufficiently small trade cost, it converges to the synchronized 2-cycle for almost all initial conditions.
- Synchronization occurs faster (i.e., with a smaller reduction of trade costs) when the two countries are more unequal in size. Furthermore, the larger country sets the tempo of global innovation cycles, with the smaller country adjusting its rhythm to the rhythm of the larger country.

The intuition behind these results should be easy to grasp. With globalization, the markets become more integrated. As a result, a big wave of innovations that took place in one country in the past discourages innovations today not only in that country, but also in the other country, causing synchronization of innovation activities across the two countries. Furthermore, the larger country determines the rhythm of fluctuations because the innovators based in the smaller country rely more heavily on the profits earned in the export market to recover the cost of innovation than those based in the larger country.

The present paper belongs to several strands of literature. First, it is related to static models of international trade, particularly those of intra-industry trade and home market effects. This is one of the core materials of international trade. We have chosen HK as one of our building blocks, because it is perhaps the most standard textbook treatment. Second, there are now a large body of literature that study the effects of globalization in innovation-driven models of

[4]In the terminology of the dynamical system theory, the set of initial conditions that converge to an attractor (e.g., an attracting steady state, an attracting period-2 cycle, a chaotic attractor, etc.) is called its *basin of attraction*.

[5]For these results, we impose the parameter conditions that ensure the existence of a unique, stable period-2 cycle in the DJ model. As pointed out above, the equilibrium trajectory in the DJ model converges to a chaotic attractor under the parameter conditions that ensure the instability of the (unique) period-2 cycle. Although we have obtained some interesting results for these cases, we have chosen not to discuss them here partly because the stable 2-cycle case is sufficient for conveying the economic intuition behind the synchronization mechanism and partly because we want to avoid making this paper more technically demanding in order to keep it accessible to a wider audience.

endogenous growth: see Grossman and Helpman (1991), Rivera-Batiz & Romer (1991), Acemoglu-Zilibotti (2001), Ventura (2005), Acemoglu (2008; Ch.19), Gancia & Zilibotti (2009) and many others. All of these examine the effects of globalization on productivity growth rates by focusing on the balanced growth path. Third, there are many closed economy models of endogenous fluctuations of innovation, which include, in addition to Judd (1985) and Deneckere-Judd (1992), Shleifer (1986), Gale (1996), Jovanovic and Rob (1990), Matsuyama (1999, 2001), Wälde (2002, 2005), Francois and Lloyd-Ellis (2003, 2008, 2009), Jovanovic (2009), and Bramoullé and Saint-Paul (2010). We have chosen DJ as one of our building blocks because of its tractability and the uniqueness of the equilibrium trajectory. Among them, Matsuyama (1999, 2001) embed the DJ mechanism into a closed economy endogenous growth model with capital accumulation.[6] In these models, there are two engines of growth, innovation and capital accumulation, which move *asynchronously*. This is because there is only one source of endogenous fluctuations; capital accumulation merely responds to the fluctuations of innovation.

To put our contribution in a broader context, we offer a new model of synchronization of *coupled oscillators*. The subject of coupled oscillators is concerned with the effects of combining two or more systems that generate self-sustained oscillation, in particular, how they adjust the rhythm of oscillations. It is a major topic in natural science, ranging from physics to chemistry to biology to engineering, with thousands of applications.[7] We are not aware of any previous example from economics.[8] To the best of our knowledge, this is the first two-country, dynamic general equilibrium model of endogenous fluctuations. Indeed, this is one of the only two dynamic general equilibrium models, whose equilibrium trajectory can be characterized by a dynamical system, which can be viewed as a coupling of two dynamical systems that generate self-sustained equilibrium fluctuations. The other one is our companion piece, Matsuyama,

[6] See also Gardini, Sushko, and Naimzada (2008) for a complete characterization of the Matsuyama (1999) model.

[7] Just to name a few, consider the Moon, with its rotation around its own axis and its revolution around the Earth. These two oscillations are perfectly synchronized in the same frequency, which is the reason why we observe only one side of the Moon from the Earth. Or consider the London Millennium Bridge. In its opening days, hundreds of pedestrians tried to adjust their footsteps to lateral movements of the bridge. In doing so, they inadvertently synchronized their footsteps among themselves, which caused the bridge to swing widely, forcing a closure of the bridge. See Strogatz (2003) for an introduction to this huge subject.

[8] Of course, there may have been attempts to borrow an existing model of coupled oscillators from science and give an economic interpretation to its variables. The problem of this approach is that it would be hard to give any structural interpretation to the parameters of the system. Importantly, we derive a system of coupled oscillators from a fully specified economic model, and we need to analyze this system, because it is new and different from any system that has been studied before. Furthermore, the country size difference has nontrivial effects in our model, and plays an important role in our analysis. We are not aware of any previous studies, which conduct a systematic analysis of the role of size difference between coupled oscillators.

Sushko, and Gardini (forthcoming), which develops a two-sector, closed economy model, where each sector produces a Dixit-Stiglitz composite of differentiated goods, as in DJ. When the consumers have Cobb-Douglas preferences over the two composites, innovation dynamics in the two sectors are decoupled. For the cases of CES preferences, it is shown that, as the elasticity of substitution between the two composites increases (decreases) from one, fluctuations in the two sectors become synchronized (asynchronized), which amplifies (dampens) the aggregate fluctuations.[9] The above two are also the first economic examples of 2D dynamic systems, defined by PWS, noninvertible maps. [10]

Matsuyama, Kiyotaki, and Matsui (1993) is also related in spirit in that they too consider globalization as a coupling of two games of strategic complementarities. They developed a two-country model of currency circulation. The agents are randomly matched together as a seller and a buyer, and currency circulation is modeled as a game of strategic complementarities, where an agent, as a seller, accepts a certain object as a means of payment, if he, as a buyer, expects, the sellers he would run into in the future to do the same. In autarky, agents are matched only within the same country, so that two countries play two separate games of strategic complementarities, hence different currencies may be circulated in the two countries. Then, globalization increases the frequency in which agents from different countries are matched together. Interestingly, the agents from the smaller country, not those from the larger country, are the first to adjust their strategies and to start accepting a foreign currency, and as a result, that the larger country's currency emerges as a vehicle currency of the world trade.

The rest of the paper is organized as follows. Section 2 develops our two-country model of endogenous fluctuations of innovation, and derives the 2D-PWS, noninvertible map that governs the equilibrium trajectory. Section 3 considers the case of autarky, where the 2D system can be decomposed into two independent 1D-PWL, noninvertible maps, which are isomorphic to the original system obtained by DJ. In Section 3.1, we offer a detailed analysis of this 1D map, thereby revisiting the DJ model. We also introduce the notion of synchronized and

[9] Some may find this result surprising, because the presence of complementary (substitutes) sectors is commonly viewed as an amplifying (moderating) factor. However, this result is not inconsistent with such a common view, which is concerned about the propagation of exogenous productivity shocks from one sector to others. This result is concerned about how productivity in various sectors responds endogenously to a change in the market condition. Sectors producing substitutes (complements) respond in the same (opposite) direction, thereby amplifying (moderating) the aggregate fluctuation.

[10] See Mira, Gardini, Barugola and Cathala (1996) for an introduction to 2D noninvertible maps in general, and see Sushko and Gardini (2010) for PWS examples.

asynchronized cycles as well as their basins of attraction in section 3.2. Section 4 then returns to the 2D system in order to study the effects of globalization, or a coupling, on innovation dynamics in the two countries. First, in Section 4.1, we briefly show that its unique steady state has the same features with the equilibrium of the HK model. For the rest of the paper, we assume the parameter condition that ensures the existence of a stable period-2 cycle in the DJ model. In Section 4.2, we consider the symmetric case where the two countries are of equal size. Then, in Section 4.3 we turn to the asymmetric cases to study the role of country size differences on the synchronization effects of globalization. We conclude in Section 5.

2. Model

Time is discrete and indexed by $t \in \{0,1,2,...\}$. The world economy consists of two countries, indexed by j or k = 1 or 2. The representative household of country j inelatically supplies the single nontradable factor, labor, by L_j (measured in its efficiency unit) at the wage rate, w_{jt}. The two countries are structurally identical, and may differ only in labor supply, so we let $L_1 \geq L_2$ without loss of generality. The household consumes the single nontradeable final good, which is competitively produced by assembling the two types of tradeable intermediate inputs, with the following Cobb-Douglas technology:

$$(1) \qquad Y_{kt} = C_{kt} = \left(\frac{X^o_{kt}}{1-\alpha}\right)^{1-\alpha}\left(\frac{X_{kt}}{\alpha}\right)^{\alpha}, \qquad (0 < \alpha < 1),$$

where X^o_{kt} is the homogeneous input, produced with the linear technology that converts one unit of labor into one unit of output. This input is competitively supplied and tradeable at zero cost so that the law of one price holds for this input. By choosing this input as the *numeraire*, we have $w_{jt} \geq 1$, and $w_{jt} = 1$ holds whenever country j produces the homogeneous input. The second type of the inputs, X_{kt}, is a composite of differentiated inputs, aggregated as

$$(2) \qquad \left[X_{kt}\right]^{1-\frac{1}{\sigma}} = \int_{\Omega_t}\left[x_{kt}(\nu)\right]^{1-\frac{1}{\sigma}} d\nu, \quad (\sigma > 1),$$

where $x_{kt}(\nu)$ is the quantity of a differentiated input variety $\nu \in \Omega_t$ used in the final goods production in country k in period t; $\sigma > 1$ is the direct partial elasticity of substitution between a pair of varieties, and Ω_t is the set of differentiated input varieties available in period t, which changes over time due to innovation as well as obsolescence. These differentiated varieties can be classified according to the location of their production as well as whether they are supplied competitively or monopolistically. Thus, $\Omega_t = \sum_j \Omega_{jt} = \sum_j (\Omega_{jt}^c + \Omega_{jt}^m)$, where $\Omega_{jt} = \Omega_{jt}^c + \Omega_{jt}^m$ is the set of differentiated inputs produced in *j* in period *t*: Ω_{jt}^c the set of competitively produced input varieties in *j* in period *t*; and Ω_{jt}^m the set of new input varieties introduced and produced in *j;* sold exclusively (and hence monopolistically) by their innovators for just one period.

Demands for Differentiated Inputs:

Assuming the balanced trade, the demand curves for these inputs by the final goods sector in *k* are derived from (1) as:

$$(3) \qquad x_{kt}(\nu) = \left(\frac{p_{kt}(\nu)}{P_{kt}}\right)^{-\sigma} X_{kt} = \left(\frac{p_{kt}(\nu)}{P_{kt}}\right)^{-\sigma} \frac{\alpha P_{kt}^Y Y_{kt}}{P_{kt}} = \left(\frac{p_{kt}(\nu)}{P_{kt}}\right)^{-\sigma} \frac{\alpha w_{kt} L_k}{P_{kt}},$$

where $p_{kt}(\nu)$ is the unit price of variety *v* in *k*; P_{kt} is the price index for differentiated inputs in *k*, given by

$$(4) \qquad [P_{kt}]^{1-\sigma} = \int_{\Omega_t} [p_{kt}(\nu)]^{1-\sigma} d\nu,$$

and $P_{kt}^Y = (P_{kt})^\alpha$ is the price of the final good. The unit price of variety *v* depends on *k*, because of the (iceberg) trade costs. That is, to supply one unit of a variety $\nu \in \Omega_{jt}$ in *k*, $\tau_{jk} \geq 1$ units need to be shipped from *j*. Then the effective unit price in *k* is $p_{kt}(\nu) = p_{jt}(\nu)\tau_{jk} \geq p_{jt}(\nu)$ for $\nu \in \Omega_{jt}$. Inserting this expression to (3), the total demand for each variety can be obtained as:

$$(5) \qquad D_{jt}(\nu) = \sum_k \tau_{jk} x_{kt}(\nu) = \alpha A_{jt}(p_{jt}(\nu))^{-\sigma}, \text{ for } \nu \in \Omega_{jt}$$

where

$$(6) \qquad A_{jt} \equiv \sum_k \frac{\rho_{jk} w_{kt} L_k}{(P_{kt})^{1-\sigma}}, \quad \text{with } \rho_{jk} \equiv (\tau_{jk})^{1-\sigma} \leq 1,$$

may be interpreted as the demand shift parameter for a variety produced in *j*, with $\rho_{jk} \equiv (\tau_{jk})^{1-\sigma}$ being the weight attached to the aggregate spending in country *k*. We follow HK and assume $\tau_{11} = \tau_{22} = 1$; $\tau_{12} = \tau_{21} = \tau > 1$ so that $\rho_{11} = \rho_{22} = 1$; $\rho_{12} = \rho_{21} = \rho \equiv (\tau)^{1-\sigma} < 1$. Thus, $\rho \in [0,1)$ measures how much the final goods producers spend on an imported variety, relative to what they would spend in the absence of the trade cost; and it is inversely related to τ , with $\rho = 0$ for $\tau = \infty$ and $\rho \to 1$ for $\tau \to 1$. This is our measure of globalization.

Differentiated Inputs Pricing:

Producing one unit of each variety of differentiated inputs requires ψ units of labor, so that the marginal cost is equal to ψw_{jt} for $\nu \in \Omega_{jt}$. Since all competitive inputs produced in the same country are priced at the same marginal cost, and they all enter symmetrically in production, we could write, from (5), as:

(7) $$p_{jt}(\nu) = \psi w_{jt} \equiv p^c_{jt}; \quad D_{jt}(\nu) = \alpha A_{jt}(p^c_{jt})^{-\sigma} \equiv y^c_{jt} \qquad \text{for } \nu \in \Omega^c_{jt} \subset \Omega_{jt}, \qquad (j = 1 \text{ or } 2),$$

where p^c_{jt} and y^c_{jt} are the (common) unit price and output of each competitive variety produced in country *j* and period *t*. Eq. (5) shows that all monopolists face the same constant price elasticity of demand, σ . Thus, they all use the same marked-up rate. Hence all monopolistic varieties produced in the same country are priced equally, and produced by the same amount because they all enter symmetrically in production. Thus,

(8) $$p_{jt}(\nu) = \frac{\psi w_{jt}}{1 - 1/\sigma} \equiv p^m_{jt}; \; D_{jt}(\nu) = \alpha A_{jt}(p^m_{jt})^{-\sigma} \equiv y^m_{jt} \quad \text{for } \nu \in \Omega^m_{jt} = \Omega_{jt} - \Omega^c_{jt},$$

where p^m_{jt} and y^m_{jt} are the (common) unit price and output of each monopolistic variety produced in country *j* and period *t*. From (7) and (8),

(9) $$\frac{p^c_{jt}}{p^m_{jt}} = 1 - \frac{1}{\sigma} < 1; \;\; \frac{y^c_{jt}}{y^m_{jt}} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma} > 1; \quad \frac{p^c_{jt}}{p^m_{jt}}\frac{y^c_{jt}}{y^m_{jt}} = \left(1 - \frac{1}{\sigma}\right)^{1-\sigma} \equiv \theta \in (1, e).$$

Thus, a competitive variety is cheaper, and hence produced and sold more than a monopolistic variety. Furthermore, the final goods producer spends more on a competitive variety than on a monopolistic variety by the factor, θ > 1. Using (7)-(9), the price indices in (4) can be written as:

(10) $$(P_{kt})^{1-\sigma} = \sum_j \left\{N^c_{jt}\left(\tau_{jk} p^c_{jt}\right)^{1-\sigma} + N^m_{jt}\left(\tau_{jk} p^m_{jt}\right)^{1-\sigma}\right\} = \sum_j \left[N^c_{jt} + N^m_{jt}\left(\frac{p^m_{jt}}{p^c_{jt}}\right)^{1-\sigma}\right]\left(\tau_{jk} p^c_{jt}\right)^{1-\sigma}$$

$$= \sum_j \rho_{jk} M_{jt}\left(\psi w_{jt}\right)^{1-\sigma},$$

where N^c_{jt} (N^m_{jt}) denote the measure of Ω^c_{jt} (Ω^m_{jt}) and

(11) $M_{jt} \equiv N^c_{jt} + N^m_{jt}/\theta$,

is the effective total input varieties produced in *j* available to the final goods producers, i.e., which captures the degree of competition that innovators would have to face upon entering. Note that the measure of monopolistic varieties is discounted by θ to convert it to the competitive variety equivalent. Thus, a unit measure of competitive varieties has the same effect with measure θ of monopolistic varieties. With $\theta > 1$, a competitive variety is more discouraging to innovators than a monopolistic variety. Note that θ is monotone increasing in σ, with $\theta \to 1$ as $\sigma \to 1$ and $\theta \to e = 2.71828...$, as $\sigma \to \infty$, and yet, it varies little with σ over an empirically relevant range, with $\theta \approx 2.37$ at $\sigma = 4$ and $\theta \approx 2.62$ at $\sigma = 14$. For this reason, we set $\theta = 2.5$ for all of our numerical demonstrations.[11]

Introduction of New Varieties:

In each period, new varieties of differentiated inputs may be introduced by using *f* units of labor per variety in each country. Following DJ, we assume that innovators hold monopoly over their innovations for only one period, the same period in which their varieties are introduced. With free entry to innovation activities, the net benefit of innovation must be equal to zero, whenever some innovations take place, and it must be negative whenever no innovation takes place. Thus, the following complementarity slackness condition holds:

$$N^m_{jt} \geq 0; \qquad \pi^m_{jt} \equiv (p^m_{jt} - w_{jt}\psi)y^m_{jt} - w_{jt} f \leq 0,$$

where one of the two inequalities holds with the equality: $\pi^m_{jt} N^m_{jt} = 0$. In other words, either *the zero profit condition* or *the non-negativity constraint* on innovation must be binding in each country. Note that the gross benefit of innovation is equal to the monopoly profit earned in the same period in which a new variety is introduced, because innovators lose its monopoly after one period. By using (7)-(9) and (11), these conditions can be further rewritten as

(12) $$N^m_{jt} = \theta(M_{jt} - N^c_{jt}) \geq 0; \quad \alpha A_{jt}(\psi w_{jt})^{-\sigma} = y^c_{jt} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma} y^m_{jt} \leq \frac{\sigma\theta f}{\psi}.$$

For the remainder of this paper, we follow HK and consider the case of non-specialization, where both countries always produce the homogeneous input, which ensures $w_{jt} = 1$**,** for all *t*. (See Appendix A for a sufficient condition for the non-specialization.) By

[11] It turns out that we need $\theta > 2$ (i.e., $\sigma > 2$) for generating endogenous fluctuations.

setting $w_{1t} = w_{2t} = 1$ in eq. (6) and (10) and using $\rho_{11} = \rho_{22} = 1$; $\rho_{12} = \rho_{21} = \rho < 1$, eq. (12), becomes

$$(13)\qquad N^m_{jt} = \theta(M_{jt} - N^c_{jt}) \geq 0;\ \frac{1}{\sigma}\left[\frac{\alpha L_j}{\theta(M_{jt} + \rho M_{kt})} + \frac{\alpha L_k}{\theta(M_{jt} + M_{kt}/\rho)}\right] \leq f, \qquad (j \neq k).$$

Thus, innovation is active in country *j*, if and only if the revenue for a new variety introduced in country *j*, given in the square bracket, is just enough to cover the cost of innovation.[12] The first term in the bracket is the revenue from its domestic market, *j*, equal to its aggregate spending on differentiated inputs, αL_j, divided by the effective competition it faces at home, $\theta(M_{jt} + \rho M_{kt})$ $= \theta N^c_{jt} + N^m_{jt} + \rho(\theta N^c_{kt} + N^m_{kt})$. Notice that the measure of competitive varieties is multiplied by $\theta > 1$, relative to the monopolistic varieties, and that the measure of the foreign varieties are multiplied by $\rho < 1$, relative to the home varieties, due to the disadvantage the foreign varieties suffer in *their* export market, *j*. The second term in the bracket is the revenue from its export market, *k*, equal to its aggregate spending on differentiated inputs, αL_k, divided by the effective competition it faces abroad, $\theta(M_{jt} + M_{kt}/\rho)$ $= \theta N^c_{jt} + N^m_{jt} + (\theta N^c_{kt} + N^m_{kt})/\rho$. Notice that the measure of the foreign varieties are multiplied by $1/\rho > 1$, relative to the home varieties, due to the advantage the foreign varieties enjoy in *their* domestic market, *k*.

Obsolescence of Old Varieties:

All new varieties, introduced and supplied monopolistically by their innovators in period *t*, are added to the existing old varieties of differentiated inputs which are competitively supplied. Each of these varieties is subject to an idiosyncratic obsolescence shock with probability, $1-\delta \in (0,1)$.[13] Thus, a fraction $\delta \in (0,1)$ of them survives and carries over to the next period and become competitively supplied, old varieties. This can be expressed as:

$$(14)\qquad N^c_{jt+1} = \delta\left(N^c_{jt} + N^m_{jt}\right) = \delta\left(N^c_{jt} + \theta(M_{jt} - N^c_{jt})\right). \qquad \delta \in (0,1). \qquad (j = 1 \text{ or } 2)$$

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[12] Note that, from eq. (8), the gross profit per unit of the revenue is $(p_{jt} - \psi w_{jt})/p_{jt} = 1/\sigma$.

[13] Alternatively, we could assume that labor supply in each country may grow at a common, constant factor, $G > 1$; $L_{jt} = L_{j0}(G)^t$, and set $\delta = 1/G < 1$. Then, the measures of varieties per labor would follow the same dynamics. It turns out that we need $\delta > 1/(e-1) = 0.582...$ or $G < e-1 = 1.71828...$ for generating endogenous fluctuations. To see what this implies, let *T* be the period length in years and *g* the annual growth rate of the exogenous component of TFP. Then, $\log G = T\log(1+g) \approx gT < \log(e-1) = 0.235...$.

Dynamical System:

To proceed further, let us introduce normalized measures of varieties as:

$$n_{jt} \equiv \frac{\theta\sigma f N^c_{jt}}{\alpha(L_1+L_2)};\ i_{jt} \equiv \frac{\theta\sigma f N^m_{jt}}{\alpha(L_1+L_2)} \text{ and } m_{jt} \equiv \frac{\theta\sigma f M_{jt}}{\alpha(L_1+L_2)} = n_{jt} + \frac{i_{jt}}{\theta}$$

Then, eqs .(13) can be rewritten as:

(15) $$i_{jt} = \theta(m_{jt} - n_{jt}) \geq 0;\ m_{jt} \geq h_j(m_{kt}),$$

where $h_j(m_k) > 0$ is implicitly defined by

$$\frac{s_j}{h_j(m_k) + \rho m_k} + \frac{s_k}{h_j(m_k) + m_k/\rho} = 1,$$

with $s_j \equiv L_j/(L_1 + L_2)$, the share of country *j*. Eq.(14) can be written as:

(16) $$n_{jt+1} = \delta\left(n_{jt} + i_{jt}\right) = \delta\left(n_{jt} + \theta(m_{jt} - n_{jt})\right) = \delta\left(\theta m_{jt} + (1-\theta)n_{jt}\right)$$

Notice that eq.(15) may be interpreted as the equilibrium conditions of the innovation games played simultaneously in the two countries. Conditional on the current global market condition, $n_t = (n_{1t}, n_{2t}) \in R^2_+$, which shows how saturated the two markets are from past innovations, the innovators in each country decide whether to introduce new varieties, and the outcomes of these games in period *t* determine, by eq.(16), the market condition in the next period, $n_{t+1} = (n_{1t+1}, n_{2t+1}) \in R^2_+$. For any $\rho \in [0,1)$, eq.(15) can be solved for its unique solution, $m_t = (m_{1t}, m_{2t}) \in R^2_+$ as a function of $n_t = (n_{1t}, n_{2t}) \in R^2_+$.[14] By inserting that solution in eq. (16), we obtain the 2D-dynamical system that governs the equilibrium law of motion of our two-country world economy, which we state formally as follows.

[14] One may wonder what happens if $\rho = 1$. Then, the two markets become fully integrated, and there will no home market advantage; the location of innovation no longer matters. As a result, eq.(15) no longer has a unique solution; and $m_t = (m_{1t}, m_{2t}) \in R^2_+$, and hence $i_t = (i_{1t}, i_{2t}) \in R^2_+$ become indeterminate. However, $m_{1t} + m_{2t}$, and hence $i_{1t} + i_{2t}$, is uniquely determined by $n_{1t} + n_{2t}$, and hence the dynamics of the world aggregates follows the same 1D-dynamics obtained by DJ. Effectively, the world economy becomes a single closed economy.

Theorem: For each initial condition, $n_0 \equiv (n_{10}, n_{20}) \in R^2_+$, the equilibrium trajectory, $\{n_t\}_{t=0}^{+\infty}$ = $\{(n_{1t}, n_{2t})\}_{t=0}^{+\infty}$, is obtained by iterating the 2D-dynamical system, $n_{t+1} = F(n_t)$; $F: R^2_+ \to R^2_+$, given by:

$$
(17)\begin{cases}
n_{1t+1} = \delta(\theta s_1(\rho) + (1-\theta)n_{1t}) & \text{for } n_t \in D_{LL} \equiv \{(n_1, n_2) \in R^2_+ | n_j \le s_j(\rho)\} \\
n_{2t+1} = \delta(\theta s_2(\rho) + (1-\theta)n_{2t}) & \\
n_{1t+1} = \delta n_{1t} & \text{for } n_t \in D_{HH} \equiv \{(n_1, n_2) \in R^2_+ | n_j \ge h_j(n_k)\} \\
n_{2t+1} = \delta n_{2t} & \\
n_{1t+1} = \delta n_{1t} & \text{for } n_t \in D_{HL} \equiv \{(n_1, n_2) \in R^2_+ | n_1 \ge s_1(\rho); n_2 \le h_2(n_1)\} \\
n_{2t+1} = \delta(\theta h_2(n_{1t}) + (1-\theta)n_{2t}) & \\
n_{1t+1} = \delta(\theta h_1(n_{2t}) + (1-\theta)n_{1t}) & \text{for } n_t \in D_{LH} \equiv \{(n_1, n_2) \in R^2_+ | n_1 \le h_1(n_2); n_2 \ge s_2(\rho)\} \\
n_{2t+1} = \delta n_{2t} &
\end{cases}
$$

where $s_1(\rho) = 1 - s_2(\rho) = \min\left\{\frac{s_1 - \rho s_2}{1-\rho}, 1\right\}$, with $0.5 \le s_1 = 1 - s_2 < 1$ and $h_j(n_k) > 0$ defined implicitly by $\frac{s_j}{h_j(n_k) + \rho n_k} + \frac{s_k}{h_j(n_k) + n_k/\rho} = 1$.

See Appendix B for the derivation of eq.(17). Once we obtain the equilibrium trajectory for $n_t = (n_{1t}, n_{2t}) \in R^2_+$ by iterating this 2D-system, it is straightforward to obtain the equilibrium trajectory for many other variables of interest. For example, from eq.(15) and eq.(17), the dynamics of innovations, in their normalized form, $i_{jt} = \theta(m_{jt} - n_{jt}) = (n_{jt+1} - \delta n_{jt})/\delta$ can be derived as:

$$
(18)\begin{cases}
i_{1t} = \theta(s_1(\rho) - n_{1t}); & i_{2t} = \theta(s_2(\rho) - n_{2t}) & \text{for } n_t \in D_{LL}, \\
i_{1t} = 0 & i_{2t} = 0 & \text{for } n_t \in D_{HH}, \\
i_{1t} = 0 & i_{2t} = \theta(h_2(n_{1t}) - n_{2t}) & \text{for } n_t \in D_{HL}, \\
i_{1t} = \theta(h_1(n_{2t}) - n_{1t}); & i_{2t} = 0 & \text{for } n_t \in D_{LH}.
\end{cases}
$$

Total factor productivities (TFPs), $Z_{kt} \equiv Y_{kt}/L_k = w_{kt}/(P_{kt})^{-\alpha}$, are, under the non-specialization, $w_{jt} = w_{kt} = 1$, expressed as $Z_{1t} = (\psi)^{-\alpha}(M_{1t} + \rho M_{2t})^{\frac{\alpha}{\sigma-1}}$ and $Z_{2t} = (\psi)^{-\alpha}(\rho M_{1t} + M_{2t})^{\frac{\alpha}{\sigma-1}}$, or, Z_{1t}

$= Z_0(m_{1t} + \rho m_{2t})^{\frac{\alpha}{\sigma-1}}$ and $Z_{2t} = Z_0(\rho m_{1t} + m_{2t})^{\frac{\alpha}{\sigma-1}}$, in their normalized form, with $Z_0 = [\alpha(L_1 + L_2)/\theta\sigma f]^{\frac{\alpha}{\sigma-1}}(\psi)^{-\alpha}$. From this, we can show that TFPs move as:

$$(19)\quad \begin{cases} Z_{1t} = Z_0((1+\rho)s_1)^{\frac{\alpha}{\sigma-1}} & Z_{2t} = Z_0((1+\rho)s_2)^{\frac{\alpha}{\sigma-1}} & \text{for } n_t \in D_{LL}, \\ Z_{1t} = Z_0(n_{1t} + \rho n_{2t})^{\frac{\alpha}{\sigma-1}} & Z_{2t} = Z_0(\rho n_{1t} + n_{2t})^{\frac{\alpha}{\sigma-1}} & \text{for } n_t \in D_{HH}, \\ Z_{1t} = Z_0(n_{1t} + \rho h_2(n_{1t}))^{\frac{\alpha}{\sigma-1}} & Z_{2t} = Z_0(\rho n_{1t} + h_2(n_{1t}))^{\frac{\alpha}{\sigma-1}} & \text{for } n_t \in D_{HL}, \\ Z_{1t} = Z_0(h_1(n_{2t}) + \rho n_{2t})^{\frac{\alpha}{\sigma-1}} & Z_{2t} = Z_0(\rho h_1(n_{2t}) + n_{2t})^{\frac{\alpha}{\sigma-1}} & \text{for } n_t \in D_{LH}. \end{cases}$$

Starting from the next section, we will conduct a step-by-step analysis of the 2D-system, eq. (17). However, it is worth offering some preliminary observations about this system. First, it is characterized by the four parameters: $\theta \in (1, e)$; $\delta \in (0,1)$; $\rho \in [0,1)$; and $s_1 = 1 - s_2 \in [0.5,1)$. (The first two come from DJ, and the second two from HK.) Second, it is a continuous, piece-wise smooth system, consisting of four smooth maps defined over four domains, depending on which of the two inequalities in eq.(15) hold with the equalities in each country. Third, n_{1t+1} is decreasing in n_{1t} in D_{LH} and D_{LL} and increasing in n_{1t} in D_{HH} and D_{HL}. Similarly, n_{2t+1} is decreasing in n_{2t} in D_{LL} and D_{HL} and increasing in n_{2t} in D_{LH} and D_{HH}. This suggests, among others, that the map is noninvertible. Fourth, if $n_{1t}/n_{2t} = s_1(\rho)/s_2(\rho)$, $n_{1t+1}/n_{2t+1} = s_1(\rho)/s_2(\rho)$. Thus, the ray, $\{(n_1, n_2) \in R_+^2 \mid n_1/n_2 = s_1(\rho)/s_2(\rho)\}$, is *forward-invariant*. Once the trajectory reaches there, it stays there forever. However, it is not *backward-invariant*, because the map is noninvertible.[15]

Figure 1 illustrates the four domains and their boundaries for $0 < \rho < s_2/s_1 \leq 1$. For $n_t \in D_{HH}$, both markets are so saturated that there is no innovation, $i_{1t} = 0$ and $i_{2t} = 0$. Due to the obsolescence shocks, $n_{1t+1} = \delta n_{1t}$ and $n_{2t+1} = \delta n_{2t}$, so that the map is contracting toward the origin in this domain. For $n_t \in D_{LL}$, neither market is saturated that innovation is active and the zero profit condition holds in both markets. Due to the obsolescence shocks, the unique steady state of this system is located in this domain, $n^* = (n_1^*, n_2^*) \in D_{LL}$. For $n_t \in D_{HL}$, the non-

[15]A set, $S \subset R_+^2$, is *forward-invariant*, if $F(S) \subset S$, and is *backward-invariant*, if $F^{-1}(S) \subset S$.

negativity constraint is binding in country 1 and the zero-profit condition is binding in country 2. Innovation is thus active only in country 2, given by $i_{2t} = \theta(h_2(n_{1t}) - n_{2t})$. Because $\rho > 0$, which implies $h_2'(n_{1t}) < 0$, innovation in country 2 is discouraged by the competitive varieties based in country 1 (a higher n_{1t}), but not as much as by the competitive varieties based in country 2 (a higher n_{2t}), because $\rho < 1$, which implies $h_2'(n_{1t}) > -1$. Hence, the iso-innovation curves for country 1 in this domain, $n_2 = h_2(n_1) + \bar{i}/\theta$ for $\bar{i} > 0$ (not drawn in Figure 1), are downward-sloping with their slopes less than one in absolute value. So is the border between D_{HL} and D_{LL}, $n_2 = h_2(n_1)$. Likewise, in D_{LH}, the iso-innovation curves for country 2, $n_1 = h_1(n_2) + \bar{i}/\theta$ for $\bar{i} > 0$ (not drawn in Figure 1), are downward-sloping with their slopes greater than one in absolute value. So is the border between D_{LH} and D_{LL}, $n_1 = h_1(n_2)$.

Before proceeding, we offer some words of caution to those accustomed to see the 2D-phase diagram for an ordinary differential equation in two variables. Our model is in discrete time, so that a trajectory generated by iterating eq.(17) can be represented as a *sequence* of points, which hop around in the state space. It cannot be represented as a continuous flow. This is why we did not draw any isocline curves nor any arrows indicating the direction of movements. They are not particularly useful for understanding the dynamics; indeed, they could be misleading.

3. Autarky and Decoupled Innovation Dynamics

We begin our analysis of eq.(17) with the case of autarky, $\rho = 0$. Then, $s_j(\rho) = s_j$ and $h_j(m_k) = s_j$. Hence, eq. (17) becomes:

$$
\begin{cases}
n_{1t+1} = \delta\left(\theta s_1 + (1-\theta)n_{1t}\right) & \text{for } n_t \in D_{LL} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 \le s_1; n_2 \le s_2\right\} \\
n_{2t+1} = \delta\left(\theta s_2 + (1-\theta)n_{2t}\right) & \\
 & \\
n_{1t+1} = \delta n_{1t} & \text{for } n_t \in D_{HH} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 \ge s_1; n_2 \ge s_2\right\} \\
n_{2t+1} = \delta n_{2t} & \\
 & \\
n_{1t+1} = \delta n_{1t} & \text{for } n_t \in D_{HL} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 \ge s_1; n_2 \le s_2\right\} \\
n_{2t+1} = \delta\left(\theta s_2 + (1-\theta)n_{2t}\right) & \\
 & \\
n_{1t+1} = \delta\left(\theta s_1 + (1-\theta)n_{1t}\right) & \text{for } n_t \in D_{LH} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 \le s_1; n_2 \ge s_2\right\} \\
n_{2t+1} = \delta n_{2t}, &
\end{cases}
$$

as illustrated in Figure 2. Not surprisingly, the dynamics of the two countries are unrelated in autarky, and hence the 2D system can be decoupled to two independent 1D systems:

$$(20) \qquad n_{jt+1} = f_j(n_{jt}) = \begin{cases} f_{jL}(n_{jt}) \equiv \delta(\theta s_j + (1-\theta)n_{jt}) & \text{for } n_{jt} \le s_j; \\ f_{jH}(n_{jt}) \equiv \delta n_{jt} & \text{for } n_{jt} \ge s_j. \end{cases} \qquad (0 < \delta < 1;\ 1 < \theta < e)$$

From (18) and (19), innovation and TFP move as:

$$i_{jt} = \theta \max\{s_j - n_{jt}, 0\}; \qquad Z_{jt} = Z_0\left(\max\{s_j, n_{jt}\}\right)^{\frac{\alpha}{\sigma-1}}.$$

3.1 *1D-Analysis of The Skew Tent Map: Revisiting Deneckere-Judd (1992)*

Figure 3 illustrates the 1D-system that governs the dynamics of each country, eq. (20), which is isomorphic to the original DJ system. (We drop the country indices in this subsection.) It is a PWL, noninverible map with the following two branches:[16]

- The ***H*-branch**, defined over $n_t \ge s$, is upward-sloping, and located below the 45º line. With too many competitive varieties, the market is too saturated for innovation. Hence, the non-negativity constraint is binding, $i_t = 0$. With no innovation and $\delta < 1$, the map is contracting over this range.
- The ***L*-branch**, defined over $n_t < s$, is downward-sloping. Without too many competitive varieties, there is active innovation, so that the zero-profit condition is binding. Notice that it is downward sloping because $\theta > 1$. Because old, competitive varieties are more discouraging than new monopolistic varieties, unit measure of additional competitive varieties this period would crowd out $\theta > 1$ measure of new varieties so that the economy will be left with fewer competitive varieties in the next period. This effect is stronger when differentiated varieties are more substitutable (a higher σ and hence, a higher θ).

Since $\delta < 1$, the unique steady state,

$$n^* = \frac{\delta\theta s}{1+(\theta-1)\delta} < s,$$

[16] The map of this form is called the *skew tent map,* which has been fully characterized in the applied math literature: see, e.g., Sushko and Gardini (2010, Section 3.1) and the references therein.

is located in *L*-branch, where the slope of the map is equal to $-\delta(\theta-1)$. Hence, the unique steady state is stable and indeed globally attracting for $\delta(\theta-1)<1$. For $\delta(\theta-1)>1$, it is unstable. For this case, there exists an *absorbing interval*, $J=[\delta s,\ f_L(\delta s)]$, indicated by the red box in Figure 4. Inside the red box, there exists a unique *period 2-cycle*,

$$n_L^* = \frac{\delta^2\theta s}{1+(\theta-1)\delta^2} \leftrightarrow n_H^* = \frac{\delta\theta s}{1+(\theta-1)\delta^2},$$

that alternates between the *L*- and the *H*-branches. This is also illustrated in Figure 4. The graph of the 2nd iterate of the map, $n_{t+2} = f \circ f(n_t) = f^2(n_t)$, shown in blue, crosses the 45° line three times. The red dot indicates the unstable steady state, n^*, where the slope of the 2nd iterate is $f^{2}{}'(n^*) = \left(f'(n^*)\right)^2 = \delta^2(\theta-1)^2 > 1$. The two blue dots, one in the *L*-branch and the other in the *H*-branch, indicate the two points on the period-2 cycle, $n_L^* = f_H(n_H^*) = f_H\left(f_L(n_L^*)\right)$ and $n_H^* = f_L(n_L^*) = f_L\left(f_H(n_H^*)\right)$. The slope of the 2nd iterate at these points is $f'\left(n_L^*\right)f'\left(n_H^*\right) = -\delta^2(\theta-1)$. Hence, for $\delta^2(\theta-1)<1$, the period 2-cycle is stable and attracting from almost all initial conditions (i.e., unless the initial condition is equal to n^* or one of its pre-images).[17] Thus, the attracting 2-cycle exists if and only if $\delta^2(\theta-1)<1<\delta(\theta-1)$. In words, it exists if and only if the survival rate of the existing varieties is high enough that innovation this period is discouraged by high innovation one period ago, but not high enough that it is not discouraged by high innovation two periods ago.

For $\delta^2(\theta-1)>1$, the unique period 2-cycle is unstable. For this range, DJ noted that the 2nd iterate of the map is *expansive* over the absorbing interval, i.e., $\left|f^{2}{}'(n)\right|>1$ for all differentiable points in J, from which they observed in their Theorem 2 that the system has ergodic chaos by appealing to Lasota and York (1973; Theorem 3). In fact, we can say more. From the existing results on the skew tent map, it can be shown that this system has a robust chaotic attractor that consists of one interval, two intervals, four intervals, or more generally, 2^m-

[17] The pre-images of a point, *n*, are all the points that map into *n* after a finite number of iterations. Note that the unstable steady state, n^*, has countably many pre-images because our map is noninvertible. One of them, $n_{-1}^* \equiv f_H^{-1}(n^*)$, is shown in Figure 4.

intervals, (m = 0, 1, 2, …).[18] Figure 5 summarizes the asymptotic behavior of the equilibrium trajectory governed by eq. (20) in the (δ, σ)-plane. Notice that endogenous fluctuations occur with a higher σ (hence a higher θ), which makes competitive varieties even more discouraging to innovators than monopolistic varieties, and with a higher δ, which makes more competitive varieties survive to discourage current innovators.

3.2 *A 2D-View of Autarky: Synchronized vs. Asynchronized 2-Cycles*

Although the innovation dynamics of the two countries in autarky can be independently analyzed, it is useful to view them jointly as a 2D-system to provide a benchmark against which to observe the effects of globalization studied in the next section.

We focus on the case where $\delta(\theta-1) > 1 > \delta^2(\theta-1)$, so that the 1D system of each country has an *unstable* steady state, $n_j^* = \dfrac{\theta\delta s_j}{1+(\theta-1)\delta}$ and a *stable* period 2-cycle,

$n_{jL}^* = \dfrac{\delta^2\theta s_j}{1+(\theta-1)\delta^2} \leftrightarrow n_{jH}^* = \dfrac{\delta\theta s_j}{1+(\theta-1)\delta^2}$, which alternates between the L- and H-branches (i.e., it alternates between the period of active innovation and the period of no innovation). As a 2D-system, the two-country world economy has:

- **An unstable steady state**, $(n_1^*, n_2^*) \in D_{LL}$;
- **A pair of stable period 2-cycles**:
  - **Synchronized 2-cycle:** $(n_{1L}^*, n_{2L}^*) \in D_{LL} \leftrightarrow (n_{1H}^*, n_{2H}^*) \in D_{HH}$, along which innovation in the two countries are active and inactive at the same time. Furthermore, n_{jt}, i_{jt}, and Z_{jt}, move in the *same* direction across the two countries. For this reason, we shall call it the *synchronized* 2-cycle.

---

[18] In contrast, many existing examples of chaos in economics are *not* attracting, particularly those relying on the Li-York theorem of "period-3 implies chaos." The existence of a period-3 cycle on the system defined by a continuous map on the interval merely suggests that the trajectory exhibits an aperiodic (chaotic) fluctuation for *some* initial conditions. For such a chaotic fluctuation to be observable, at least a positive measure of initial conditions must converge to it. Furthermore, most examples of chaotic attractors in economics are *not* robust (i.e., they do *not* exist for an open region of the parameter space), and a transition from period-2 cycle to chaos requires an infinite cascade of bifurcations, as these are general features of a system generated by everywhere smooth maps. Our system can generate a chaotic attractor, which is robust (i.e., existing for an open region of the parameter space) and a transition for the stable 2-cycle to chaos is immediate, because our system is piecewise linear. Sushko and Gardini (2010) discuss more on these issues.

- **Asynchronized 2-cycle:** $\left(n_{1L}^*, n_{2H}^*\right) \in D_{LH} \leftrightarrow \left(n_{1H}^*, n_{2L}^*\right) \in D_{HL}$, along which innovation is active only in one country. Furthermore, n_{jt}, i_{jt}, and Z_{jt}, move in the *opposite* direction across the two countries. For this reason, we shall call it the *asynchronized* 2-cycle. [19]

- **A pair of saddle 2-cycles**: $\left(n_{1L}^*, n_2^*\right) \in D_{LL} \leftrightarrow \left(n_{1H}^*, n_2^*\right) \in D_{HL}$ and $\left(n_1^*, n_{2H}^*\right) \in D_{LH} \leftrightarrow \left(n_1^*, n_{2L}^*\right) \in D_{LL}$.

In Figure 6, the light green dot indicates the unstable steady state, the dark green dots the two saddle 2-cycles, and the black dots the two stable 2-cycles. The red area illustrates the basin of attraction for the synchronized 2-cycle and the white area the basin for the asynchronized 2-cycle. Notice that neither basin of attraction is connected, which is one of the features of a noninvertible map.[20] The boundaries of these basins are formed by the *closure* of the *stable* sets of the two saddle 2-cycles.[21]

4. Globalization and Interdependent Innovation Dynamics: 2D Analysis

We now turn to the case $\rho > 0$ to study the effects of globalization.

4.1 A Brief Look at the Unique Steady State: Reinterpreting Helpman-Krugman (1985)

First, we look at the unique steady state of eq.(17),

$$\left(n_1^*, n_2^*\right) = \frac{\delta\theta}{1+\delta(\theta-1)}\left(s_1(\rho), s_2(\rho)\right),$$

which is stable and globally attracting if $\delta(\theta-1) < 1$. At this steady state, innovations and the effective measures of the varieties produced in each country are given by:

---

[19] Later we will call any 2-cycle that alternates between D_{HH} and D_{LL} *synchronized* and any 2-cycle that alternates between D_{HL} and D_{LH} *asynchronized* also in asymmetric cases.

[20] To see why the two basins of attraction show the chess board patterns in Figure 6, consider the dynamical system defined by the 2nd iterate of the map, eq.(20), whose graph is shown in blue in Figure 4. It has two stable fixed points, n_L^* and n_H^*, whose basins of attraction are given by alternating intervals, which are separated by its unstable fixed point, n^*, its immediate pre-image, $n_{-1}^* \equiv f_H^{-1}(n^*)$, and all of its pre-images. If both countries start from the basin of attraction for n_L^* (n_H^*), they converge to the synchronized 2-cycle in which they both innovate in every even (odd) period. On the other hand, if one country starts from the basin of attraction for n_L^* and the other starts from the basin of attraction for n_H^*, they converge to the asynchronized 2-cycle in which one country innovates in every even period and the other innovates in every odd period.

[21] The *stable set* of an invariant set (say, a fixed point, a cycle, etc.) is the set of all initial conditions that converge to it. It is necessary to take the closure in order to include the unstable steady state and all of its pre-images.

$$\left(i_1^*, i_2^*\right) = \frac{(1-\delta)\theta}{1+\delta(\theta-1)}\left(s_1(\rho), s_2(\rho)\right); \qquad \left(m_1^*, m_2^*\right) = \left(s_1(\rho), s_2(\rho)\right)$$

Figure 7a shows how the share of country 1 in these variables depends on its size at the steady state. In the interior, it is equal to:

$$s_n \equiv \frac{n_1^*}{n_1^* + n_2^*} = \frac{i_1^*}{i_1^* + i_2^*} = \frac{m_1^*}{m_1^* + m_2^*} = s_1(\rho) = \frac{(1+\rho)s_1 - \rho}{1-\rho}.$$

Notice that the slope is $(1+\rho)/(1-\rho) > 1$. Thus, a disproportionately larger share of input varieties is produced and a disproportionately large share of innovation is done in the country that has the larger domestic market and hence the larger country becomes the net exporter of the differentiated inputs varieties (*Home Market Effect*), with the smaller country becoming the net exporter of the homogeneous input. Furthermore, this effect becomes *magnified* if the trade cost become *smaller* (i.e. with a *larger* ρ), as shown in Figure 7b.[22] Thus, the steady state of our model shares the same properties with the equilibrium of the static HK model.

One might think that the comparative steady state analysis of this kind would make sense only if the steady state is stable, i.e., $\delta(\theta-1) < 1$. In fact, the above comparative analysis is also informative even when the steady state is unstable, because globalization causes synchronized cycles and the share of country 1 asymptotically converges to the same steady state value, s_n, as will be shown in Section 4.3.

For the remainder of this paper, we assume that the unique steady state is unstable, $\delta(\theta-1) > 1$. Indeed, we will focus on the cases where the dynamics of each country converges to the stable period-2 cycle in autarky, $\delta(\theta-1) > 1 > \delta^2(\theta-1)$.

4.2 Synchronization Effects of Globalization: Symmetric Cases

In this section, we assume that the two countries are of equal size ($s_1 = 1/2$), so that the 2D-system defined by eq.(17), becomes symmetric as follows.

[22]Note that the graph in Figure 7b is a correspondence at $\rho = 1$ (the lack of lower hemi-continuity), because the equilibrium allocation is indeterminate if $\rho = 1$, as pointed out earlier.

$$
(21)\begin{cases}
n_{1t+1} = \delta\left(\theta/2 + (1-\theta)n_{1t}\right) & \text{for } n_t \in D_{LL} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_j \le 1/2\right\} \\
n_{2t+1} = \delta\left(\theta/2 + (1-\theta)n_{2t}\right) & \\
n_{1t+1} = \delta n_{1t} & \text{for } n_t \in D_{HH} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_j \ge h(n_k)\right\} \\
n_{2t+1} = \delta n_{2t} & \\
n_{1t+1} = \delta n_{1t} & \text{for } n_t \in D_{HL} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 \ge 1/2; n_2 \le h(n_1)\right\} \\
n_{2t+1} = \delta\left(\theta h(n_{1t}) + (1-\theta)n_{2t}\right) & \\
n_{1t+1} = \delta\left(\theta h(n_{2t}) + (1-\theta)n_{1t}\right) & \text{for } n_t \in D_{LH} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 \le h(n_2); n_2 \ge 1/2\right\} \\
n_{2t+1} = \delta n_{2t} &
\end{cases}
$$

where $h(n) > 0$ is defined implicitly by $\dfrac{1}{h(n) + \rho n} + \dfrac{1}{h(n) + n/\rho} = 2$.

Figure 8 shows the symmetric 2D system, with the blue arrows illustrating how the four domains change with ρ. First, the diagonal, $\Delta \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 = n_2\right\}$, is forward-invariant, and the dynamics on Δ is independent of ρ. In fact, it is the skew tent map, given by eq. (20) with $s_j = 1/2$. Second, ρ has no effect on D_{LL}. Third, in D_{LH}, a higher ρ reduces innovation in 1, given by $i_1 = \theta(h(n_2) - n_1)$, as the competitive varieties produced in 2, n_2, discourages innovators in 1. This also causes D_{LH} to shrink and D_{HH} to expand, with the boundary, $n_1 = h(n_2)$, initially vertical (as $n_1 = 1/2$) at $\rho = 0$, tilts counter-clockwise as ρ increases, and approaching to $n_1 = 1 - n_2$ as $\rho \to 1$. A higher ρ also tilts the iso-innovation curves in D_{LH}, $n_1 = h(n_2) + \bar{i}_1/\theta$ (not drawn; horizontally paralleled to the boundary between D_{LH} and D_{HH}), in the same way. Likewise, a higher ρ reduces innovation in 2 in D_{HL}. This causes D_{HL} to shrink and D_{HH} to expand, with the boundary, $n_2 = h(n_1)$, initially horizontal (as $n_2 = 1/2$) at $\rho = 0$, tilting clockwise as ρ increases, and approaching to $n_2 = 1 - n_1$ as $\rho \to 1$. It has the same tilting effect on the iso-innovation curves in D_{HL}, $n_2 = h(n_1) + \bar{i}_2/\theta$ (not drawn; vertically paralleled to the boundary between D_{HL} and D_{HH}). Taken together, this implies that a higher ρ causes the alignment of innovation incentives across the two countries, in the sense that both a higher n_1 and a higher n_2 have similar discouraging effects on the innovators in both countries.

For $\delta(\theta-1) > 1 > \delta^2(\theta-1)$, each country would have an *unstable* steady state, $n_j^* = n^*$ $\equiv \dfrac{\theta\delta/2}{1+(\theta-1)\delta}$ and a *stable* 2-cycle, $n_{jL}^* = n_L^* \equiv \dfrac{\delta^2\theta/2}{1+(\theta-1)\delta^2} \leftrightarrow n_{jH}^* = n_H^* \equiv \dfrac{\delta\theta/2}{1+(\theta-1)\delta^2}$ in autarky, $\rho = 0$. Thus, as already pointed out in Section 3.2, the world economy consisting of the two countries in autarky has the two stable 2-cycles. One of them is the synchronized 2-cycle, $(n_L^*, n_L^*) \in D_{LL} \leftrightarrow (n_H^*, n_H^*) \in D_{HH}$. The other is the symmetric asynchronized cycle, $(n_L^*, n_H^*) \in D_{LH} \leftrightarrow (n_H^*, n_L^*) \in D_{HL}$.

Now, let ρ rise. Since the diagonal is invariant, and ρ has no effect on the dynamics in D_{LL} and D_{HH}, **the synchronized 2-cycle**, $(n_L^*, n_L^*) \in D_{LL} \leftrightarrow (n_H^*, n_H^*) \in D_{HH}$, exists for all $\rho \in (0,1)$. Indeed, it is independent of ρ and its local stability is not affected.

In addition, there exists a unique **symmetric asynchronized 2-cycle**, $(n_L^a, n_H^a) \in D_{LH} \leftrightarrow (n_H^a, n_L^a) \in D_{HL}$, for all $\rho \in (0,1)$. To see this, if it exists, n_L^a and n_H^a must satisfy, from eq. (21), $n_H^a = \delta(\theta h(n_H^a) + (1-\theta)n_L^a) = \delta(\theta h(n_H^a) + (1-\theta)\delta n_H^a)$, which can be written more compactly as:

$$h(n_H^a) = \beta n_H^a, \text{ where } \beta \equiv \frac{1+\delta^2(\theta-1)}{\delta\theta} \in (\delta, 1).$$

By inserting this expression into the definition of h, we obtain

$$(22) \qquad n_L^a = \delta n_H^a = \frac{\delta}{2}\left(\frac{1}{\beta+\rho} + \frac{1}{\beta+1/\rho}\right).$$

Note that $\beta < 1$ implies $n_H^a = \dfrac{1}{2}\left(\dfrac{1}{\beta+\rho} + \dfrac{1}{\beta+1/\rho}\right) > \dfrac{1}{2}\left(\dfrac{1}{1+\rho} + \dfrac{1}{1+1/\rho}\right) = \dfrac{1}{2}$ and that $\beta > \delta$ implies $n_L^a = \delta n_H^a < \beta n_H^a = h(n_H^a)$. This proves the existence and the uniqueness of the symmetric asynchronized 2-cycle, $(n_L^a, n_H^a) \in D_{LH} \leftrightarrow (n_H^a, n_L^a) \in D_{HL}$.

For $\rho = 0$, this 2-cycle is equal to $(n_L^*, n_H^*) \leftrightarrow (n_H^*, n_L^*)$. However, it moves continuously as ρ varies, and is not equal to $(n_L^*, n_H^*) \leftrightarrow (n_H^*, n_L^*)$, for $\rho > 0$. Furthermore, it becomes unstable for a sufficiently large ρ. More formally,

Proposition: For all $\rho \in (0,1)$, there exists a unique symmetric asynchronized 2-cycle, $\left(n_L^a, n_H^a\right) \in D_{LH} \leftrightarrow \left(n_H^a, n_L^a\right) \in D_{HL}$, given by

$$n_H^a = \frac{1}{2}\left(\frac{1}{\beta+\rho} + \frac{\rho}{\rho\beta+1}\right) > \frac{1}{2}; \qquad n_L^a = \frac{\delta}{2}\left(\frac{1}{\beta+\rho} + \frac{\rho}{\rho\beta+1}\right) = \delta n_H^a < \beta n_H^a = h(n_H^a)$$

where $\beta \equiv \dfrac{1+\delta^2(\theta-1)}{\delta\theta} \in (\delta,1)$ and $h(n) > 0$ solves $\dfrac{1}{h(n)+\rho n} + \dfrac{1}{h(n)+n/\rho} = 2$. Furthermore,

i) For $0 < \gamma(\rho) < 2\sqrt{\theta-1}/\theta$, it is a stable focus;

ii) For $2\sqrt{\theta-1}/\theta < \gamma(\rho) < \beta$, it is a stable node;

iii) For $\beta < \gamma(\rho) < 1$, it is a saddle,

where $\gamma(\rho) \equiv \dfrac{(\beta+1/\rho)^2\rho + (\beta+\rho)^2/\rho}{(\beta+1/\rho)^2 + (\beta+\rho)^2}$ is a continuous, increasing function with $\gamma(0) = 0$ and $\gamma(1) = 1$.

See Appendix C for the proof. This proposition says that the unique symmetric asynchronized 2-cycle exists for all $\rho \in (0,1)$, but it is stable for $\rho \in (0, \rho_c)$ and unstable for $\rho \in (\rho_c, 1)$, where $\rho_c \in (0,1)$ is given by $\gamma(\rho_c) = \beta$. Thus, for a sufficiently large ρ (or a sufficiently small trade cost), the stable asynchronized 2-cycle disappears.

Furthermore, even before its disappearance, a higher ρ expands the basin of attraction for the synchronized 2-cycle and reduces that for the asynchronized 2-cycle for $\rho \in (0, \rho_c)$. Figures 9a-c show this numerically with three different values of δ = 0.7, = 0.75, and = 0.8.[23] In all three cases, an increase in ρ cause the red area (the basin of attraction for the synchronized 2-cycle) to expand and the white area (the basin of attraction for the symmetric asynchronized 2-cycle) to shrink. These figures show that the red area fills most of the state space at ρ = 0.8. However, the symmetric asynchronized 2-cycle is still stable at ρ = 0.8, so that the white area still occupies a positive (though very small) measure of the state space. Only at a higher value of $\rho = \rho_c$, the symmetric asynchronized 2-cycle loses its stability. For $\rho > \rho_c$, the red area covers a full measure of the state space (i.e., the synchronized 2-cycle becomes the

[23]Recall that the stable 2-cycle exists in autarky for $\delta(\theta-1) > 1 > \delta^2(\theta-1)$, which implies $\delta \in (0.666...,0.816...)$ for θ = 2.5.

unique attractor and the equilibrium trajectory converges to the synchronized 2-cycle for almost all initial conditions).[24]

4.3 Synchronization Effects of Globalization: Asymmetric Cases

We now turn to the cases where the two countries differ in size; $s_1 > 0.5 > s_2$. We continue to assume $\delta(\theta-1) > 1 > \delta^2(\theta-1)$ so that, in autarky, each country has an *unstable* steady state, and a *stable* period 2-cycle. Thus, viewed as a 2D-system, the world economy has an unstable steady state, a pair of stable 2-cycles, one synchronized and one asynchronized, whose basins of attraction are already shown in Figure 6 as Red and White, and the boundaries of the two basins are given by the closure of the stable sets of a pair of saddle 2-cycles, as already pointed out in Section 3.2.

Now, let ρ rise. The blue arrows in Figure 10a illustrate the effects of a higher ρ, which are absent in the symmetric case. That is, these effects are *in addition to* those illustrated by the blue arrows in Figure 8 for the symmetric case. With unequal country sizes, $s_1 > 0.5 > s_2$, a higher ρ increases $s_1(\rho) = 1 - s_2(\rho)$, which is nothing but the magnification of the home market effect in the HK model. This causes the ray, $n_{1t}/n_{2t} = s_1(\rho)/s_2(\rho)$, to rotate clockwise, and the border point of the four domains, $(s_1(\rho), s_2(\rho))$, to move southeast. This continues until $\rho = s_2/s_1$, when D_{LL} and D_{HL}, vanish. For $\rho > s_2/s_1$, there is no innovation in country 2, as shown in Figure 10b.

[24]Techinically speaking, the symmetric asynchronized 2-cycle, $(n_L^a, n_H^a) \in D_{LH} \leftrightarrow (n_H^a, n_L^a) \in D_{HL}$, undergoes a *subcritical pitchfork bifurcation* at $\rho = \rho_c$. Recall that the closure of the stable sets of the symmetric pair of saddle 2-cycles form the boundaries of the red and white areas. At $\rho = 0$, this symmetric pair of saddle 2-cycles are given by $(n_L^*, n^*) \in D_{LL} \leftrightarrow (n_H^*, n^*) \in D_{HL}$ and $(n^*, n_H^*) \in D_{LH} \leftrightarrow (n^*, n_L^*) \in D_{LL}$. As ρ rises, they move and simultaneously cross the boundary of D_{LL} at $\rho = \rho_{cc} < \rho_c$, after which they become a symmetric pair of saddles of the form, $(n_L^{'}, n_H^{''}) \in D_{LH} \leftrightarrow (n_H^{'}, n_L^{''}) \in D_{HL}$ and $(n_L^{''}, n_H^{'}) \in D_{LH} \leftrightarrow (n_H^{''}, n_L^{'}) \in D_{HL}$. Thus, for $\rho \in (\rho_{cc}, \rho_c)$, there exist three asynchronized 2-cycles; a symmetric pair of asymmetric asynchronized 2-cycles, which are saddles, and the symmetric asynchronized 2-cycle, which is stable. Then, as $\rho \to \rho_c$, the symmetric pair of the saddle 2-cycles merge with the symmetric asynchronized 2-cycle and disappear, after which the latter becomes a saddle. However, the interval, $\rho \in (\rho_{cc}, \rho_c)$, seems very narrow. According to our calculation, $0.87735830 < \rho_{cc} < 0.87735831 < \rho_c < 0.87735832$ for $\delta = 0.7$; $0.8333226 < \rho_{cc} < 0.8333227 < \rho_c < 0.8333228$ for $\delta = 0.75$; and $0.8189858 < \rho_{cc} < 0.8189859$; $0.8189860 < \rho_c < 0.8189861$ for $\delta = 0.8$.

As long as $0 < \rho < s_2 / s_1 < 1$, innovation will never stop in neither country. For this range, there always exists the stable synchronized 2-cycle, $\left(n_{1L}^s, n_{2L}^s\right) \in D_{LL} \leftrightarrow \left(n_{1H}^s, n_{2H}^s\right) \in D_{HH}$, where

$$n_{jL}^s \equiv \frac{\delta^2 \theta s_j(\rho)}{1+(\theta-1)\delta^2}; \qquad n_{jH}^s = \frac{\delta\theta s_j(\rho)}{1+(\theta-1)\delta^2}.$$

Along this synchronized 2-cycle, the world economy alternates between D_{LL} and D_{HH} , and stays on the ray, $n_{1t} / n_{2t} = s_1(\rho)/ s_2(\rho)$, and hence the share of country 1 is equal to $s_1(\rho)$.

There also exists a stable asynchronized 2-cycle, $\left(n_{1L}^a, n_{2H}^a\right) \in D_{LH} \leftrightarrow \left(n_{1H}^a, n_{2L}^a\right) \in D_{HL}$, for a small enough $\rho < \rho_c$. For $\rho > \rho_c$, it disappears.[25] Furthermore, even before its disappearance, a higher ρ causes the basin of attraction for the synchronized 2-cycle to expand and the basin of attraction for the asynchronized 2-cycle to shrink. Furthermore, this occurs more rapidly with a higher s_1. Figures 11a-d illustrate these numerically, for four different values of s_1 = 0.55, = 0.6, = 0.7, and = 0.8, for δ = 0.75. Notice that the red becomes dominant faster for a higher s_1 = 0.6. These figures also show a sudden appearance of infinitely many red islands inside the white area just before the disappearance of the asynchronized 2-cycle.[26] The results are very similar for δ = 0.7 and δ = 0.8.

We have also estimated ρ_c, the critical value at which the stable asynchronized 2-cycle disappears, leaving the synchronized 2-cycle as the unique attractor. This is reported in this Table ($\theta = 2.5$ for all).

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[25] At $\rho = \rho_c$, the stable asynchronized 2-cycle collides with one of the (no longer symmetric) pair of saddle 2-cycles co-existing for $\rho < \rho_c$, and they both disappear via a *fold (border collision) bifurcation*.

[26] This is due to a *contact bifurcation*, where a critical curve crosses the basin boundary, after which a new set of countably infinite pre-images are created, another common occurrence in systems with noninvertible maps.

TABLE

		δ		
		0.7	0.75	0.8
s_1	0.5	0.8773	0.8333	0.8189
	0.505	0.6416	0.6341	0.6310
	0.51	0.5749	0.5697	0.5676
	0.53	0.4513	0.4486	0.4475
	0.55	0.3871	0.3852	0.3845
	0.6	0.2929	0.2918	0.2913
	0.65	0.2325	0.2317	0.2314
	0.7	0.1860	0.1854	0.1851
	0.8	0.1126	0.1122	0.1120
	0.9	0.0525	0.0523	0.0522

Notice that it declines very rapidly as s_1 increases from 0.5, but it hardly changes with δ**.** Figure 12 show the graph of the critical value as a function of s_1 for δ = 0.7, = 0.75, and = 0.8.[27] Each shows that the critical value declines sharply, as s_1 increases from 0.5. Thus, even a small difference in country sizes would cause synchronization to occur very rapidly.

An interesting question is this. Suppose that the two countries are initially out of sync in autarky. And when globalization causes them to synchronize, which country sets the tempo of global innovation cycles. Or to put it differently, which country adjusts its rhythm to synchronize? Is it the smaller country or the larger country?[28] To answer this question, we look at the 2nd iterate of the map, $n_{t+2} = F \circ F(n_t) \equiv F^2(n_t)$, and its *four* stable steady states, which are the four points on the two stable 2-cycles. In Figure 13, we use the following four colors to indicate the four basins of attraction for the four stable steady states of the 2nd iterate.

- **Red:** Basin of attraction for the stable steady state in D_{LL}. This corresponds to the set of initial conditions that converges to the synchronized 2-cycle along which the trajectory visits D_{LL} in even periods and D_{HH} in odd periods.
- **Azure:** Basin of attraction for the stable steady state in D_{HH}. This corresponds to the set of initial condtions that converges to the synchronized 2-cycle along which the trajectory visits D_{HH} in even periods and D_{LL} in odd periods.

[27]The three graphs vary little with δ. We would not be able to tell them apart, if we were to superimpose them.

[28] We thank Gadi Barlevy for posing this question to us.

- **White:** Basin of attraction for the stable steady state in D_{LH}. This corresponds to the set of initial conditions that converges to the asynchronized 2-cycle along which the trajectory visits D_{LH} in even periods and D_{HL} in odd periods.
- **Gray:** Basin of attraction for the stable steady state in D_{HL}. This corresponds to the set of initial conditions that converge to the asynchronized 2-cycle along which the trajectory visits D_{HL} in even periods and D_{LH} in odd periods.

Synchronization means that Red and Azure expand, while White and Gray shrink. Figure 13 shows that, as ρ goes up, and synchronization occurs by Red invading White and Azure invading Gray, instead of Red invading Gray and Azure invading White, and we observe the emergence of vertical slips of Red and Azure. We have experimented with many different values of parameters, but this pattern has been always observed. This means that the tempo of synchronized fluctuations is dictated by the rhythm of country 1, which is the larger country and that country 2, the smaller country, adjusts its rhythm to the rhythm of the larger country.

5. Concluding Remarks

We proposed and analyzed a two-country model of endogenous innovation cycles, built on the work of Deneckere-Judd (1992) and Helpman-Krugman (1985). In Autarky, innovation dynamics of the two countries are decoupled. As trade cost falls and intra-industry trade rise, they become more synchronized. This is because globalization leads to the alignment of innovation incentives across innovators based in different countries, as they operate in the increasingly global (hence common) market environment. Synchronization occurs faster when the two countries are more unequal in size. And it is the larger country that dictates the tempo of global innovation cycles, with the smaller country adjusting its rhythm to the rhythm of the larger country. This is because the innovators based in the smaller country rely more heavily on the profit earned in its larger export market to recover the cost of innovation than those based in the larger country. Our results suggest that adding endogenous sources of fluctuations would help improve our understanding of why countries that trade more with each other have more synchronized business cycles.

One obvious next step is to use different models of international trade to examine the effects of globalization as a coupling of two DJ mechanisms of endogenous fluctuations of innovation. For example, what if the two countries are vertically specialized through some types of global supply chains? Our conjecture is that, if globalization takes the form of greater vertical integration, it would lead to *asynchronization* of innovation cycles. This is because, unlike the two countries in the HK model, which produce and trade highly substitutable, horizontally differentiated goods, vertical chains make the production structure of the two countries complementary. Then, as the goods innovated in the past in one country lose their monopoly, they become cheaper, which discourage the innovators in that country, but encourages the innovators in the other country, which produces their complementary goods.[29] If this conjecture is confirmed, it is certainly empirically not inconsistent, because the evidence for the synchronizing effect of trade is strong among developed countries, but *less so* between developed and developing countries.

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[29] This may come as a surprise to those familiar with the existing studies that try to explain synchronization of business cycles with vertical specialization. However, it is not contradictory, because these studies look at the propagation effects of a country specific productivity shock from one country to another. Here, we are considering how productivity of different countries responds endogenously to a change in the global market condition. In this paper, we showed that productivity movements synchronize when the two countries produce highly substitutable goods. We conjecture that productivity movements would be asynchronized when the two countries produce complements.

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Appendices:

Appendix A: The sufficient condition for the non-specialization

Country *j* produces the homogeneous input if and only if the total labor demand by its differentiated inputs sector falls short of its labor supply. That is, $L_j > N^c_{jt}(\psi y^c_{jt}) + N^m_{jt}(\psi y^m_{jt} + f)$ $= \dfrac{N^c_{jt}(p^c_{jt}y^c_{jt}) + N^m_{jt}(p^m_{jt}y^m_{jt})}{w_{jt}} = \left[N^c_{jt} + N^m_{jt}\left(\dfrac{p^m_{jt}x^m_{jt}}{p^c_{jt}x^c_{jt}}\right)\right]\left(\dfrac{p^c_{jt}x^c_{jt}}{w_{jt}}\right) = M_{jt}\psi y^c_{jt}$, or $L_j / M_{jt} > \psi y^c_{jt}$. From eq.(12), this inequality is guaranteed if $1 > \sigma\theta f M_{jt}/L_j \equiv \alpha m_{jt}/s_j$. Thus, both countries always produce the homogenous input if $0 < \alpha < \min\left\{\dfrac{s_1}{m_{1t}}, \dfrac{s_2}{m_{2t}}\right\}$ along the sequence, satisfying eqs. (15) and (16), which is bounded so that the upper bound is strictly positive.

Appendix B: Derivation of eq.(17) from eqs.(15) and eq. (16)

We discuss only the case of $0 < \rho < s_2/s_1 \le 1$, which implies $0.5 \le s_1(\rho) = 1 - s_2(\rho) <$ 1. The case of $s_2/s_1 \le \rho < 1$, which implies $s_1(\rho) = 1 - s_2(\rho) = 1$, is similar (and simpler).

First, note that $h_j(m_k) > 0$, defined by $\dfrac{s_j}{h_j(m_k) + \rho m_k} + \dfrac{s_k}{h_j(m_k) + m_k/\rho} = 1$, has the following properties, as seen in Figure 14.

- They are hyperbole, monotone decreasing with $h_j(m_k) \to 1$ as $m_k \to 0$ and $h_j(0) = 1$ and $h_j(m_k) \to 0$ as $m_k \to s_j/\rho + \rho s_k$.
- $m_1 = h_1(m_2)$ and $m_2 = h_2(m_1)$ intersect at $(m_1, m_2) = (s_1(\rho), s_2(\rho))$ in the positive quadrant.
- $m_1 > h_1(h_2(m_1))$ implies $m_1 > s_1(\rho)$ and $m_2 > h_2(h_1(m_2))$ implies $m_2 > s_2(\rho)$.

We now consider each of the four cases in eq.(15).

i) Suppose $m_{jt} > n_{jt}$ for both j = 1 and 2. Then, from (15), $m_{1t} = h_1(m_{2t})$ and $m_{2t} = h_2(m_{1t})$, hence $n_{jt} < m_{jt} = s_j(\rho)$. Inserting these expressions in eq. (16) yields the map for the interior of D_{LL}.

ii) Suppose $m_{1t} > h_1(m_{2t})$ and $m_{2t} > h_2(m_{1t})$. Then, from (15), $m_{jt} = n_{jt}$ for both *j* = 1 and 2, hence $n_{1t} > h_1(n_{2t})$ and $n_{2t} > h_2(n_{1t})$. Inserting these expressions in (16) yields the map for the interior of D_{HH}.

iii) Suppose $m_{1t} > h_1(m_{2t})$ and $m_{2t} > n_{2t}$. Then, from (15), $m_{1t} = n_{1t}$ and $m_{2t} = h_2(m_{1t})$, hence $n_{1t} > h_1(h_2(n_{1t}))$, which implies $n_{1t} > s_1(\rho)$ and $n_{2t} < h_2(n_{1t})$. Inserting these expressions in (16) yields the map for the interior of D_{HL}.

iv) Supposing $m_{1t} > n_{1t}$ and $m_{2t} > h_2(m_{1t})$ similarly yields the map for the interior of D_{LH}.

Finally, it is straightforward to show that the map is continuous at the boundaries of these four domains.

Appendix C: Proof of Proposition

Since the unique existence of the symmetric asynchronized 2-cycle has been shown in the text, we only need to investigate its local stability properties. From $(n_H^a, n_L^a) = F(n_L^a, n_H^a)$ and $(n_L^a, n_H^a) = F(n_H^a, n_L^a)$, the Jacobian matrix at the asynchronized 2-cycle can be calculated as:

$$J = \delta\begin{bmatrix} 1 & 0 \\ -\theta\gamma & 1-\theta \end{bmatrix}\delta\begin{bmatrix} 1-\theta & -\theta\gamma \\ 0 & 1 \end{bmatrix} = \delta^2\begin{bmatrix} 1-\theta & -\theta\gamma \\ -(1-\theta)\theta\gamma & 1-\theta+\theta^2\gamma^2 \end{bmatrix}$$

where $\gamma \equiv -h'(n_H^a) > 0$. Its eigenvalues are the roots of its characteristic function,

$$F(\lambda) \equiv \lambda^2 - tra(J)\lambda + \det(J) = \lambda^2 - \delta^2\{2(1-\theta)+\theta^2\gamma^2\}\lambda + \delta^4(1-\theta)^2 = 0.$$

They are complex conjugated if $[tra(J)]^2 < 4\det(J) \Leftrightarrow \delta^4\{2(1-\theta)+\theta^2\gamma^2\}^2 < 4\delta^4(1-\theta)^2 \Leftrightarrow$

$$0 < \gamma < \frac{2\sqrt{\theta-1}}{\theta} < 1.$$

Its modulus is $\sqrt{\det(J)} = \delta^2(\theta-1) < 1$, hence the 2-cycle is a stable focus in this range.

For $\frac{2\sqrt{\theta-1}}{\theta} < \gamma < 1$, $[tra(J)]^2 \geq 4\det(J)$, so that $F(\lambda) = 0$ has two real roots. At $\gamma = \frac{2\sqrt{\theta-1}}{\theta}$, they are both equal to $\lambda = \delta^2(\theta-1) < 1$. For a higher γ, the two real roots are distinct, and satisfy $0 < \lambda_1 < \delta^2(\theta-1) < \lambda_2 < 1$, if $F(1) = 1-\delta^2\{2(1-\theta)+\theta^2\gamma^2\}+\delta^4(1-\theta)^2 > 0$ $\Leftrightarrow \gamma^2 < [1-\delta^2(1-\theta)]^2/\delta^2\theta^2 \equiv \beta^2$. That is, for

$$\frac{2\sqrt{\theta-1}}{\theta} < \gamma < \frac{1+\delta^2(\theta-1)}{\delta\theta} \equiv \beta,$$

the 2-cycle is a stable node. For $\gamma > \beta$, $F(1) < 0$ and $0 < \lambda_1 < 1 < \lambda_2$, so that the 2-cycle is a saddle.

To obtain γ, differentiate the definition of h,

$$\frac{1}{h(n)+\rho n} + \frac{1}{h(n)+n/\rho} = 2,$$

with respect to n to have

$$\frac{h'(n)+\rho}{(h(n)+\rho n)^2} + \frac{h'(n)+1/\rho}{(h(n)+n/\rho)^2} = 0.$$

By evaluating this expression at $n = n_H^a$, and using $\gamma \equiv -h'(n_H^a)$ and $\beta n_H^a = h(n_H^a)$,

$$\frac{\rho-\gamma}{(\beta+\rho)^2} + \frac{1/\rho-\gamma}{(\beta+1/\rho)^2} = 0$$

from which,

$$\gamma \equiv -h'(n_H^a) = \frac{(\beta+1/\rho)^2\rho + (\beta+\rho)^2/\rho}{(\beta+1/\rho)^2 + (\beta+\rho)^2} \equiv \gamma(\rho).$$

Figure 1: The State Space and The Four Domains of the 2D System (for $0 < \rho < s_2/s_1 \le 1$).

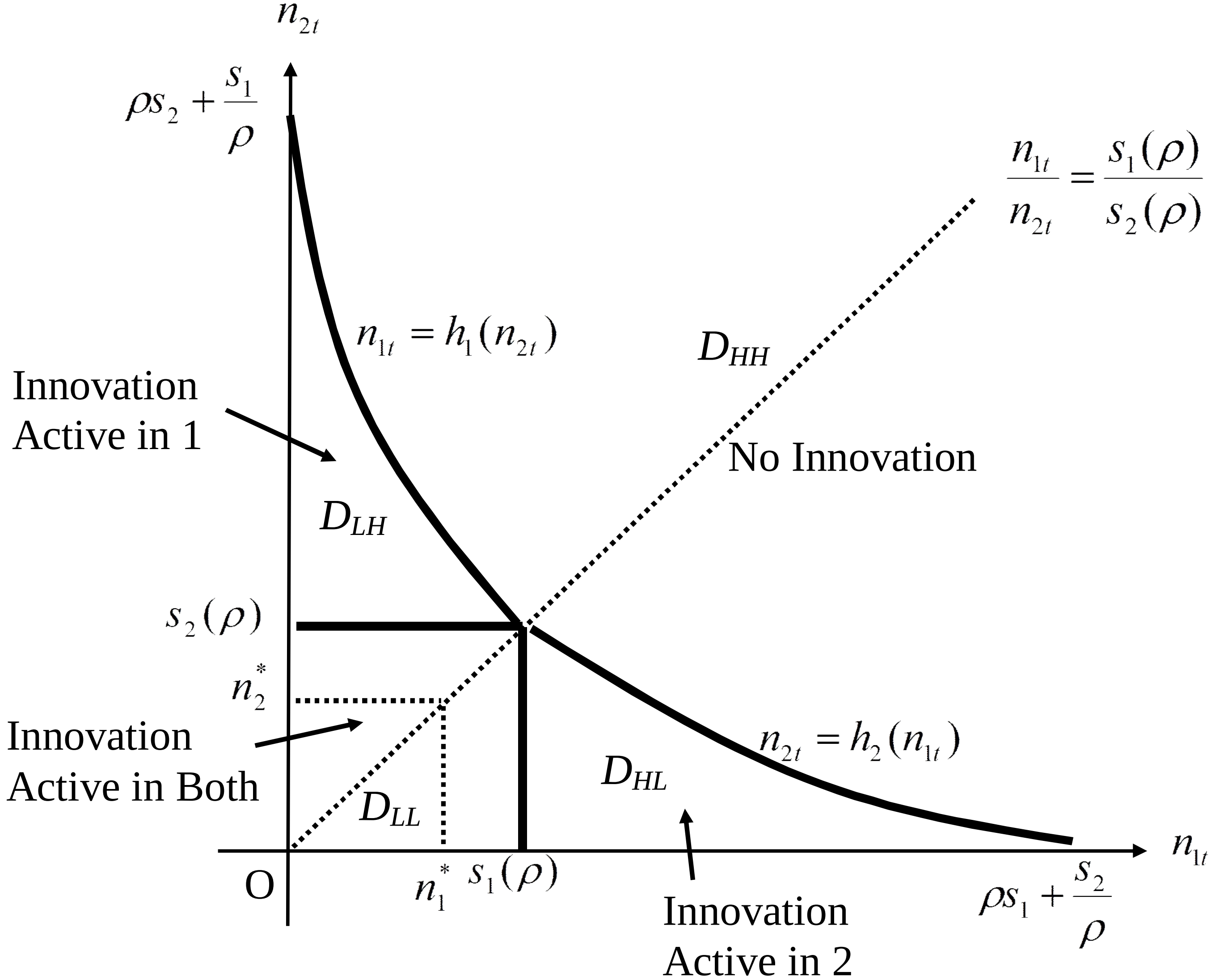


Figure 2: The State Space and The Four Domains of the 2S-System in Autarky ($\rho = 0$).

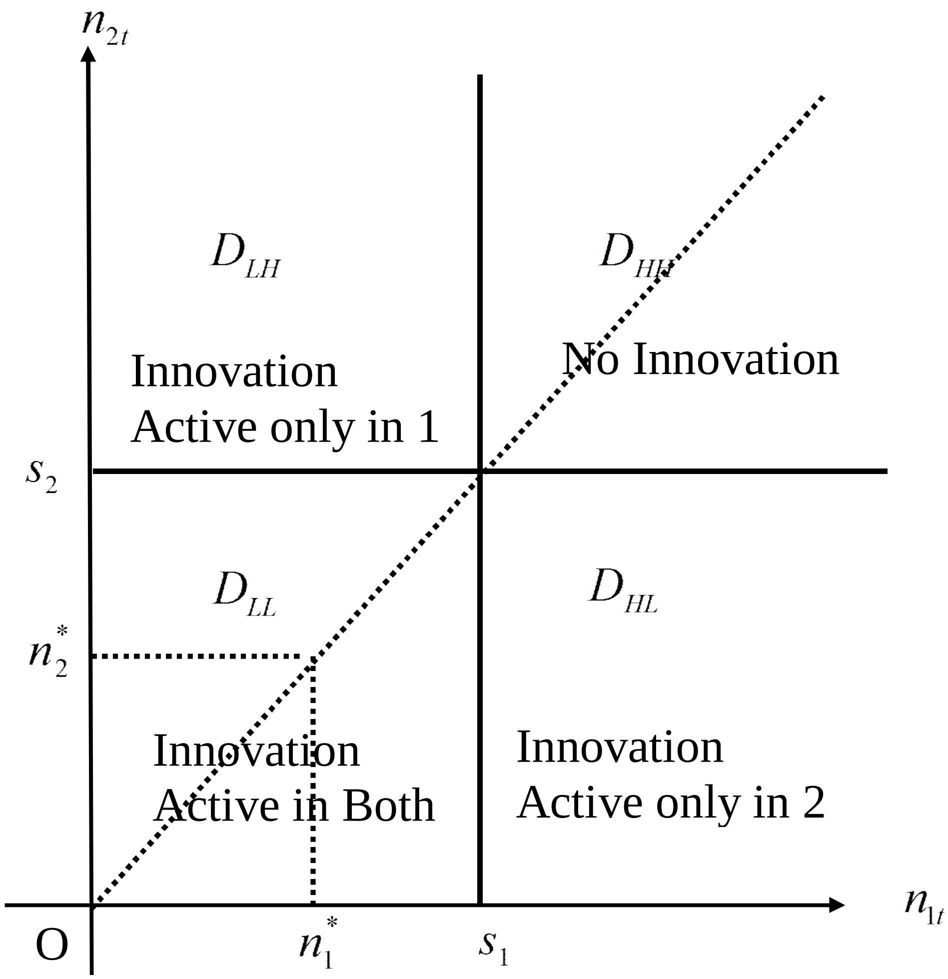


Figure 3: 1D-System: The Skew Tent Map

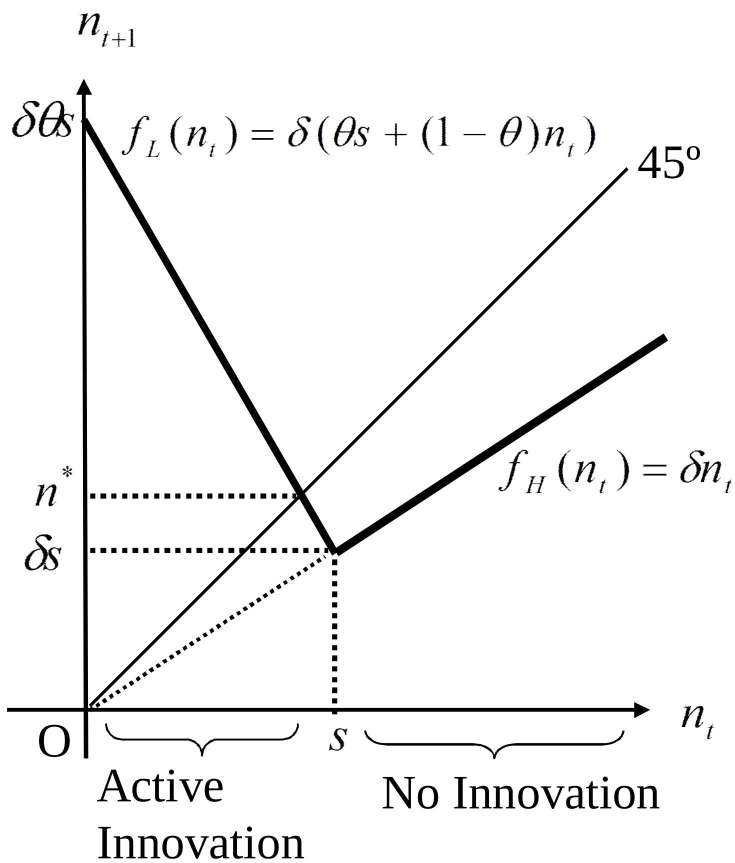
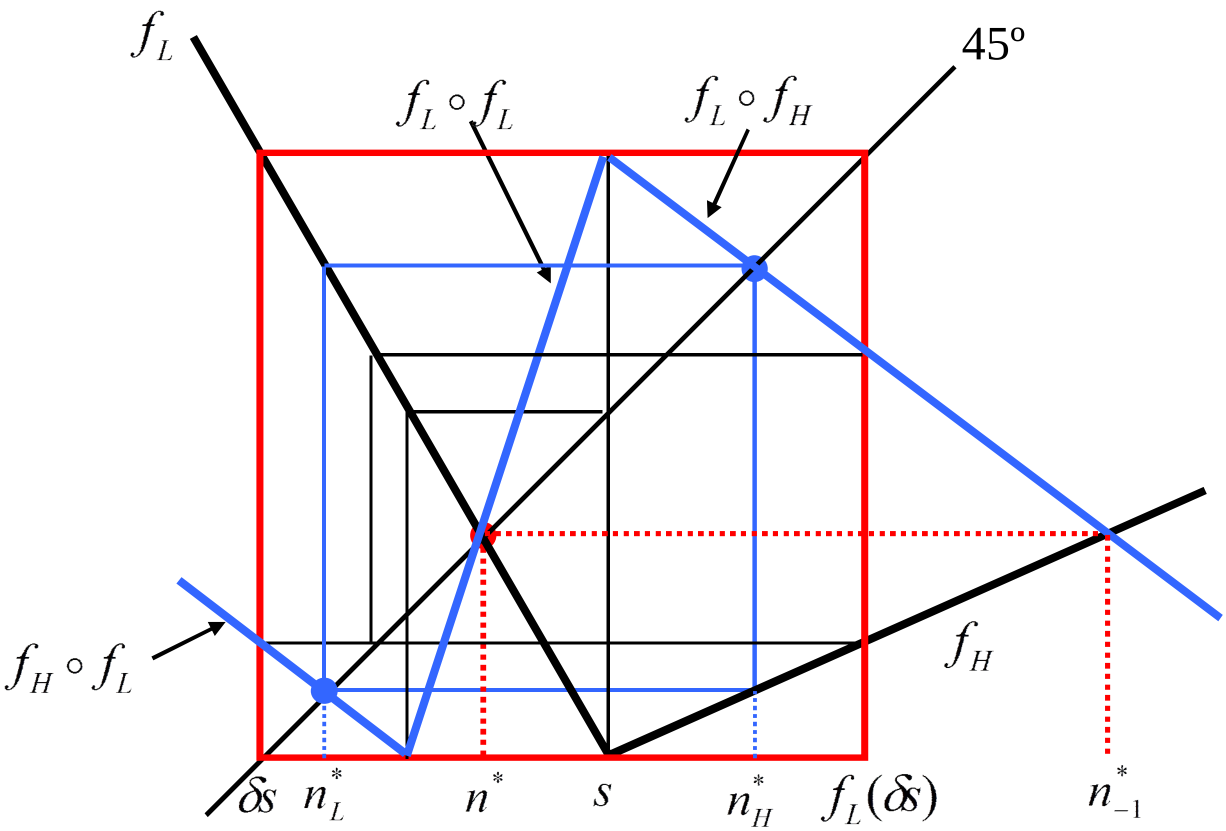


Figure 4: The Unstable Steady State, The Absorbing Interval, and the Stable 2-Cycle for $\delta^2(\theta-1)<1<\delta(\theta-1)$



The thick black lines show the graph of the skew tent map, f, eq.(20). The thin black lines show how the graph of the 2nd iterate of the map, f^2, shown in the thick blue lines, can be constructed from the graph of f. The red dot is the steady state, n^*, which is unstable for $\delta(\theta-1)>1$. The red box indicates the absorbing interval, which exists for $\delta(\theta-1)>1$. The blue box indicates the period-2 cycle (with the blue dots indicating the two points on the period-2 cycle, n_L^* and n_H^*), which is stable for $\delta^2(\theta-1)<1$. Notice that n_L^* and n_H^* are the two stable steady states under f^2. Note that n^* has two immediate pre-images under f, given by n^* < $n_{-1}^* = f_H^{-1}(n^*)$. Likewise, n^* has four immediate pre-images under f^2, given by $f_L^{-1}(n_{-1}^*)$ < n^* < n_{-1}^* < $f_H^{-1}(n_{-1}^*)$. The two intervals, ($f_L^{-1}(n_{-1}^*)$, n^*) and (n_{-1}^*, $f_H^{-1}(n_{-1}^*)$), belong to the basin of attraction for n_L^* under f^2. The interval, (n^*, n_{-1}^*), as well as an interval immediately below $f_L^{-1}(n_{-1}^*)$ and an interval immediately above $f_H^{-1}(n_{-1}^*)$, belong to the basin of attraction for n_H^* under f^2. This way, we can see why the two basins are not connected, given by alternating intervals, and their boundaries are formed by the pre-images of the unstable steady state, n^*.

Figure 5: Bifurcation diagram in the (δ, σ)-plane and Its Magnification

$\tilde{Q}_{2^m}$ (m = 0, 1, 2,…) indicate the parameter regions for the existence of a chaotic attractor that consists of 2^m intervals. The bottom figure is a magnification of the red box area in the top figure.

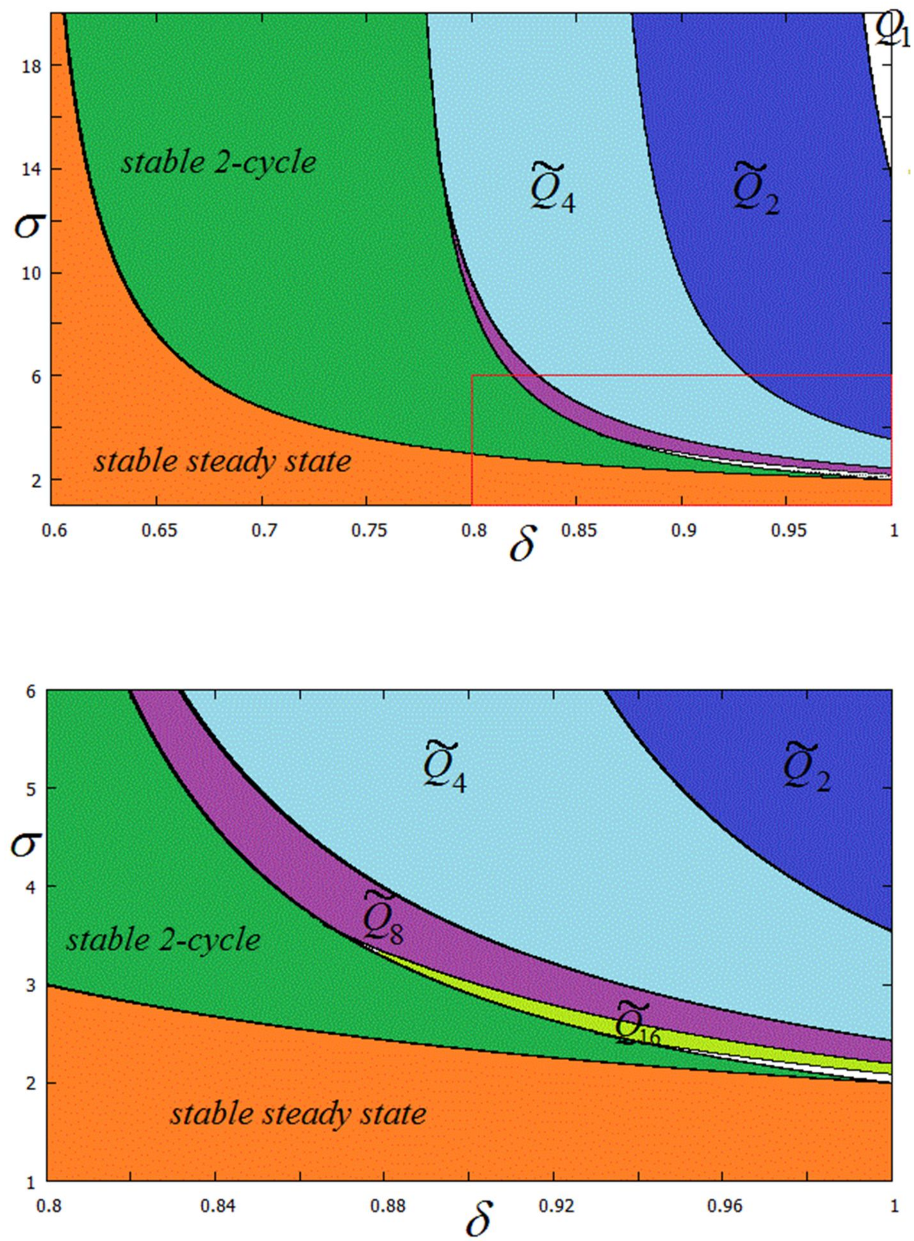


Figure 6 : Synchronzied vs. Asynchronized 2-Cycles: A 2D-view of the World Economy with the two-countries in autarky

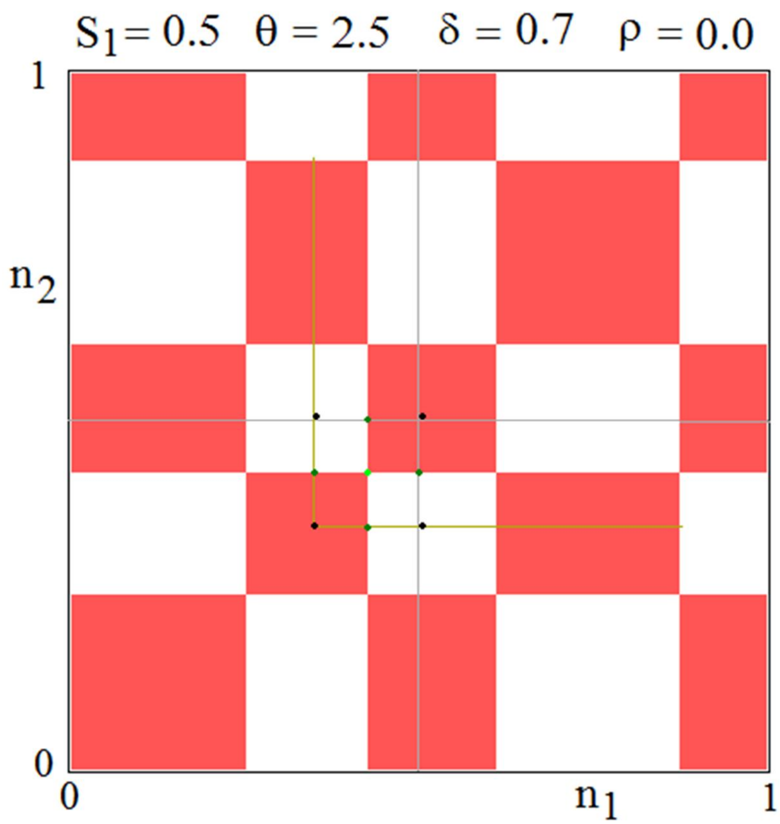


Figure 7: Steady State Analysis with $1 > s_1 = 1 - s_2 > 0.5$

Figure 7a: Home Market Effect

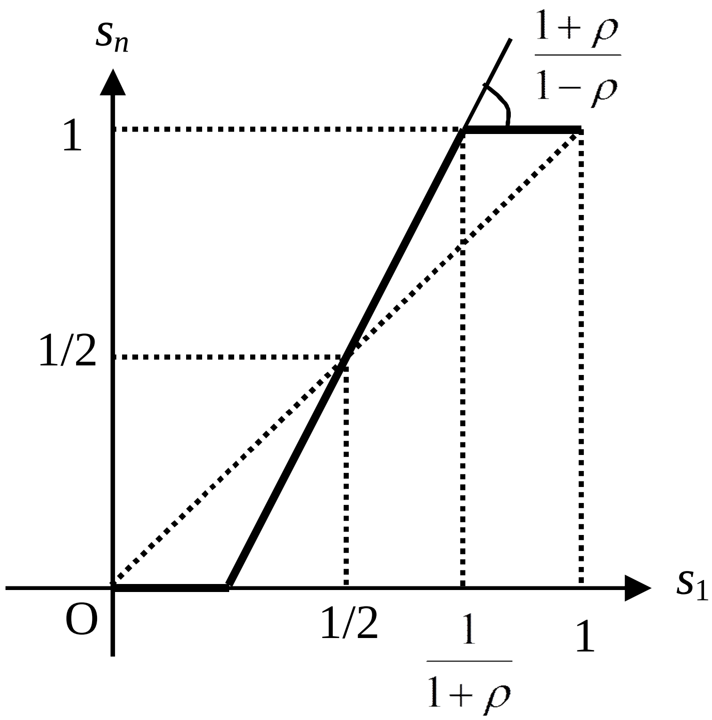


Figure 7b: Globalization and Magnification of the Home Market Effect

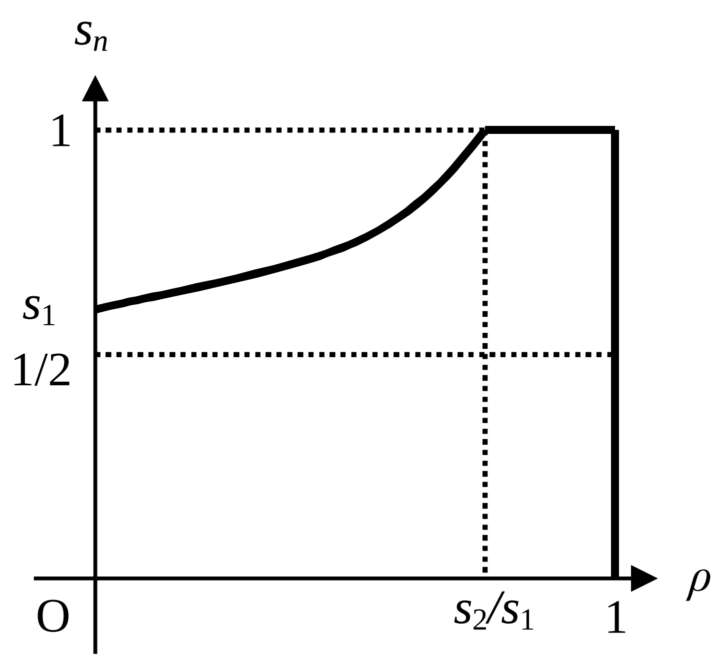


Figure 8: Symmetric ($s_1 = 1/2$) 2D System

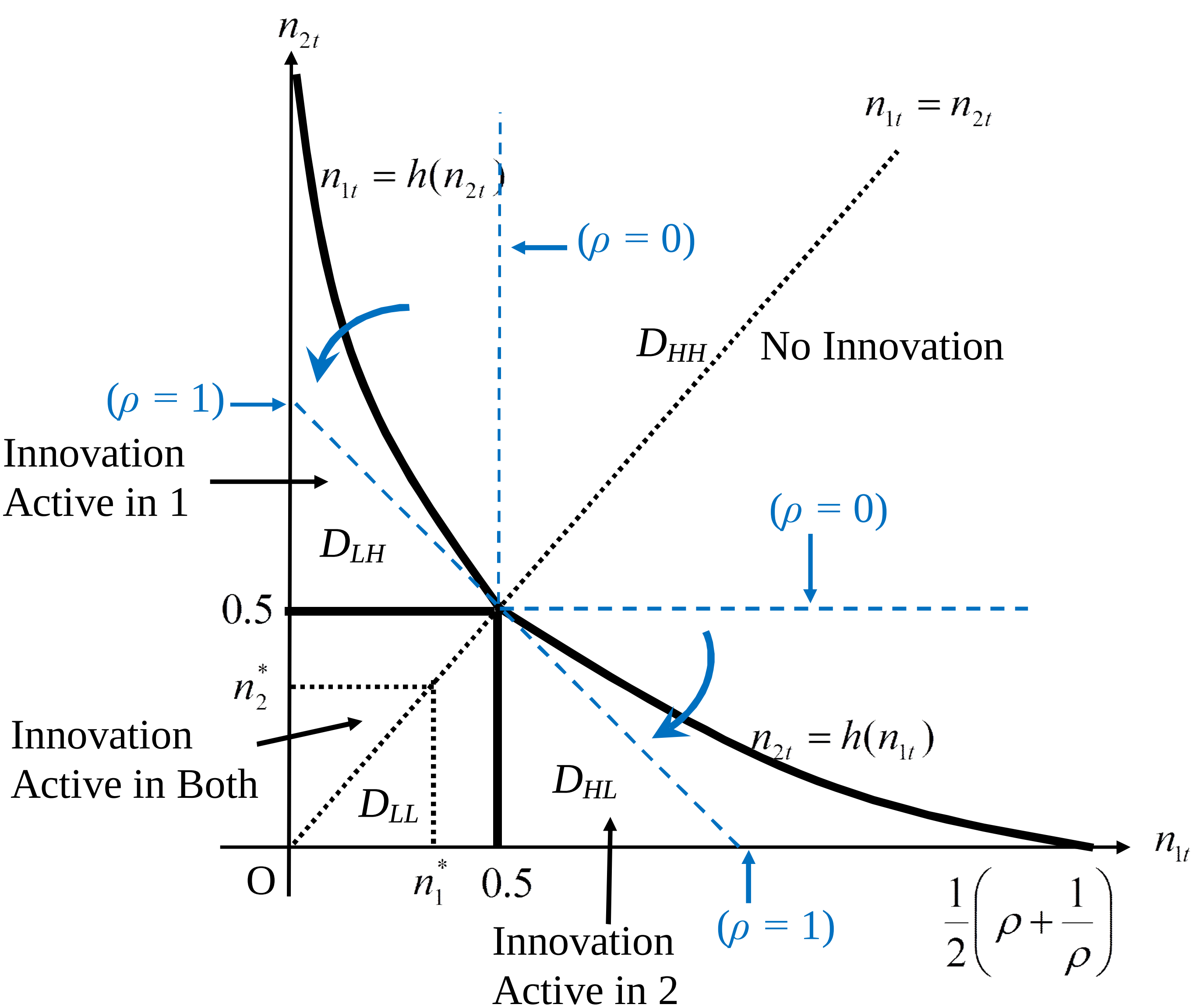
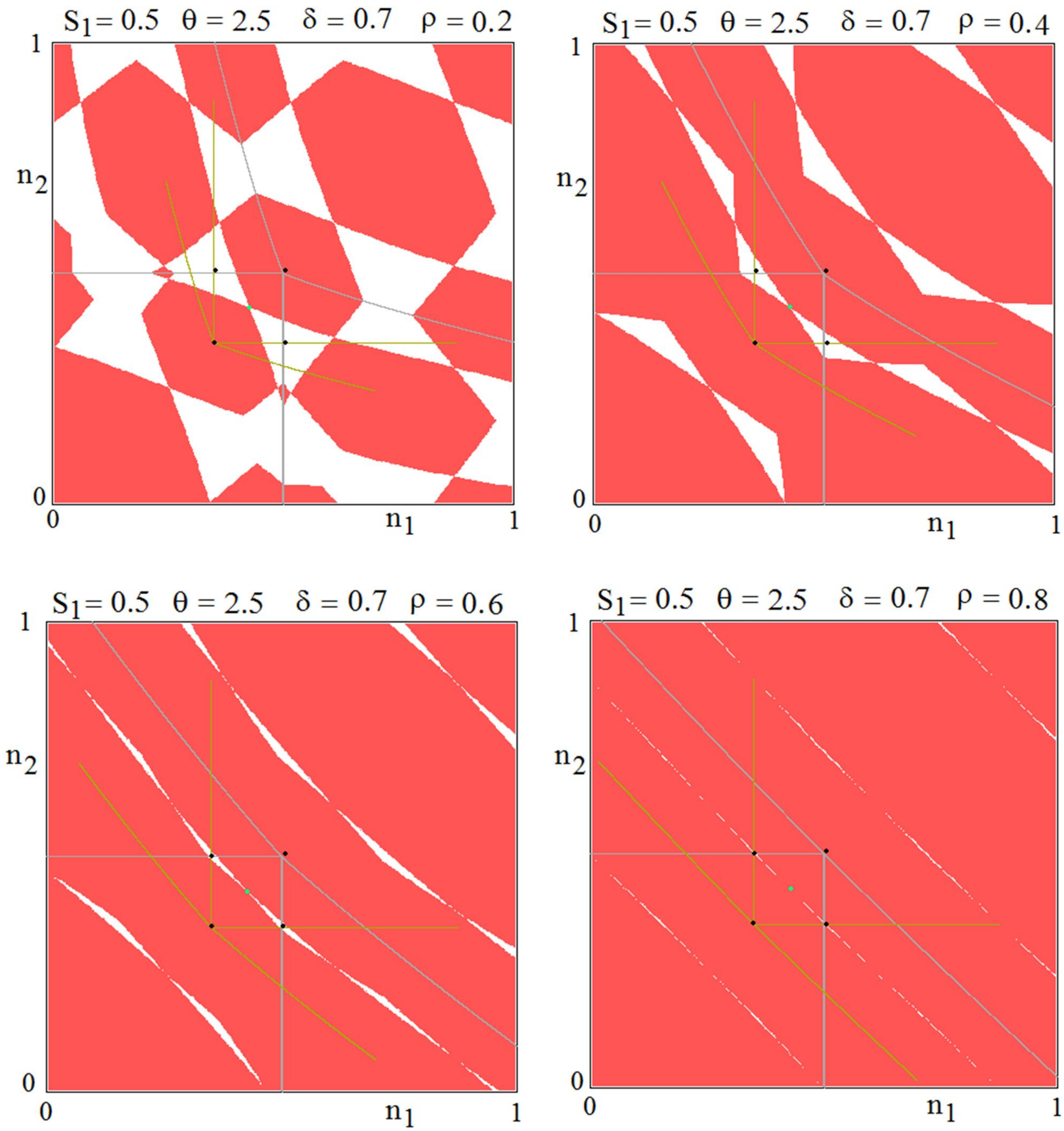
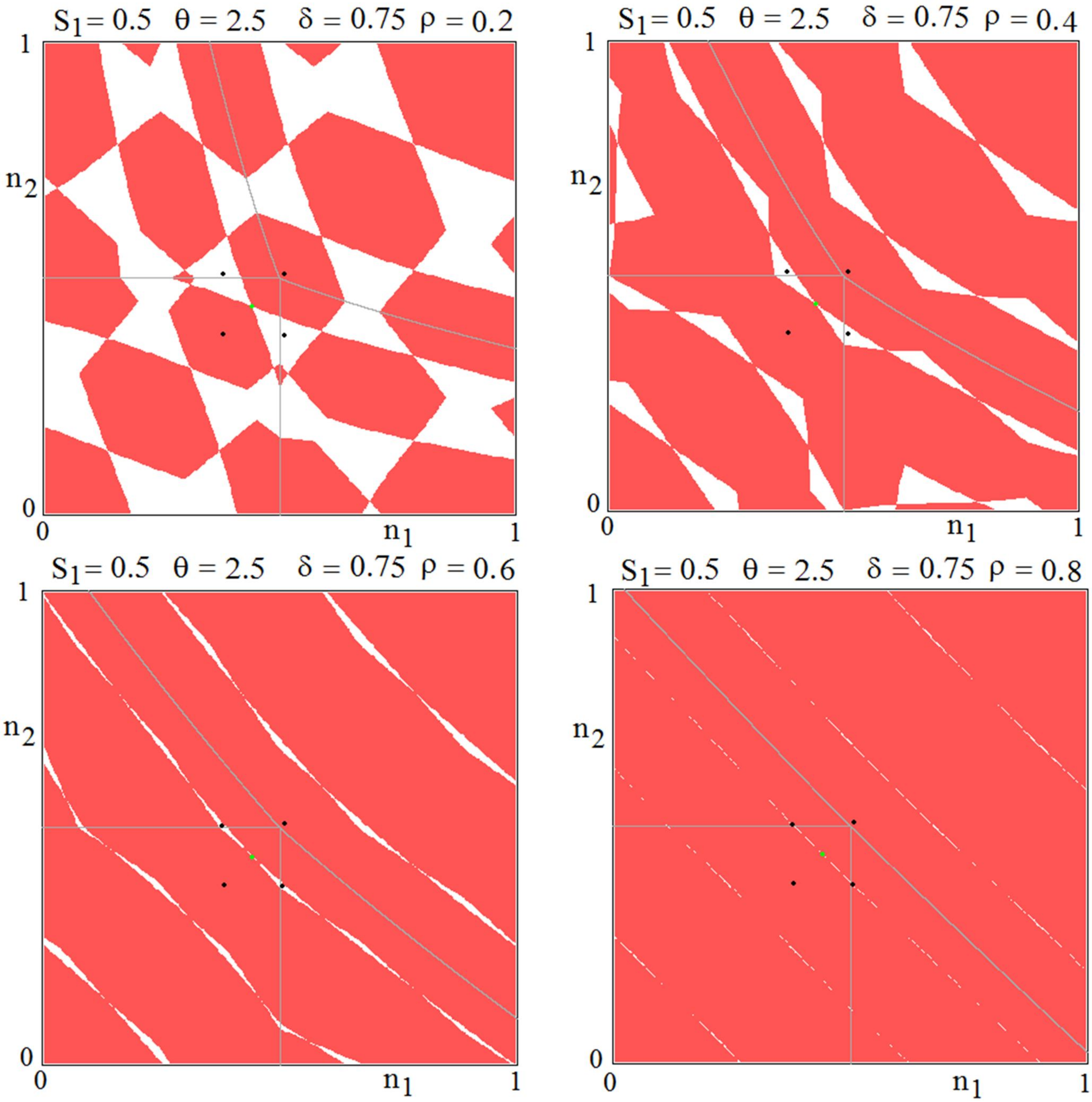


Figure 9a: Synchronized versus Asynchronized 2-Cycles: $s_1 = 0.5$, $\theta = 2.5$, $\delta = 0.7$



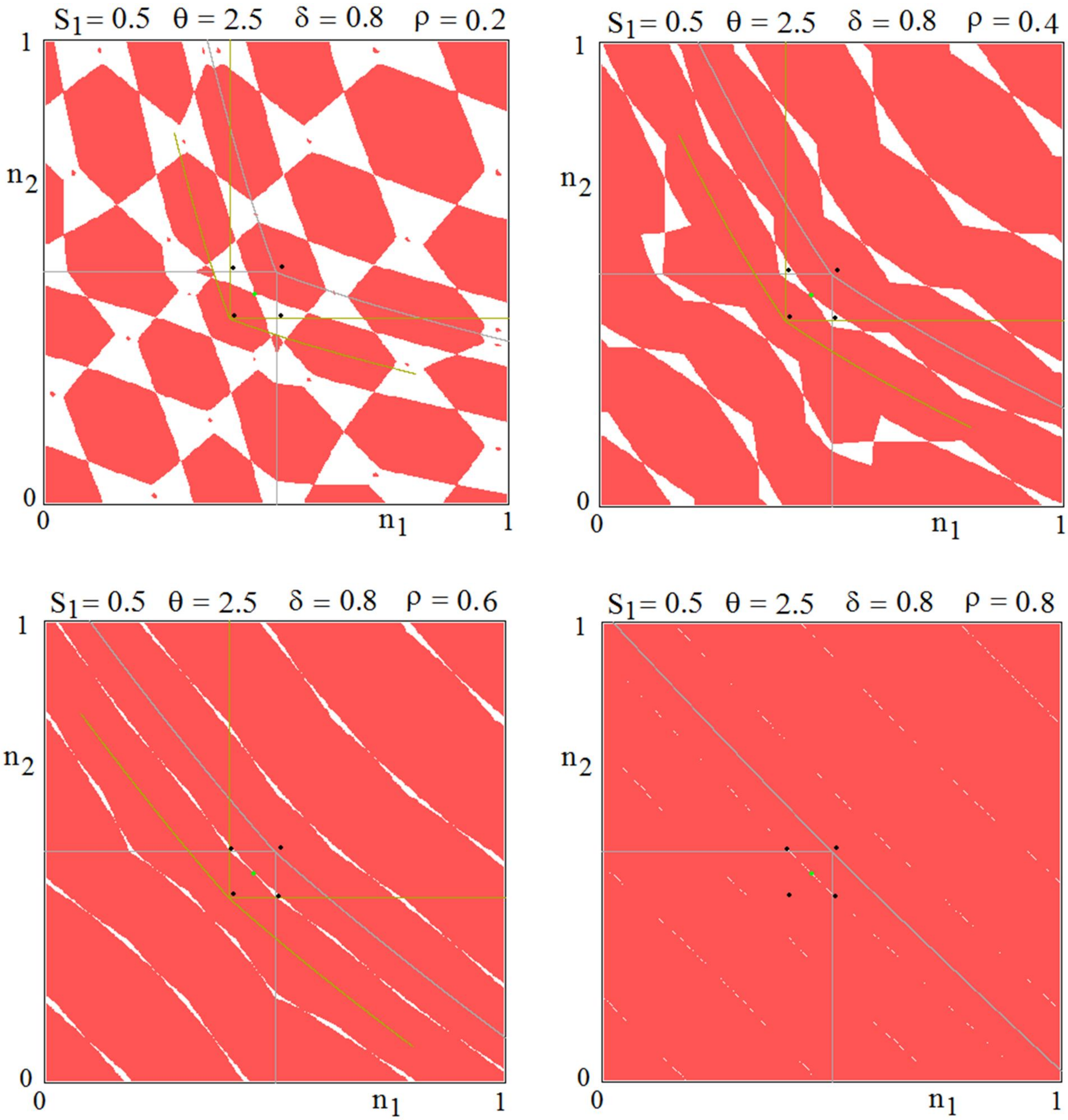
Red (the basin for the synchronized 2-cycle) becomes dominant.
The symmetric asynchronized 2-cycle becomes a stable node at ρ = .817202; and a saddle at ρ = .877358.

Figure 9b: Synchronized versus Asynchronized 2-Cycles: $s_1 = 0.5$, $\theta = 2.5$, $\delta = 0.75$



Red (the basin for the synchronized 2-cycle) becomes dominant.
The symmetric asynchronized 2-cycle becomes a stable node at ρ = .817867, and a saddle at ρ = .833323.

Figure 9c: Synchronized versus Asynchronized 2-Cycles: $s_1 = 0.5$, $\theta = 2.5$, $\delta = 0.8$



Red (the basin for the synchronized 2-cycle) becomes dominant.
The symmetric asynchronized 2-cycle becomes a stable node at ρ = .81814; a saddle at ρ = .818986.

Figure 10a: Asymmetric ($s_1 > 1/2$) 2D System: $0 < \rho < s_2 / s_1 < 1$

A higher ρ has *additional* effects of shifting innovation towards 1 (and away from 2), shown by blue arrows.

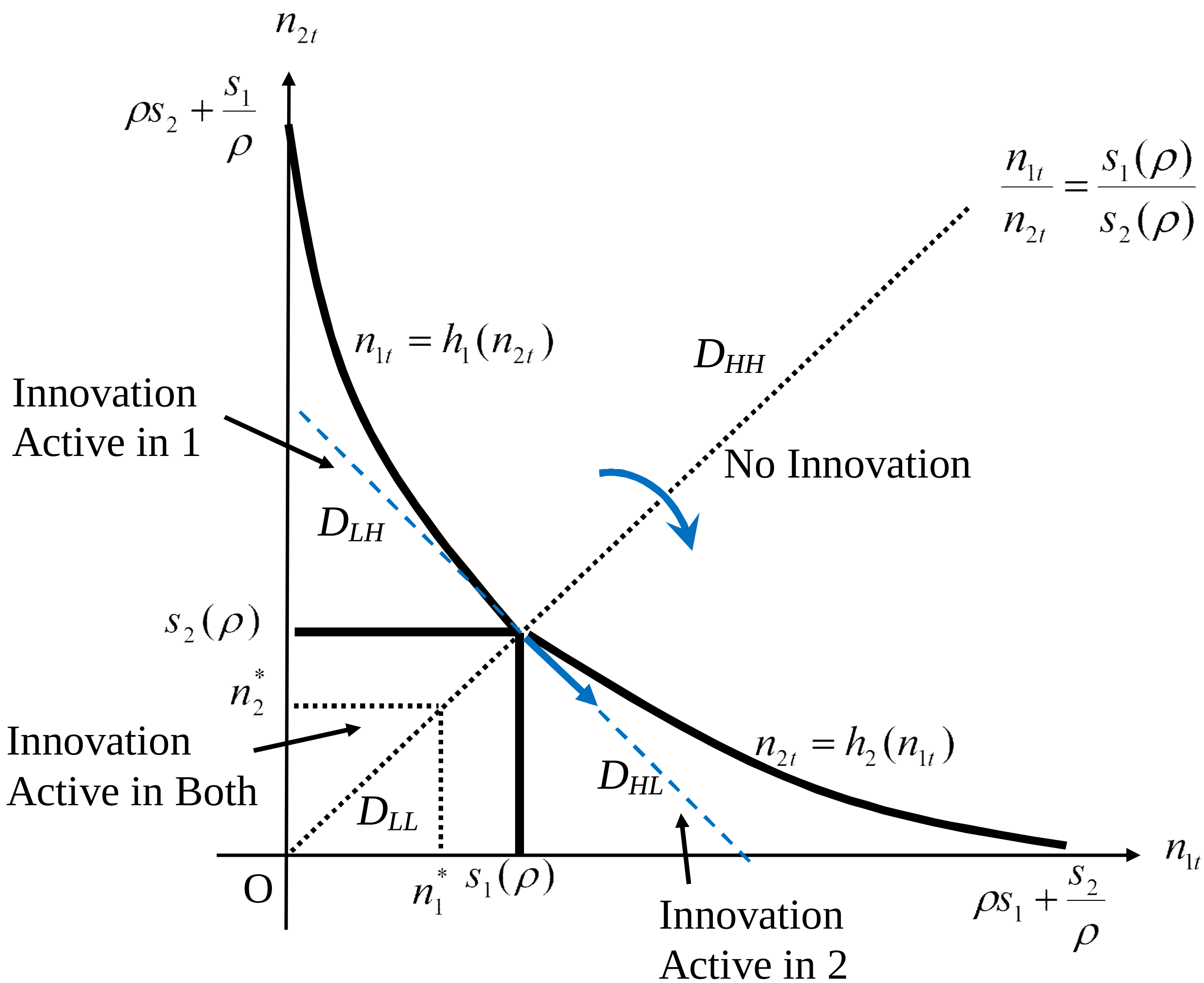


Figure 10b: Asymmetric ($s_1 > 1/2$) 2D System: for $s_2 / s_1 < \rho < 1$.

No innovation in 2; $n_{2t+1} = \delta n_{2t}$ and $n_{2t} \to 0$.
Innovation in 1: $n_{1t+1} = \delta\left(\theta \max\{h_1(n_{2t}), n_{1t}\} + (1-\theta)n_{1t}\right) \to \delta\left(\theta \max\{1, n_{1t}\} + (1-\theta)n_{1t}\right)$.
Asymptotically, the dynamics is given by a **1D-skew tent map** on the horizontal axis.

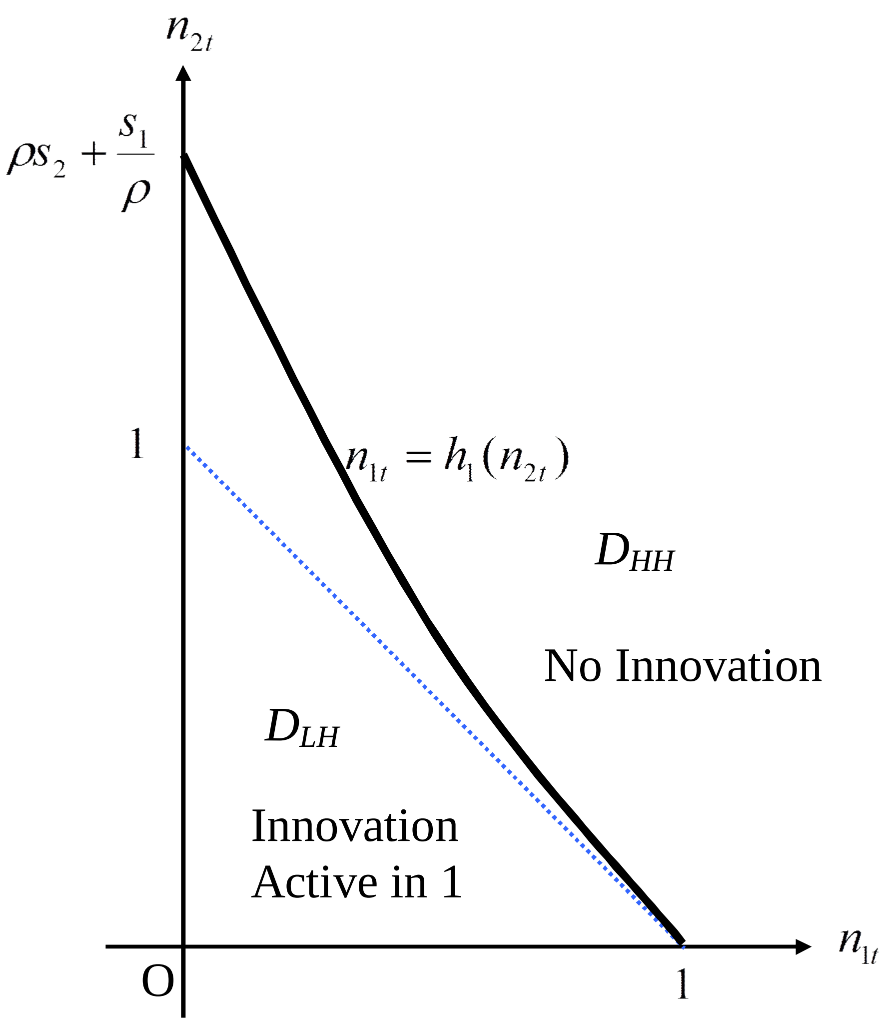
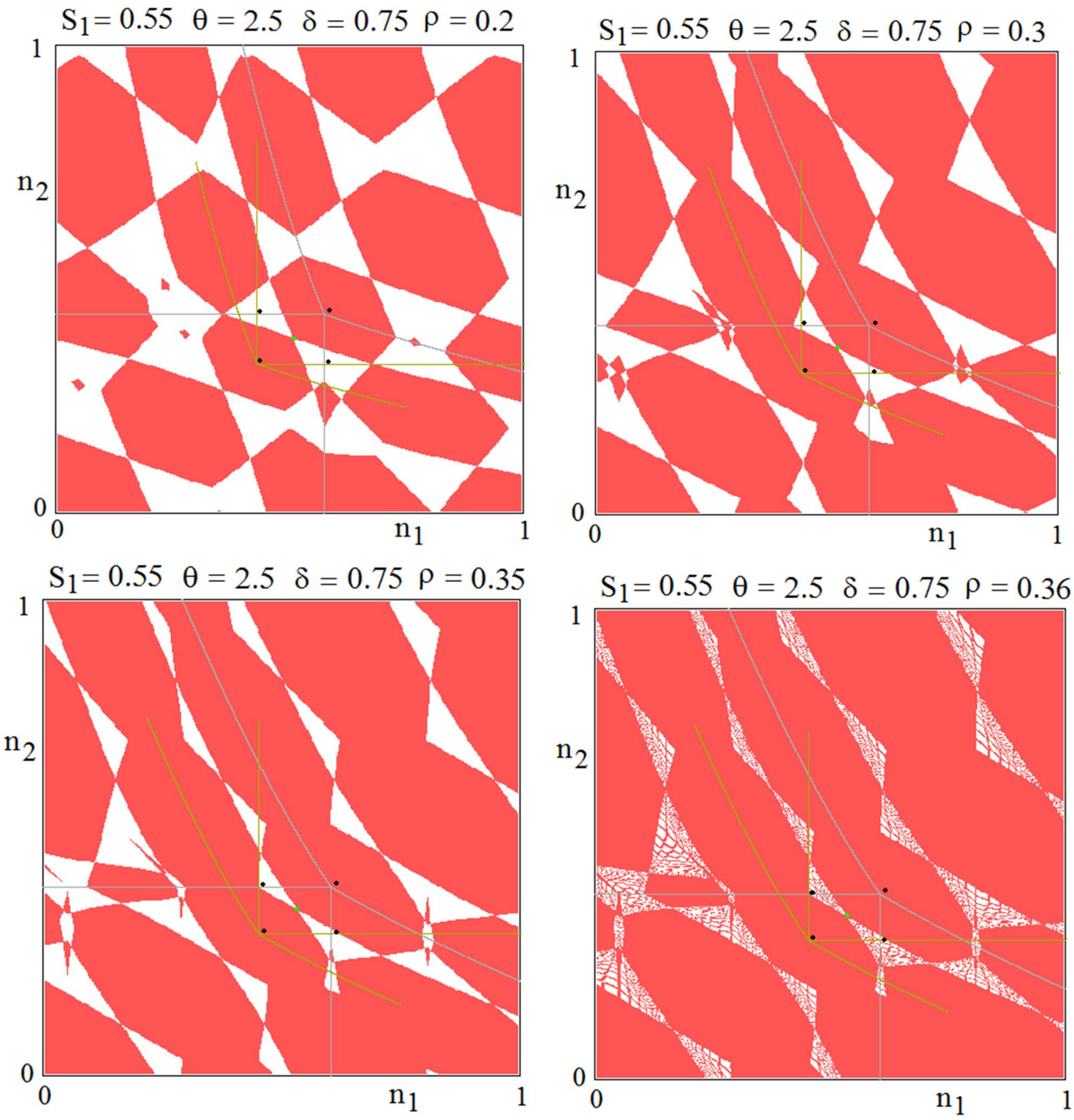
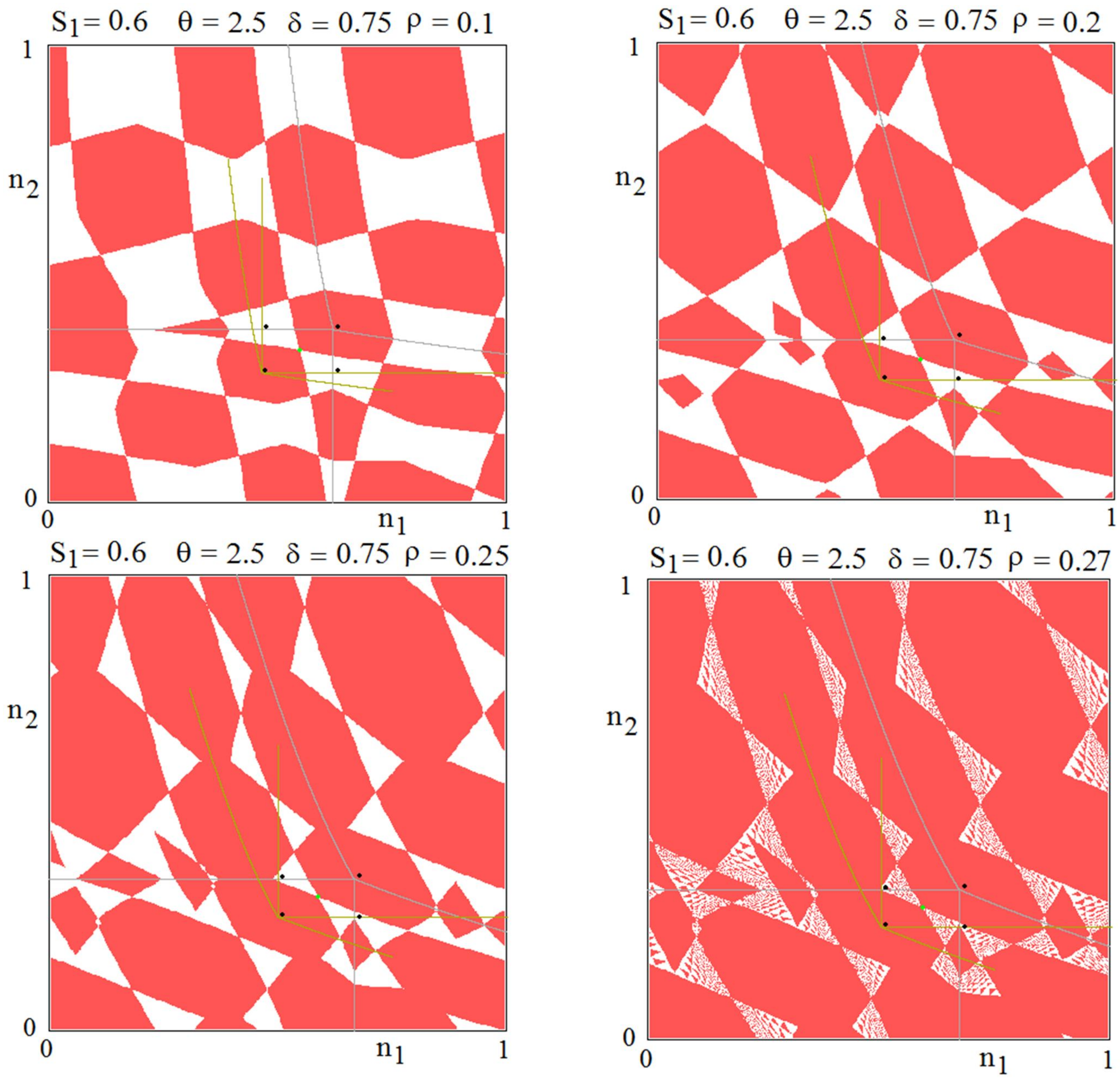


Figure 11a:Asymmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.55,\ \theta = 2.5,\ \delta = 0.75$



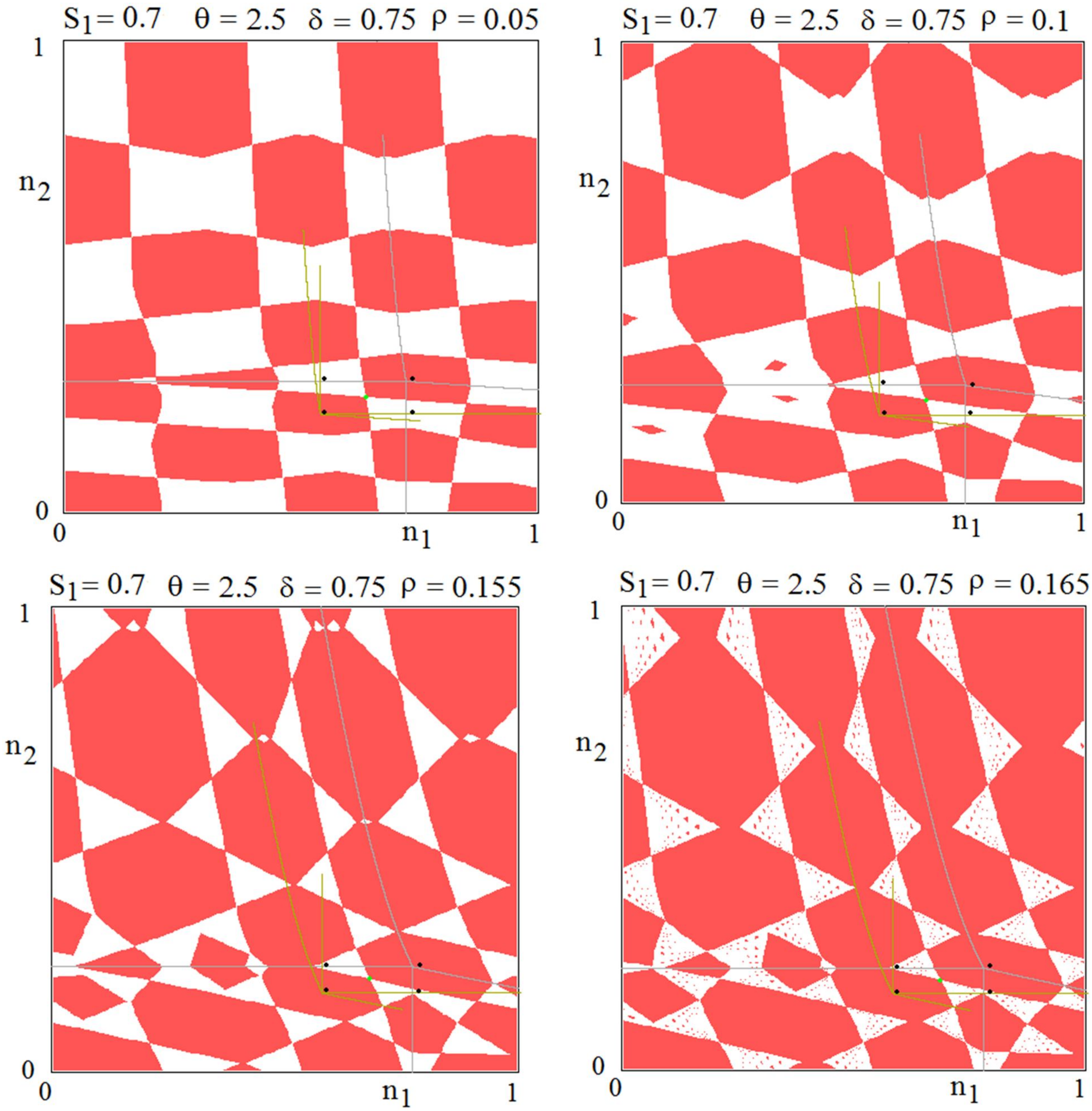
By ρ = .36, infinitely many Red islands appear inside White.
By ρ = .39, the stable asynchronized 2-cycle collides with the basin boundary and disappears, leaving **the Synchronized 2-cycle as the unique attractor.**

Figure 11b: Asymmetric Synchronized & Asynchronized 2-Cycles : $s_1 = 0.6$, $\theta = 2.5$, $\delta = 0.75$



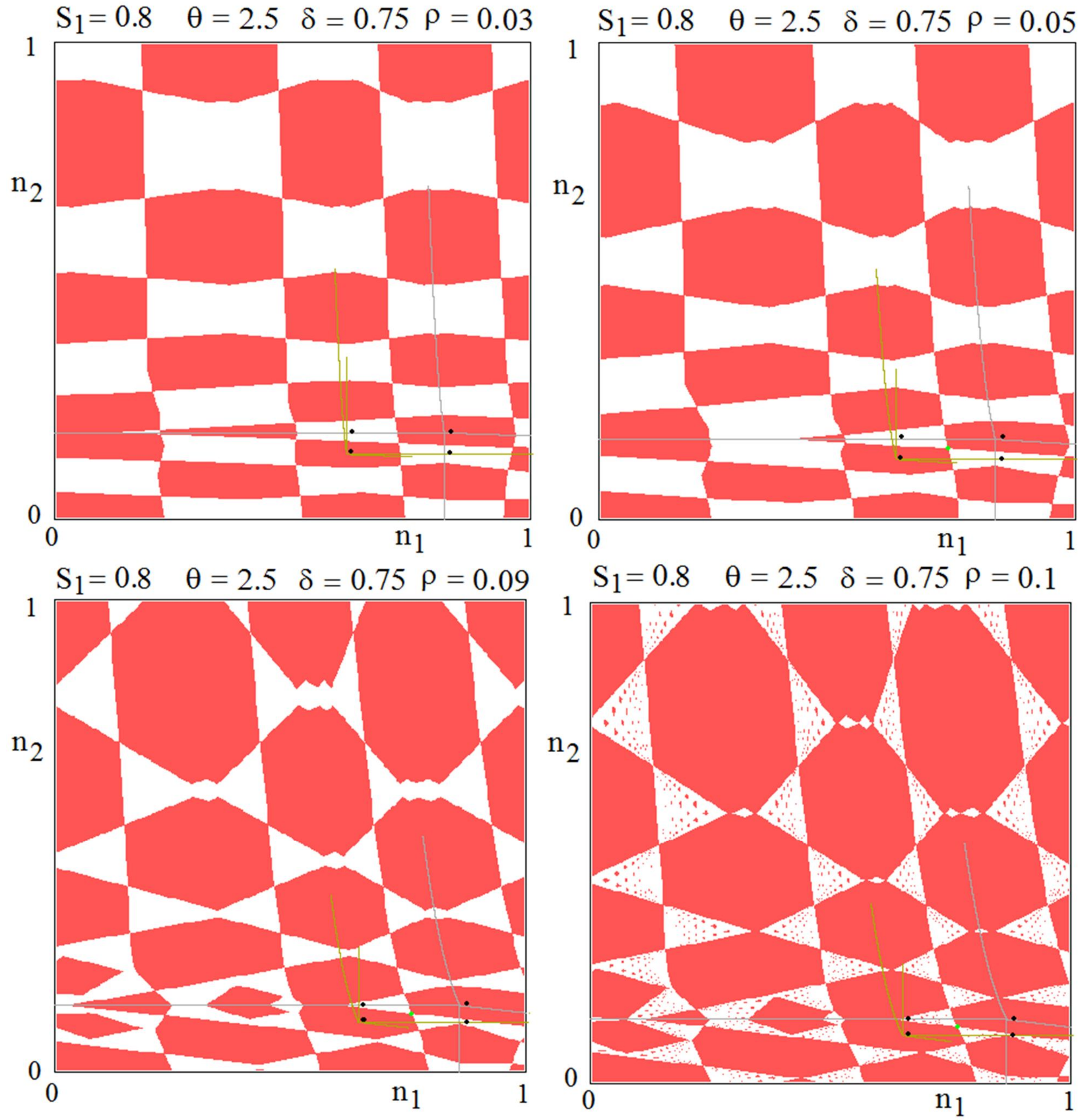
By ρ = .27, infinitely many Red islands appear inside White region.
By ρ = .30, the stable asynchro. 2-cycle collides with its basin boundary and disappears, leaving **the Synchronized 2-cycle as the unique attractor**.

Figure 11c: Asymmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.7,\ \theta = 2.5,\ \delta = 0.75$



By ρ = .165, infinitely many Red islands appear inside White.
By ρ =.19, the stable asynchronized 2-cycle collides with its basin boundary and disappears, leaving **the Synchronized 2-cycle as the unique attractor**.

Figure 11d: Asymmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.8$, $\theta = 2.5$; $\delta = 0.75$



By ρ = .10, infinitely many Red islands appear inside White.
By ρ = .12, the stable asynchro. 2-cycle collides with its basin boundary and disappears, leaving **the Synch. 2-cycle as the unique attractor.**

Figure 12: Critical Value of ρ at which the Stable Asynchronized 2-cycle disappears (as a function of s_1)

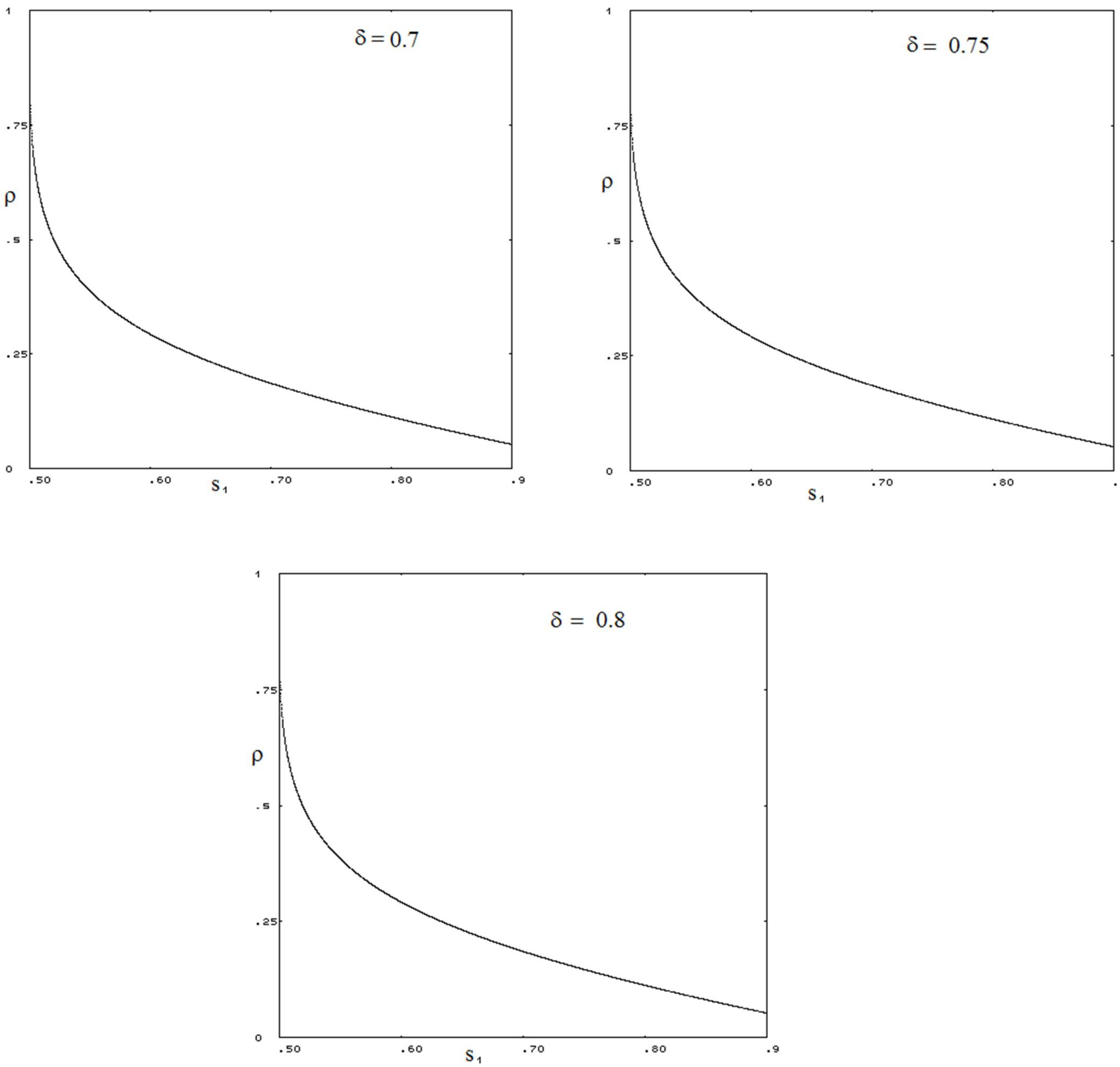
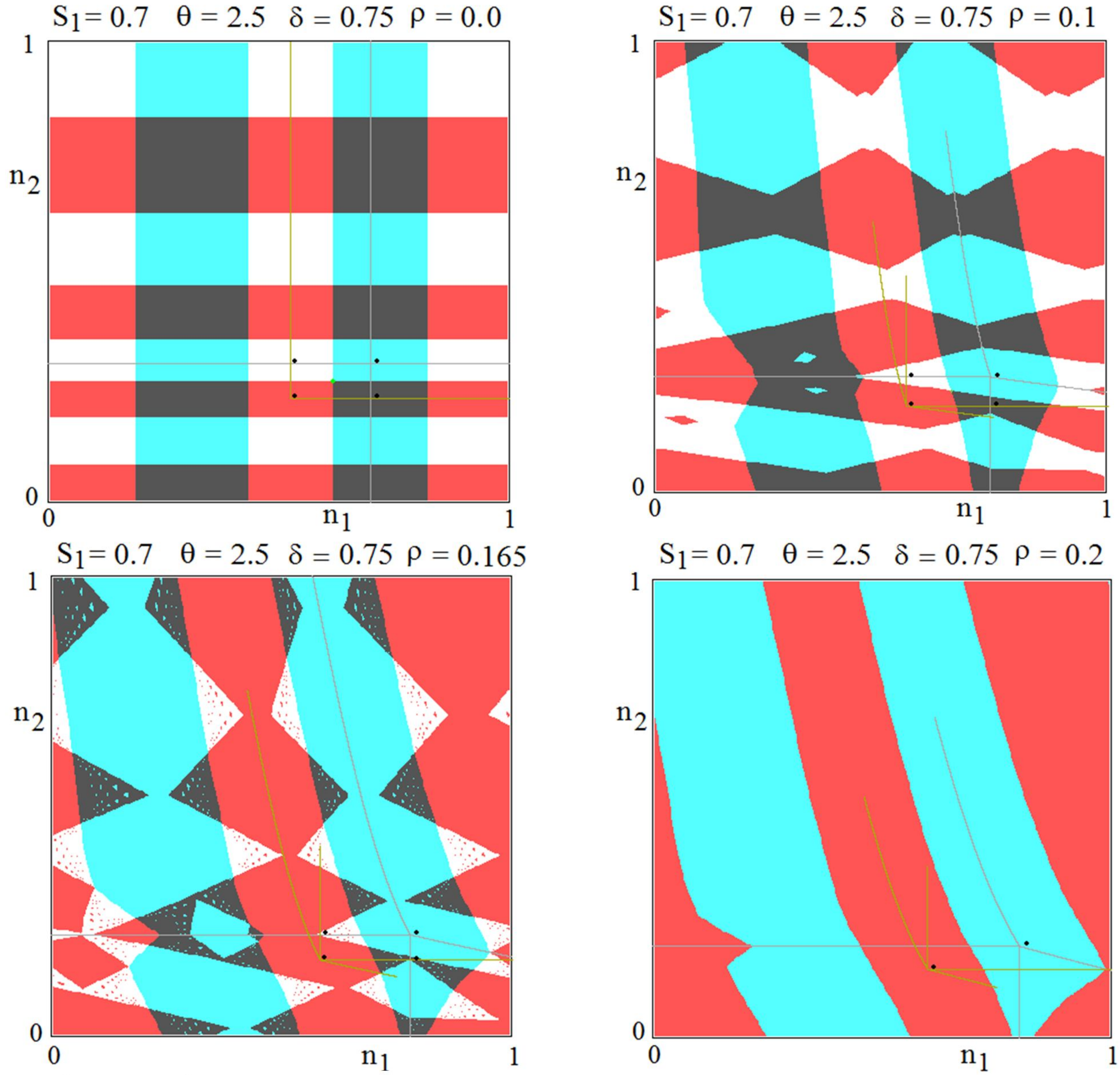
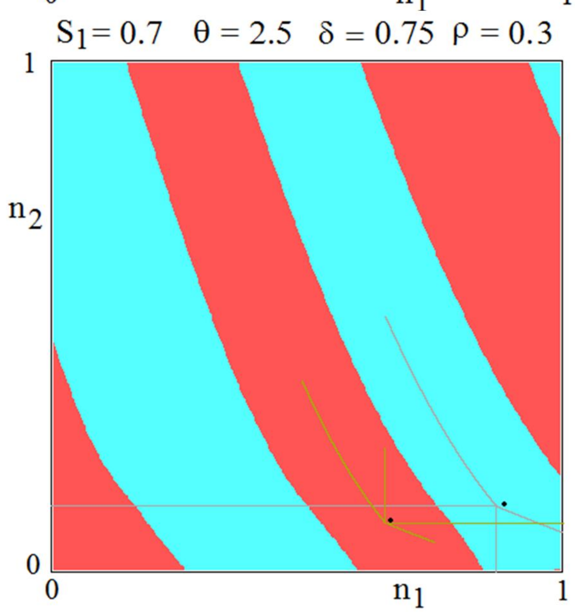


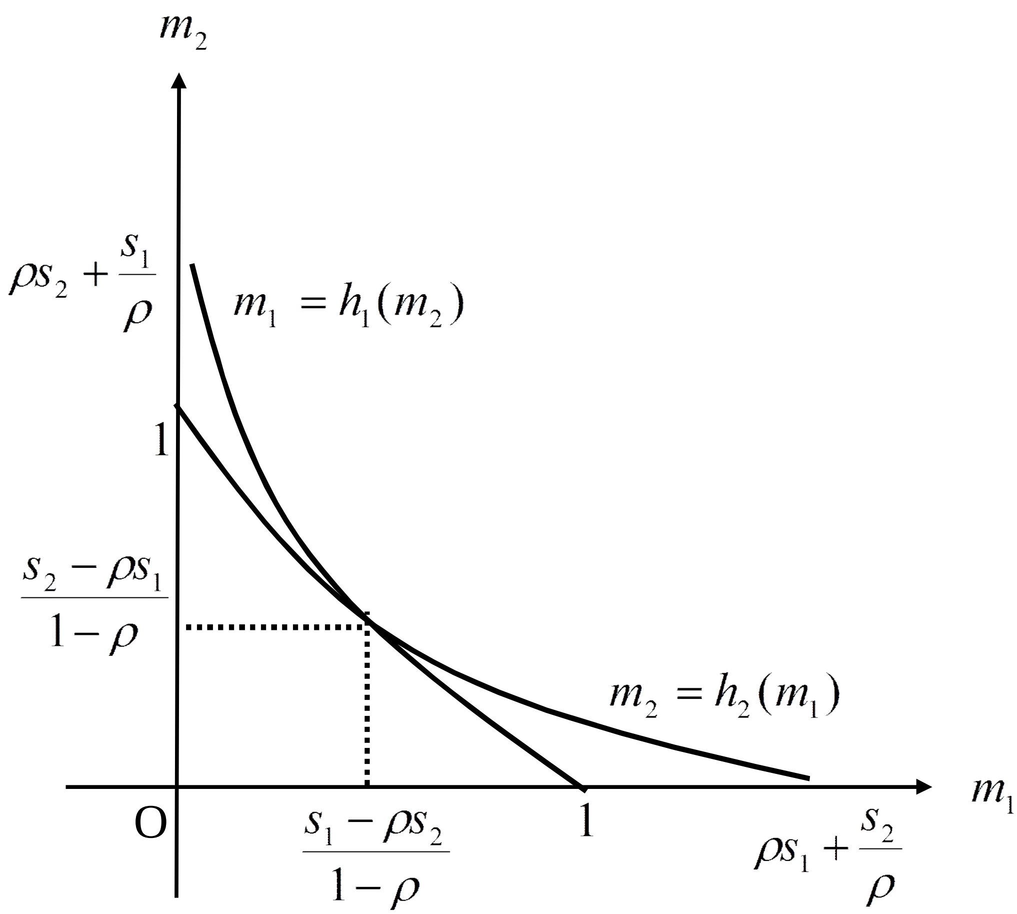
Figure 13: Four Basins of Attraction: $s_1 = 0.7$, $\theta = 2.5$, $\delta = 0.75$





As ρ rises, Red invades White, and Azure invades Gray, and vertical slips of Red and Azure emerge.

Figure 14:


m_2
$\rho s_2 + \frac{s_1}{\rho}$
$m_1 = h_1(m_2)$
1
$\frac{s_2 - \rho s_1}{1-\rho}$
$m_2 = h_2(m_1)$
m_1
O
$\frac{s_1 - \rho s_2}{1-\rho}$
1
$\rho s_1 + \frac{s_2}{\rho}$