Imitation Dynamics in Cournot Games with Heterogenous Players

Daan Lindeman
Univ. Amsterdam

Marius Ochea
Univ. Amsterdam

MDEF, Urbino, 2014

Background

Convergence to and *stability of* the Cournot-Nash equilibrium, in oligopoly games:

- *"two is few, three is many"*: Theocharis (1960)
- behavior adjustment process: Cournot adjustment, gradient dynamics
- non-monotonic inverse demand-cost structure: 'exotic' chaotic dynamics of Rand (1978) and Kopel (1996)
- expectations: naive, adaptive, fictitious play, rational

Cournot Equilibrium: Instability thresholds

- Cournot players only: $n < 3$ (Theocharis, 1960)
- Cournot vs. equilibrium (Nash) players: $n < 5$ (Hommes et. al. 2011, WP)
- Cournot vs. Rational with no information costs: always stable (Hommes et. al. 2011, WP)
- Cournot vs. Rational with info costs: various thresholds depending on magnitude of the costs and of the evolutionary pressure
- Imitation vs. equilibrium play: $n < 7$
- Imitation vs. Cournot vs. Rational: various thresholds function of the *relative costs* of the stable and unstable heuristics

Literature on Imitation

- imitate-the-best and optimizers in Cournot oligopoly: stochastic stability, long-run distribution imitators better off (Schipper, 2009, JEDC)
- unbeatable imitation? Yes! if imitate-the-best not subjected to a money pump, i.e. game is not of Rock-Scissors-Paper variety (Duersch et. al., 2012, GEB)
- unconditional imitation (tit-for-tat variety), essentially unbeatable in class of potential games (Duersch et. al., 2014, IJGT)
- experiments:
  - Huck et. al. (2002): process where participants mix between the Cournot adjustment heuristic an imitating the previous period's average quantity gives the best description of behaviour
  - Duersch et. al. (2009): Cournot duopoly, subjects earn on average higher profits when playing against "best-response" computers than against "imitate" computers

Cournot analysis

- homogeneous goods, Cournot oligopoly with n firms
- Inverse demand $P(Q)$: continuously diff. $P(Q) \geq 0, P'(Q) \leq 0$
- $Q = \sum_{i=1}^{n} q_i$, q_i is production of firm i
- cost function $C(q_i)$: twice continuously diff. $C(q_i) \geq 0\,, C'(q_i) \geq 0$
- first order condition of firm i:

$$P(Q_{-i} + q_i) + q_i P'(Q_{-i} + q_i) - C'(q_i) = 0$$

- second order condition:

$$2P'(Q_{-i} + q_i) + q_i P''(Q_{-i} + q_i) - C''(q_i) \leq 0.$$

- best response correspondence: $q_i = R(Q_{-i}), i = 1, n$
- assume that a symmetric equilibrium q^*,aggregate output $Q^* = nq^*$

Learning Rules: Introspection vs. Adaptation

- how does firm i learn to play q^*?
- what does i believe about Q_{-i} at the time when the production decision has to be made?
- dynamical system: $q_i(t) = R(Q^e_{-i}(t)), i = 1, n$

$$R'(Q_{-i}) = -\frac{P'(Q) + q_i P''(Q)}{2P'(Q) + q_i P''(Q) - C''(q_i)}.$$

- rest points: existence, uniqueness, stability

Learning Rules: Introspection vs. Adaptation

- Naive or "Cournot" play:

$$q_{i,t+1} = R(Q_{-i,t}),\ i = 1, ...n$$

- Unconditional imitation:

$$q_{i,t+1} = \frac{Q_{-i,t}}{n-1},\ i = 1, ..., n.$$

- Rational play:

$$q_{i,t+1} = R(Q_{-i,t+1}),\ i = 1, ...n$$

Instability under Cournot dynamics

- Jacobian evaluated at steady state:

$$\begin{pmatrix} 0 & R'\left(Q^*_{-1}\right) & \cdots & R'\left(Q^*_{-1}\right) \\ R'\left(Q^*_{-2}\right) & 0 & & \vdots \\ \vdots & & \ddots & R'\left(Q^*_{-(n-1)}\right) \\ R'\left(Q^*_{-n}\right) & \cdots & R'\left(Q^*_{-n}\right) & 0 \end{pmatrix}.$$

- $n-1$ eigenvalues equal to $-R'\left(\left(n-1\right)q^*\right)$, one e.v. $\left(n-1\right)R'(\left(n-1\right)q^*)$
- local stability: $\xi_n\left(q^*\right) \equiv \left(n-1\right)\left|R'(\left(n-1\right)q^*)\right| < 1$
- typically, $\xi_n\left(q^*\right)$ increases in n and a threshold value $\hat{n}$ of n exists such that $\xi_{\hat{n}}\left(q^*\right) \leq -1 < \xi_{\hat{n}-1}\left(q^*\right)$.
- example, linear-linear: $R'\left(Q_{-i}\right) = -\frac{1}{2}$,CNE stable for $n = 2$, bounded oscillations, for $n = 3$,exploding fluctuations for $n > 3$

Stability under imitation dynamics

- Jacobian evaluated at steady state:

$$J\big|_{q^*} = \begin{pmatrix} 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & & \vdots \\ \vdots & & \ddots & \frac{1}{n-1} \\ \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 \end{pmatrix}.$$

- $n-1$ eigenvalues equal to $-\frac{1}{n-1}$, one e.v. 1

Heterogeneous Heuristics in Cournot oligopolies

- study the aggregate behavior of a heterogeneous set of interacting quantity-setting heuristics
- consider a large population of firms from which in each period groups of n firms are sampled randomly and matched to play the one-shot n-player Cournot game
- assume that a fixed fraction $\eta \in [0, 1]$ of the population uses one heuristic, and a fraction $1 - \eta$ uses another
- after each one-shot Cournot game, the random matching procedure is repeated, leading to new combinations of the two types of firms

Imitation vs. Cournot play

$$\begin{aligned} q^C_{t+1} &= R((n-1)(\eta q^C_t + (1-\eta)q^I_t)) \\ q^I_{t+1} &= \eta q^C_t + (1-\eta)q^I_t. \end{aligned}$$

Lemma

The Cournot-Nash equilibrium,where all firms produce the Cournot-Nash quantity (q^*, q^*), is a locally stable fixed point for the model with exogenous fractions of Cournot and imitation firms if and only if $|1-\eta+\eta(n-1)(R'^*)| < 1$.

- linear-linear case: $n < \frac{4-\eta}{\eta}$
  - $\eta = 1$ is stable if $n < 3$
  - $\eta = \frac{1}{2}$ is stable if $n < 7$
  - $\eta \to 0$, always stable

Imitation vs. Rational play

- fully rational player knows the number of non-rational firms and knows exactly what they will produce in period t, namely q_t
- assume they do not know the identity of the firms in its market when production decision is made:

$$\max_{q_i} \Pi_R^e = E\left[P\left(Q_{-i} + q_i\right) q_i - C\left(q_i\right)\right].$$

- Rational firm i chooses q_i to maximize:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \eta^k (1-\eta)^{n-1-k} [P((n-1-k)q_t^I + kq_t^R + q_{t,i})q_{t,i} - C(q_{t,i}$$

- Let the solution to f.o.c be given by $q_t^R = H^R(q_t^I, \eta)$

Imitation vs. Rational play

$$q^R_{t+1} = H^R(q^I_{t+1}, \eta) = H^R(\eta q^R_t + (1-\eta)q^I_t, \eta)$$
$$q^I_{t+1} = \eta q^R_t + (1-\eta)q^I_t.$$

Lemma

The Cournot-Nash equilibrium, where all firms produce the Cournot-Nash quantity (q^*, q^*), is a locally stable fixed point for the model with exogenous fractions of rational and imitation firms if and only if $|\eta H_q(q*, \eta) + 1 - \eta| < 1$

- linear-linear case: $\lambda_2 = 1 - \eta - \frac{(n-1)(1-\eta)\eta}{2+(n-1)\eta} < 1$, always holds

Evolutionary competition between heuristics

- heuristics involve *information cost* $C_k \geq 0$, that may differ across heuristics
- Fitness of a heuristic is then given by the average profits generated in the game minus the information costs, $U_k = \Pi_k - C_k$
- fractions evolve according to an evolutionary dynamic

$$\eta_{t+1} = K(U_{1,t} - U_{2,t}) = K(\Delta U_{1,t}).$$

- The map $K : \mathbb{R} \to [0, 1]$ is a continuously differentiable, monotonically increasing function with $K(0) = \frac{1}{2}$

Imitation vs. Cournot with heuristics switching

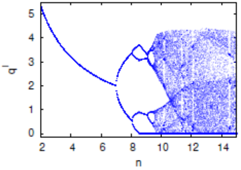
$$
\begin{aligned}
q^C_{t+1} &= R((n-1)(\eta q^C_t + (1-\eta)q^I_t)) \\
q^I_{t+1} &= \eta q^C_t + (1-\eta)q^I_t. \\
\eta_{t+1} &= K(\Delta U_t)
\end{aligned}
$$

Lemma

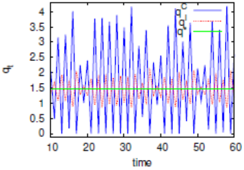
The Cournot-Nash equilibrium $(q^*, q^*, \eta^* = K(-C))$ is a locally stable fixed point for the model with endogenous fractions of Cournot and imitators where all firms produce the Cournot-Nash quantity, firms if and only if $\eta^* R((n-1)q^*)(n-1) - \eta^* > -2$.

- linear-linear case: (q^*, q^*, η^*) is stable when $n < \frac{4-\eta^*}{\eta^*}$

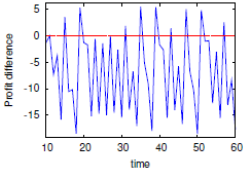
Imitation vs. Cournot: simulations



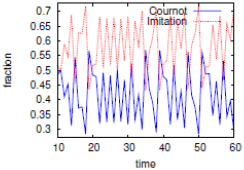
(a) Bifurcation diagram (q_t, n)



(b) Time path of Cournot and imitatio quantities



(c) Cournot profit differential



(d) Time path Cournot fraction

Game parameters: $n = 10,\ a = 17,\ b = 1,\ c = 1,\ C^C = 0,\ C^I = 0,\ \beta = 0.05$

Initial conditions: $q_0^C = 0.8,\ q_0^I = 0.8,\ \eta_0 = 0.5$

Imitation vs. Rational play with switching

$$q^R_{t+1} = H^R(q^I_{t+1}, \eta) = H^R(\eta q^R_t + (1-\eta)q^I_t, \eta)$$
$$q^I_{t+1} = \eta q^R_t + (1-\eta)q^I_t$$
$$\eta_{t+1} = K(\Delta U_t)$$

Lemma

The Cournot-Nash equilibrium $(q^*, q^*, \eta^* = K(-C))$ is a locally stable fixed point for the model with endogenous fractions of rational and imitation firms, where all firms produce the Cournot-Nash quantity, if and only if $|\eta^* H_q(q^*, \eta^*) + 1 - \eta^*| < 1$

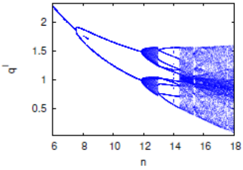
- linear-linear example: (q^*, q^*, η^*) is stable for all n regardless of C!

Imitation vs. Cournot vs. Rational with switching

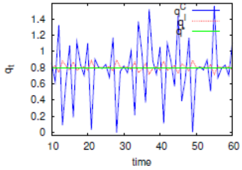
$$
\begin{aligned}
q^R_{t+1} &= H^R(q^C_{t+1}, q^I_{t+1}, \eta^R_{t+1}, \eta^C_{t+1}) \\
q^C_{t+1} &= R((n-1)(\eta^R_t q^R_t + \eta^C_t q^C_t + (1-\eta^R_t - \eta^C_t) q^I_t) \\
q^I_{t+1} &= \eta^R_t q^R_t + \eta^C_t q^C_t + (1-\eta^R_t - \eta^C_t) q^I_t \\
\eta_{R,t+1} &= K^R(\Delta U^R_t, \Delta U^C_t) \\
\eta_{C,t+1} &= K^C(\Delta U^R_t, \Delta U^C_t).
\end{aligned}
$$

- ($q^*, \eta^{R*} = \frac{e^{\beta(C^C - C^R)}}{e^{\beta(C^C-C^R)}+1+e^{-\beta(C^I-C^C)}}, \eta^{C*} = \frac{e^{\beta(C^I-C^C)}}{e^{\beta(C^I-C^R)}+e^{-\beta(C^I-C^C)}+1}$)
- Stability:
  - $C^R > 0, C^I = C^C = 0 : \lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4(n, C^R\beta) = \frac{3e^{C^R\beta} - ne^{C^R\beta}}{n+4e^{C^R\beta}+1}$
  - instability threshold: $n < \psi(C^{R*}\beta) = \frac{7e^{C^R\beta}+1}{e^{C^R\beta}-1}$
  - general $C^I, C^C \neq 0$: stability depends on the relative costs of the stable heuristic(imitation) to costs of the unstable heuristic (Cournot)

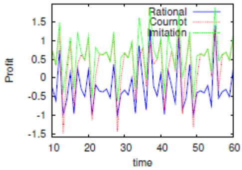
Imitation vs. Cournot vs. Rational: simulations



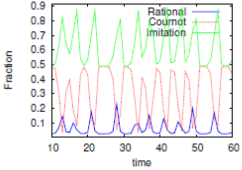
(a) Bifurcation diagram (q_t, n)



(b) Time path of Cournot and imitation quantities



(c) Cournot profit differential



(d) Time path Cournot fraction

Parameters: $n = 19,\ a = 17,\ b = 1,\ c = 1,\ C^R = 1,\ C^C = 0,\ C^I = 0,\ \beta = 3.$

Initial conditions: $q_0^R = 0.3,\ q_0^C = 0.1,\ q_0^I = 0.25,\ \eta_0^R = 0.5,\ \eta_0^C = 0.2.$

Conclusions

- imitation: stabilizing role
- ecology of rules critical: cheaper stable (imitation) vs. cheaper unstable (best-reply) rule
- conditional imitation?: imitate-the-best (the better)
- super(sub)modular games