Self-Similar Measures in Multi-Sector Endogenous Growth Models

Davide La Torre*, Simone Marsiglio† and Fabio Privileggi‡

*Dept. of Economics, Management and Quantitative Methods – Università di Milano
†School of Business – James Cook University, Cairns (QLD, Australia)
‡Dept. of Economics and Statistics "Cognetti de Martiis" – Università di Torino

8th Workshop "Dynamic Models in Economics and Finance"
September, 18-20, 2014 – Urbino (Italy)

Introduction

- We analyze 2 types of stochastic discrete time multi-sector endogenous growth models:

1. a basic Lucas-Uzawa (1988) model and
2. an extended three sector version as in La Torre and Marsiglio (2010)

- In both models we explicitly compute the optimal dynamics which, as the models may exhibit sustained growth, can diverge in the long-run
- Thus we focus on the dynamics of (different types of capital) ratio variables
- Through a log-transformation, they become linear Iterated Function Systems (IFS) converging to some self-similar invariant measure, possibly supported on a fractal set
- We determine parameters' configurations under which such measures turn out to be singular or, in some special cases, absolutely continuous.

The standard 2-sector Lucas-Uzawa model (1988)

$$V(k_0, h_0, z_0) = \max_{\{c_t, u_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$\text{s.t.} \quad \begin{cases} k_{t+1} = z_t k_t^{\alpha} (u_t h_t)^{1-\alpha} - c_t \\ h_{t+1} = b(1-u_t) h_t \\ k_0 > 0, \; h_0 > 0, \; z_0 \in \{q, 1\} \text{ are given} \end{cases}$$

- $0 < \beta < 1$ rate of time preference
- c_t consumption
- k_t physical capital
- h_t human capital
- $0 < \alpha < 1$ physical capital share
- $0 < u_t < 1$ proportion of human capital employed in physical production
- $b > 0$ productivity coefficient of (linear) human capital production
- z_t *iid* exogenous shock multiplicatively affecting final production, it takes on two values, $z_t \in \{q, 1\}$, $0 < q < 1$

- Educational choices are not affected by eventual shocks

Uncertainty

Assumption

Only 2 shock values can occur with positive probability, $z_t \in \{q, 1\}$, $0 < q < 1$, *each with (constant) probability* p *and* $1 - p$, *respectively*

- Interpretation: at any time, given the realization of the random shocks, the economy may be in 2 alternative situations:

1. an economic crisis due to a supply shock, $z_t = q$, lowering physical productivity
2. a business-as-usual scenario with no shocks, $z_t = 1$, in which the whole economy evolves along its full capacity

Optimal dynamics computation

- Thanks to the log-Cobb-Douglas specification, we can apply the Verification principle to the Bellman equation and analytically obtain the value function $V(k, h, z)$ plus the optimal dynamics of control and state variables:

$$
\begin{aligned}
c_t &= (1-\alpha\beta)(1-\beta)^{1-\alpha} z_t k_t^{\alpha} h_t^{1-\alpha} \\
u_t &\equiv 1-\beta \qquad \forall t \\
k_{t+1} &= \alpha\beta(1-\beta)^{1-\alpha} z_t k_t^{\alpha} h_t^{1-\alpha} \\
h_{t+1} &= b\beta h_t
\end{aligned}
$$

- Consumption is proportional to output; *i.e.*, the saving rate is constant (as in Solow, 1956)
- The share of human capital employed in final production is constant (as in Bethmann, 2007)

Detrended dynamics

- k_t and h_t are diverging whenever $b > 1/\beta$
- Hence, we take **physical to human capital ratio**,

$$\chi_t = \frac{k_t}{h_t},$$

which reduces the 2-dimensional system into a 1-dimensional dynamic that evolves over time according to:

$$\chi_{t+1} = \sigma z_t \chi_t^{\alpha},$$

$$\text{with } \sigma = \frac{\alpha \left(1 - \beta\right)^{1-\alpha}}{b}$$

The associated nonlinear IFS

- The associated nonlinear IFS is defined by the two maps

$$\begin{cases} f_0(\chi) = \sigma q \chi^{\alpha} & \text{with probability } p \\ f_1(\chi) = \sigma \chi^{\alpha} & \text{with probability } 1-p \end{cases} \tag{1}$$

- which eventually is being trapped into (a subset of) the compact interval $[\chi_0^*, \chi_1^*]$ (χ_0^* and χ_1^* are the fixed points of f_0 and f_1)
- If (1) converges to an invariant measure supported over (a subset of) $[\chi_0^*, \chi_1^*]$, then we have a **stochastic balanced growth path (SBGP)** equilibrium, the stochastic equivalent of a typical equilibrium in deterministic endogenous growth theory

Non-contractive IFS

- Whenever $\alpha > q$ the IFS (1) turns out to be **non-contractive**, as there exists a right neighborhood of the left fixed point χ_1^* on which $f_1' > 1$.
- In this case, the general theory on IFS establishing convergence to a unique invariant measure cannot be directly applied, as it is based on the assumption that the maps of the IFS are **contractions**
- However, the next Proposition establishes the existence of a unique invariant measure for (1) indirectly.

Conjugate linear IFS

Proposition

The one-to-one logarithmic transformation $\chi_t \to \varphi_t$ defined by:

$$\varphi_t = -\frac{1-\alpha}{\ln q} \ln \chi_t + 1 + \frac{\ln \sigma}{\ln q},$$

defines a contractive linear IFS (a **similitude***) equivalent to the original nonlinear dynamics $\chi_{t+1} = \sigma z_t \chi_t^{\alpha}$, which is composed of two maps $w_0, w_1 : [0,1] \to [0,1]$ (0 and 1 are the fixed points of w_0 and w_1) given by:*

$$\begin{cases} w_0(\varphi) = \alpha\varphi & \text{with probability } p \\ w_1(\varphi) = \alpha\varphi + (1-\alpha) & \text{with probability } 1-p. \end{cases} \quad (2)$$

The IFS (2) converges weakly to a unique **self-similar** *measure supported on an attractor which is either the interval $[0,1]$ when $1/2 \leq \alpha \leq 1$ or a Cantor set when $0 < \alpha < 1/2$*

Relationship with the stochastic 1-sector growth model

- Apart from the constant $\sigma = \left[\alpha\left(1-\beta\right)^{1-\alpha}\right]/b$, our nonlinear—as well as linear—optimal dynamics turn out to be the same as those of the 1-sector stochastic optimal growth model in Mitra et al. (2003)
- The novelty here is that what converges to an invariant measure supported on a Cantor set is a transformation of the physical to human capital ratio (and not a transformation of physical capital)
- Hence, we have shown that also an economy experiencing sustained growth can exhibit a long-run pattern related to some fractal attractor
- Specifically, the SBGP equilibrium has a fractal nature.

A nonlinear non-contractive IFS that converges to a unique invariant measure

- The following Corollary establishes weak convergence of the nonlinear IFS to a unique invariant measure also when $\alpha > q$, that is, when it is *non-contractive*

Corollary

For any $0 < \alpha < 1$, $0 < \beta < 1$, $0 < q < 1$, $0 < p < 1$, *and* $b > 1/\beta$ *(the latter envisaging sustained growth), the nonlinear IFS (1) weakly converges to a unique invariant measure supported either over the full interval* $[\chi_0^*, \chi_1^*]$ *or over some subset of it.*
In the latter case, whenever $0 < \alpha < 1/2$ *the attractor of (1) is a generalized topological Cantor set—i.e., totally disconnected and perfect—contained in* $[\chi_0^*, \chi_1^*]$

An example

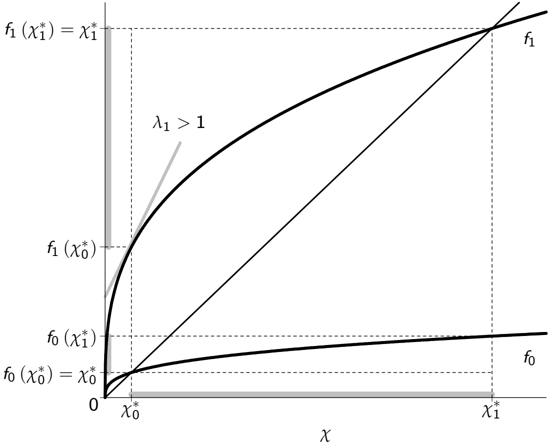

Figure: the nonlinear maps f_0 and f_1 in (1) when $\alpha = 1/3$, $q = 1/6$, $p = 2/3$, $\beta = 0.96$ and $b = 1.052 > 1/\beta$ (sustained growth). Such IFS is non-contractive, as the Lipschitz constant $\lambda_1 = f_1'(\chi_0^*) = \alpha/q = 2$ associated to f_1 is larger than 1; its attractor is a generalized topological Cantor set as $f_0(\chi_1^*) < f_1(\chi_0^*)$.

Singular vs. absolute continuous self-similar measures

Theorem (Peres & Solomyak, 1998; Mitra et al., 2003)

Let μ^* be the self-similar measure associated to the IFS (2), $(\alpha\varphi, \alpha\varphi + (1-\alpha)\,; p, (1-p))$, on $[0,1]$.

- i) *If $0 < \alpha < p^p (1-p)^{1-p}$, then μ^* is singular.*
- ii) *If $\alpha = p^p (1-p)^{1-p}$ and $p \neq 1/2$, then μ^* is singular.*
- iii) *If $\alpha = p = 1/2$, then μ^* is absolutely continuous—it is the uniform (Lebesgue) measure over $[0,1]$.*
- iv) *If $1/3 \leq p \leq 2/3$, then μ^* is absolutely continuous for Lebesgue a.e. $p^p (1-p)^{1-p} < \alpha < 1$.*
- v) *If $0.156 < p < 1/3$ or $2/3 < p < 0.844$, then μ^* is absolutely continuous for Lebesgue a.e. $p^p (1-p)^{1-p} < \alpha < 0.649$, while, for any $1 < \gamma \leq 2$ such that $\left[p^\gamma + (1-p)^\gamma\right]^{1/(\gamma-1)} < 0.649$, μ^* has density in L^γ for Lebesgue a.e. $\left[p^\gamma + (1-p)^\gamma\right]^{1/(\gamma-1)} \leq \alpha < 0.649$.*

The last result in a bifurcation diagram

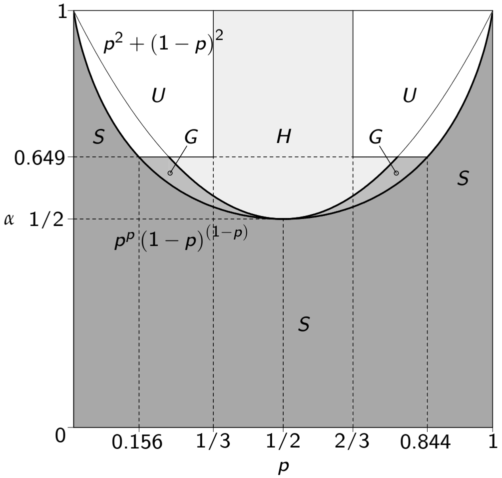

Figure: S: singular measure; H: *a.e.* absolutely continuous measures with density in L^2; G: *a.e.* absolutely continuous measures with density in L^γ, with $1 < \gamma \leq 2$; U: unknown area.

A 3-Sector Model I (La Torre and Marsiglio, 2010)

- Now human capital is endogenously allocated across three sectors: 1) *physical capital*, 2) *human capital* and 3) *knowledge (technology)*.
- (Cobb-Douglas) final production uses all 3 factors, (Cobb-Douglas) knowledge production uses knowledge and human capital, while (linear) human capital uses only itself

A 3-Sector Model II

$$V(k_0, h_0, a_0, z_0, \eta_0) = \max_{\{c_t, u_t, v_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$\text{s.t.} \quad \begin{cases} k_{t+1} = z_t k_t^{\alpha} (u_t h_t)^{\gamma} a_t^{1-\alpha-\gamma} - c_t \\ h_{t+1} = b(1 - u_t - v_t) h_t \\ a_{t+1} = \eta_t (v_t h_t)^{\phi} a_t^{1-\phi} \\ k_0 > 0, \; h_0 > 0, \; a_0 > 0, \; z_0 \in \{q_1, q_2, 1\}, \; \eta_0 \in \{r, 1\} \text{ given} \end{cases}$$

- Everything as in previous model plus:
  - $0 < \gamma < 1 - \alpha$ human capital share in final production
  - $0 < \phi < 1$ human capital share in knowldege production
  - $0 < v_t < 1$ proportion of human capital employed in knowledge production
  - η_t another *iid* exogenous shock that multiplicatively affects only knowledge production—besides z_t affecting final production
- Again educational choices are not affected by exogenous shocks

Uncertainty

- There are 2 *iid* exogenous shocks, z_t and η_t, multiplicatively affecting respectively the production of the final good and that of knowledge

Assumption

Only 3 pairs of shock values can occur with positive probability, $(z_t, \eta_t) \in \{(q_1, r), (q_2, 1), (1, 1)\}$, *with* $0 < q_1 < q_2 < 1$ *and* $0 < r < 1$, *each with (constant) probability* p_0, p_1, p_2, $0 < p_i < 1$ *for all* i, *respectively, with* $\sum_{i=0}^{2} p_i = 1$

- Interpretation: at any time the economy may be in 3 situations:

1. deep financial crisis with wide effects on the whole economy, involving both production and knowledge sectors: $(z_t, \eta_t) = (q_1, r)$
2. a sudden surge in raw materials' (oil) prices affecting only production sector but not that of knowledge: $(z_t, \eta_t) = (q_2, 1)$
3. no shocks, the economy evolves along full capacity: $(z_t, \eta_t) = (1, 1)$

Optimal dynamics computation

- Thanks to the log-Cobb-Douglas specification, we can apply the Verification principle to the Bellman equation and analytically obtain the value function $V(k, h, z)$ plus the optimal dynamics of control and state variables:

$$
\begin{aligned}
c_t &= (1-\alpha\beta)\,\overline{u}^{\gamma} z_t k_t^{\alpha} h_t^{\gamma} a_t^{1-\alpha-\gamma}, \\
u_t &\equiv \frac{\gamma(1-\beta)(1-\beta+\beta\phi)}{\gamma(1-\beta)+\beta\phi(1-\alpha)} = \overline{u} \qquad \forall t \\
v_t &\equiv \frac{\beta\phi(1-\alpha-\gamma)(1-\beta)}{\gamma(1-\beta)+\beta\phi(1-\alpha)} = \overline{v} \qquad \forall t \\
k_{t+1} &= \alpha\beta\overline{u}^{\gamma} z_t k_t^{\alpha} h_t^{\gamma} a_t^{1-\alpha-\gamma} \\
h_{t+1} &= b(1-\overline{u}-\overline{v})\,h_t \\
a_{t+1} &= \overline{v}^{\phi} \eta_t h_t^{\phi} a_t^{1-\phi}
\end{aligned}
$$

Detrended dynamics

- k_t and h_t and a_t are diverging whenever $b > 1/\left(1 - \overline{u} - \overline{v}\right)$
- Hence, we take the the physical to human capital and the knowledge to human capital ratio variables,

$$\chi_t = \frac{k_t}{h_t} \qquad \text{and} \qquad \omega_t = \frac{a_t}{h_t},$$

which reduces the 3-dimensional system into a 2-dimensional nonlinear dynamic that evolves over time according to:

$$\begin{cases} \chi_{t+1} = \Delta z_t \chi_t^{\alpha} \omega_t^{1-\alpha-\gamma} \\ \omega_{t+1} = \Theta \eta_t \omega_t^{1-\phi} \end{cases} \tag{3}$$

$$\text{with } \Delta = \frac{\alpha\beta\overline{u}^{\gamma}}{b\left(1-\overline{u}-\overline{v}\right)} \qquad \text{and} \qquad \Theta = \frac{\overline{v}^{\phi}}{b\left(1-\overline{u}-\overline{v}\right)}$$

- If this system converges to an invariant measure supported over some compact set of $\mathbb{R}^2$, then we have a SBGP equilibrium

Conjugate linear IFS I

Proposition

Assume that $\phi \neq 1-\alpha$ *and parameters* q_1, q_2 *satisfy* $q_1 < q_2^2$ *if* $\phi < 1-\alpha$ *or* $q_1 > q_2^2$ *if* $\phi > 1-\alpha$, *and let* $r = \left(q_1/q_2^2\right)^{\frac{1-\alpha-\phi}{1-\alpha-\gamma}}$.
Then, the one-to-one transformation $(\chi_t, \omega_t) \to (\varphi_t, \psi_t)$ *defined by*

$$\varphi_t = \rho_1 \ln \chi_t + \rho_2 \ln \omega_t + \rho_3$$
$$\psi_t = \rho_4 \ln \omega_t + \rho_5$$

where

$$\rho_1 = -\frac{1-\alpha}{2\ln q_2},\ \rho_2 = \frac{(1-\alpha-\gamma)(1-\alpha)}{2(1-\alpha-\phi)\ln q_2},\ \rho_3 = 1 + \frac{1}{2\ln q_2}\left(\ln \Delta - \frac{1-\alpha-\gamma}{1-\alpha-\phi}\ln \Theta\right),$$
$$\rho_4 = \frac{(1-\alpha-\gamma)\phi}{(1-\alpha-\phi)}\ln\left(\frac{q_2^2}{q_1}\right),\ \rho_5 = 1 + \frac{(1-\alpha-\gamma)}{(1-\alpha-\phi)}\ln\left(\frac{q_1}{q_2^2}\right)\ln \Theta,$$

defines a contractive linear IFS which is equivalent to the nonlinear dynamics in (3)

Conjugate linear IFS II

Proposition (... continued)

Such IFS is composed of the following 3 maps $w_0, w_1, w_2 : \mathbb{R}^2 \to \mathbb{R}^2$*:*

$$\begin{cases} w_0(\varphi, \psi) = (\alpha\varphi, (1-\phi)\psi) & \textit{with prob. } p_0 \\ w_1(\varphi, \psi) = (\alpha\varphi + (1-\alpha)/2, (1-\phi)\psi + \phi) & \textit{with prob. } p_1 \\ w_2(\varphi, \psi) = (\alpha\varphi + (1-\alpha), (1-\phi)\psi) & \textit{with prob. } p_2 \end{cases} \quad (4)$$

and converges weakly to a unique self-similar measure supported on a **generalized Sierpinski gasket** *with vertices* $(0,0)$, $(1/2, 1)$ *and* $(1,0)$

- Rewriting (4) as

$$\begin{cases} \varphi_{t+1} = \alpha\varphi_t + \zeta_t \\ \psi_{t+1} = (1-\phi)\psi_t + \vartheta_t, \end{cases}$$

one can see that the random vector $(\zeta_t, \vartheta_t) \in \mathbb{R}^2$ taking the 3 values $(0,0)$, $((1-\alpha)/2, \phi)$ and $(1-\alpha, 0)$ corresponds to the 3 values (q_1, r), $(q_2, 1)$ and $(1,1)$ for the original random variables (z_t, η_t)

Similitudes and self-similar measures

- If the contraction mappings w_i in a IFS on $\mathbb{R}^n$ are **similitudes**, *i.e.*, if there exist numbers $0 < \lambda_i < 1$ such that

$$d\left(w_i\left(x\right), w_i\left(y\right)\right) = \lambda_i d\left(x, y\right), \quad \forall x, y \in X, \quad i = 0, \ldots, m-1,$$

  the attractor A^* and the invariant measure μ^* of the IFS are said to be **self-similar**
- An IFS satisfies the **open set condition (OSC)** if there exists a nonempty open set U such that $w_i\left(U\right) \subset U$ for all $i = 0, \ldots, m-1$ and $w_i\left(U\right) \cap w_j\left(U\right) = \varnothing$ for all $i \neq j$
- The OSC requires that the image sets of the attractor, $w_i\left(A^*\right)$, have only "*small overlap*" ("just touching")

Singular vs. absolute continuous self-similar measures

Theorem (Ngai and Wang, 2005)

Let $(w; p)$ be an IFS on $\mathbb{R}^n$ with maps $w_i : \mathbb{R}^n \to \mathbb{R}^n$ defined by $w_i(x) = \lambda_i Q_i x + \xi_i$, $i = 0, \ldots, m-1$, where $0 < \lambda_i < 1$, $\xi_i \in \mathbb{R}^n$ and Q_i is an orthogonal $n \times n$ matrix, and let $p = (p_0, p_1, \ldots, p_{m-1})$ be the associated probability weights. Denote by μ^* the self-similar invariant measure defined by $(w; p)$

i) *If $\prod_{i=0}^{m-1} p_i^{p_i} \lambda_i^{-np_i} > 1$, then μ^* is singular*

ii) *If $\prod_{i=0}^{m-1} p_i^{p_i} \lambda_i^{-np_i} = 1$ but $p_i \neq \lambda_i^n$ for some i, then μ^* is singular*

iii) *If $p_i = \lambda_i^n$ for all $i = 0, \ldots, m-1$, then μ^* is absolutely continuous if and only if the IFS $(w; p)$ satisfies the open set condition (OSC). In this case μ^* is the uniform (n-dimensional Lebesgue) measure over the attractor $A^* \subset \mathbb{R}^n$*

WORK IN PROGRESS!!!

- The (critical) assumption $\phi \neq 1 - \alpha$ in our result establishing the one-to-one correpondence between the nonlinear dynamics in (3) and the linear IFS (4) implies that *the latter cannot be a similitude*
- This precludes the possibility of applying the Ngai and Wang (2005) Theorem to say something on singularity vs.absolute continuity of μ^*
- However, we believe that our result holds when $\phi = 1 - \alpha$ as well, and we trust we'll find a way to prove it

A conjecture

- Hence, through a partial application of Ngai and Wang (2005) Theorem, we conjecture at least the following Proposition
- Let $p_2 = 1 - p_0 - p_1$ and define the (exponential of the) *entropy of the Bernoulli process* underlying the exogenous shocks in our model as

$$E\left(p_0, p_1\right) = p_0^{p_0} p_1^{p_1} \left(1 - p_0 - p_1\right)^{1 - p_0 - p_1}$$

Proposition

If $\phi = 1 - \alpha$, then the self-similar measure μ^* associated to our linear iFS on the square $[0,1]^2$ is singular whenever $0 < \alpha \leq \sqrt{E\left(p_0, p_1\right)}$

- At any rate, nothing can be said on the possible absolute continuity of μ^* when $\sqrt{E\left(p_0, p_1\right)} < \alpha < 1$

An illustration of the last result

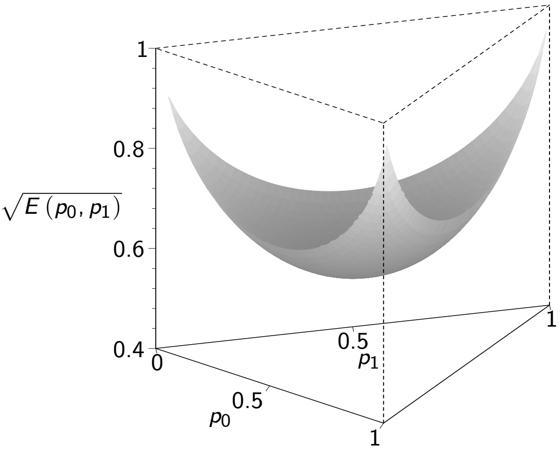

Figure: plot of the square root of of the entropy curve, $\sqrt{E(p_0, p_1)}$, on the unitary simplex. Any α-value on or below such curve characterizes an IFS that weakly converges to a singular self-similar measure supported on a generalized Sierpinski gasket with vertices $(0, 0)$, $(1/2, 1)$ and $(1, 0)$.

Examples

- We keep constant $\beta = 0.96$, $q_1 = 0.2$ and $q_2 = 0.6$, we set $\gamma = \phi$ and $b = 1/\left(1 - \overline{u} - \overline{v}\right) + 0.01$, so to have always sustained growth
- $\gamma = \phi$ implies that $r = q_1/q_2^2 \equiv 0.556$ and $q_1 = 0.2 < 0.36 = q_2^2$ holds, which implies that we must choose values for the key parameters α, ϕ satisfying $\phi < 1 - \alpha$
- We decide to link parameter ϕ to our choice of parameter α according to

$$\phi = 1 - \alpha - 0.001,$$

so that $\phi < 1 - \alpha$ holds, but at the same time we keep very close to the condition $\phi = 1 - \alpha$ required by the last Proposition

Example 1

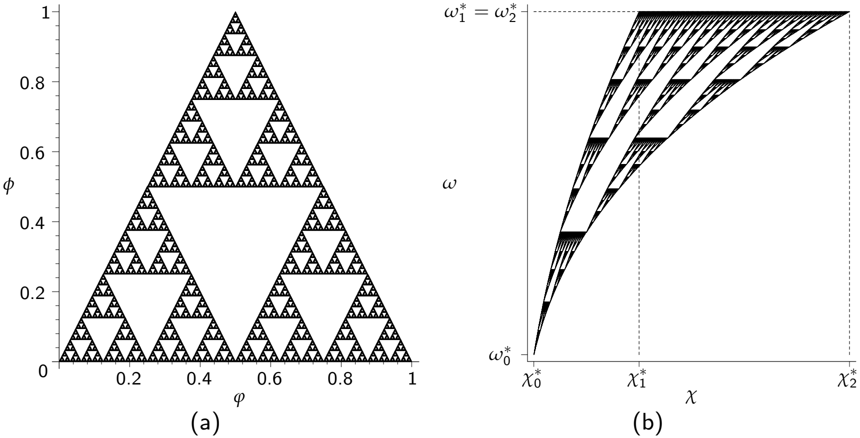

Figure: (a) first 8 iterations of the linear IFS for (a) $\alpha = 0.5$, $\phi = 0.499$, and (b) its corresponding distorted nonlinear counterpart. (χ_0^*, ω_0^*), (χ_1^*, ω_1^*) and (χ_2^*, ω_2^*) are the fixed points of the 3 maps of the nonlinear IFS. As $\alpha = 1/2 < \min \sqrt{E(p_0, p_1)} \cong 0.5806$, μ^* should be *singular*.

Example 2

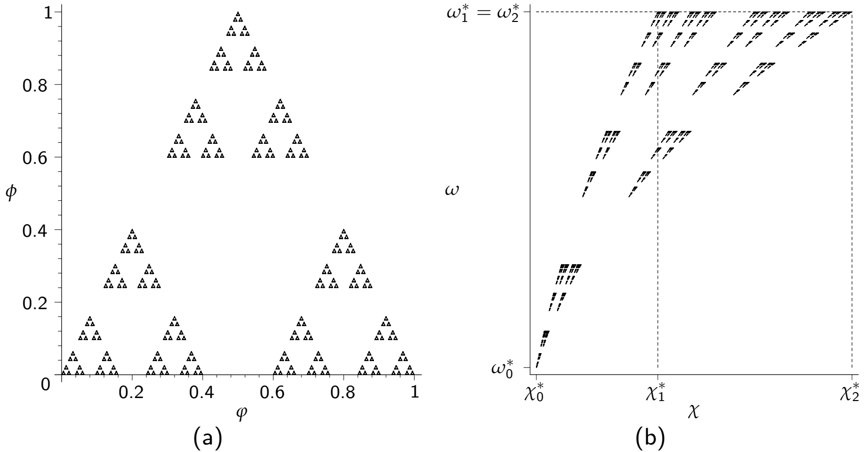

Figure: (a) first 8 iterations of the linear IFS for (a) $\alpha = 0.4$, $\phi = 0.599$, and (b) its corresponding distorted nonlinear counterpart. Again μ^* should be *singular*, as clearly confirmed by the strong no-overlapping of the prefractals.

Example 3

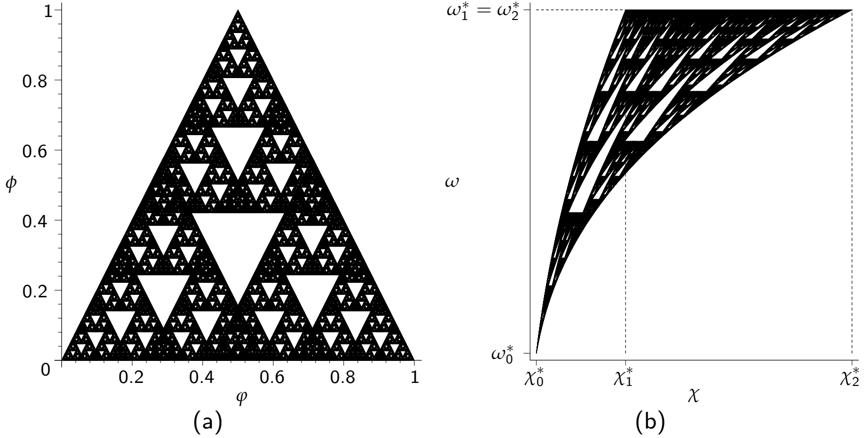

Figure: (a) first 8 iterations of the linear IFS for (a) $\alpha = 1/\sqrt{3}$, $\phi = 0.4216$, and (b) its nonlinear counterpart. Again, as $\alpha = 1/\sqrt{3} \cong 0.5774 < \min \sqrt{E(p_0, p_1)} \cong 0.5806$, μ^* should be *singular*, although the degree of overlapping of the prefractals would lead to believe that it may be absolutely continuous.