A Schelling-like Segregation Model with heterogeneous distributions of tolerance and entry restrictions

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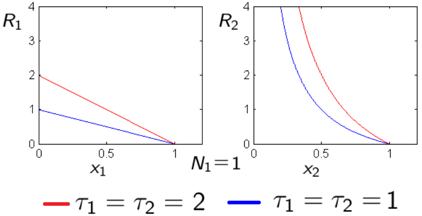
Aim of the paper

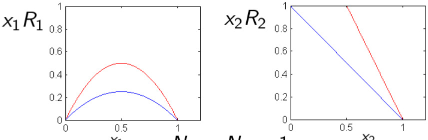
- In the seminal paper Schelling (1969), Shelling proposed two models for the description of residential segregation of a population formed by two kinds of inhabitants, differing, e.g., for racial or religious or cultural features.
- Segregation is explained in terms of the **interplay of individual choices** driven by *tolerances* on neighbors of opposite kind.
- The first model proposed by Shelling is a typical agent-based simulation model. It has inspired a flourishing stream of literature, see e.g. Epstein and Axtell (1996), Zhang (2004) and Pancs and Vriend (2007).
- The second one is formulated in term of a two-dimensional dynamical system, see also Bischi and Merlone (2011). The second approach has been rather neglected.
- With the aim to fill the gap, we study the effects of **heterogeneous distributions of tolerance** and **entry limitations** on the dynamics of the second model employing the latest developments on piecewise-smooth systems.

Assumptions

- Individuals are partitioned in two groups: (local population) C_1 of numerosity N_1 and (newcomers) C_2 of numerosity N_2.
- Let $x_1(t)$ (resp. $x_2(t)$) be the number of members of group C_1 (resp. C_2) that live in a residential area at time t.
- *Distribution of Tolerance* (Clark (1991)) $R_1(x_1) = \tau_1 \left(1 - \frac{x_1}{N_1}\right)$ gives the maximum ratio $\frac{x_2}{x_1}$ that a fraction $\frac{x_1}{N_1}$ of members of C_1 abide.
- *Distribution of Tolerance* (Clark (1991) & Bischi and Merlone (2011)) $R_2(x_2) = \tau_2 \left(\frac{N_2 - x_2}{x_2}\right)$ gives the maximum ratio $\frac{x_1}{x_2}$ that a fraction $\frac{x_2}{N_2}$ of members of C_2 abide.



Assumptions



$N_1 = N_2 = 1$

- Relative variation of individuals of group $C_{i,i=1,2}$:

$$\frac{x_i(t+1) - x_i(t)}{x_i(t)} = \gamma_i \left[x_i(t) R_i(x_i(t)) - x_j(t) \right], \quad i = 1, 2, \quad i \neq j$$

  $\gamma_i > 0$ is the *speed of adjustment*.

- $\gamma := \gamma_1 = \gamma_2$ due to general conditions of the economy.
- *Natural constraints* $0 \leq x_i(t) \leq N_i$, $i = 1, 2$. $N_1 = 1$ and $N_2 \leq N_1$.
- **Entry limitations** for newcomers $x_2(t) \leq K_2$, with $K_2 \leq N_2$.

The model

The segregation model is described by map $T: D \to D$, where $D := [0, N_1] \times [0, K_2]$, given by

$$(x_1(t+1), x_2(t+1)) = (T_1(x_1(t), x_2(t)), T_2(x_1(t), x_2(t)))$$

$$T_1(x_1, x_2) = \begin{cases} 0 & \text{if} \quad F_1(x_1, x_2) \le 0 \\ F_1(x_1, x_2) & \text{if} \quad 0 \le F_1(x_1, x_2) \le N_1; \\ N_1 & \text{if} \quad F_1(x_1, x_2) \ge N_1 \end{cases}$$

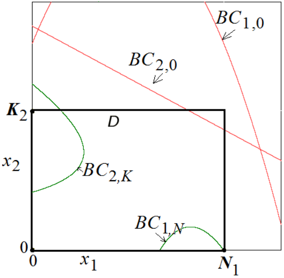
$$T_2(x_1, x_2) = \begin{cases} 0 & \text{if} \quad F_2(x_1, x_2) \le 0 \\ F_2(x_1, x_2) & \text{if} \quad 0 \le F_2(x_1, x_2) \le K_2; \\ K_2 & \text{if} \quad F_2(x_1, x_2) \ge K_2 \end{cases}$$

where

$$\begin{aligned} F_1(x_1, x_2) &= x_1 [1 - \gamma x_2 + \gamma x_1 R_1(x_1)] \\ F_2(x_1, x_2) &= x_2 [1 - \gamma x_1 + \gamma x_2 R_2(x_2)] \end{aligned}$$

Then the **following curves are of non-differentiability** for T:

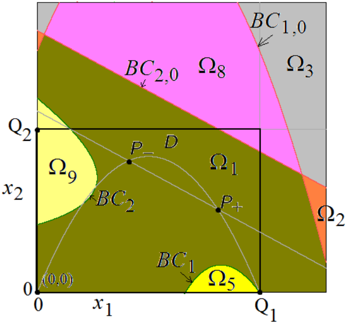
$$\begin{aligned} BC_{1,N} &: \quad F_1(x_1, x_2) = N_1; \\ BC_{2,K} &: \quad F_2(x_1, x_2) = K_2; \\ BC_{1,0} &: \quad F_1(x_1, x_2) = 0, \; x_1 \neq 0; \\ BC_{2,0} &: \quad F_2(x_1, x_2) = 0, \; x_2 \neq 0; \\ x_1 = 0 &; \quad x_2 = 0. \end{aligned}$$



Remarks on the equilibria of the model

The phase plane is divided in 9 regions where T is defined by different functions:

$$
\begin{array}{ll}
(x_1,x_2)\in\Omega_1 & :(x_1',x_2')=(F_1(x_1,x_2),F_2(x_1,x_2))\\
(x_1,x_2)\in\Omega_2 & :(x_1',x_2')=(0,F_2(x_1,x_2))\\
\mathbf{(x_1,x_2)\in\Omega_3} & :\mathbf{(x_1',x_2')=(0,0)}\\
\mathbf{(x_1,x_2)\in\Omega_4} & :\mathbf{(x_1',x_2')=(0,K_2)}\\
(x_1,x_2)\in\Omega_5 & :(x_1',x_2')=(N_1,F_2(x_1,x_2))\\
\mathbf{(x_1,x_2)\in\Omega_6} & :\mathbf{(x_1',x_2')=(N_1,0)}\\
\mathbf{(x_1,x_2)\in\Omega_7} & :\mathbf{(x_1',x_2')=(N_1,K_2)}\\
(x_1,x_2)\in\Omega_8 & :(x_1',x_2')=(F_1(x_1,x_2),0)\\
(x_1,x_2)\in\Omega_9 & :(x_1',x_2')=(F_1(x_1,x_2),K_2)
\end{array}
$$



Property (fixed points of T)

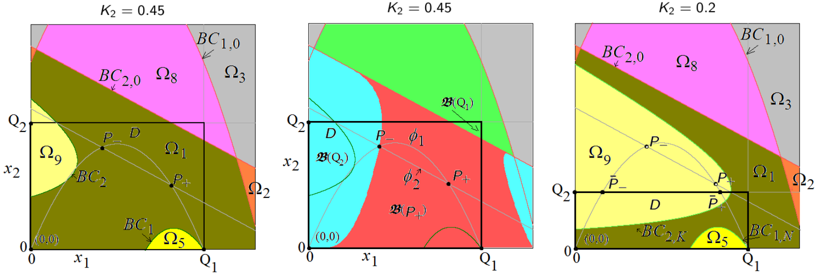
1. **Equilibria of segregation**: *$Q_1 = (N_1, 0)$ and $Q_2 = (0, K_2)$ always exist. If $\Omega_6 \cap D$ has positive measure, then Q_1 is superstable, otherwise either stable or unstable. If $\Omega_4 \cap D$ has positive measure, then Q_2 is superstable, otherwise stable.*
2. **Equilibrium of extinction**: *$(0, 0)$ is always locally unstable though its basin of attraction $\mathcal{B}(0,0) = \Omega_3 \cap D$ can have a positive measure.*

Remarks on the equilibria of the model

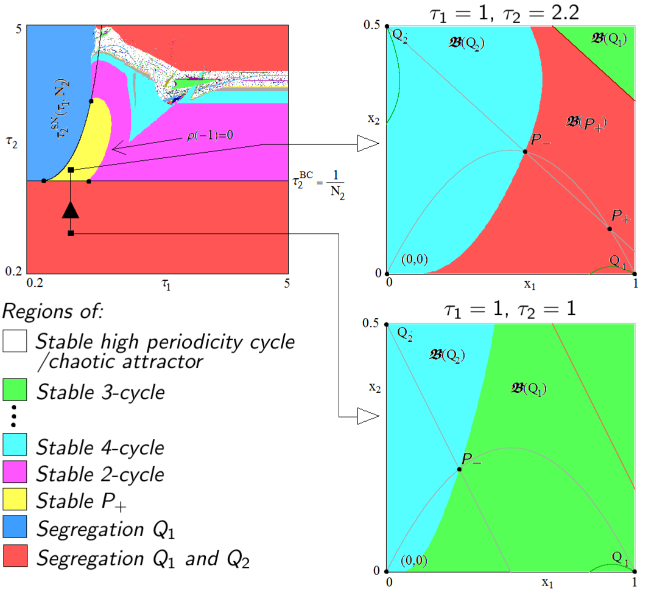
Property

1. **Natural equilibria of nonsegregation**: *solve $F_1 = x_1$ and $F_2 = x_2$. They are at most two, P_- and P_+ and are feasible in $\Omega_1 \cap D$. P_- is either a saddle or a repellor while P_+ is either a saddle or an attractor.*
2. **Artificial equilibria of nonsegregation**: *$\bar{P}_-$ and $\bar{P}_+$ are feasible iff $\bar{P}_{+,-} \in \Omega_9 \cap D$. When feasible $\bar{P}_-$ is a saddle and $\bar{P}_+$ is an attractor.*

Parameters' values: $N_1 = 1$, $N_2 = 0.5$, $\tau_1 = 1.5$, $\tau_2 = 3$ and $\gamma = 1.5$.



The dynamics for $K_2 = N_2 = 0.5$ and $\gamma = 1.5$



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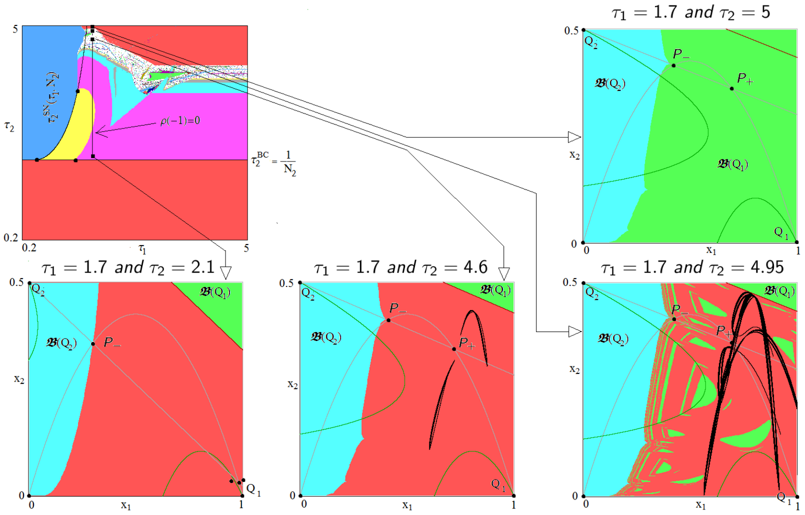

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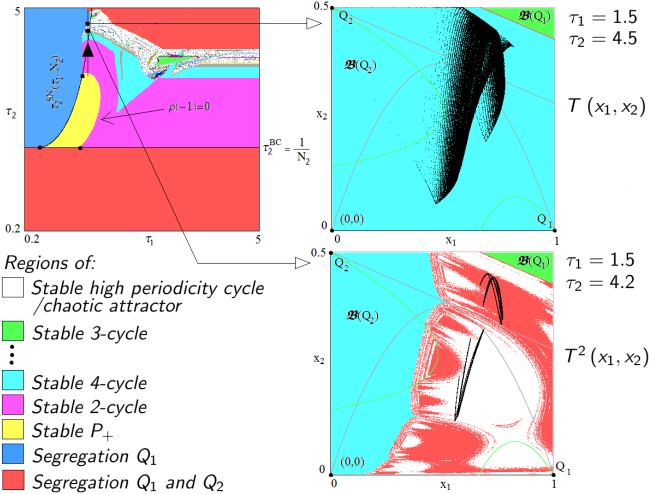
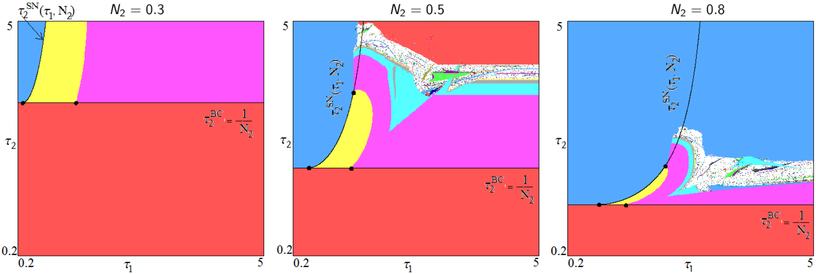


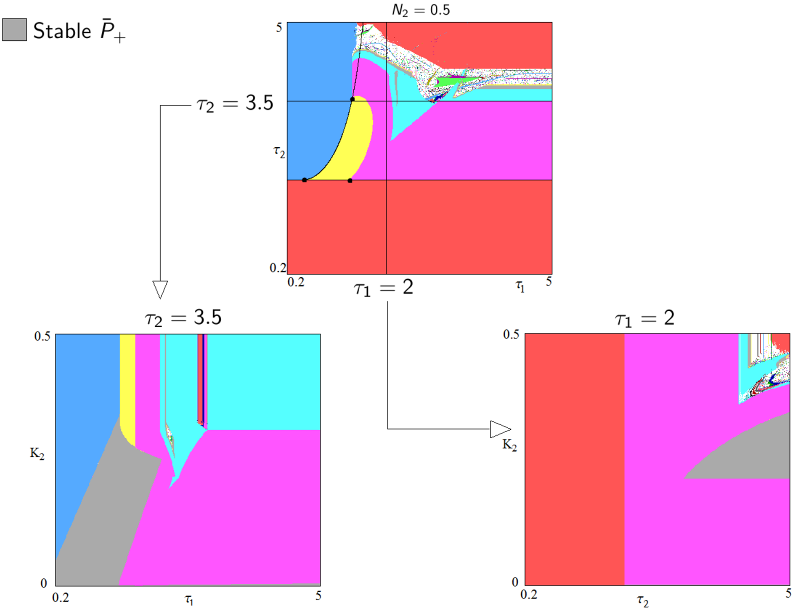
The dynamics for $K_2 = N_2$ and $\gamma = 1.5$



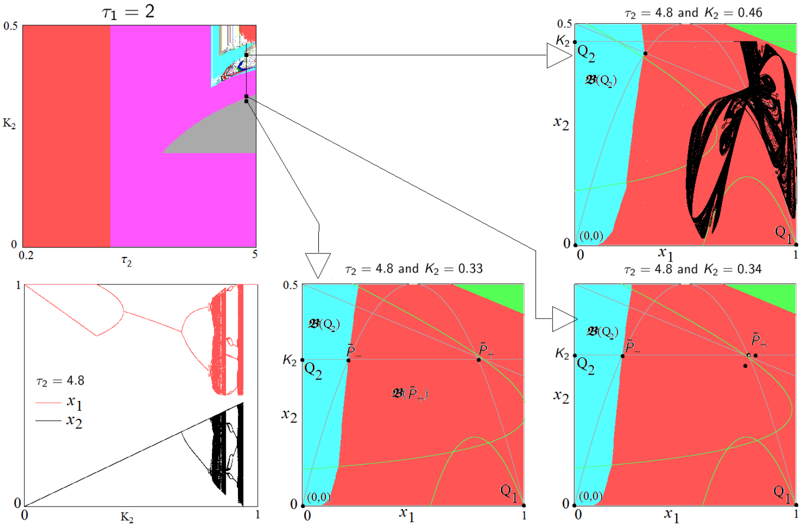
Remark

- *Similar 2D bifurcation diagrams for* $\gamma = 2, 1, 0.7$.
- *Low levels of tolerance* $(\tau_2 < \tau_2^{TR})$ *of the newcomers lead to segregation.*
- *The region of "segregation due to non tolerance":* $\{(\tau_1, \tau_2) \,|\tau_2 < \tau_2^{BR} \text{ or } \tau_2 > \tau_2^{SN}(\tau_1, N_2)\}$

Effects of entry limitations: $K_2 \le N_2 = 0.5$ and $\gamma = 1.5$



Effects of entry limitations: $K_2 \leq N_2 = 0.5$ and $\gamma = 1.5$



Conclusions

- The investigation shows that heterogeneous distributions of tolerance can lead to segregation.
- **Entry limitations are a useful policy measure to avoid segregation**.
- Entry limitations reduce the overshooting effects due to "emotional behaviors", see Schelling (1969) and Bischi and Merlone (2011).
- Entry limitations are responsible for border collision bifurcations through which periodic and chaotic solutions disappear.
- Recent contributions show that similar results hold true even with homogeneous distributions of tolerance, see Radi *et al.* (2014a) and Radi *et al.* (2014b).

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