Modeling of economic dynamics under stochastic noise:

Confidence domains in the analysis of noise-induced transition for Goodwin model

Tatyana Ryazanova

with

Jochen Jungeilges

Urbino, Italy
2014

Classical Goodwin model

$$\varepsilon\theta\ddot{Y}(t)+[\varepsilon+(1-\alpha)\theta]\dot{Y}(t)-\varphi(\dot{Y}(t))+(1-\alpha)Y(t)=O^{*}(t) \quad [1]$$

where $Y(t)$ – income at time t,
ε – saving rate, $\varepsilon \in (0,1)$
θ – the investment outlay

$$\ddot{Y}(t)+a\frac{Y^{2}(t)-1}{Y^{2}(t)+1}\dot{Y}(t)-bY(t)+cY^{3}(t)=O^{*}(t) \quad [2]$$

First question is: How did Lorenz and Nusse take this form of Goodwin model?

Second question is: How are new parameters *a*, *b* and *c* identified in terms of the parameters in classical model?

[1] R.Goodwin. The Nonlinear Accelerator and the Persistence of Business Cycles. // Econometrica, 19, 1 (1951), 1-17.
[2] H.W. Lorenz, H.E. Nusse. Chaotic attractors, chaotic saddles, and fractal basin boundaries: Goodwin's nonlinear accelerator model reconsidered. // Chaos Solit. Fract.,13, (2002), 975-965

From Goodwin to van der Pol

$$\ddot{Y}(t)+\frac{\varepsilon+\theta[1-c'(Y(t))]}{\varepsilon\theta}\dot{Y}(t)+\frac{Y(t)-c(Y(t))}{\varepsilon\theta}=\frac{\phi(Y(t),\dot{Y}(t))}{\varepsilon\theta}+\frac{\beta+\lambda-R}{\varepsilon\theta}$$

where β – automonous consumption,
λ – automonous investment,
$C(t)$ – consumption at time t,
$\phi(t)$ – accelerator function at time t,
R – the sum of the remainder terms for the respective Taylor expansions.

Assumption 1. $C(t)=\alpha Y+\gamma Y^3$, where $\alpha\in(0,1)$, $\gamma\in(-1,0)$

linear part → αY; nonlinear term → γY^3

Assumption 2. $\phi(Y(t),\dot{Y}(t))=\psi(Y(t))\dot{Y}(t)$

$$\psi(Y)=\frac{\theta(1+Y^2)(1-\alpha-3\gamma Y^2)+2\varepsilon}{1+Y^2}$$

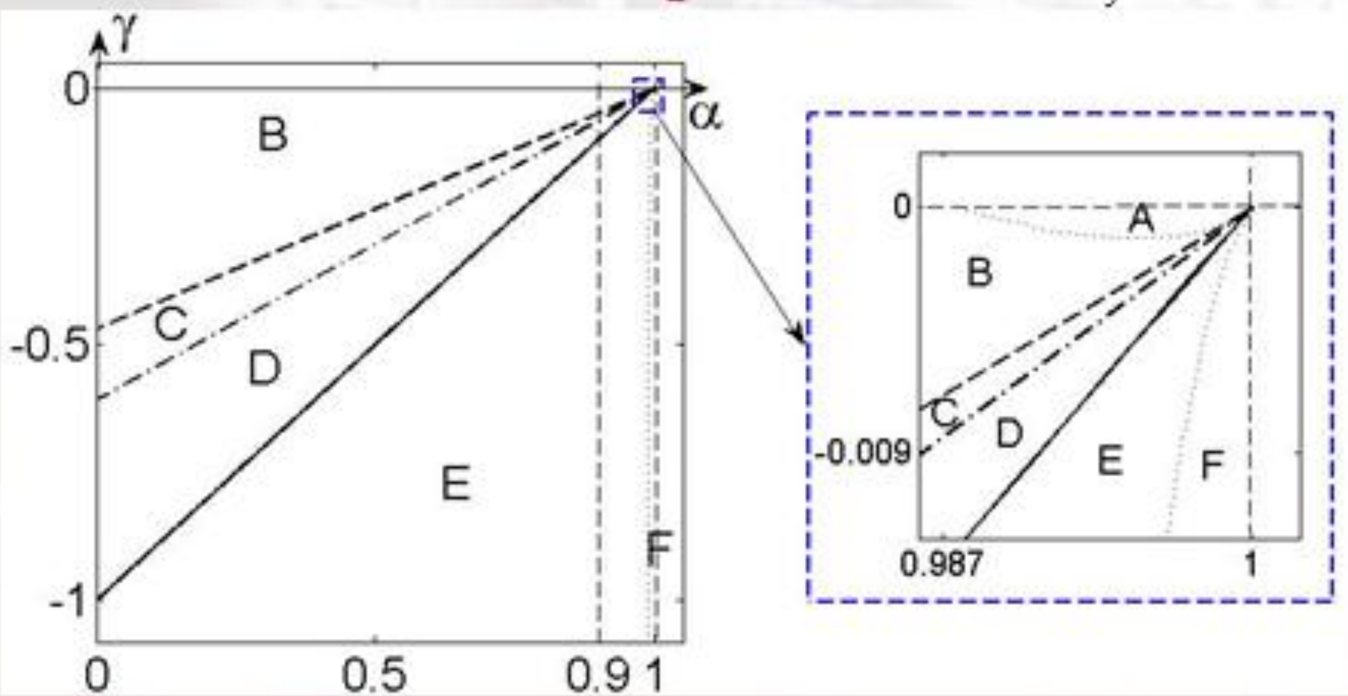
Interpretation: Instead of modeling the accelerator as constant we allow the accelerator to vary in levels of income.

Deterministic attractors

$$\ddot{Y}(t)+\frac{1}{\theta}\cdot\frac{Y^2(t)-1}{Y^2(t)+1}\dot{Y}(t)-\frac{1-\alpha}{\varepsilon\theta}Y(t)+\frac{-\gamma}{\varepsilon\theta}Y^3(t)=0$$

3 equilibria $M_{1,2}=\left(\pm\sqrt{\frac{\alpha-1}{\gamma}},0\right)$ and $M_3=\left(0,0\right)$ $\qquad M_3$ – saddle

Bifurcation diagram for $\theta=2,\ \ \varepsilon=0.2,$ for $M_{1,2}$

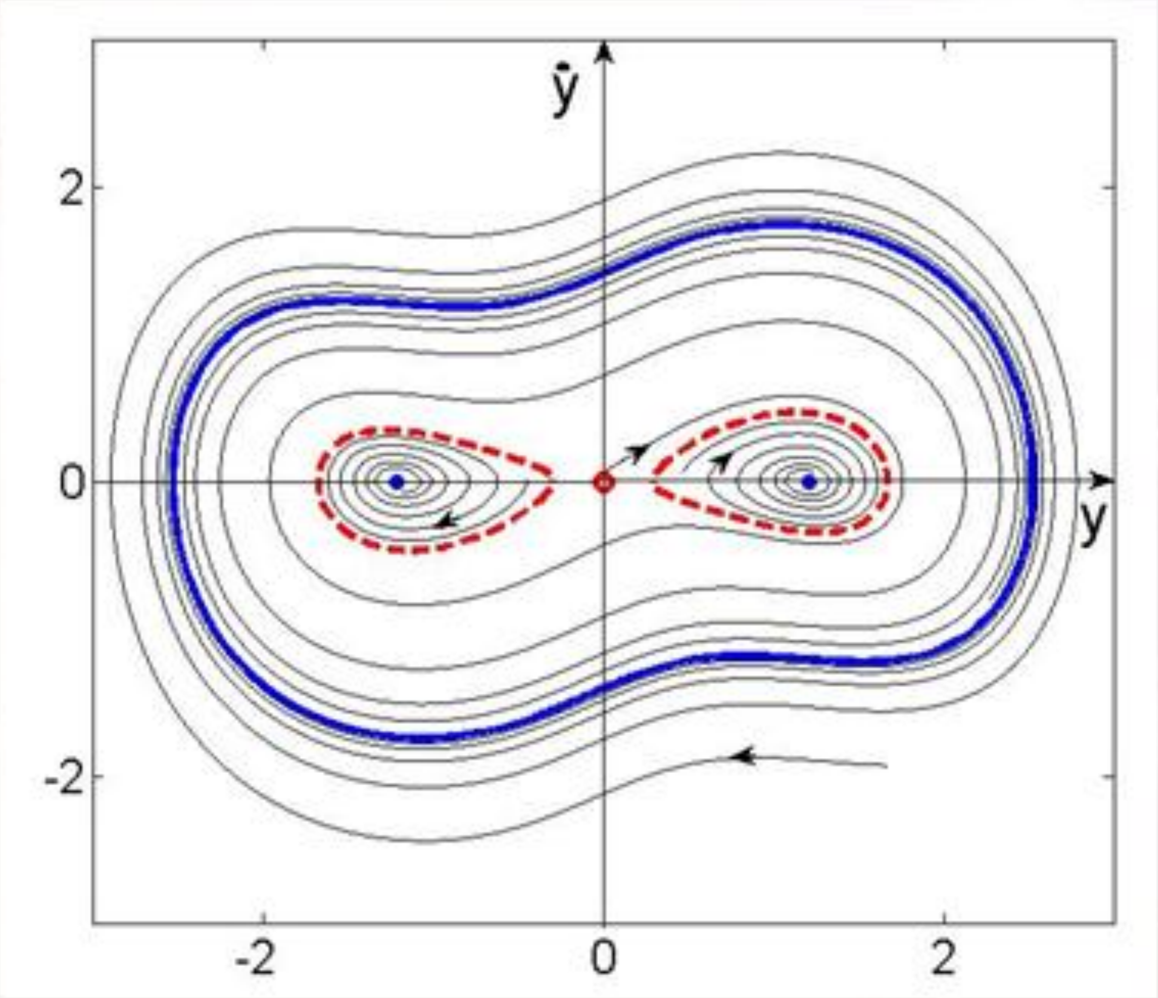


A – stable node,
B, C, D – stable focus,
E – unstable focus,
F – unstable node.

Numerically: C – one stable cycle and one unstable cycle,
D – one stable cycle and two unstable cycle.

Deterministic attractors

Phase portrait for $\theta = 2,\ \varepsilon = 0.2,\ \alpha = 0.9,\ \gamma = -0.07$



Stochastic sensitivity function

$$\dot{x} = f(x) + \varepsilon\sigma(x)\dot{w},$$ **- Ito equation**

$$\frac{\varepsilon^2}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\rho) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}(f_i\rho) = 0,$$ **Stationary Fokker-Plank equation**

$$a_{ij} = \left[\sigma\sigma^T\right]_{ij}$$

$$v(x) = -\lim_{\varepsilon\to 0}\varepsilon^2 \ln \rho(x,\varepsilon),$$ **- quasipotential** [3]

$$\rho(x,\varepsilon) \approx K \exp\left(-\frac{v(x)}{\varepsilon^2}\right)$$ **- asymptotic of the stationary distribution**

Approximation quasipotential $v(x)$ [4]

for equilibrium $\bar{x}$

$$v(x) \approx \tfrac{1}{2}\left(x - \bar{x}, W^{-1}(x - \bar{x})\right)$$

for T-periodic solution $x = \xi(t)$

$$v(x) \approx \tfrac{1}{2}\left(x - \xi(t), W^{+}(t)(x - \xi(t))\right)$$

[3] Freidlin M.I., Wentzell, A.D. Random Perturbations of Dynamical Systems. // Springer, New York (1984).

[4] I.A.Bashkirtseva, L.B.Ryashko Sensitivity and chaos control for the forced nonlinear oscillations. // Chaos Solit.Fract., 26, 1437-1451 (2005).

Stochastic sensitivity function

I.Bashkirtseva, L.Ryashko [5]

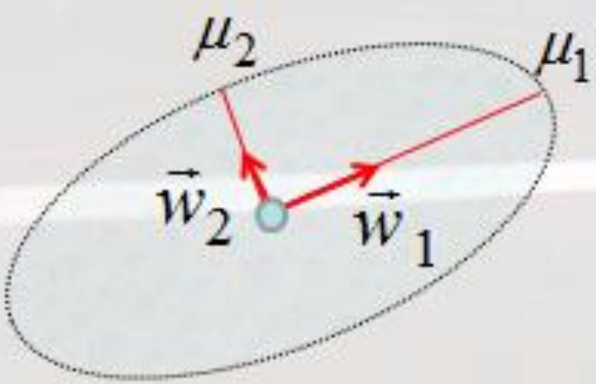
1. equilibrium

$\bar{x}$ - an exponentially stable equilibrium

W - stochastic sensitivity matrix

$$FW + WF^T = -S,$$

where $F = \dfrac{\partial f}{\partial x}(\bar{x}),\ \ S = GG^T,\ \ G = \sigma(\bar{x}).$



$\vec{w}_1, \vec{w}_2$ – normalized eigenvectors of W

μ_1, μ_2 – eigenvalues of W

[5] I.A.Bashkirtseva, L.B.Ryashko Sensitivity analysis of stochastic attractors and noise-induced transitions for population model with Allee effect. // *Chaos*, 21, 047514

Stochastic sensitivity function
2. Limit cycle

$x = \xi(t)$ - an exponentially stable limit cycle corresponding T-periodic solution

$$\begin{cases} \dot{W} = F(t)W + WF^T(t) + P(t)S(t)P(t), \\ W(0) = W(T), \\ W(t)f(\xi(t)) \equiv 0, \end{cases}$$

where $F(t) = \frac{\partial f}{\partial x}(\xi(t)), \ S(t) = \sigma(\xi(t))\sigma^T(\xi(t)),$

$P(t)$ – a matrix of the orthogonal projection onto the hyperplane Π_t.

for 2d systems $W(t) = m(t)P(t)$, where $P(t) = p(t)p^T(t)$

$p(t)$ – a normalized vector orthogonal to $f(\xi(t))$,
$m(t) > 0$ – a T - periodic scala function satisfying

$$\begin{cases} \dot{m} = a(t)m + b(t), \\ m(0) = m(T), \end{cases}$$

where $a(t) = p^T(t)(F^T(t) + F(t))p(t),$
$b(t) = p^T(t)S(t)p(t).$

$m(t)$ - stochastic sensitivity function (SSF).

Stochastic Goodwin model

$$\ddot{Y}(t) + \frac{1}{\theta}\cdot\frac{Y^2(t)-1}{Y^2(t)+1}\dot{Y}(t) - \frac{1-\alpha}{\varepsilon\theta}Y(t) + \frac{-\gamma}{\varepsilon\theta}Y^3(t) = \sigma\dot{w}$$

w – standart Wiener procces, σ – intensity of noise.

Stochastic sensitivity matrix for equilibrium

$$W = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix},$$

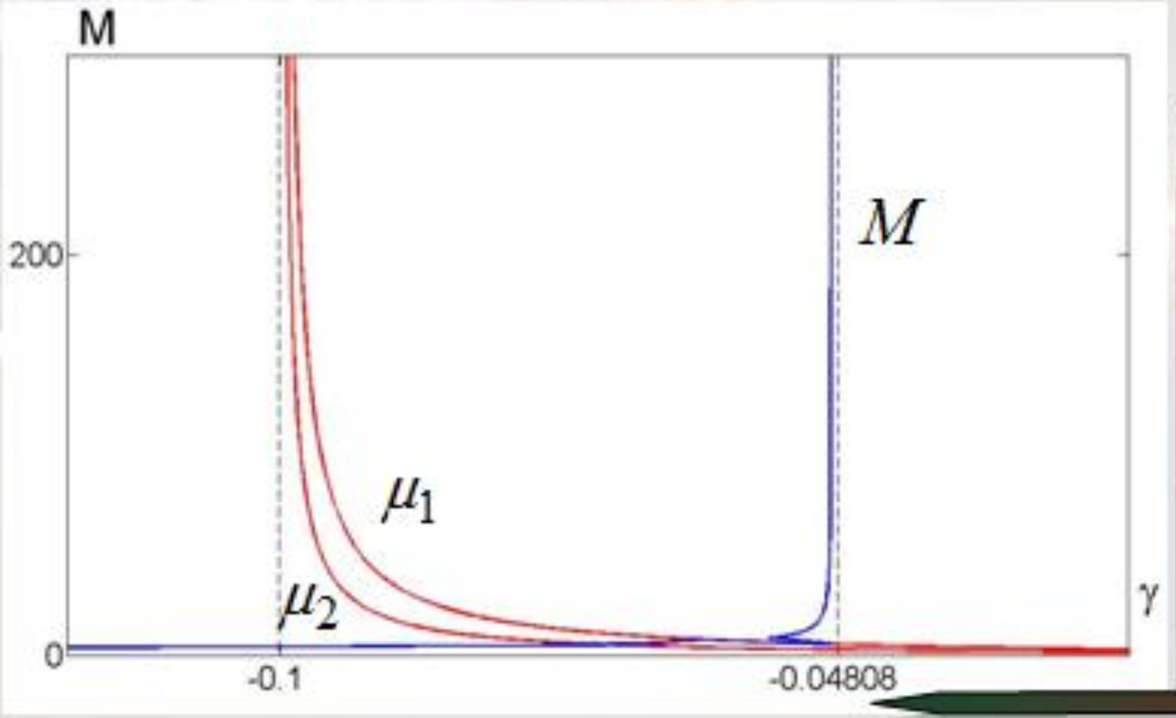
$$\mu_1 = \frac{\varepsilon\theta^2(1-\alpha-\gamma)}{4(1-\alpha)(1-\alpha+\gamma)},$$

$$\mu_2 = \frac{\theta(1-\alpha-\gamma)}{2(1-\alpha+\gamma)}$$

$$\theta = 2,\quad \varepsilon = 0.2,\quad \alpha = 0.9$$

Stochastic factor for limit cycle

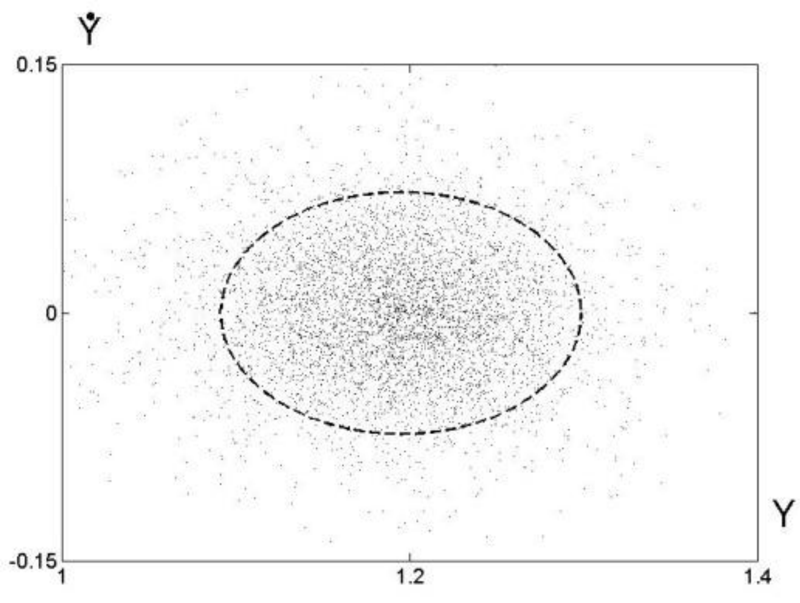
$$M = \max_{x\in[0,T]} m(t)$$



Stochastic Goodwin model

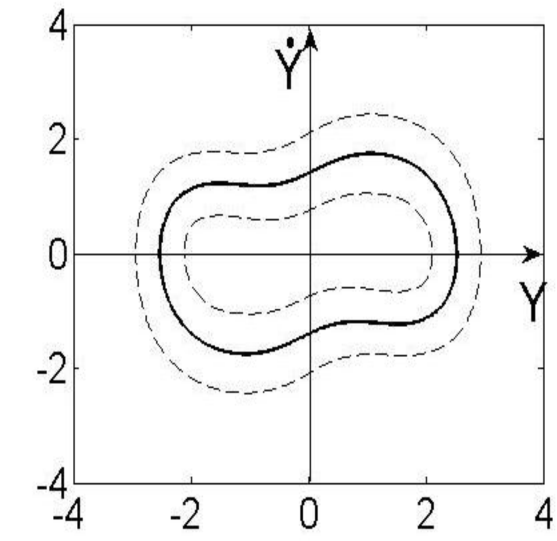
Confidence ellipse for equilibrium

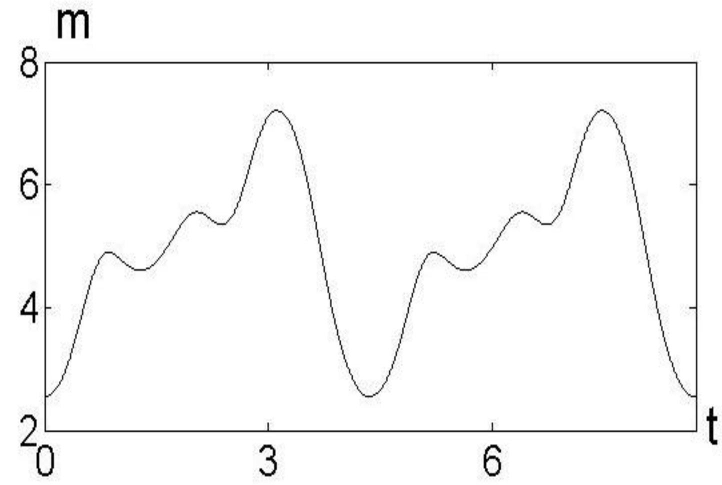
$$\theta = 2, \quad \varepsilon = 0.2, \quad \alpha = 0.9,$$
$$\gamma = -0.07, \quad \sigma = 0.01$$



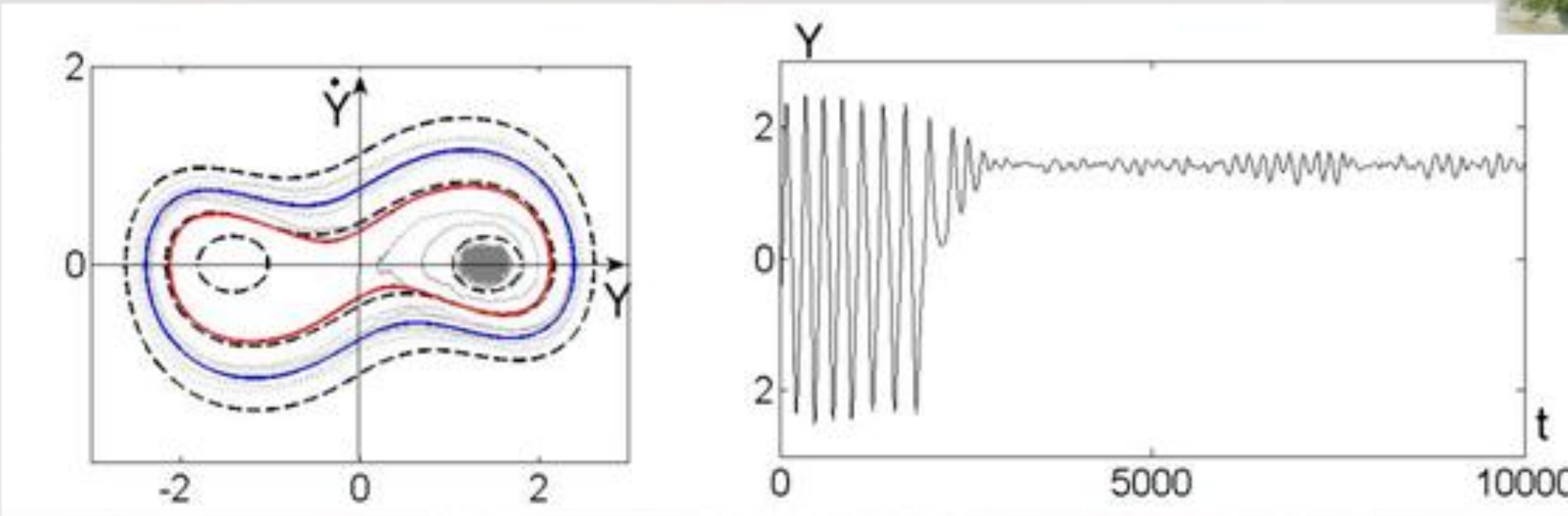
Confidence band for limit cycle

$$\theta = 2, \quad \varepsilon = 0.2,$$
$$\alpha = 0.9, \quad \gamma = -0.07$$
$$\sigma = 0.1$$

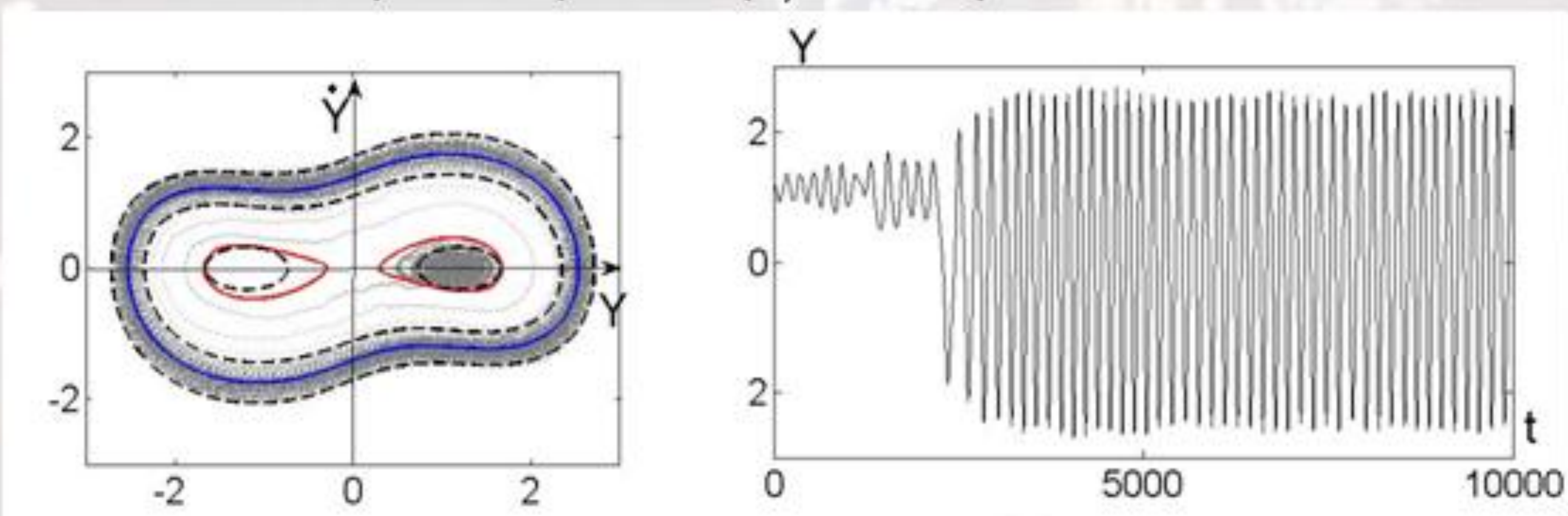




Noise-induced transitions in Goodwin model



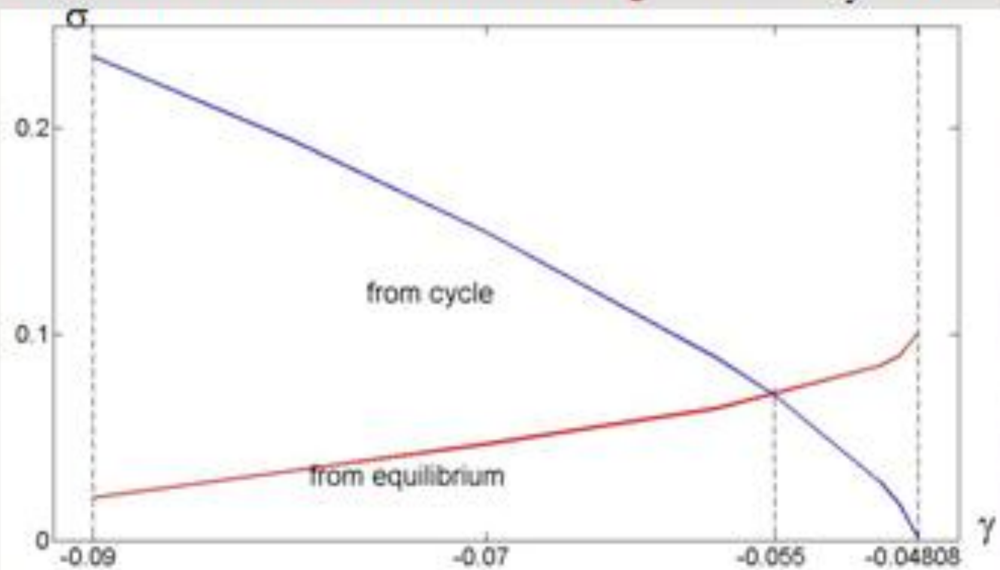
$\theta = 2, \ \varepsilon = 0.2, \ \alpha = 0.9, \ \gamma = -0.05, \ \sigma = 0.03$

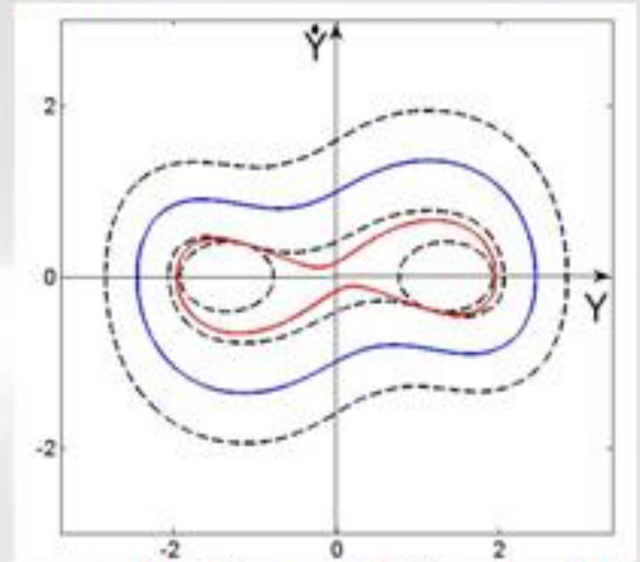


$\theta = 2, \ \varepsilon = 0.2, \ \alpha = 0.9, \ \gamma = -0.07, \ \sigma = 0.045$

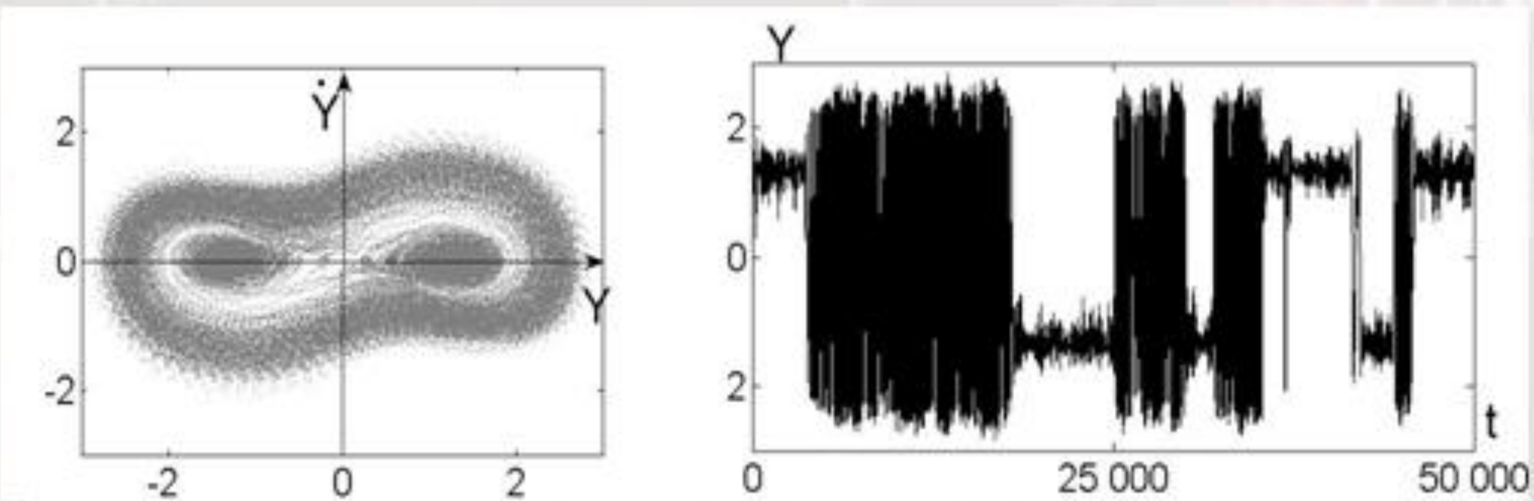
Noise-induced transition for Goodwin model

Critical values of intensity $\theta = 2, \ \alpha = 0.9, \ \varepsilon = 0.2$





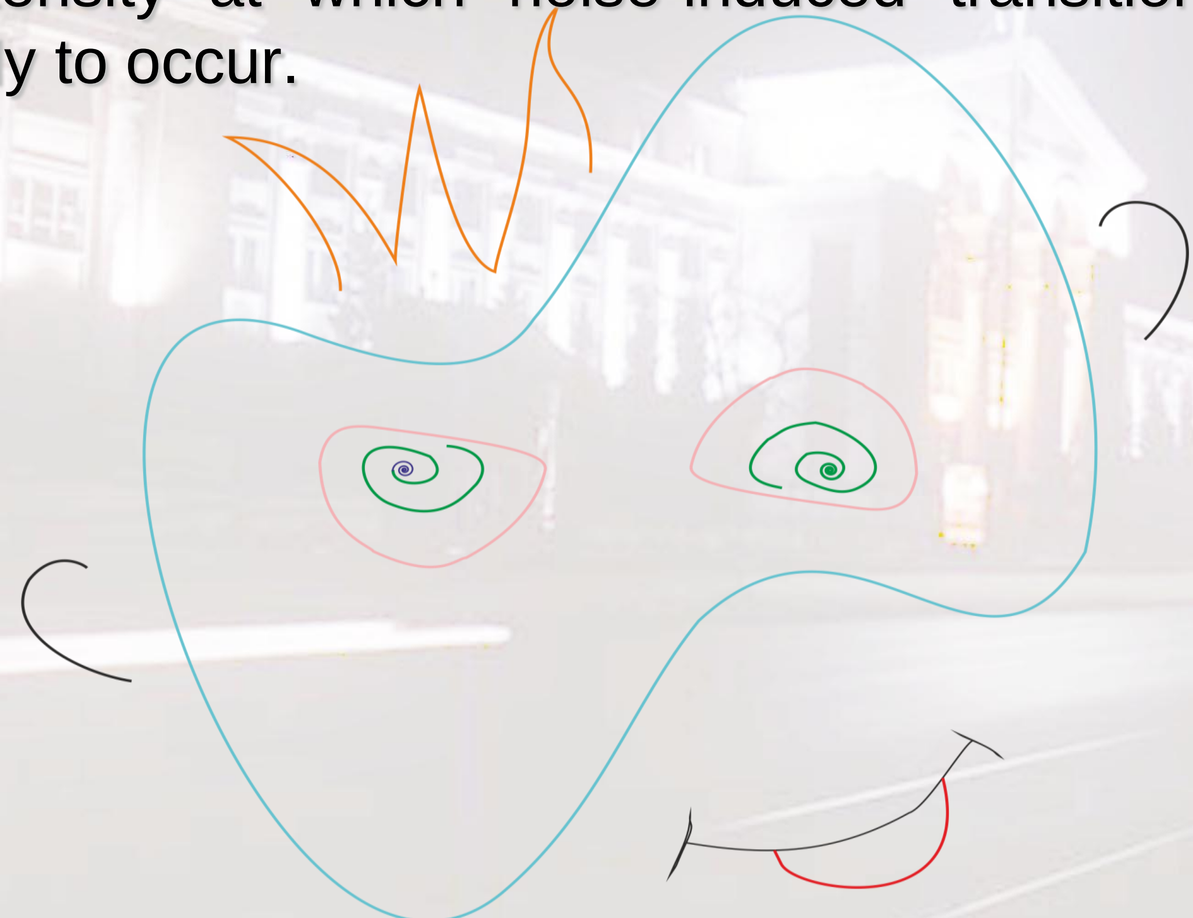
$\theta = 2, \ \varepsilon = 0.2, \ \alpha = 0.9,$
$\gamma = -0.055,$
$\sigma = 0.07$



$\theta = 2, \ \varepsilon = 0.2, \ \alpha = 0.9, \ \gamma = -0.055, \ \sigma = 0.07$

Conclusion

We have demonstrated how the confidence domain method is used to understand the qualitative changes in stochastic dynamics and predict a critical value of the noise intensity at which noise-induced transitions are most likely to occur.



Thank you for your attention!