A continuous time Cournot duopoly with delays

Luca Guerrini

Department of Management,
Polytechnic University of Marche

Luca Gori

Department of Law,
University of Genoa

Mauro Sodini

Department of Economics and
Management, University of Pisa,

Outline

- Motivations of the work;
- Brief discussion on existing literature;
- The mathematical model;
- Dynamics of the model;
- Conclusions;

Motivations

- (Philosophical) Debate on Continuous-time Vs Discrete-time framework.
- Huge differences in mathematical properties of the models, especially if characterized by bounded rationality (Dixit Vs Bischi et Al.)....

Possible compromise: continuous time framework...with delays. In this setting it is possible to have important nonlinearities even if the system is described by two equations

Related literature:

- Dixit, A.K., 1986, Comparative statics for oligopoly. International Economic Review 27
- Puu, T., 1991, Chaos in duopoly Pricing, Chaos, Solitons & Fractals 1, 573-581.
- Bischi, G.I., Naimzada A, 1999 Global Analysis of a Dynamic Duopoly Game with Bounded Rationality, Advances in Dynamic Games and applications, 5, 361-385
- Matsumoto, A., Szidarovszky, F., 2014. Discrete and continuous dynamics in nonlinear monopolies. Applied Mathematics and Computation 232, 632-642.

Economic models with (fixed) lags

- Time to built:

  Asea, P. Zak, P., 1999, Time to build and cycles, J. Econ. Dyn. Contr. 23 1155–1175.
  Bambi, M. , Gori, F.,2013, Unifying time-to-build theory,Macroeconomic Dynamics, 1-13.

- Solow models:

  Matsumoto, A., Szidarovszky, F., 2011 Delay differential neoclassical growth model, J. Econ. Behav. Organ., 78:272–289.
  Guerrini L., and Sodini, M., 2013, Dynamic properties of the Solow model with increasing or decreasing population and time-tobuild technology, Abstract and Applied Analysis.
  Ferrara, M., Guerrini, L., Sodini, M., 2014. Nonlinear dynamics in a Solow model with delay and non-convex technology, Applied Mathematics and Computation 228 (1), August, 1-12.

Economic models with (fixed) lags

- Cobweb Dynamics:

  Ferrara, M., Guerrini, L., Sodini, M., 2014 Equilibrium and disequilibrium dynamics in cobweb, working paper

- Dual models

  Guerrini, L., Sodini, M., 2014. Persistent fluctuations in a dual model with frictions: The role of delays. Applied Mathematics and Computation 241 (1), February, 371-379.

Our approach

- We consider a Cournot duopoly for a single homogeneous product with normalised linear inverse demand given by $p = 1-X$, where p is the market price of product Q, and $X < 1$ is the sum of output $x_1 \geq 0$ and output $x_2 \geq 0$ produced by firm 1 and firm 2, respectively.
- The (average and marginal) cost of producing an additional unit of output is $0 < w < 1$ for every firm.
- The technology of production of firm $i = 1, 2$ has constant marginal returns to labour and it is equal to $xi = Li$, where Li represents the labour force employed in that firm

- Profits of firm i are given by $\Pi i = (p - w)\, x_i$, that can alternatively be written as follows: $\Pi i = (1 - x_i - x_j - w)\, x_i$, $i = 1, 2$, $i = j$.
- Firm i’s marginal profits are then given by:

$$\frac{\partial \Pi_i}{\partial x_i} = 1 - 2x_i - x_j - w$$

- Each firm has limited information about rival’s decision variables. In particular, following Matsumoto and Szidarovszky (2014), we adapt the Bischi et Al. adjustment mechanism in the following way (Berezowski, 2001)

Dynamical System

The two-dimensional dynamic system with different time delays is the following:

$$\begin{cases} \sigma_1 \dot{x}_1(t) + x_1(t) = x_1(t-\tau_1) + \alpha x_1(t-\tau_1)\left[1 - w - 2x_1(t-\tau_1) - x_2(t-\tau_1)\right], \\ \sigma_2 \dot{x}_2(t) + x_2(t) = x_2(t-\tau_2) + \alpha x_2(t-\tau_2)\left[1 - w - 2x_2(t-\tau_2) - x_1(t-\tau_2)\right], \end{cases} \tag{3}$$

where $0 < w < 1$, $\sigma_1, \sigma_2 \geq 0$ and $\tau_1, \tau_2 \geq 0$ are two parameters that capture time delays.

- tau_1=tau_2=1, sigma_1=sigma_2=0------> Classical Bischi et Al. Model without heterogeneity

The symmetric case

Let $x_1 = x_2$, $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2$.

$$x_1^* = (1 - w)/3$$

$$\sigma_1 \dot{x}_1(t) + x_1(t) = x_1(t - \tau) + \alpha x_1(t - \tau)[1 - w - 3x_1(t - \tau)]$$

Theorem 1 *Let* τ_0 *be defined as in* (4).

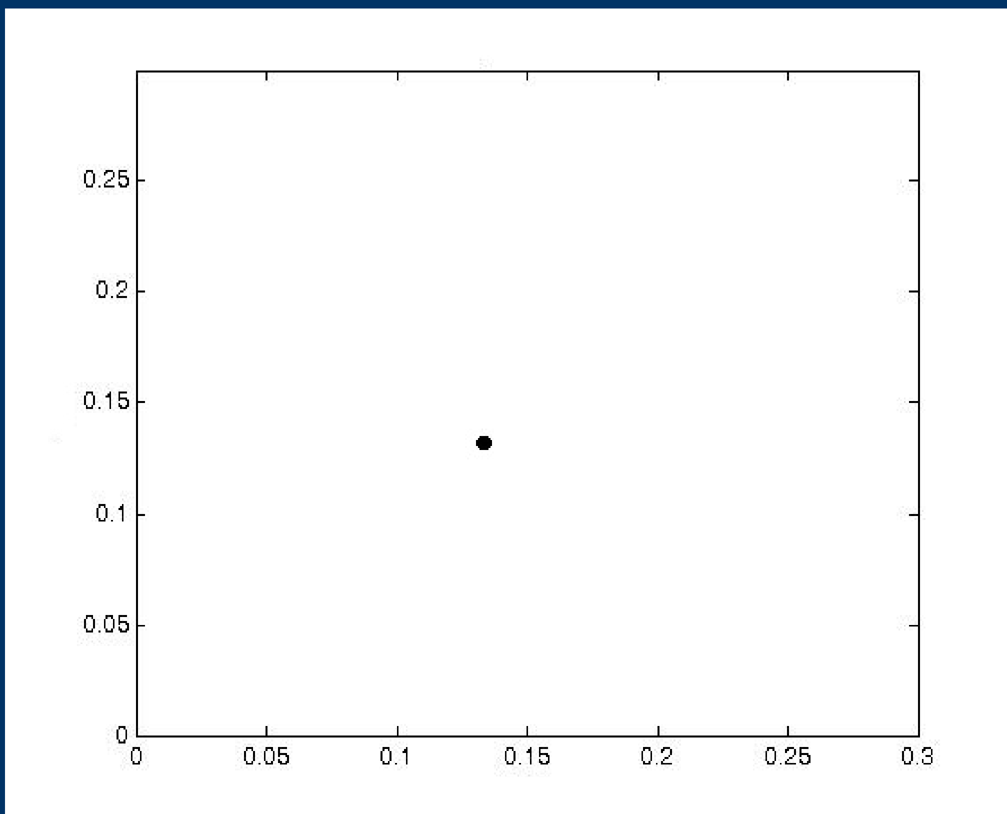
1) *If* $\alpha \leq 2/(1 - w)$, *then the equilibrium* x_1^* *is locally asymptotically stable for all* $\tau \geq 0$.

2) *If* $\alpha > 2/(1 - w)$, *then the equilibrium* x_1^* *is locally asymptotically stable for* $\tau \in [0, \tau_0)$ *and unstable for* $\tau > \tau_0$. *Furthermore, Eq* (1) *undergoes a Hopf bifurcation at* x_1^* *when* $\tau = \tau_0$.

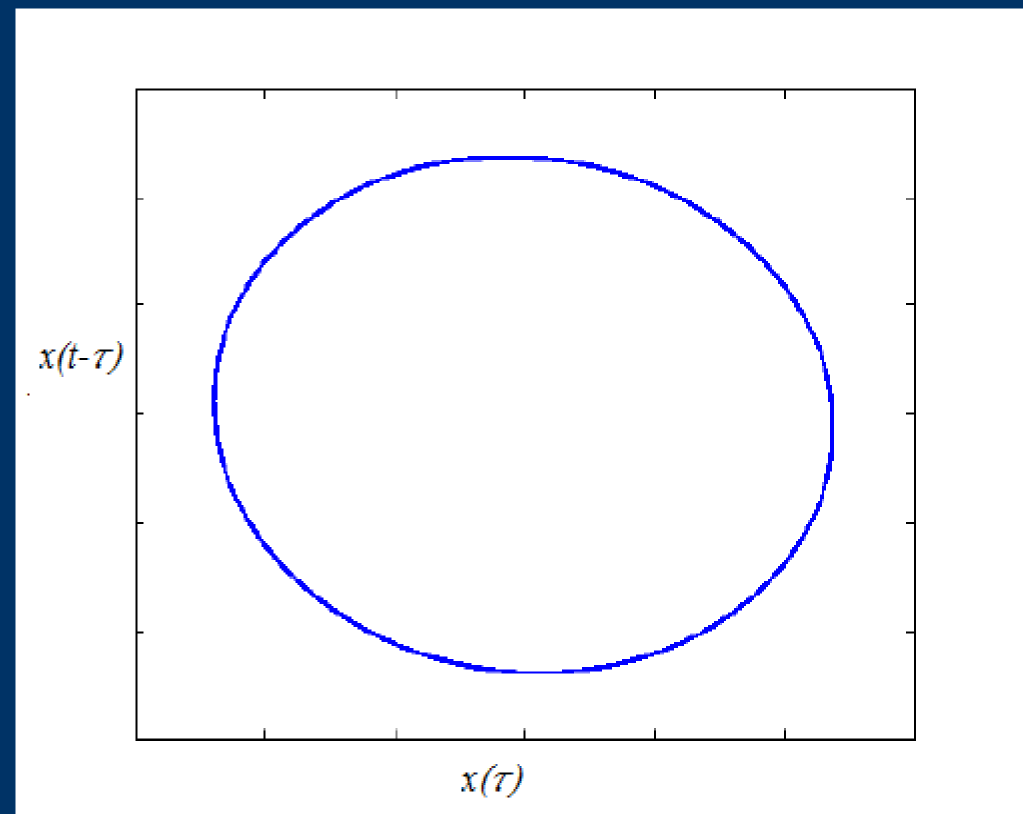
where

$$\tau_0 = \frac{1}{\omega_0}\left\{\tan^{-1}\left(-\frac{\omega_0}{\sigma_1}\right) + 2\pi\right\}, \tag{4}$$

$$\omega_0 = \frac{1}{\sigma_1}\sqrt{3\alpha x_1^*(3\alpha x_1^* - 2)}.$$

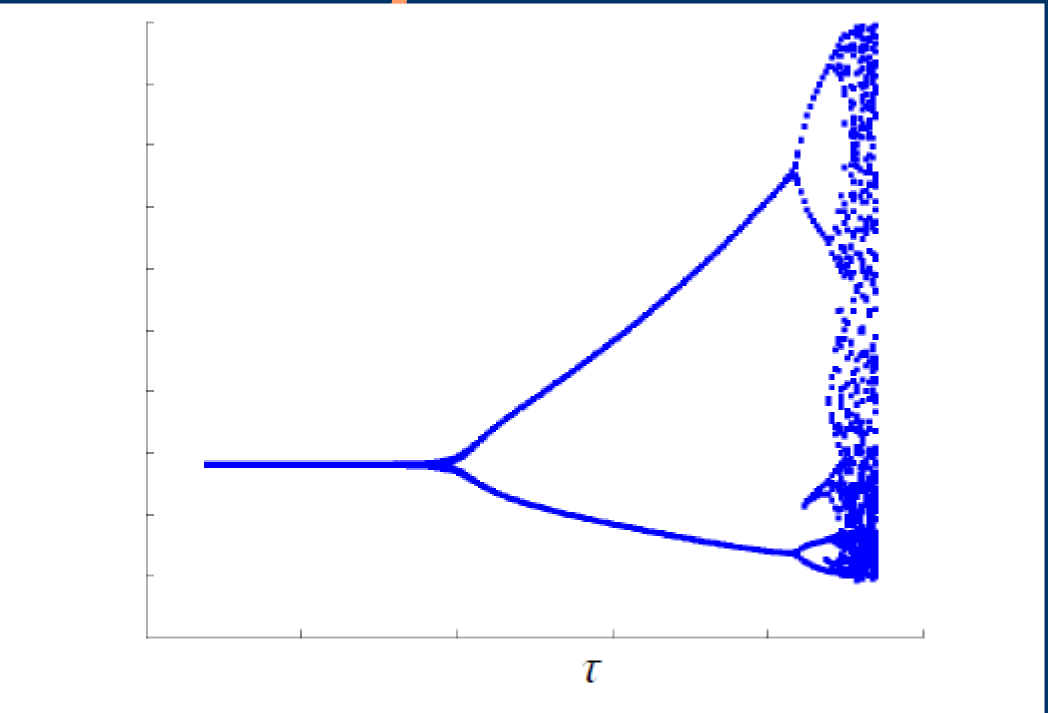


$\tau \in [0, \tau_0)$



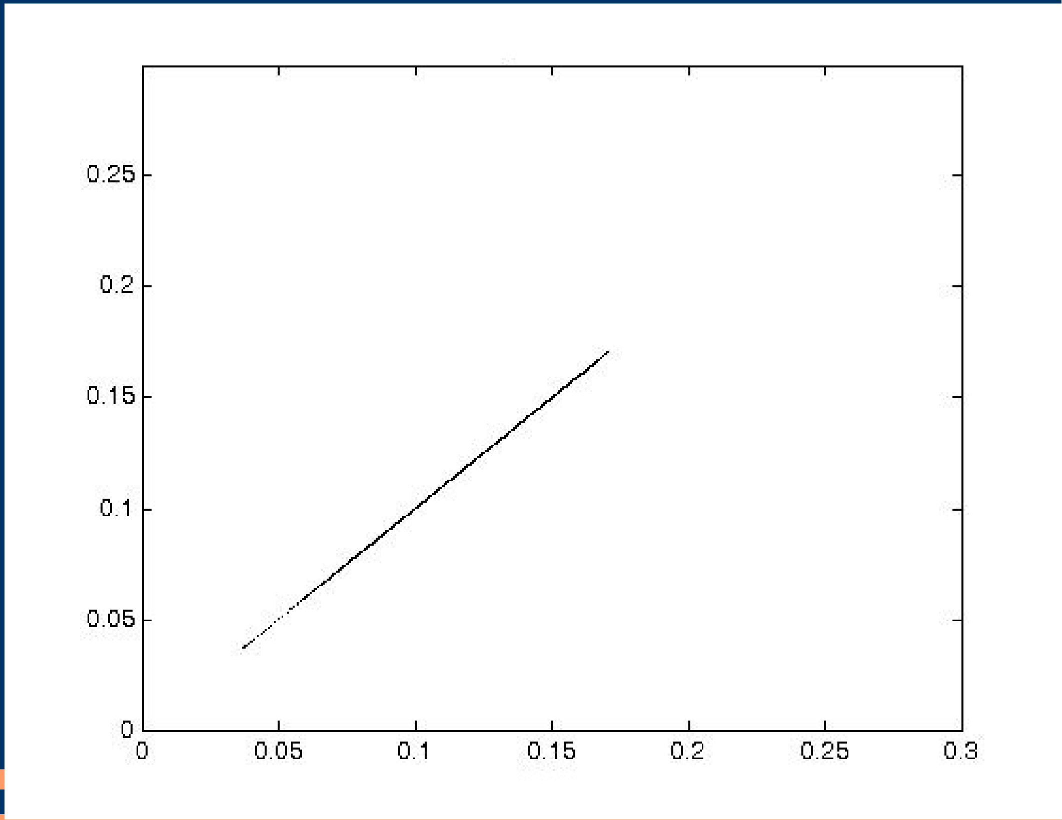
$\tau > \tau_0$

Complex behaviour on the diagonal

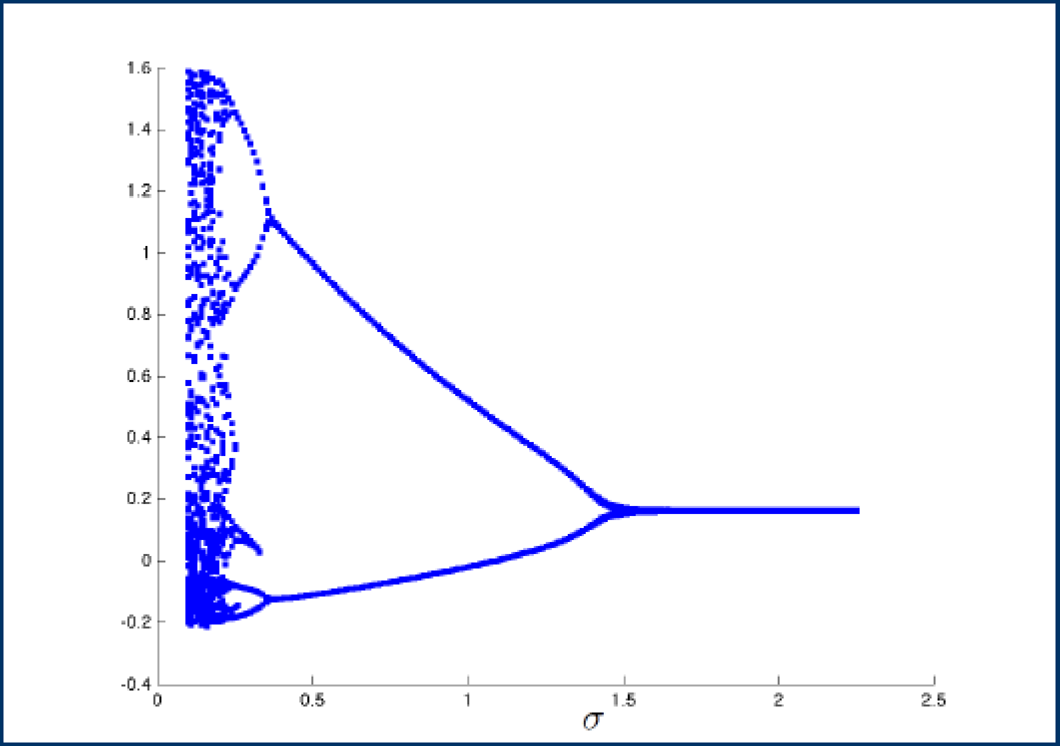


Bifurcation diagram with respect to tau

Dynamic in x1-x2 plane



Complex behaviour on the diagonal



Existence and stability of positive equilibrium for the two equations system

- System (3) has a unique positive equilibrium (x_1*, x_1*), where x_1* = (1 − w)/3.

Stability

Setting $x = \sigma_1(x_1 - x_1^*)$, $y = \sigma_2(x_2 - x_1^*)$, and linearizing at $(0,0)$, we have

$$\begin{cases} \dot{x}(t) = -\dfrac{1}{\sigma_1}x(t) + \dfrac{(1-2\alpha x_1^*)}{\sigma_1}x(t-\tau_1) - \dfrac{\alpha x_1^*}{\sigma_1}y(t-\tau_1), \\ \dot{y}(t) = -\dfrac{1}{\sigma_2}y(t) - \dfrac{\alpha x_1^*}{\sigma_2}x(t-\tau_2) + \dfrac{(1-2\alpha x_1^*)}{\sigma_2}y(t-\tau_2). \end{cases}$$

The characteristic equation associated with (4) is given by

$$\begin{vmatrix} -\dfrac{1}{\sigma_1} - \lambda + \dfrac{(1-2\alpha x_1^*)}{\sigma_1}e^{-\lambda\tau_1} & -\dfrac{\alpha x_1^*}{\sigma_1}e^{-\lambda\tau_1} \\ -\dfrac{\alpha x_1^*}{\sigma_2}e^{-\lambda\tau_2} & -\dfrac{1}{\sigma_2} - \lambda + \dfrac{(1-2\alpha x_1^*)}{\sigma_2}e^{-\lambda\tau_2} \end{vmatrix} = 0,$$

Characteristic polynomial

$$\lambda^2 + \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)\lambda + \frac{1}{\sigma_1\sigma_2} + \left[-\frac{(1-2\alpha x_1^*)}{\sigma_1\sigma_2} - \frac{(1-2\alpha x_1^*)}{\sigma_1}\lambda\right] e^{-\lambda\tau_1}$$
$$+ \left[-\frac{(1-2\alpha x_1^*)}{\sigma_1\sigma_2} - \frac{(1-2\alpha x_1^*)}{\sigma_2}\lambda\right] e^{-\lambda\tau_2} + \left[\frac{(1-2\alpha x_1^*)^2 - (\alpha x_1^*)^2}{\sigma_1\sigma_2}\right] e^{-\lambda(\tau_1+\tau_2)} = 0. \quad (5)$$

Lemma 2 *Let* $\tau_1 = \tau_2 = 0$*. The the equilibrium point of system* (3) *is locally asymptotically stable.*

The case tau_1 = 0, tau_2 > 0

The characteristic equation (5) takes the form

$$\lambda^2 + p\lambda + r + (s\lambda + q)\, e^{-\lambda\tau_2} = 0, \tag{6}$$

where

$$p = \frac{1}{\sigma_2} + \frac{2\alpha x_1^*}{\sigma_1}, \quad r = \frac{2\alpha x_1^*}{\sigma_1\sigma_2}, \quad s = -\frac{(1-2\alpha x_1^*)}{\sigma_2} \quad \text{and} \quad q = \frac{(3\alpha x_1^* - 2)\alpha x_1^*}{\sigma_1\sigma_2}.$$

where

$$p = \frac{1}{\sigma_2} + \frac{2\alpha x_1^*}{\sigma_1}, \quad r = \frac{2\alpha x_1^*}{\sigma_1\sigma_2}, \quad s = -\frac{(1-2\alpha x_1^*)}{\sigma_2} \quad \text{and} \quad q = \frac{(3\alpha x_1^* - 2)\alpha x_1^*}{\sigma_1\sigma_2}.$$

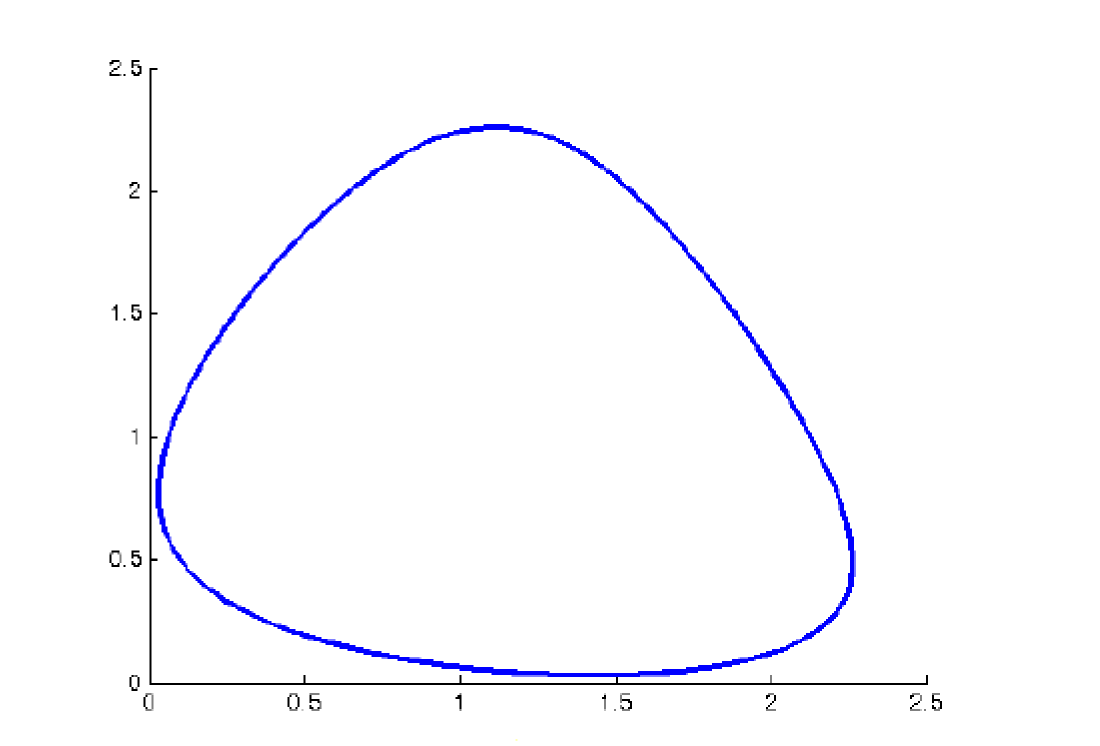
let

$$M = \left(4\sigma_1^4 + 4\sigma_2^4 + \sigma_1^2\sigma_2^2\right)(\alpha x_1^*)^2 - 4\sigma_1^4\left(2\sigma_1^4 + \sigma_2^2\right)\alpha x_1^* + 4\sigma_1^4.$$

$$\tau_{2_n} = \frac{1}{\omega_0}\cos^{-1}\left\{\frac{(q-ps)\,\omega_0^2 - rq}{s^2\omega_0^2 + q^2}\right\} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \ldots$$

$$\tau_{2_j}^{\pm} = \frac{1}{\omega_\pm}\cos^{-1}\left\{\frac{(q-ps)\,\omega_\pm^2 - rq}{s^2\omega_\pm^2 + q^2}\right\} + \frac{2j\pi}{\omega_\pm}, \quad j = 0, 1, 2, \ldots$$

Hopf bifurcation in x1-x2 plane



Theorem 6 *Let $M, \tau_{2_n}, \tau^{\pm}_{2_j}$ be defined as in (11), (13) and (14), respectively.*

1) *If $\alpha x_1^* = 4/3$ and $\sigma_1 \leq 2\sigma_2$, or $\alpha x_1^* < 4/3$ and $\sigma_1 = \sigma_2$, or $\alpha x_1^* < 4/3$ and $\sigma_1 \leq 2\sigma_2$, or $\alpha x_1^* < 4/3$, $\sigma_1 \geq 2\sigma_2$ and $\alpha x_1^* \leq \sigma_1^2/(\sigma_1^2 - \sigma_2^2)$, or $\alpha x_1^* < 4/3$, $\sigma_1 > 2\sigma_2$, $\sigma_1^2/(\sigma_1^2 - \sigma_2^2) < \alpha x_1^*$ and $M < 0$ hold, then the equilibrium (x_1^*, x_1^*) is locally asymptotically stable for all $\tau_2 \geq 0$.*

2) *If $\alpha x_1^* = 4/3$ and $\sigma_1 > 2\sigma_2$, or $\alpha x_1^* > 4/3$, or $\alpha x_1^* < 4/3$, $\sigma_1 > 2\sigma_2$, $\sigma_1^2/(\sigma_1^2 - \sigma_2^2) < \alpha x_1^*$ and $M = 0$ hold, then the equilibrium (x_1^*, x_1^*) is locally asymptotically stable for $\tau_2 < \tau_{2_0}$ and unstable for $\tau_2 > \tau_{2_0}$. Furthermore, system (3) undergoes a Hopf bifurcation at (x_1^*, x_1^*) when $\tau_2 = \tau_{2_0}$ if the corresponding root $\lambda = i\omega_0$ of (6) is simple.*

3) *If $\alpha x_1^* < 4/3$, $\sigma_1 > 2\sigma_2$, $\sigma_1^2/(\sigma_1^2 - \sigma_2^2) < \alpha x_1^*$ and $M > 0$ hold, then there is a positive integer m such that the equilibrium (x_1^*, x_1^*) is locally asymptotically stable when $\tau_2 \in [0, \tau^+_{2_0}) \cup (\tau^-_{2_0}, \tau^+_{2_1}) \cup \cdots \cup (\tau^-_{2_{m-1}}, \tau^+_{2_m})$ and unstable when $\tau_2 \in (\tau^+_{2_0}, \tau^-_{2_0}) \cup (\tau^+_{2_1}, \tau^-_{2_1}) \cup \cdots \cup (\tau^+_{2_{m-1}}, \tau^-_{2_{m-1}}) \cup (\tau^+_{2_m}, \infty)$. Furthermore, system (3) undergoes a Hopf bifurcation at (x_1^*, x_1^*) when $\tau_2 = \tau^{\pm}_{2m}$, $m = 0, 1, 2, \ldots$, if the corresponding root $\lambda = i\omega_{\pm}$ of (6) is simple.*

The case tau_1 > 0 and tau_2 fixed in the interval [0, tau_2_0)

Characteristic polynomial:

$$\lambda^2 + A\lambda + B + (C + D\lambda)\, e^{-\lambda\tau_1} + (C + E\lambda)\, e^{-\lambda\tau_2} + F e^{-\lambda(\tau_1+\tau_2)} = 0,$$

where:

$$A = \frac{1}{\sigma_1} + \frac{1}{\sigma_2}, \qquad B = \frac{1}{\sigma_1\sigma_2}, \qquad C = -\frac{(1-2\alpha x_1^*)}{\sigma_1\sigma_2}, \qquad D = -\frac{(1-2\alpha x_1^*)}{\sigma_1},$$

$$E = -\frac{(1-2\alpha x_1^*)}{\sigma_2}, \qquad F = \frac{(1-2\alpha x_1^*)^2 - (\alpha x_1^*)^2}{\sigma_1\sigma_2}.$$

We introduce:

$$g(\omega) = \omega^4 + 2\left\{\frac{\sigma_1^2 + 2\left(\sigma_1^2 - \sigma_2^2\right)\left[(\alpha x_1^*)^2 - 1\right]}{\sigma_1^2\sigma_2^2}\right\}\omega^2 + \frac{\alpha x_1^*\left(-3\alpha x_1^* + 4\right)\left[3(\alpha x_1^*)^2 - 4\alpha x_1^* + 2\right]}{\sigma_1^2\sigma_2^2}$$

$$+ \frac{2(1 - 2\alpha x_1^*)}{\sigma_2}\left[\frac{(-3\alpha x_1^* + 4)\alpha x_1^*}{\sigma_1^2} + \omega^2\right]\omega \sin \omega\tau_2$$

$$- \frac{2(1 - 2\alpha x_1^*)}{\sigma_2^2}\left[\frac{(-3\alpha x_1^* + 4)\alpha x_1^*}{\sigma_1^2} + \omega^2\right]\cos \omega\tau_2.$$

We have the following:

Theorem 10 *Let* $\tau_2 \in [0, \tau_{2_0})$.

1) *If* $g(\omega)$ *has no positive zero, then the equilibrium* (x_1^*, x_1^*) *of system* (3) *is locally asymptotically stable for* $\tau_1 \geq 0$.

2) *Let* $\alpha x_1^* > 4/3$. *Then there exists a positive number* τ_{1_0} *such the equilibrium* (x_1^*, x_1^*) *of system* (3) *is locally asymptotically stable for* $\tau_1 \in [0, \tau_{1_0})$ *and unstable for* $\tau_1 > \tau_{1_0}$. *System* (3) *undergoes Hopf bifurcation at the equilibrium* (x_1^*, x_1^*) *for* $\tau_1 = \tau_{1_0}$ *if the corresponding root* $\lambda = i\tilde{\omega}$ *of* (16) *is simple and condition* (18) *holds.*

The case tau_1 =tau_2 =tau

The characteristic equation (5) becomes

$$\lambda^2 + a\lambda + b + (d + c\lambda)\, e^{-\lambda\tau} + h e^{-2\lambda\tau} = 0, \tag{19}$$

with

$$a = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} > 0, \qquad b = \frac{1}{\sigma_1\sigma_2} > 0, \qquad c = -\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right)(1 - 2\alpha x_1^*),$$

$$d = -\frac{2(1 - 2\alpha x_1^*)}{\sigma_1\sigma_2}, \qquad h = \frac{(1 - 2\alpha x_1^*)^2 - \alpha^2 x_1^{*2}}{\sigma_1\sigma_2}.$$

We can apply some recent results provided in:
Chen, S., Shi, J., Wei, J., 2013. Time delay-induced instabilities and Hopf bifurcations in general reaction—diffusion systems. Journal of Nonlinear Science 23, 1—38.

Let $\lambda = i\omega$ $(\omega > 0)$ be a root of (19). Then, we have

$$-\omega^2 + ai\omega + b + (d + ci\omega)\, e^{-i\omega\tau} + he^{-2i\omega\tau} = 0,$$

If $(\omega\tau)/2 \neq (\pi/2) + j\pi$, $j \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, then we have $e^{-i\omega\tau} = (1 - i\theta)/(1 + i\theta)$, with $\theta = \tan\left[(\omega\tau)/2\right]$. Separating the real and imaginary parts, one has that θ satisfies

$$\begin{cases} \left(\omega^2 - b + d - h\right)\theta^2 - 2a\omega\theta = \omega^2 - b - d - h, \\ (c - a)\,\omega\theta^2 + \left(-2\omega^2 + 2b - 2h\right)\theta = -(c + a)\,\omega. \end{cases} \tag{20}$$

Define

$$D(\omega) = \begin{vmatrix} \omega^2 - b + d - h & -2a\omega \\ (c - a)\,\omega & -2\omega^2 + 2b - 2h \end{vmatrix}, \tag{21}$$

$$E(\omega) = \begin{vmatrix} \omega^2 - b - d - h & -2a\omega \\ -(c + a)\,\omega & -2\omega^2 + 2b - 2h \end{vmatrix},$$

$$F(\omega) = \begin{vmatrix} \omega^2 - b + d - h & \omega^2 - b - d - h \\ (c - a)\,\omega & -(c + a)\,\omega \end{vmatrix}.$$

Theorem 18 *Let $D(\omega)$ be defined as in (21).*

1) *Let $\alpha x_1^* > 4/3$, $\sigma_1 = \sigma_2$ and $D(\omega) = 0$. Then the equilibrium (x_1^*, x_1^*) of system (3) is locally asymptotically stable for all $\tau \geq 0$.*

2) *Let $\alpha x_1^* > 4/3$, $\sigma_1 = \sigma_2$ and $D(\omega) \neq 0$, or $\alpha x_1^* > 4/3$ and $\sigma_1 \neq \sigma_2$, or $\alpha x_1^* \leq 4/3$ and $3\alpha x_1^* - 1 \neq 0$, $\alpha x_1^* - 1 \neq 0$ and $1 - 2\alpha x_1^* \neq 0$.*

i) *The quartic polynomial equation (22) has a root ω_N^2 for $\omega_N > 0$.*

ii) *The characteristic equation (19) has a pair of roots $\pm i\omega_N$ when $\tau = \tau_N^j$, $j \in \mathbb{N}^0$, with τ_N^j defined as in (23).*

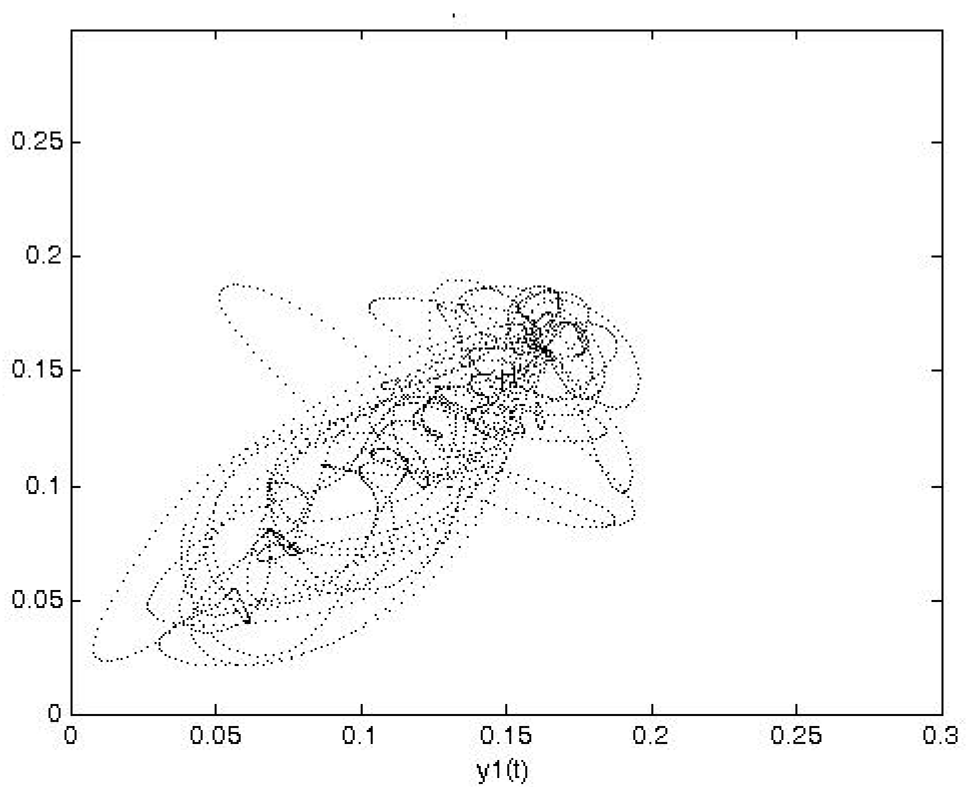
iii) *Let*

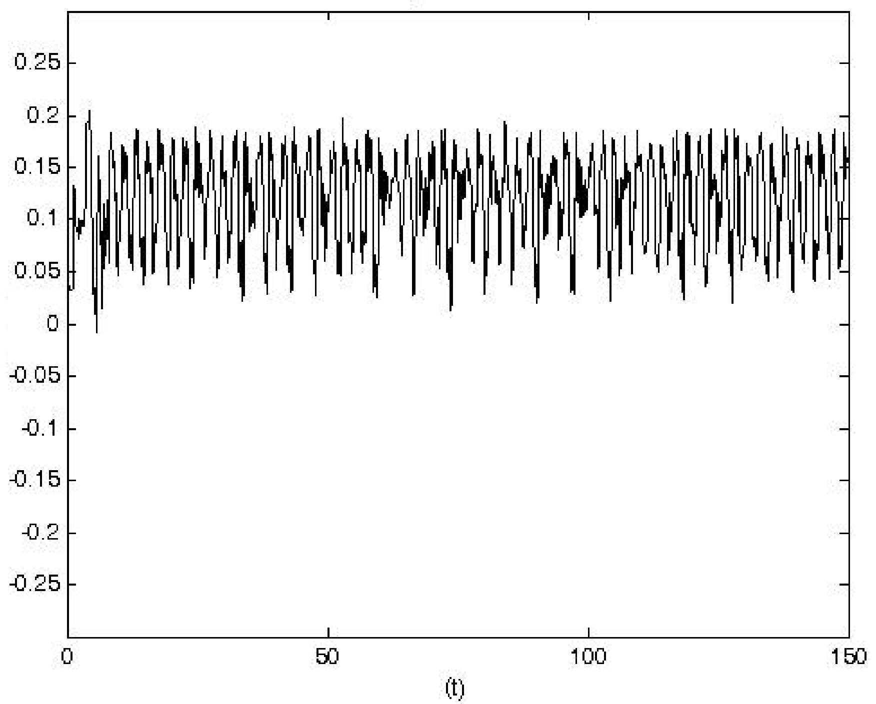
$$\mathcal{G}(\omega, \theta) = \left[d(1+\theta^2) + 2h(1-\theta^2)\right]\left[2\omega(1-\theta^2) + 2a\theta\right] - \left[c\omega(1+\theta^2) - 4h\theta\right]\left[a(1-\theta^2) - 4\omega\theta + c(1+\theta^2)\right].$$

If $\mathcal{G}(\omega_N, \theta_N) > 0$, then $i\omega_N$ is a simple root of the characteristic equation for $\tau = \tau_N^j$ and there exists $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ which is the unique root for $\tau \in (\tau_N^j - \varepsilon, \tau_N^j + \varepsilon)$ for some small $\varepsilon > 0$ satisfying $\nu(\tau_N^j) = 0$, $\omega(\tau_N^j) = \omega_N$ and $\nu'(\tau_N^j) > 0$.

iv) *If $\mathcal{G}(\omega_N, \theta_N) > 0$, then there exists $\tau_* > 0$ such that the equilibrium (x_1^*, x_1^*) of system (3) is locally asymptotically stable when $\tau \in [0, \tau_*)$ and it is unstable when $\tau \in (\tau_*, \tau_* + \varepsilon)$ for $\varepsilon > 0$ and small. Furthermore, a Hopf bifurcation occurs at $\tau = \tau_*$.*

Synchronization Failure





Thank you