Bull and Bear market with different entry thresholds

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in collaboration with

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Background models

Day and Huang, 1990:

financial market model generating Bull and Bear (BB) market dynamics (randomly alternating periods of generally rising or generally falling prices). Three types of agent:

- a *market maker* who adjusts prices with respect to excess demand;
- *chartists* who believe in the persistence of BB markets;
- *fundamentalists* who bet on mean reversion.

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Tramontana, Westerhoff, Gardini, 2013:

based on Huang and Day, 1993 (**1D PWL continuous map**), it is assumed that

- a number of chartists and fundamentalists are always active in the market;
- additional chartists and fundamentalists may enter when the distance between the price and its fundamental value exceeds a critical level;
- new traders' demand may be non-zero at the market entry level.

→ **1D PWL discontinuous map**: entry thresholds are *symmetric*.

The set up of the model

The log-linear price adjustment rule:

$$P_{t+1} = P_t + a\left(D_t^{C,1} + D_t^{F,1} + D_t^{C,2} + D_t^{F,2}\right)$$

$a > 0$ is a price adjustment parameter; $a = 1$.

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are always active, and when prices are above the fundamental value F they buy:

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wait for a stronger price signal before entering the market:

$$D_t^{C,2} = \begin{cases} c_2(P_t - F) - c_3, & P_t - F \leq -z^- \\ 0, & -z^- < P_t - F < z^+ \\ c_2(P_t - F) + c_3, & P_t - F \geq z^+ \end{cases}$$

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$c_1 > 0$ and $c_2 > 0$ measure the reactivity to the signal of type 1 and type 2 chartists;
$z^+ > 0$, $z^- > 0$ are entry thresholds for type 2 chartists in the bull and bear region;
$c_3 \geq \max\left[-c_2 z^+; -c_2 z^-\right]$ permits to adjust the transaction to have non-negative order in the bull market and non-positive order in the bear market.

The set up of the model

The orders of type 1 and 2 fundamentalists

are similar except for the fact that they buy when the price is below the fundamental value, while they sell when price is higher than the fundamental value:

$$D_t^{F,1} = f_1(F - P_t)$$

$$D_t^{F,2} = \begin{cases} f_2(F - P_t) + f_3, & \text{if} \quad P_t - F \leq -z^- \\ 0, & \text{if} \quad -z^- < P_t - F < z^+ \\ f_2(F - P_t) - f_3, & \text{if} \quad P_t - F \geq z^+ \end{cases}$$

$f_1 > 0$ and $f_2 > 0$ are reaction parameters; $f_3 \geq \max\left[f_2 z^+; f_2 z^-\right]$ helps to ensure positive (resp. negative) orders in the bull (resp. bear) region.

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Price adjustment:

$$P_{t+1} = \begin{cases} P_t + (c_1 + c_2 - f_1 - f_2)(P_t - F) - c_3 + f_3 & \text{if} \quad P_t - F \leq -z^- \\ P_t + (c_1 - f_1)(P_t - F) & \text{if} \quad -z^- < P_t - F < z^+ \\ P_t + (c_1 + c_2 - f_1 - f_2)(P_t - F) + c_3 - f_3 & \text{if} \quad P_t - F \geq z^+ \end{cases}$$

The set up of the model

The model in terms of deviations from the fundamental value:

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$$x_t := P_t - F, \quad S_1 := c_1 - f_1, \quad S_2 := c_2 - f_2, \quad m := c_3 - f_3$$

we get the following family of 1D discontinuous PWL maps f with three linear branches:

$$f: \ x_{t+1} = \begin{cases} f_L(x) = (1 + S_1 + S_2)x_t - m & \text{if} \quad x_t \le -z^- \\ f_M(x) = (1 + S_1)x_t & \text{if} \quad -z^- < x_t < z^+ \\ f_R(x) = (1 + S_1 + S_2)x_t + m & \text{if} \quad x_t \ge z^+ \end{cases}$$

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Our aim

To describe *bifurcation structure* of the parameter space of the map f, in particular, to understand how parameter regions related to attracting cycles (*periodicity regions*) are ordered.
TWG, 2013: symmetric case $z^- = z^+ = 1$. How bifurcation structure changes if symmetry is broken?

1D discontinuos piecewise linear (PWL) maps

with one discontinuity point:

$$g : x \to g(x) = \begin{cases} g_L(x) = a_L x + \mu_L, & x < 0 \\ g_R(x) = a_R x + \mu_R, & x > 0 \end{cases}$$

$g_L(0) \neq g_R(0)$;

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$g_L(d_L) \neq g_M(d_L)$, $g_M(d_R) \neq g_R(d_R)$.

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Boundaries of periodicity regions

Suppose g has an attracting cycle of period $n \geq 3$. Then a boundary of the related periodicity region corresponds to either *border collision bifurcation* (BCB) of the cycle, or to its *degenerate flip bifurcation* (DFB).

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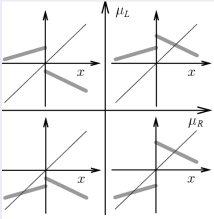

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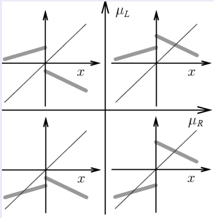

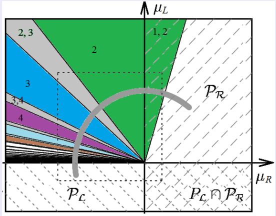

Bifurcation structures in 1D discontinuous PWL maps

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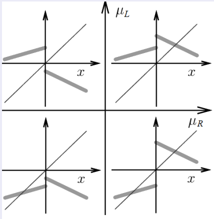

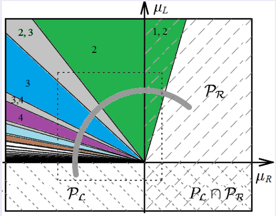

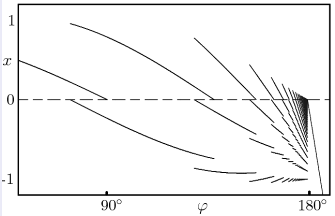

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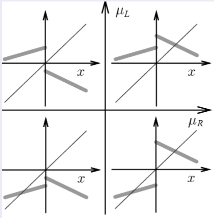

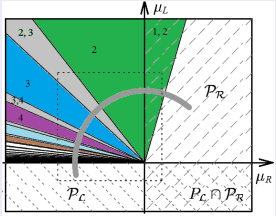

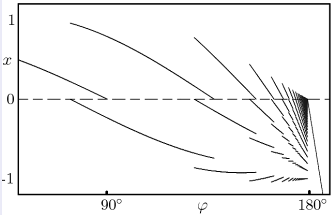

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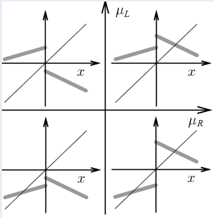

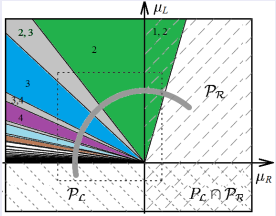

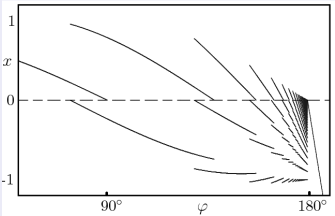

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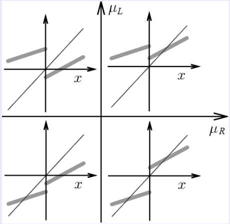

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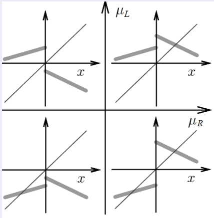

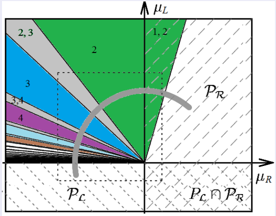

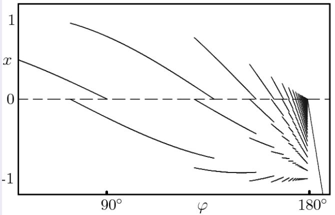

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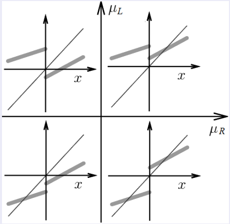

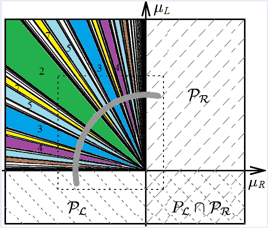

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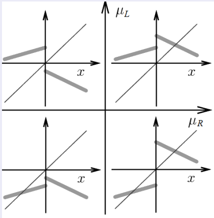

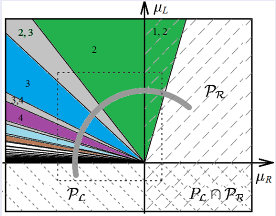

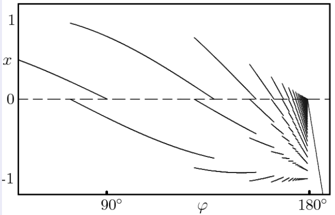

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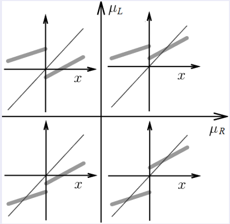

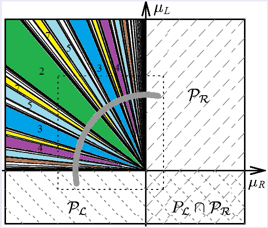

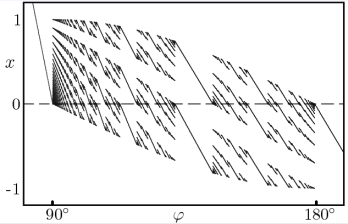

Symbolic sequences and complexity levels

Leonov, 1959, Avrutin *et al.*, 2010, Gardini *et al.*, 2010

First complexity level (basic cycles):

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Second complexity level ($m = n_2$):

$$\Sigma_{1,2} = \left\{LR^{n_2}\left(RLR^{n_2}\right)^{n_1}\right\}_{n_1,n_2=1}^{\infty}, \quad \Sigma_{2,2} = \left\{LRL^{n_2}\left(RL^{n_2}\right)^{n_1}\right\}_{n_1,n_2=1}^{\infty}$$

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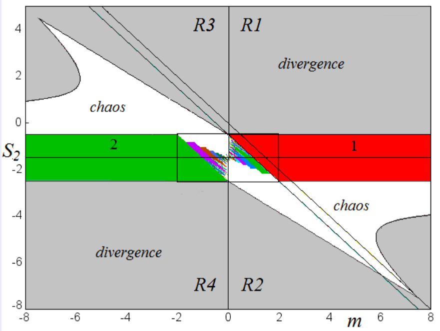
Applying the replacements with $m = n_3$ to the families of complexity level two we obtain 2^3 families $\Sigma_{j,3}$, $j = 1, \ldots, 2^3$, of *complexity level three*, and so on.

BB model, $z^- = z^+ = 1$ (TWG, 2013)

Periodicity regions in the (m, S_2)-parameter plane, $S_1 = 0.5$



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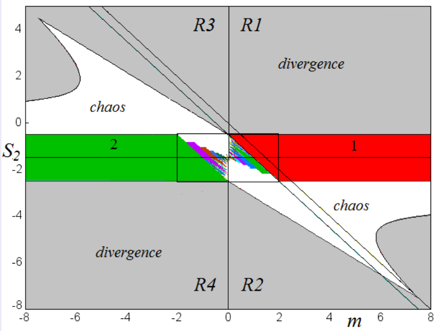

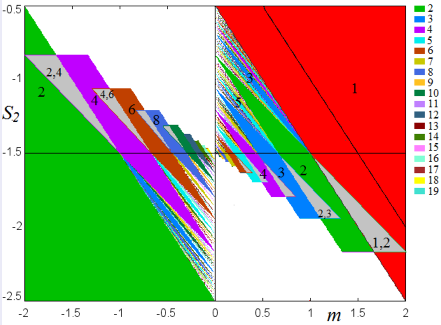

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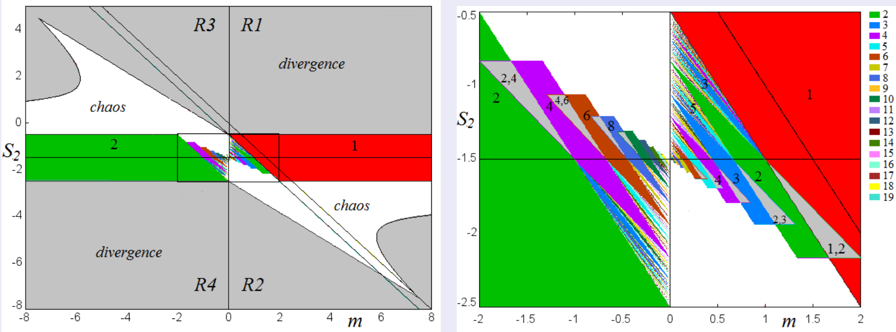

- R1: period adding structure and disjoint symmetric attractors;
- R2: period incrementing structure and chaotic intervals;
- R3: even-period incrementing structure and chaotic intervals;
- R4: even-period adding structure and bistability.

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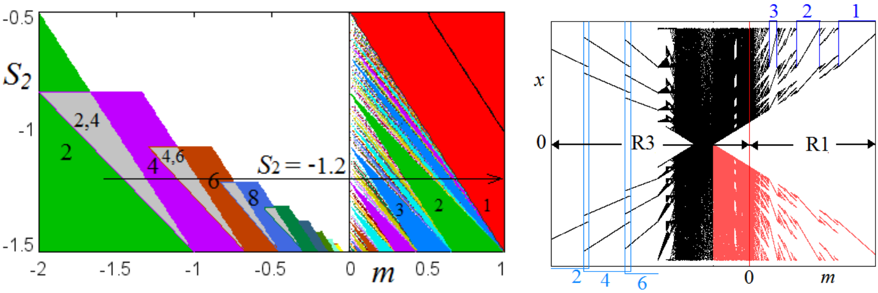

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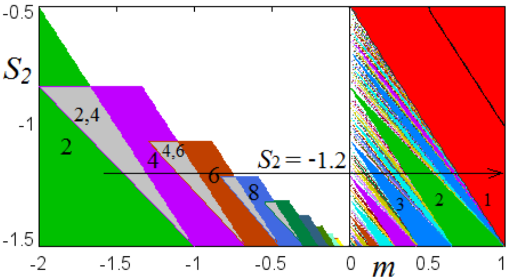
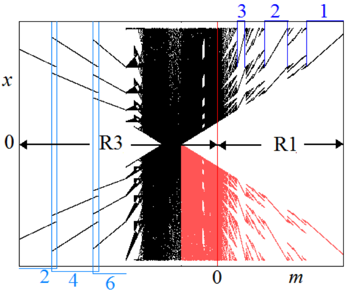
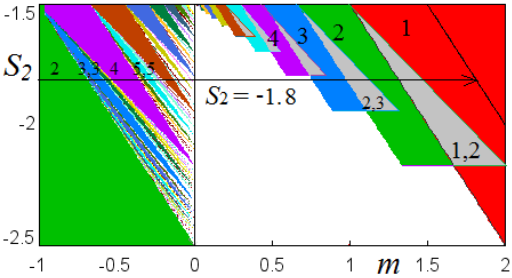


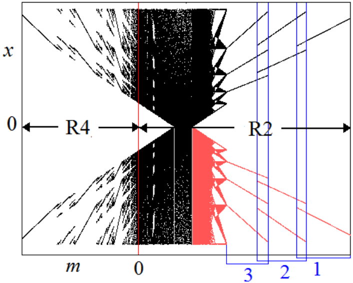
-0.5
S_2
-1
-1.5
2,4
2
4
4,6
6
8
S_2 = -1.2
3
2
1
-2
-1.5
-1
-0.5
0
m
0.5
1



3
2
1
x
0
R3
R1
2
4
6
0
m

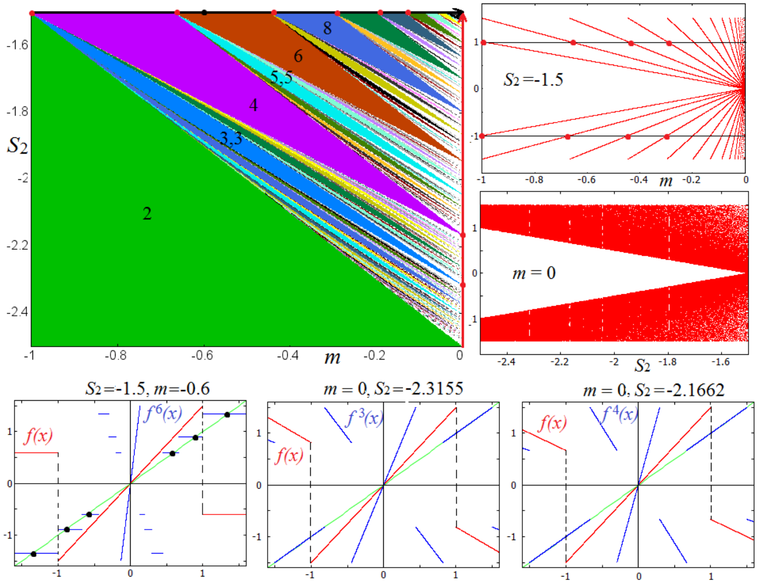


-1.5
S_2
-2
-2.5
2
3,3
4
5,5
4
3
2
1
S_2 = -1.8
2,3
1,2
-1
-0.5
0
0.5
1
m
1.5
2



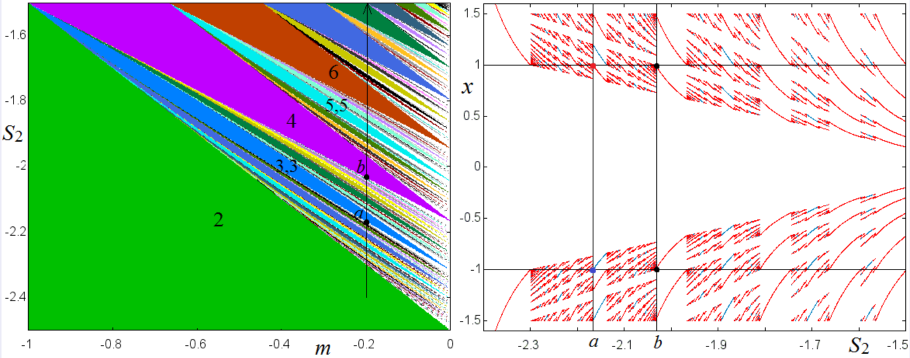
x
0
R4
R2
m
0
3
2
1

BB model, $z^- = z^+ = 1$: the region R_4



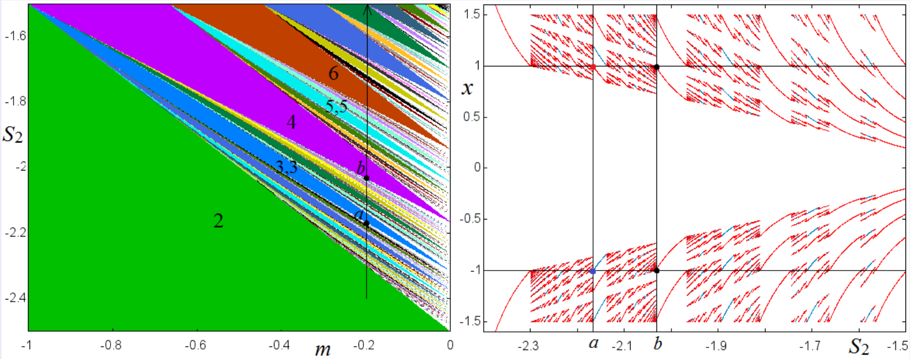
BB model, $z^- = z^+ = 1$: the region R_4

R_4: period adding structure



BB model, $z^- = z^+ = 1$: the region R_4

R_4: period adding structure



Symmetry of f wrt to the origin

$\Rightarrow$ an invariant set A is either symmetric itself wrt to 0, or $\exists$ A' symmetric to A;

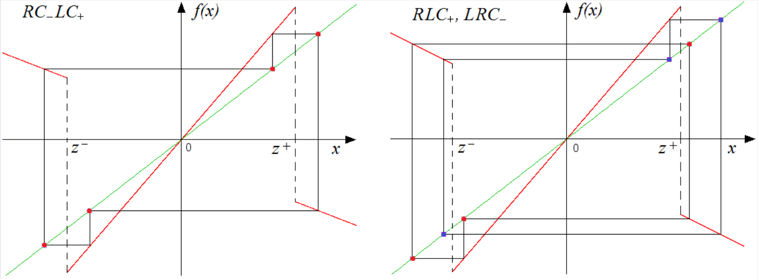
$\Rightarrow$ if f has a cycle γ_n with odd n then $\exists$ γ'_n symmetric to γ_n.

BB model, $z^- = z^+ = 1$: the region R_4

Basic cycles

$\Sigma_{1,1} = \left\{RC_-^k LC_+^k\right\}_{k\geq 0}$ ((2k + 2)-cycles);

$\Sigma_{2,1} = \left\{RC_-^k LC_+^{k+1}, LC_+^k RC_-^{k+1}\right\}_{k\geq 0}$ ((2k + 3)-cycles).



The region R_4: breaking symmetry

BCB boundaries

$$ll = \{p : f^{n-1}(f_L(z^-)) = z^-\},\ lr = \{p : f^{n-1}(f_M(z^-)) = z^-\}$$

$$rl = \{p : f^{n-1}(f_M(z^+)) = z^+\},\ rr = \{p : f^{n-1}(f_R(z^+)) = z^+\}$$

p is a point in the parameter space (in symmetric case $ll = rr$, $lr = rl$).

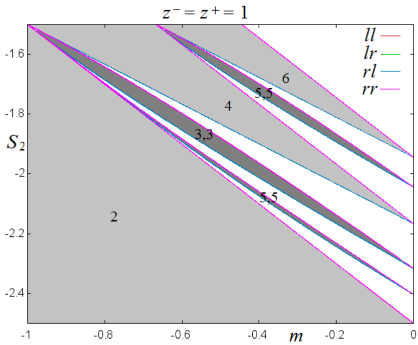
The region R_4: breaking symmetry

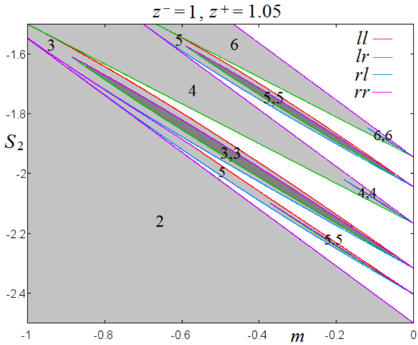
BCB boundaries

$$ll = \{p : f^{n-1}(f_L(z^-)) = z^-\},\ lr = \{p : f^{n-1}(f_M(z^-)) = z^-\}$$

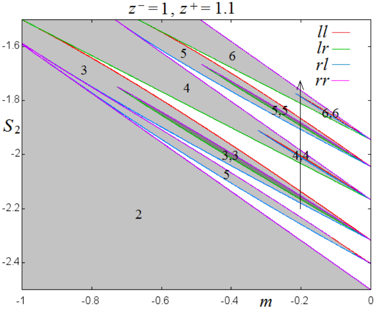
$$rl = \{p : f^{n-1}(f_M(z^+)) = z^+\},\ rr = \{p : f^{n-1}(f_R(z^+)) = z^+\}$$

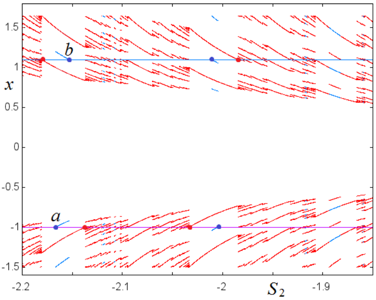
p is a point in the parameter space (in symmetric case $ll = rr$, $lr = rl$).



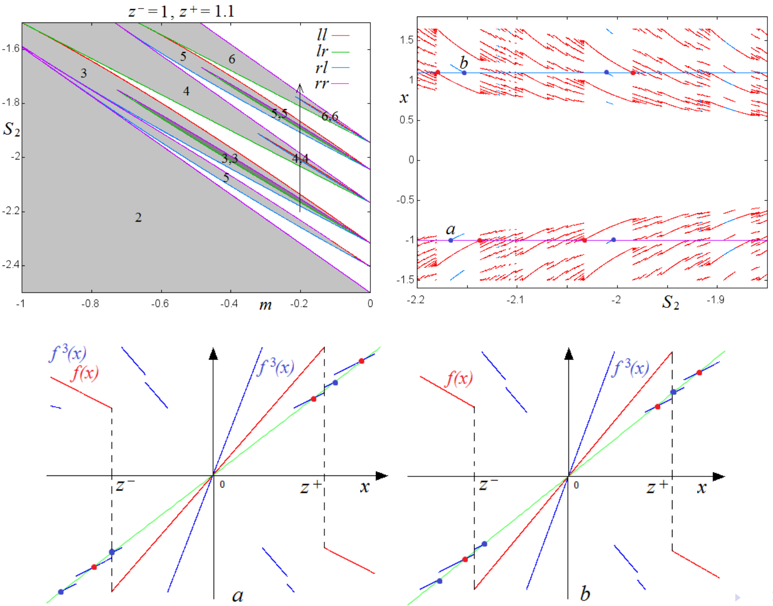


The region R_4: breaking symmetry

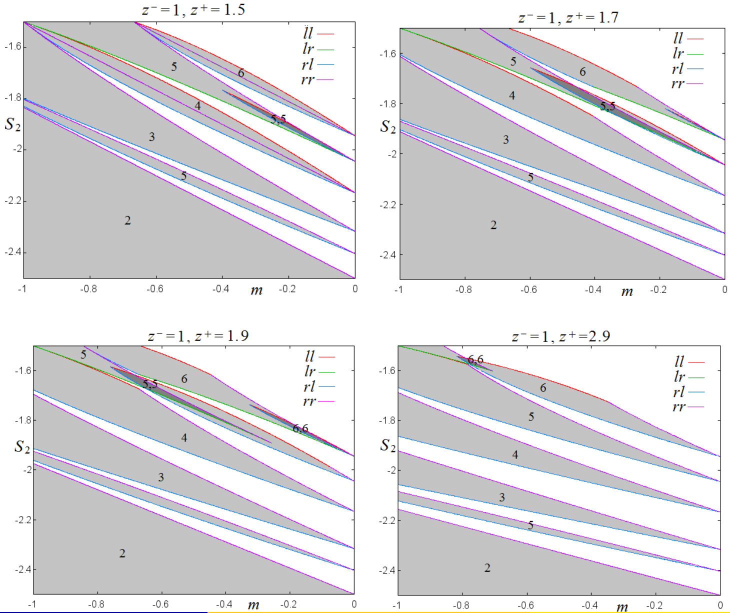




The region R_4: breaking symmetry

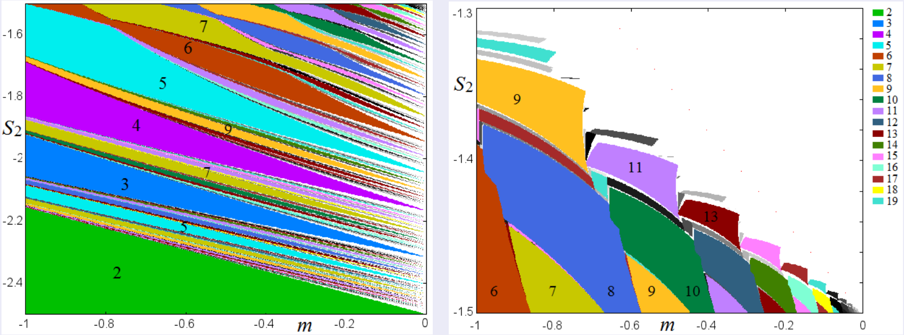


The region R_4: breaking symmetry



The regions R_4 and R_3: breaking symmetry

$z^- = 1$, $z^+ = 2.9$



References

- Avrutin, V., Schanz, M. and Gardini, L. (2010): Calculation of bifurcation curves by map replacement. Int. J. Bifurcation and Chaos 20(10), 3105-3135.
- Avrutin, V., Sushko, I. and Gardini, L. (2014): Cyclicity of chaotic attractors in one-dimensional discontinuous maps. Mathematics and Computers in Simulation, in press.
- Brock, W. and Hommes, C. (1997): A rational route to randomness. Econometrica, 65, 1059-1095.
- Brock, W. and Hommes, C. (1998): Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics Control, 22, 1235-1274.
- Chiarella, C., He, X.-Z., Huang, W. and Zheng, H. (2012): Estimating behavioural heterogeneity under regime switching. Journal of Economic Behavior and Organization, in press.
- Chiarella, C., Dieci, R. and He, X.-Z. (2009): Heterogeneity, market mechanisms, and asset price dynamics. In: Hens, T. and Schenk-Hoppé, K.R. (eds.): Handbook of Financial Markets: Dynamics and Evolution. North-Holland, Amsterdam, 277-344.
- Day, R. and Huang, W. (1990): Bulls, bears and market sheep. Journal of Economic Behavior and Organization, 14, 299-329.
- Day, R. 1997: Complex dynamics, market mediation and stock price behavior. North American Actuarial Journal, 1, 6-21.

References

- Gardini, L., Tramontana, F., Avrutin, V. and Schanz, M. (2010): Border Collision Bifurcations in 1D PWL map and Leonov's approach. Int. J. Bifurcation and Chaos, 20(10), 3085-3104.
- Gardini, L. and Tramontana, F. (2010): Border Collision Bifurcations in 1D PWL map with one discontinuity and negative jump. Use of the first return map. Int. J. Bifurcation and Chaos, 20(11), 3529-3135.
- Hommes, C. and Wagener, F. (2009): Complex evolutionary systems in behavioral finance. In: Hens, T. and Schenk-Hoppé, K.R. (eds.): Handbook of Financial Markets: Dynamics and Evolution. North-Holland, Amsterdam, 217-276.
- Hommes, C. (2011): The heterogeneous expectations hypothesis: Some evidence from the lab. Journal of Economic Dynamics and Control, 35, 1-24.
- Huang, W. and Day, R. (1993): Chaotically switching bear and bull markets: the derivation of stock price distributions from behavioral rules. In: Day, R. and Chen, P. (Eds.): Nonlinear Dynamics and Evolutionary Economics, Oxford University Press, Oxford, 169-182.
- Keener J.P. (1980): Chaotic behavior in piecewise continuous difference equations. Trans. Amer. Math. Soc. 261(2) 589-604.
- Leonov, N. N. (1959): On a pointwise mapping of a line into itself. Radiofisisica 2(6), 942-956.

References

- Leonov, N. N. (1962): On a discontinuous pointwise mapping of a line into itself. Dolk. Acad. Nauk SSSR 143(5), 1038-1041.
- Lux, T. (2009): Stochastic behavioural asset-pricing models and the stylized facts. In: Hens, T. and Schenk-Hoppé, K.R. (eds.): Handbook of Financial Markets: Dynamics and Evolution. North-Holland, Amsterdam, 161-216.
- Manzan, S. and Westerhoff, F. (2007): Heterogeneous expectations, exchange rate dynamics and predictability. Journal of Economic Behavior and Organization, 64, 111-128.
- Sushko I., and Gardini L. (2010): Degenerate Bifurcations and Border Collisions in Piecewise Smooth 1D and 2D Maps. International Journal of Bifurcation Chaos, 20(7) 2045-2070.
- Tramontana, F., Westerhoff, F. and Gardini, L. (2010a): On the complicated price dynamics of a simple one-dimensional discontinuous financial market model with heterogeneous interacting traders. Journal of Economic Behavior and Organization, 74, 187-205.
- Tramontana, F., Gardini, L. and Westerhoff, F. (2010b): Intricate asset price dynamics and one-dimensional discontinuous maps. In: Puu, T. and Panchuck, A. (eds): Nonlinear economic dynamics. Nova Science Publishers, 43-57.
- Tramontana, F., Gardini, L. and Westerhoff, F. (2011): Heterogeneous speculators and asset price dynamics: further results from a one-dimensional discontinuous piecewise-linear map. Computational Economics, Vol. 38, 329-347.

References

- Tramontana, F., Westerhoff, F. and Gardini, L., (2012a): A simple financial market model with chartists and fundamentalists: market entry levels and discontinuities. Working Paper.
- Tramontana, F., Westerhoff, F. and Gardini, L. (2012b): One-dimensional maps with two discontinuity points and three linear branches: mathematical lessons for understanding the dynamics of financial markets. Working Paper.
- Tramontana, F., Gardini L., Avrutin V. and Schanz M. (2012c): Period Adding in Piecewise Linear Maps with Two Discontinuities. International Journal of Bifurcation Chaos , 22(3) (2012) 1250068 (1-30).
- Tramontana, F. and Westerhoff, F. (2013): One-dimensional discontinuous piecewise-linear maps and the dynamics of financial markets. In: Bischi, G.I., Chiarella, C., Sushko, I. (eds.): Global Analysis of Dynamic Models in Economics and Finance. Springer, New York, 205-227.
- Westerhoff, F. (2009): Exchange rate dynamics: a nonlinear survey. In: Rosser, J.B., Jr. (ed): Handbook of Research on Complexity. Edward Elgar, Cheltenham, 287-325.
- Westerhoff, F. and Franke, R. (2012): Converse trading strategies, intrinsic noise and the stylized facts of financial markets. Quantitative Finance, Vol. 12, 425-436.
- Wieland, C. and Westerhoff, F. (2005): Exchange rate dynamics, central bank intervention and chaos control methods. Journal of Economic Behavior and Organization 58, 117-132.