Nonlinear dynamics and global analysis of a heterogeneous Cournot duopoly

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Outline

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Complexity in oligopolies

- Since the pionnering works of Rand (1978) and Poston and Stewart (1978) it is well known that by relaxing the degree of rationality of the players, even simple duopoly games may give rise to complex dynamic phenomena;
- Among all the contributions, particularly important is the work of Puu (1991), who proposed a duopoly model based on unimodal reaction functions derived solving an optimization problem for profit functions;
- Moving from the hypothesis of Cobb-Douglas preferences of the consumers, he obtained an isoelastic demand function and showed that players endowed with naïve expectations may fail to converge to the Cournot-Nash equilibrium.

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Heterogeneous oligopolies

- A lot of different research strands started from the work of Puu. Our work is related to the one dealing with the investigation of decisional mechanisms based on low degrees of informational and/or computational abilities.
- The literature on heterogeneous oligopolies is relatively recent (Leonard and Nishimura 1999, Den-Haan 2001, Agiza and Elsadany 2003, Angelini et al. 2009, Tramontana 2010, Dubiel-Teleszynski 2011).
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Related literature

- The economic structure of our duopoly is the same of Angelini et al. (2009) and Tramontana (2010), that is the demand function is isolestic and marginal costs are constant;
- One of the two firms is the same as Tramontana (2010) but we don't have the best response player. This permits us to compare the results and to find the role played by a lower degree of market knowledge;
- The presence of the isoelastic demand function gives rise to a map characterzed by a denominator that can vanishes. By deepening such a global feature our work is also related to the paper studying the so-called maps with denominator (see Bischi et al. 1999, 2001, 2003, 2005, Bischi and Tramontana 2005, Naimzada and Tramontana 2009).

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The gradient-like mechanism

- Our first player is endowed with a classical gradient-like decisional mechamism;

$$q_1(t+1) = q_1(t) + \alpha q_1(t) \frac{\partial \Pi_1(t)}{\partial q_1(t)} \tag{1}$$

  where $\alpha q_1(t)$ denotes the quantity-dependent speed of adjustment;
- This player does not have a complete knowledge of the demand and cost functions but he uses a local estimate of the marginal profit;
- The version with constant reactivity is discussed in Varian (1992) and Corchon and Mas-Colell (1996) and in the heterogeneous duopoly model of Angelini et al. (2009). The version with endogenous reactivity is studied in Bischi and Naimzada (1999), Bischi et al. (2001, 2007), Agiza et al. (2002), Tramontana (2010) and Askar (2014)

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Local Monopolistic Approssimation

Our second player adopts a Local Monopolistic Approssimation (LMA), mechanism introduced by Tuinstra (2004), Bischi et al. (2007), Naimzada and Sbragia (2006) and applied in a monopolistic setting by Naimzada and Ricchiuti (2011).

Player 2 does not know the market demand function and he conjectures it is linear, so he proceeds by estimating such a linear function through the local knowledge of the true demand curve and the knowledge of the current market state in terms of quantities and price.

$$q_2(t+1) = \arg \max_{q_2(t+1)} [p_2^e(t+1)q_2(t+1) - c_2 q_2(t+1)] \qquad (2)$$

where c_2 is the constant marginal cost of the second player and:

$$p_2^e(t+1) = p_2(t) + p'(Q(t))(Q^e(t+1) - Q(t)), \qquad (3)$$

with $Q^e(t+1) = q_1^e(t+1) + q_2(t+1)$.

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The map

Putting together the assumptions and considering static expectations on the rival's output, we get the following two-dimensional dynamical system:

The 2D nonlinear dynamical system

$$T(q_1,q_2):\begin{cases} q_1' = q_1 + \alpha q_1 \left[\frac{q_2}{(q_1+q_2)^2} - c_1\right] \\ q_2' = \frac{1}{2}q_2 + \frac{1}{2}\left[1 - c_2\left(q_1+q_2\right)\right]\left(q_1+q_2\right) \end{cases} \tag{4}$$

where "′" denotes the unit-time advancement operator.

whose only interior Nash equilibrium is given by:

$$E = (q_1^N, q_2^N) = \left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2}\right) \tag{5}$$

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Stability conditions

The Nash Equilibrium E is locally stable provided that:

Stability conditions

$$\begin{array}{ccc} 0 < \frac{c_1}{c_2} \leq \frac{1}{4} & \textit{and} & \alpha < \alpha_{ns} \\ \frac{c_1}{c_2} > \frac{1}{4} & \textit{and} & \alpha < \alpha_f \end{array} \quad (6)$$

with:

$$\alpha_{ns} = \frac{2}{c_2 - 3c_1} \quad ; \quad \alpha_f = \frac{8c_1 + 4c_2}{c_2(7c_1 - c_2)}$$

denoting the values of the speed of adjustment leading to a Neimark-Sacher or a Flip bifurcation, respectively.

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Best reply vs LMA

The duopoly model of Tramontana (2010) has the same interior Nash Equilibrium but different stability conditions:

Stability conditions (Tramontana, 2010)

$$\begin{array}{lll} 0 < \frac{c_1}{c_2} \leq \frac{1}{3} \cup \frac{c_1}{c_2} > 3 & and & \alpha < \alpha_{ns}^T \\ \frac{1}{3} < \frac{c_1}{c_2} < 3 & and & \alpha < \alpha_f^T \end{array} \quad (7)$$

with:

$$\alpha_{ns}^T = \frac{2(c_1+c_2)}{(c_2-c_1)^2} \quad ; \quad \alpha_f^T = \frac{4(c_1+c_2)}{4c_1c_2-(c_2-c_1)^2}$$

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Best reply vs LMA

Comparing the two sets of stability conditions we have the following result:

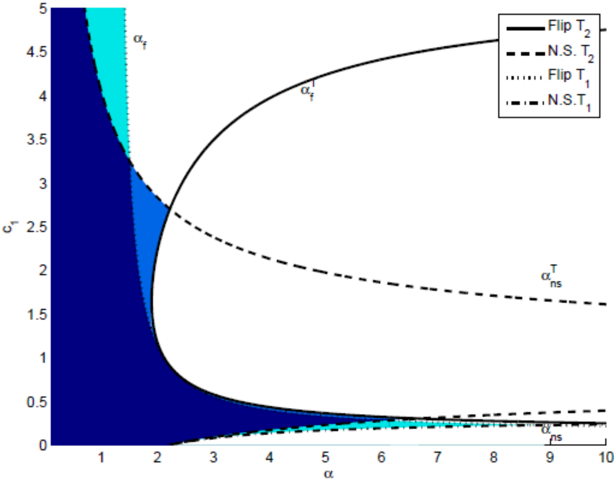
BR versus LMA

The stability region with LMA is larger than the stability region with BR provided that:

$$\frac{c_1}{c_2} < s_1 \cup \frac{c_1}{c_2} > s_2 \tag{8}$$

with $s_1 = 0.3108$ and $s_2 = 3.6081$.

Graphical comparison ($c_2 = 0.9$)



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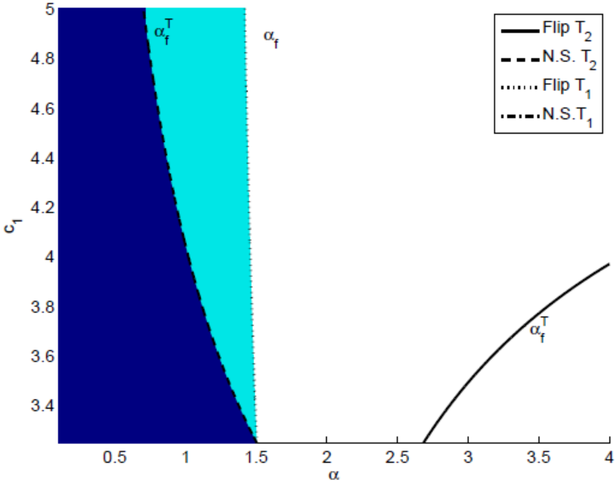

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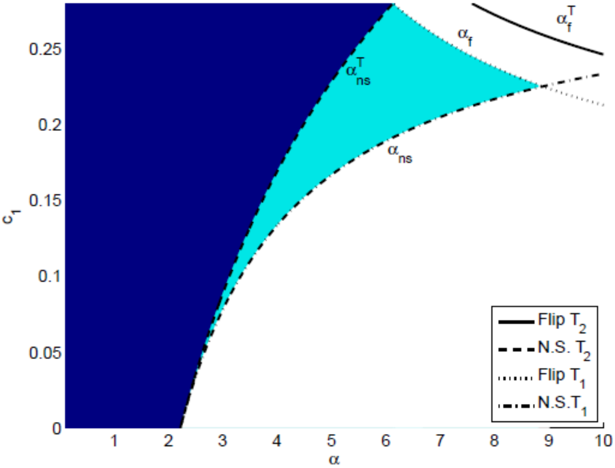

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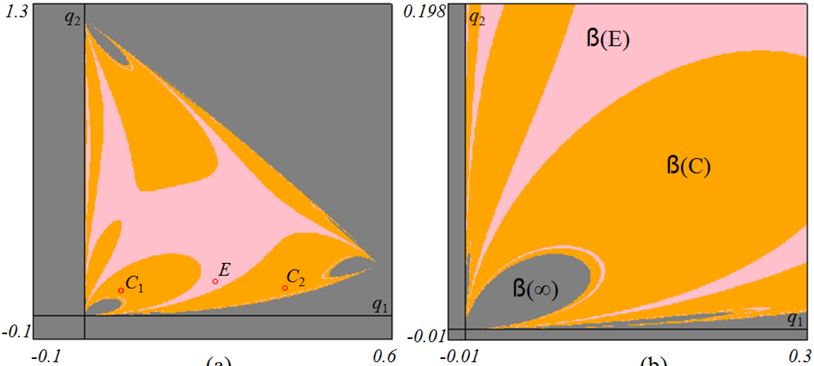
A typical scenario ($c_1 = 0.9$; $c_2 = 1.62$ *and* $\alpha = 1.8$)

The NE undergoes a flip bifurcation at $\alpha = \alpha_f = 1.804$, so it is still locally stable. Here is how the phase space appears:



(a) (b)

Causes

The nonlinearity of the map may cause this kind of scenarios characterized by multistability and particular basins' configurations.
In order to perform a global analysis, two characteristics of the map must be taken into account:

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Noninvertible maps

- A map T is *non-invertible* if one of its points can be characterized by several rank-1 preimages (i.e. points mapped in that point after one iteration);
- In such cases, the inverse map T^{-1} is in general the union of more than one inverse maps of the phase plane;
- Map T *folds and pleats* the phase plane, while map T^{-1} *unfolds* it;
- The phase plane can be subdivided into several Z_i regions, where i is the number of preimages of the points of that region (Mira et al., 1996 and Abraham et al. 1996)

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Z_i regions (Mira et al. 1996, Abraham et al. 1996)

- Two contiguous Z_i regions are characterized by a number of preimages the differs by two;
- The boundary between two regions is called LC curve, is a generalization of the one-dimensional *critical value* and is the locus of points where *two (or more) rank-1 preimages are coincident*;
- The coincident preimages are located on a set called LC_{-1}, that generalizes the notion of *critical point* (i.e. local extremum point);
- For a continuosly differentiable map, the set LC_{-1} is included in the set of points where the determinant of the Jacobian matrix vanishes:

$$LC_{-1} \subseteq \left\{ (q_1, q_2) \in \mathbb{R}^2 \,|\, detJ(q_1, q_2) = 0 \right\}$$

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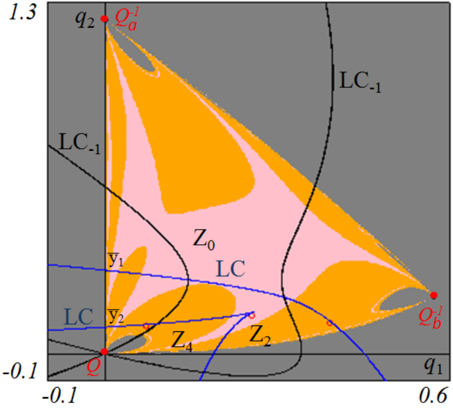
Z_i regions

Here is our numerically computed Z_i regions:



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A map with denominator

$$T(q_1, q_2) : \begin{cases} q_1' = q_1 + \alpha q_1 \left[\frac{q_2}{(q_1+q_2)^2} - c_1 \right] \\ q_2' = \frac{1}{2} q_2 + \frac{1}{2} \left[1 - c_2 (q_1 + q_2) \right] (q_1 + q_2) \end{cases}$$

- Our map T is defined in the whole plane except fo the line $\delta_S : q_2 = -q_1$ (and its preimages of any order), where *the denominator of the first difference equation vanishes*;
- In particular in the origin $Q(0,0)$ *a component of the map takes the form 0/0;*
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- To each focal point ia associated a *prefocal curve* δ_Q, made up by points that are mapped into the focal point by at least one of the inverses of the map T;
- In our case the prefocal curve is a prefocal line: the horizontal axis:

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- If the focal point is *simple* (see Bischi et al. 1999, 2003, 2005) there is a *one-to-one correspondence* between the slope m of an arc γ thorugh Q and the point in which its image crosses the prefocal line δ_Q.

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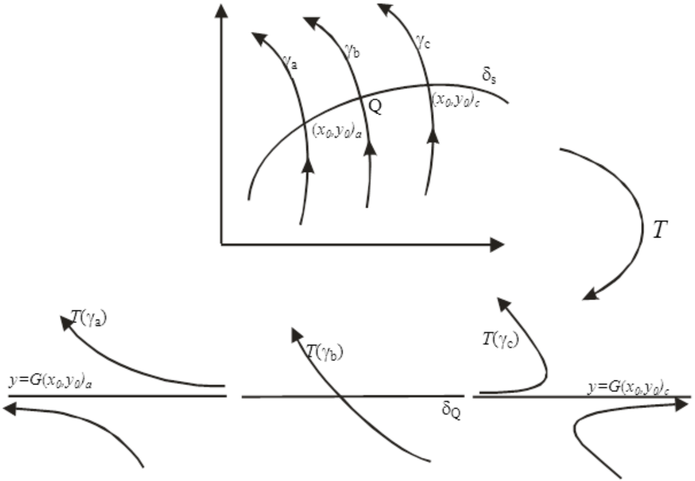
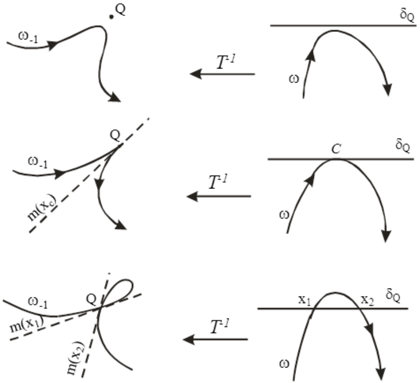


Noninvertible maps

If we consider an arc crossing δ_Q in two points, then there exists one rank-1 preimage forming a *loop* with a knot in Q



Focal point not simple

In our case our focal point Q is *not simple* and this implies that the correspondence between arcs through Q and points of the prefocal line δ_Q is two-to-one instead of one-to-one

Proposition: For any given point $(q_1, 0) \in \delta_Q$, the two slopes of the arcs through Q are:

$$m_{\pm}(q_1) = \frac{\alpha - 2q_1 \pm \sqrt{\alpha^2 - 4\alpha q_1}}{2q_1}$$

Moreover

Q belongs to the prefocal line δ_Q

Corollary: Any arc transverse to δ_Q has infinitely many preimages which are arcs through Q

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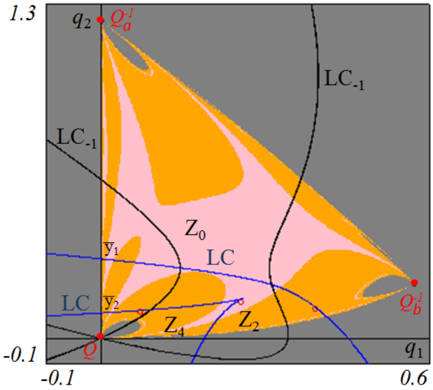
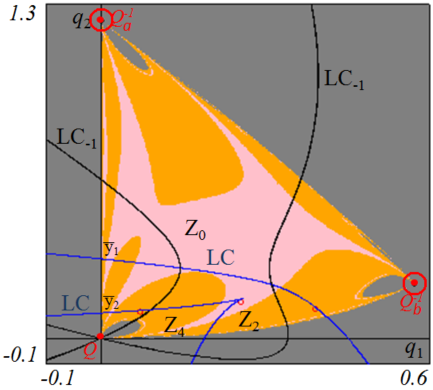
Explaining our lobes



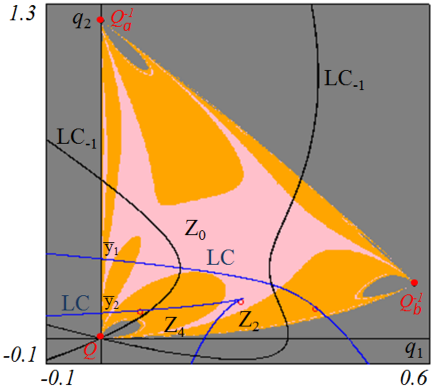
Let us focus on the grey lobe (denoting divergence) issuing from the focal point Q

Explaining our lobes



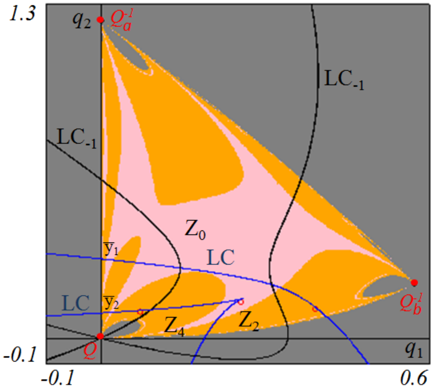
Q has two preimages different from Q itself: Q_a^{-1}, that is always on the vertical axis, and Q_b^{-1}

Explaining our lobes



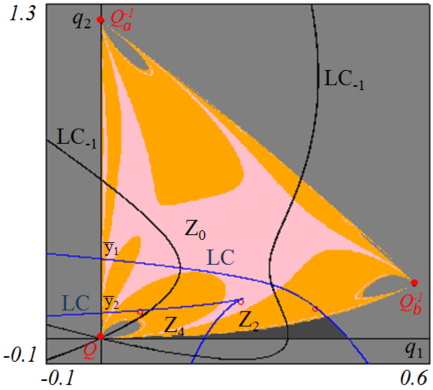
The vertical axis is invariant (a trapping set), in fact
$T(0, q_2) = \left(0, q_2 - \frac{1}{2}c_2 q_2^2\right)$.

Explaining our lobes



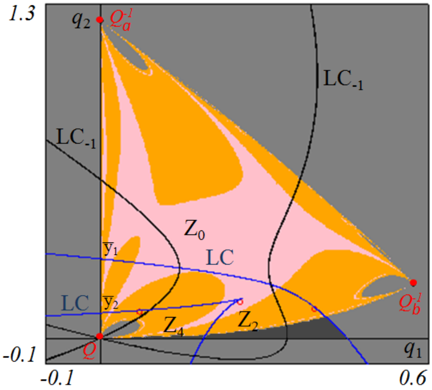
The vertical axis and its preimages of any rank separate the basin of diverging trajectories $B(\infty)$ from the basin of feasible trajectories $B(f)$

Explaining our lobes



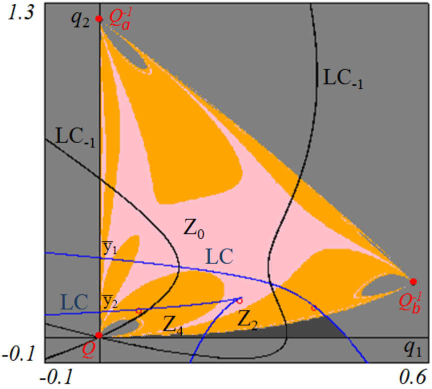
The lower curve side of the pseudo-triangle is one of the preimages of the vertical axis and the gray lobe are the preimages of the dark gray region

Explaining our lobes



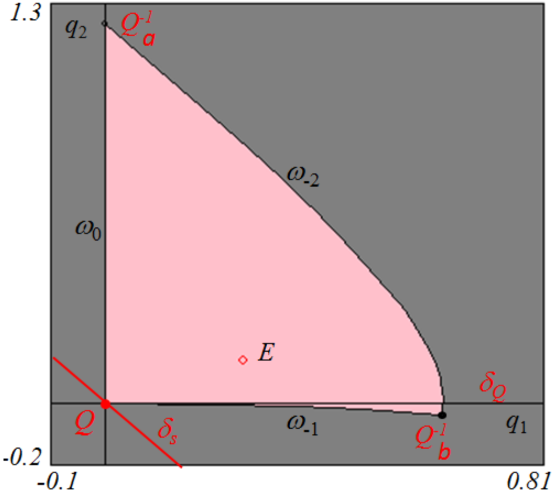
When Q_b^{-1} is in the forth quadrant there must be no grey lobes. The grey lobe is created when the q_2 coordinate of Q_b^{-1} is equal to 0.

Explaining our lobes

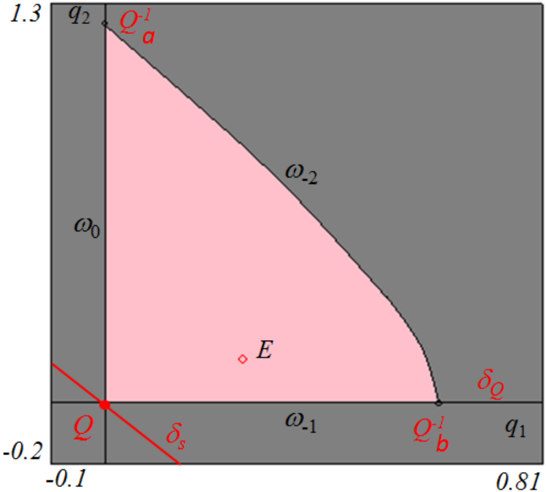


The q_2 coordinate of Q_b^{-1} is $\frac{\alpha(\alpha c_1 - 1)}{[1-\alpha(c_1-c_2)]^2}$ and is equal to 0 when $\alpha = \alpha_g = 1/c_1$

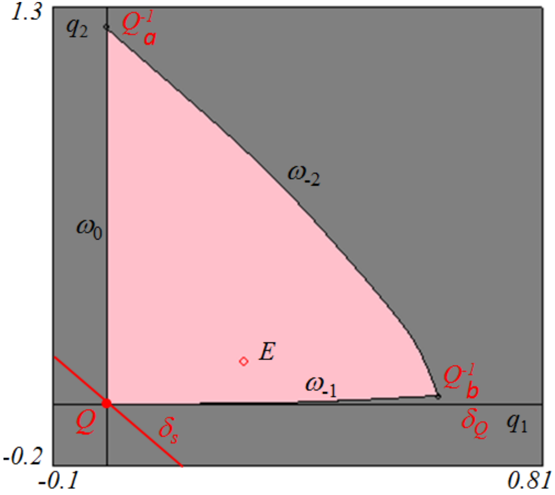
$\alpha_g = 1.\overline{1};\ \alpha = 0$



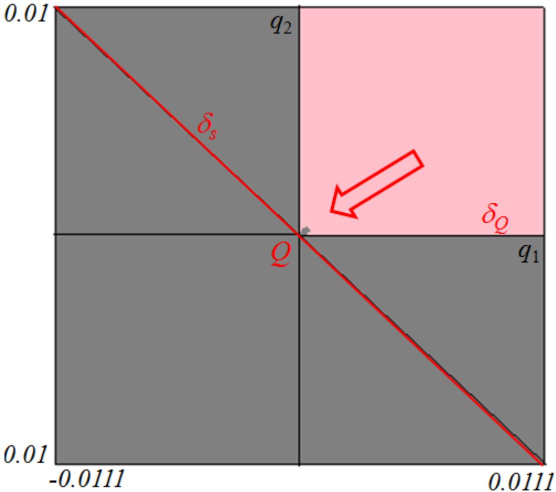
$\alpha = \alpha_g = 1.\overline{1}$



$\alpha_g = 1.\overline{1};\ \alpha = 1.2$



$\alpha_g = 1.\overline{1}$; $\alpha = 1.2$ (enlargement)



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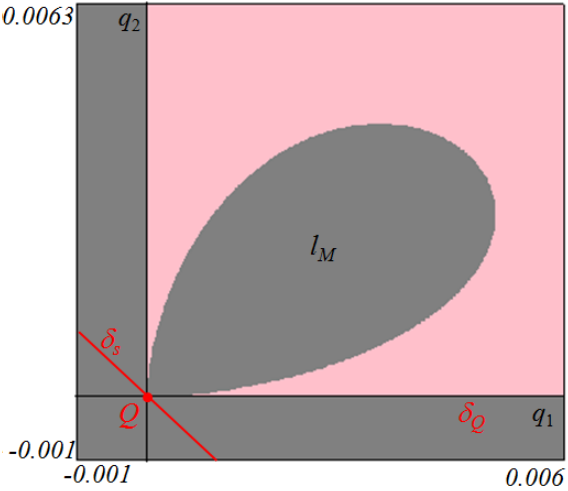
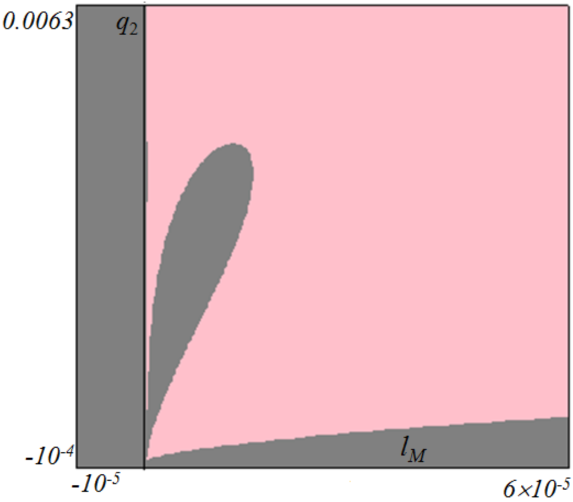


$\alpha_g = 1.\bar{1}$; $\alpha = 1.2$ another lobe



Conclusions

Two main messages:

1. The effects of the introduction of heterogeneity and different degrees of rationality in an economic model may be contrary to the common sense and deserve further investigation;
2. The local analysis is not the whole story, in particular maps with denominator may give rise to particular shapes of the basins of attraction (note that by using the isoelastic demand function we easily obtain a map with denominator)

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