Applications to financial markets: Higher-dimensional discrete dynamical systems in heterogeneous-agent asset pricing models

Speculation, diversification and price dynamics in a multiple-asset framework with heterogeneous investors

Roberto Dieci (*roberto.dieci@unibo.it*)
Department of Mathematics for Economics and Social Sciences
University of Bologna, Italy

(adapted from joint work with:

Carl Chiarella (*carl.chiarella@uts.edu.au*)
School of Finance and Economics, UTS Sydney, Australia

Laura Gardini (*laura.gardini@uniurb.it*)
Dipartimento di Economia e Metodi Quantitativi, Università di Urbino)

Tutorial Workshop “Discrete Dynamical Systems and Applications”
University of Urbino, June 30 - July 3, 2010

1 Model framework

- Two-risky assets, one risk-free asset
- Agents adopt simple rules to form and update first- and second-moment beliefs on the joint distribution of asset returns
- Heterogeneous investors, *fundamentalists* and *chartists*
- ‘Market maker’ price setting mechanism
- Focus on the ‘behavioral’ determinants of the co-movements of asset prices: trend extrapolation and dynamic updating of variance / covariance beliefs determine the coupling of otherwise independent subsystems (asset markets)

2 One risky asset (and the risk-free asset)

- Wealth dynamics

$$\Omega_{t+1} - \Omega_t = \Omega_t r_f + \zeta_t \pi_{t+1}$$

- Asset demand of myopic mean-variance investors

$$\zeta_t = \frac{E_t\left[\pi_{t+1}\right]}{\alpha Var_t\left[\pi_{t+1}\right]}$$

Ω_t: wealth at time t

ζ_t: amount of wealth to be invested in the risky asset in period $(t, t+1)$

π_{t+1}: excess return on the risky asset over the same period

r_f: risk-free return

α: CARA coefficient

E_t, Var_t: agent's conditional expectation / variance

- Using log-returns (P_t denotes log price):

$$\zeta_t = \frac{m_t + d - r}{\alpha V_t}$$

$m_t := E_t(P_{t+1} - P_t)$: expected 'price return'

$d := E_t(d_{t+1})$: (constant) expected 'dividend return'

$V_t := Var_t\left[P_{t+1} - P_t + d_{t+1}\right]$: variance of the (excess) return

- Common (constant) beliefs about first and second moment of d_{t+1} (uncorrelated with price changes)

- Heterogeneous beliefs about price component of return

Heterogeneous agents' beliefs and demands

- *Fundamentalists* (superscript f): mean reversion, constant variance

$$\begin{aligned} m_t^f &:= E_t^f(P_{t+1} - P_t) = \eta(W - P_t) \qquad \eta > 0 \\ V_t^f &:= Var_t^f\left[P_{t+1} - P_t + d_{t+1}\right] = \sigma^2 \end{aligned}$$

W: log-fundamental value

- Fundamentalist demand

$$\begin{aligned} \zeta_t^f &= a(W - P_t) + h \\ a &:= \frac{\eta}{\alpha^f \sigma^2}, \quad h := \frac{d - r}{\alpha^f \sigma^2} \end{aligned}$$

- *Chartists* (superscript c): extrapolation through exponentially weighted moving averages (EWMA) results in the following adaptive rules, see, e.g. Chiarella et al. (Appl. Math. Finance 2005)

$$\begin{aligned} m_t^c &= E_t^c(P_{t+1} - P_t) := \xi_t \\ \xi_t &= (1-c)\xi_{t-1} + c(P_t - P_{t-1}) \end{aligned}$$

$$\begin{aligned} V_t^c &: = Var_t^c\left[P_{t+1} - P_t + d_{t+1}\right] = \sigma^2 + v_t \\ v_t &= (1-c)v_{t-1} + c(1-c)(P_t - P_{t-1} - \xi_{t-1})^2 \end{aligned}$$

- Extrapolation parameter c, $0 < c < 1$: weight given to the most recent observation

- Chartist demand

$$\zeta_t^c = \frac{\xi_t + d - r}{\alpha^c(v_t + \sigma^2)}$$

Price setting rule and price dynamics

- (log-)price change $P_{t+1} - P_t$ directly related to excess demand (desired asset holding)

$$P_{t+1} = P_t + \beta[\zeta_t^f + \zeta_t^c - Z] \qquad (\beta > 0)$$

  Z: a threshold related to existing supply of asset and some 'target' level of market maker inventory

  Z set equal to steady state asset holding for simplicity

- Dynamical system driven by the 3D map ($q := P - W$)

$$T_1 : \begin{cases} q' = F(q, \xi, v) := q + \beta[-aq + (\zeta^c - \overline{\zeta}^c)] \\ \xi' = G(q, \xi, v) := (1-c)\xi + c(F(q, \xi, v) - q) \\ v' = H(q, \xi, v) := (1-c)v + c(1-c)(F(q, \xi, v) - q - \xi)^2 \end{cases}$$

  where

$$\zeta^c = \frac{\xi + d - r}{\alpha^c(v + \sigma^2)}, \quad \overline{\zeta}^c := \frac{d - r}{\alpha^c \sigma^2}$$

Steady state stability in the single-risky-asset model

- $O := (0, 0, 0)$ unique steady state
- Jacobian matrix is block diagonal

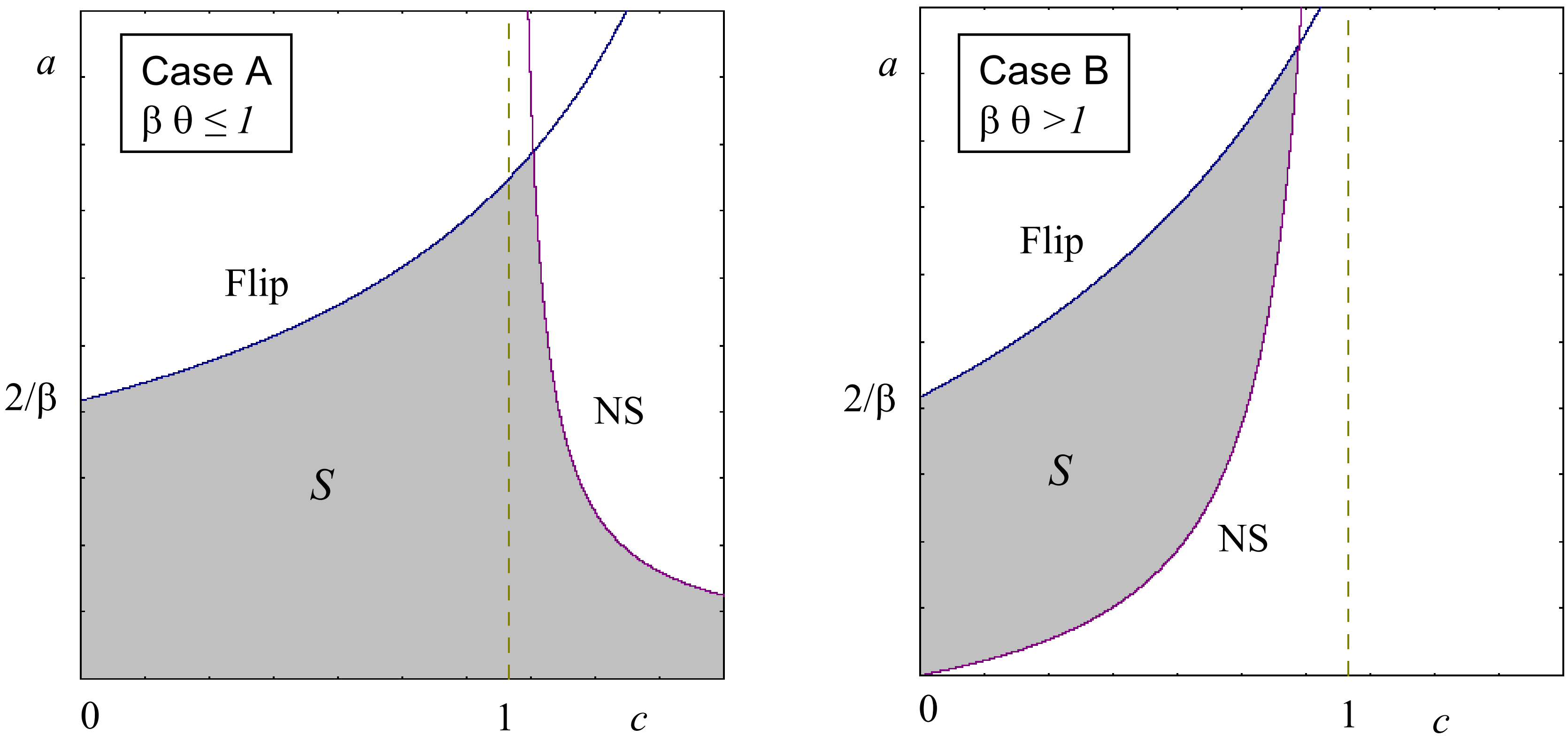
$$DT_1(O) = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & 1 - c \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 - a\beta & \beta\theta \\ -c\beta a & 1 - c + c\beta\theta \end{bmatrix}$$

$\theta := 1/(\alpha^c \sigma^2)$: *strength of chartist demand* (at the steady state)

- Eigenvalue $\lambda_3 = 1 - c$, $0 < \lambda_3 < 1$, local stability / bifurcations investigated based on the characteristic roots λ_1 and λ_2 associated with 2D block $\mathbf{A}$

Single-risky-asset model
Stability region in the space of parameters (*c*,*a*)



3 Two risky assets (and the risk-free asset)

- Risky assets (and asset-specific quantities) indexed with $i = 1, 2$
- Beliefs $m_{i,t}$, $V_{i,t}$, d_i with the same meaning as the single-risky-asset case, formed / updated in a similar way
- In addition, beliefs about returns covariance and correlation matter

$$S_t \quad : \quad = Cov_t(P_{1,t+1} - P_{1,t} + d_{1,t+1}, P_{2,t+1} - P_{2,t} + d_{2,t+1})$$

$$\rho_t \quad = \quad \frac{S_t}{\sqrt{V_{1,t}V_{2,t}}}$$

- Wealth dynamics

$$\Omega_{t+1} - \Omega_t = \Omega_t r_f + \sum_{i=1}^{2} \zeta_{i,t} \pi_{i,t+1}$$

$\zeta_{i,t}$: amount of wealth to be invested in the ith risky asset in period $(t, t+1)$

$\pi_{i,t+1}$: excess return on the ith risky asset over the same period

- Asset demands of myopic mean-variance investors

$$\zeta_{1,t} = \frac{1}{(1-\rho_t^2)} \frac{(m_{1,t} + d_1 - r)}{\alpha V_{1,t}} - \frac{\rho_t \sqrt{V_{2,t}}}{(1-\rho_t^2)\sqrt{V_{1,t}}} \frac{(m_{2,t} + d_2 - r)}{\alpha V_{2,t}}$$

$$\zeta_{2,t} = \frac{1}{(1-\rho_t^2)} \frac{(m_{2,t} + d_2 - r)}{\alpha V_{2,t}} - \frac{\rho_t \sqrt{V_{1,t}}}{(1-\rho_t^2)\sqrt{V_{2,t}}} \frac{(m_{1,t} + d_1 - r)}{\alpha V_{1,t}}$$

- Demand for each asset consists of a *direct* component and a *hedging* component

Heterogeneous agents' beliefs and demands

- *Fundamentalists*: mean reversion, constant second-moment beliefs

$$m_{i,t}^f = \eta_i(W_i - P_{i,t}), \quad V_{i,t}^f = \sigma_i^2, \quad \rho_t^f = \delta, \quad S_t^f = \delta\sigma_1\sigma_2$$

- Fundamentalist demand

$$\begin{aligned} \zeta_{1,t}^f &= a_1(W_1 - P_{1,t}) - b_2(W_2 - P_{2,t}) + h_1 \\ \zeta_{2,t}^f &= a_2(W_2 - P_{2,t}) - b_1(W_1 - P_{1,t}) + h_2 \end{aligned}$$

where

$$\begin{aligned} a_i &:= \frac{\eta_i}{\alpha^f(1-\delta^2)\sigma_i^2}, \qquad b_i = \frac{\delta\eta_i}{\alpha^f(1-\delta^2)\sigma_1\sigma_2} \\ h_1 &:= \frac{1}{(1-\delta^2)}\frac{d_1 - r}{\alpha^f\sigma_1^2} - \frac{\delta\sigma_2}{(1-\delta^2)\sigma_1}\frac{d_2 - r}{\alpha^f\sigma_2^2} \\ h_2 &:= \frac{1}{(1-\delta^2)}\frac{d_2 - r}{\alpha^f\sigma_2^2} - \frac{\delta\sigma_1}{(1-\delta^2)\sigma_2}\frac{d_1 - r}{\alpha^f\sigma_1^2} \end{aligned}$$

- *Chartists*: adaptive rules resulting from EWMA

$$m^c_{i,t} = \xi_{i,t} = (1-c)\xi_{i,t-1} + c(P_{i,t} - P_{i,t-1})$$

$$V^c_{i,t} = \sigma_i^2 + v_{i,t}, \quad v_{i,t} = (1-c)v_{i,t-1} + c(1-c)(P_{i,t} - P_{i,t-1} - \xi_{i,t-1})^2$$

$$\begin{aligned} S^c_t &= \delta\sigma_1\sigma_2 + K_t \\ K_t &= (1-c)K_{t-1} + c(1-c)(P_{1,t} - P_{1,t-1} - \xi_{1,t-1})(P_{2,t} - P_{2,t-1} - \xi_{2,t-1}) \\ \rho^c_t &= \frac{K_t + \delta\sigma_1\sigma_2}{\sqrt{(v_{1,t}+\sigma_1^2)(v_{2,t}+\sigma_2^2)}} \end{aligned}$$

- Chartist demand

$$\zeta^c_{1,t} = \frac{1}{(1-\rho^{c\,2}_t)} \frac{(\xi_{1,t} + d_1 - r)}{\alpha^c(v_{1,t}+\sigma_1^2)} - \frac{\rho^c_t\sqrt{v_{2,t}+\sigma_2^2}}{(1-\rho^{c\,2}_t)\sqrt{v_{1,t}+\sigma_1^2}} \frac{(\xi_{2,t}+d_2-r)}{\alpha^c(v_{2,t}+\sigma_2^2)}$$

$$\zeta^c_{2,t} = \frac{1}{(1-\rho^{c\,2}_t)} \frac{(\xi_{2,t} + d_2 - r)}{\alpha^c(v_{2,t}+\sigma_2^2)} - \frac{\rho^c_t\sqrt{v_{1,t}+\sigma_1^2}}{(1-\rho^{c\,2}_t)\sqrt{v_{2,t}+\sigma_2^2}} \frac{(\xi_{1,t}+d_1-r)}{\alpha^c(v_{1,t}+\sigma_1^2)}$$

Price dynamics

Dynamics of asset prices / beliefs driven by a 7D map ($q_i := P_i - W_i$)

$$T_2 : \begin{cases} q_1' = q_1 + \beta_1[-a_1 q_1 + b_2 q_2 + (\zeta_1^c - \overline{\zeta}_1^c)] \\ q_2' = q_2 + \beta_2[-a_2 q_2 + b_1 q_1 + (\zeta_2^c - \overline{\zeta}_2^c)] \\ \xi_i' = (1-c)\xi_i + c(q_i' - q_i) \quad i = 1,2 \\ v_i' = (1-c)v_i + c(1-c)(q_i' - q_i - \xi_i)^2 \quad i = 1,2 \\ K' = (1-c)K + c(1-c)(q_1' - q_1 - \xi_1)(q_2' - q_2 - \xi_2) \end{cases}$$

where

$$\zeta_1^{(c)} = \frac{(v_2 + \sigma_2^2)(\xi_1 + d_1 - r) - (K + \delta\sigma_1\sigma_2)(\xi_2 + d_2 - r)}{\alpha^c[(v_1 + \sigma_1^2)(v_2 + \sigma_2^2) - \delta^2\sigma_1^2\sigma_2^2 - K^2 - 2K\delta\sigma_1\sigma_2]}$$

$$\zeta_2^{(c)} = \frac{(v_1 + \sigma_1^2)(\xi_2 + d_2 - r) - (K + \delta\sigma_1\sigma_2)(\xi_1 + d_1 - r)}{\alpha^c[(v_1 + \sigma_1^2)(v_2 + \sigma_2^2) - \delta^2\sigma_1^2\sigma_2^2 - K^2 - 2K\delta\sigma_1\sigma_2]}$$

$$\overline{\zeta}_1^c := \frac{1}{(1-\delta^2)}\frac{d_1 - r}{\alpha^c\sigma_1^2} - \frac{\delta\sigma_2}{(1-\delta^2)\sigma_1}\frac{d_2 - r}{\alpha^c\sigma_2^2}$$

$$\overline{\zeta}_2^c := \frac{1}{(1-\delta^2)}\frac{d_2 - r}{\alpha^c\sigma_2^2} - \frac{\delta\sigma_1}{(1-\delta^2)\sigma_2}\frac{d_1 - r}{\alpha^c\sigma_1^2}$$

Steady state stability in the two-risky-asset model

- $O = \mathbf{0}$ unique steady state
- Jacobian matrix (at the steady state) is upper block triangular

$$DT_2(O) = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & (1-c)\mathbf{I} \end{bmatrix}$$

  A: four-dimensional matrix

- Eigenvalues $\lambda_5 = \lambda_6 = \lambda_7 = 1 - c$, $0 < 1 - c < 1$
- Local stability / bifurcations investigated based on characteristic roots λ_1, λ_2, λ_3, λ_4 associated with 4D block **A**.

Steady state stability (continued)

- Particular case of zero exogenous correlation ($\delta = 0$), matrix $\mathbf{A}$ block diagonal

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}$$

  Matrices $\mathbf{A}_i$, $i = 1, 2$, as in the one-asset case

$$\mathbf{A}_i = \begin{bmatrix} 1 - a_i\beta_i & \beta_i\theta_i \\ -c\beta_i a_i & 1 - c + c\beta_i\theta_i \end{bmatrix}$$

  $\theta_i := 1/(\alpha^c\sigma_i^2)$: *strength of chartist demand* in the ith market

- Asset markets decouple from each other in the linearized system around the steady state, local behavior of asset 1 (asset 2) uniquely associated with the eigenvalues of $\mathbf{A}_1$ ($\mathbf{A}_2$)

- *Increasing extrapolation* (parameter c), combined with differences in asset-specific parameters (d_i, σ_i^2, β_i), results in asynchronous loss of stability of the 'fundamental' steady states in the two asset markets, via a Neimark-Sacker bifurcation followed by a 'secondary' bifurcation.

Numerical example

- At the steady state, asset 2 has higher expected excess return and higher volatility than asset 1, $d_2 - r > d_1 - r$, $\sigma_2 > \sigma_1$

- Therefore, strength of chartist demand ($\theta_i := 1/(\alpha^c \sigma_i^2)$) is higher in market 1 than in market 2.

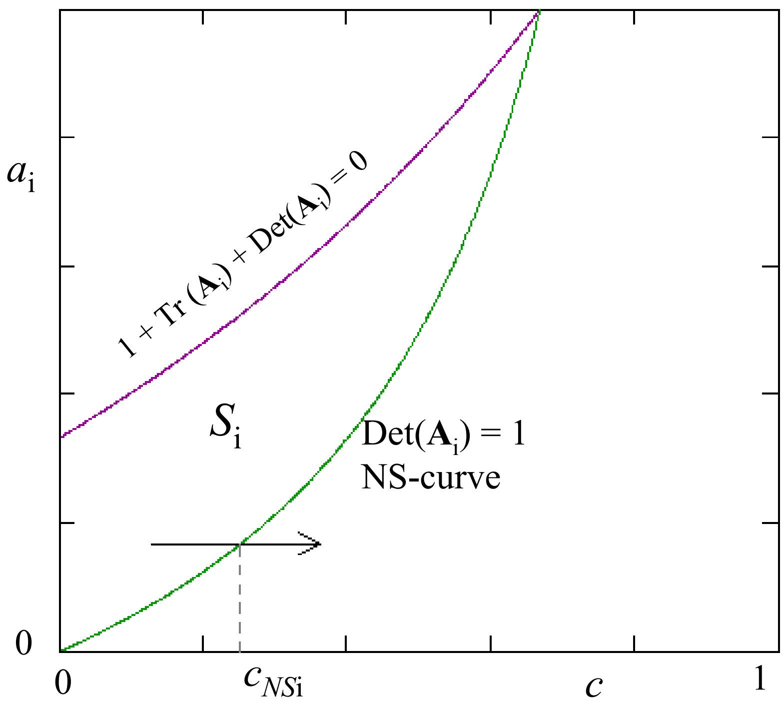
- Chartists are less risk averse than fundamentalists ($\alpha^c < \alpha^f$).

- Joint representation of the regions S_i of the parameter plane (c, a_i) where the two eigenvalues associated with market i are of modulus smaller than 1

- For a given choice of c, a_1, a_2, the steady state O is locally asymptotically stable when (c, a_1) lies inside S_1 and (c, a_2) lies inside S_2

- 3 dynamic scenarios, *associated with increasing levels of extrapolation*: (i) both prices stable; (ii) fluctuations of price 1 while market 2 stable; (iii) both markets destabilized

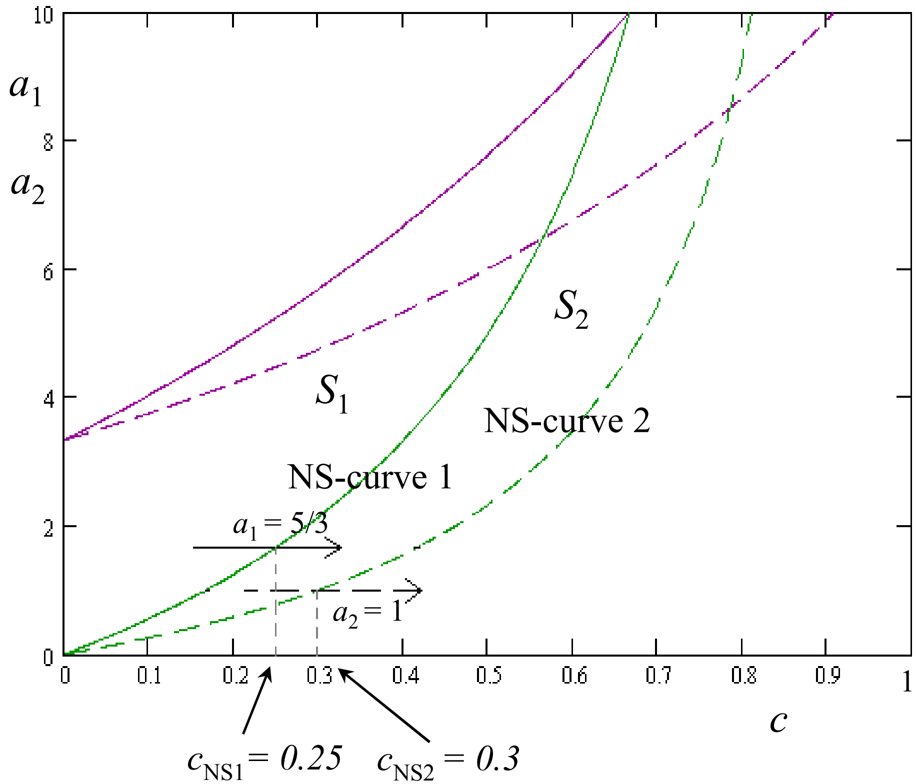
Parameter region of 'stability' of the eigenvalues associated with market *i*



c: chartist extrapolation parameter

a_i: "strength" of fundamentalist demand of asset *i*

Parameter regions of ‘stability’ of the eigenvalues associated with market 1 (solid boundary) and market 2 (dashed boundary)



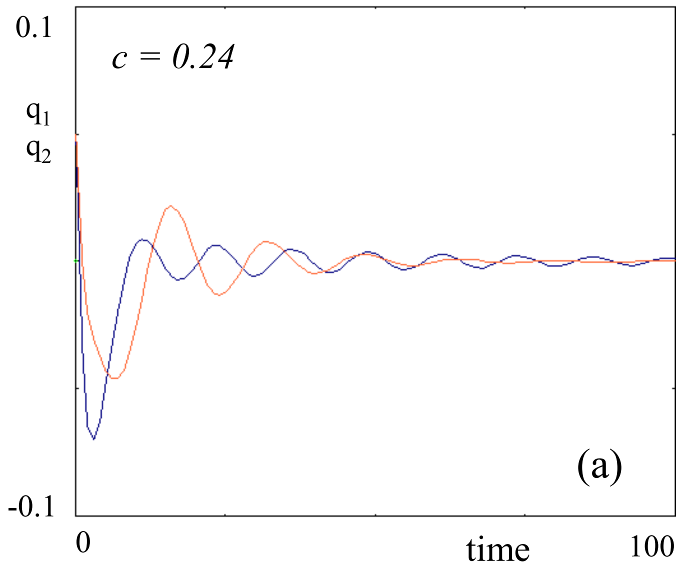
parameters

$\theta_1 = 20/3 \cong 6.66667$

$\theta_2 = 4$

$\beta_1 = \beta_2 = 0.6$

Different stability scenarios



parameters

$\delta = 0 \quad \eta_1 = \eta_2 = 0.5$

$\alpha^f = 100 \quad \alpha^c = 50$

$\sigma_1^2 = 0.003 \quad \sigma_2^2 = 0.005$

$d_1 - r = 0.01 \quad d_2 - r = 0.02$

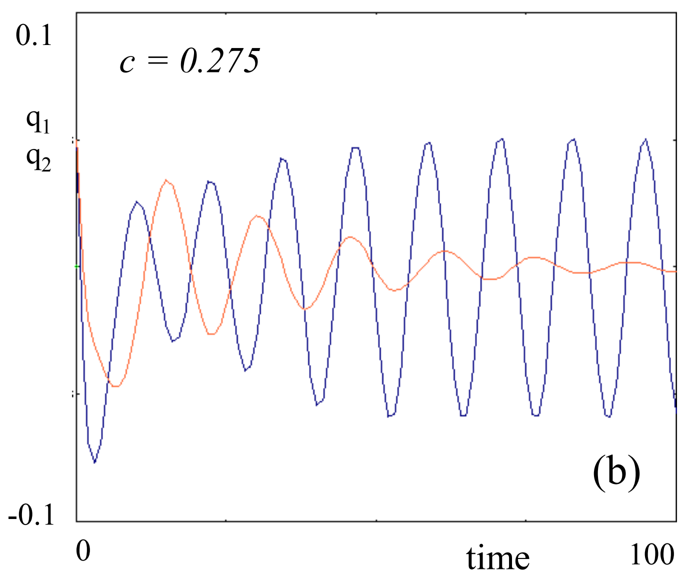
$a_1 := \eta_1 / (\alpha^f \sigma_1^2) = 1.66667$

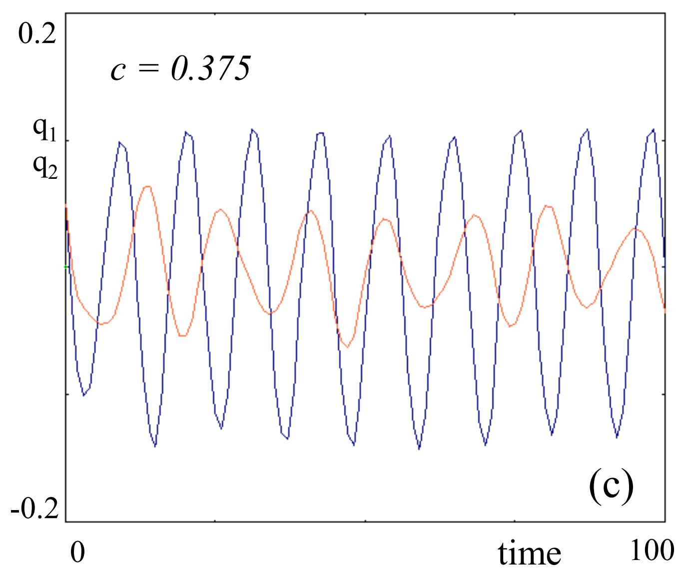
$a_2 := \eta_2 / (\alpha^f \sigma_2^2) = 1$

$\theta_1 := 1 / (\alpha^c \sigma_1^2) = 6.66667$

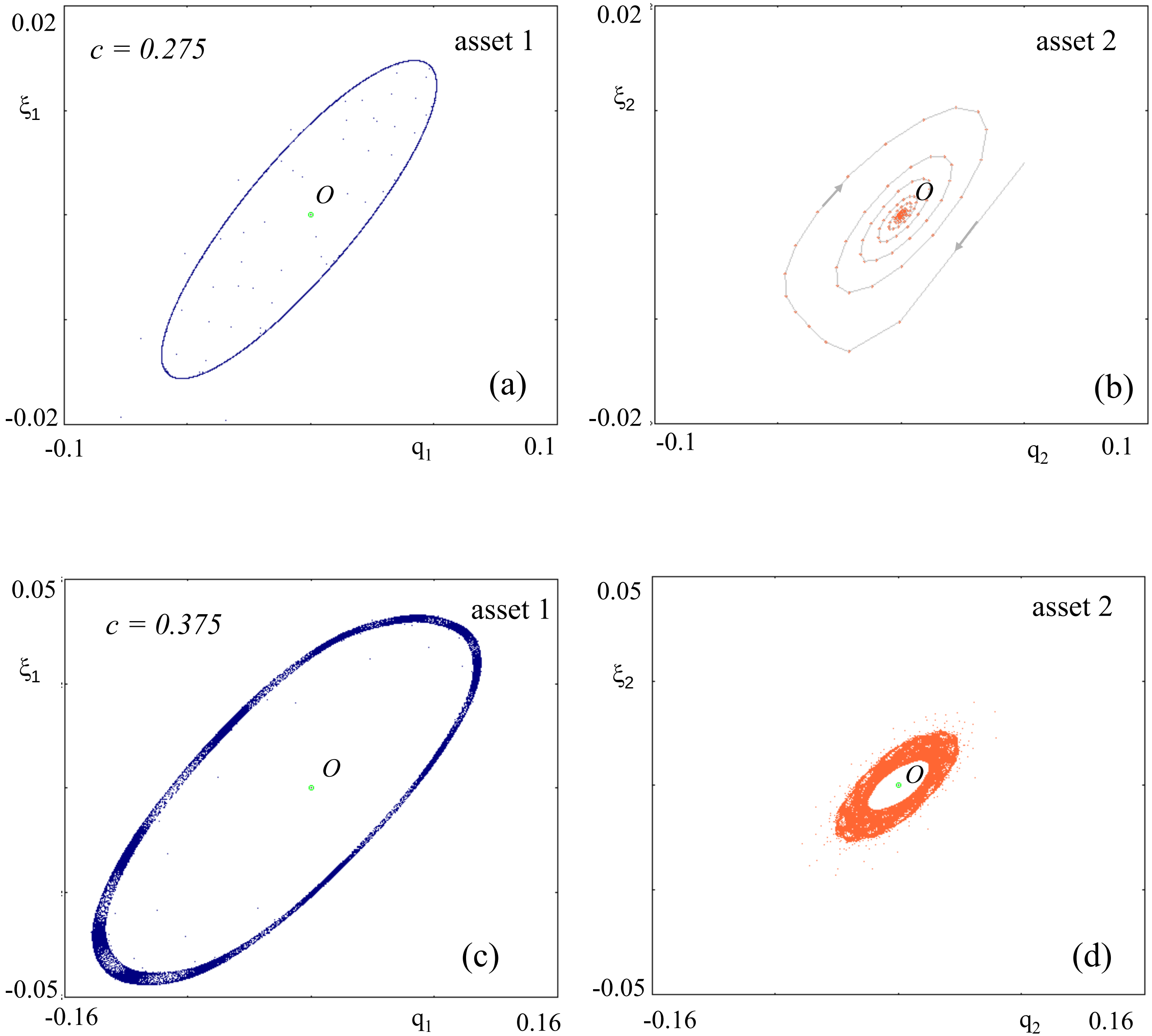
$\theta_2 := 1 / (\alpha^c \sigma_2^2) = 4$

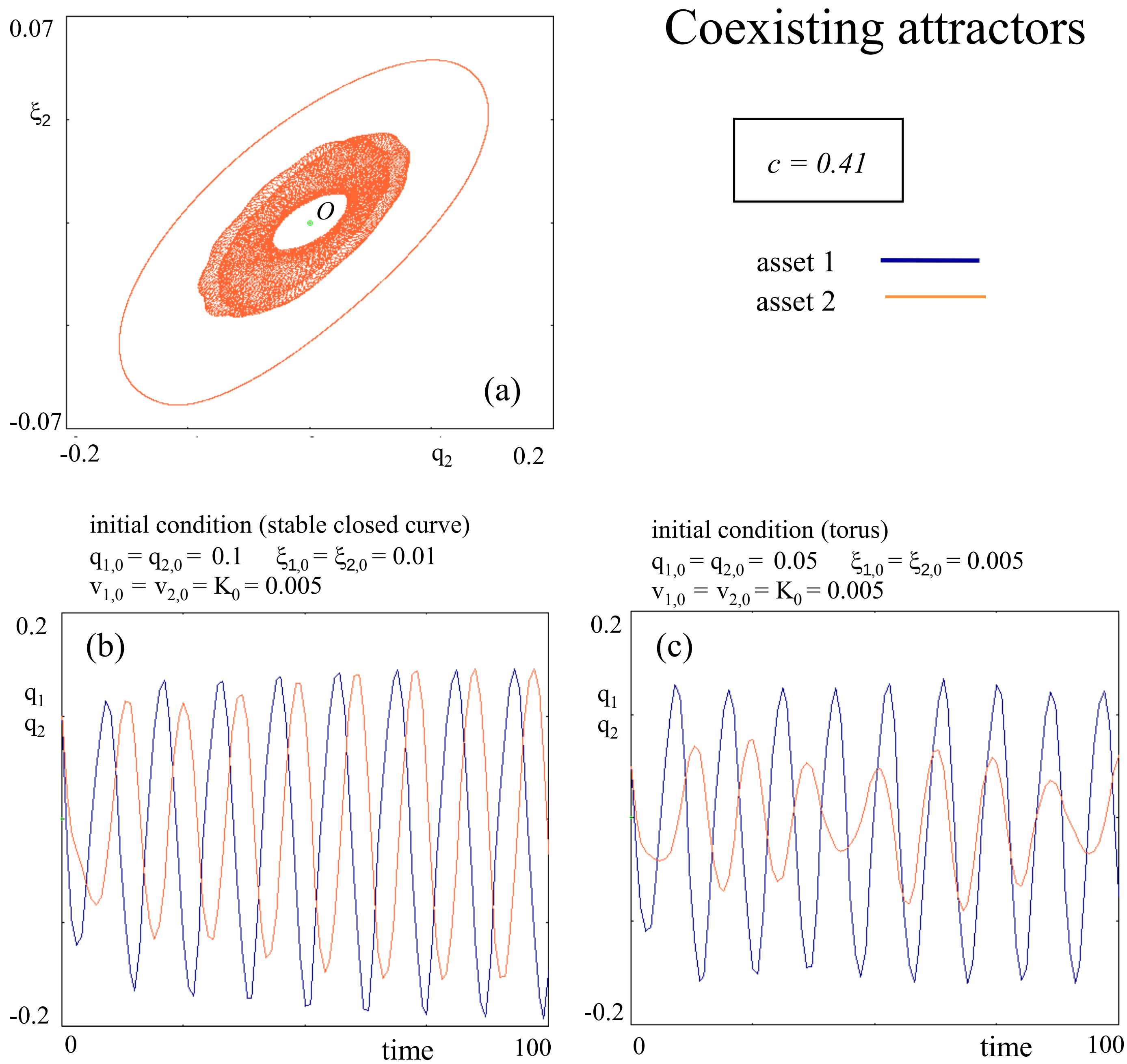
$\beta_1 = \beta_2 = 0.6$





Effect of the ‘secondary’ bifurcation





Coexisting attractors
c = 0.41
asset 1
asset 2
0.07
ξ2
-0.07
-0.2
q2
0.2
O
(a)
initial condition (stable closed curve)
q1,0 = q2,0 = 0.1 ξ1,0 = ξ2,0 = 0.01
v1,0 = v2,0 = K0 = 0.005
0.2
(b)
q1
q2
-0.2
0
time
100
initial condition (torus)
q1,0 = q2,0 = 0.05 ξ1,0 = ξ2,0 = 0.005
v1,0 = v2,0 = K0 = 0.005
0.2
(c)
q1
q2
-0.2
0
time
100