Applications to Economics:
Modeling interacting markets with
higher-dimensional discrete dynamical systems

Interdependent 'cobweb' markets

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1 Introduction

The basic linear 'cobweb' model

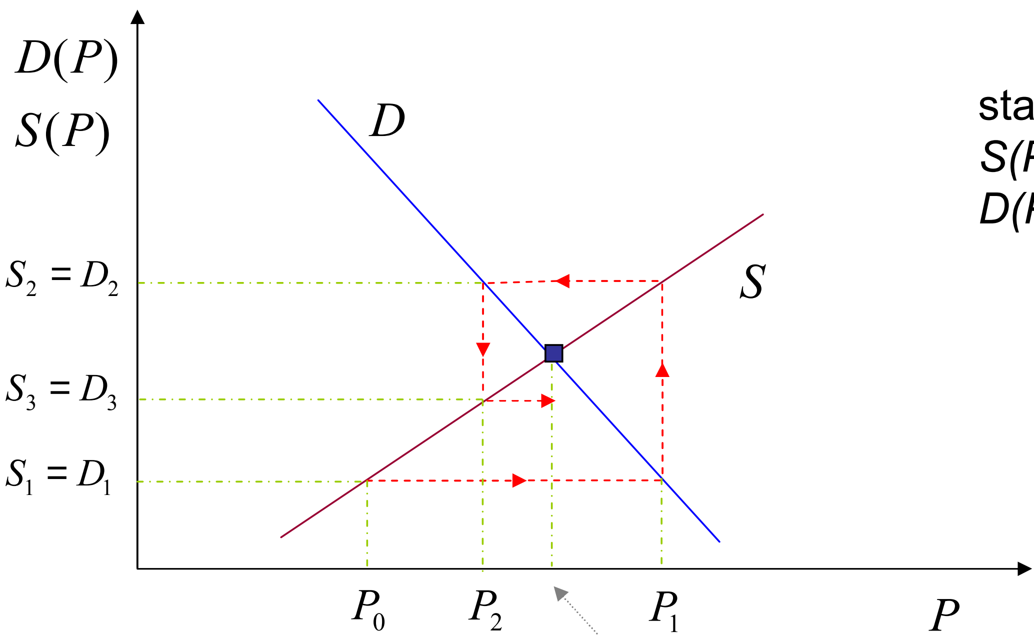
- price dynamics in a market of a non-storable good with a production lag
- suppliers must form price expectations and undertake output decisions one time period ahead (e.g. agricultural markets)
- *linear* demand and supply curves, *naïve* expectations ($P_t^e = P_{t-1}$)
- explosive, constant, or damped oscillations according as to the slope of the supply schedule is larger than, equal to, or smaller than the slope of the demand schedule (in modulus)

Cobweb model

$$D_t = \frac{a - P_t}{b} \quad \text{demand} \qquad S_t = \frac{P_{t-1} - c}{d} \quad \text{supply}$$

$$D_t = S_t \quad \text{market clearing}$$

$$\frac{b}{d} < 1 \qquad \text{(stable case)}$$



$$P^* = \frac{ad + bc}{b + d}$$

stable market if
S(P) relatively flat (large *d*) or
D(P) relatively sloped (small *b*)

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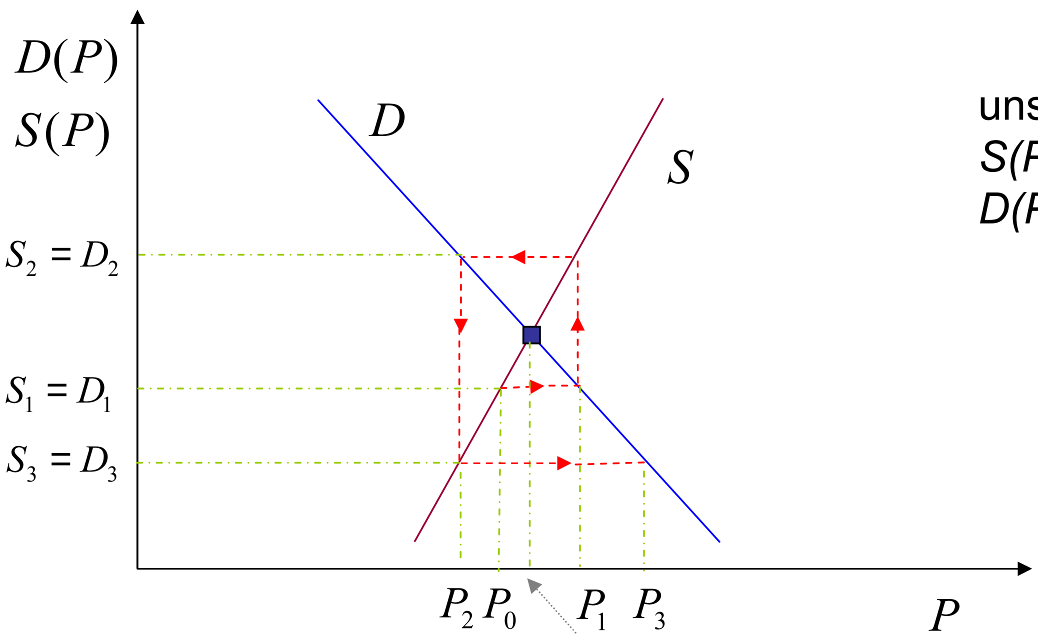

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S(P) relatively sloped (small *d*) or
D(P) relatively flat (large *b*)

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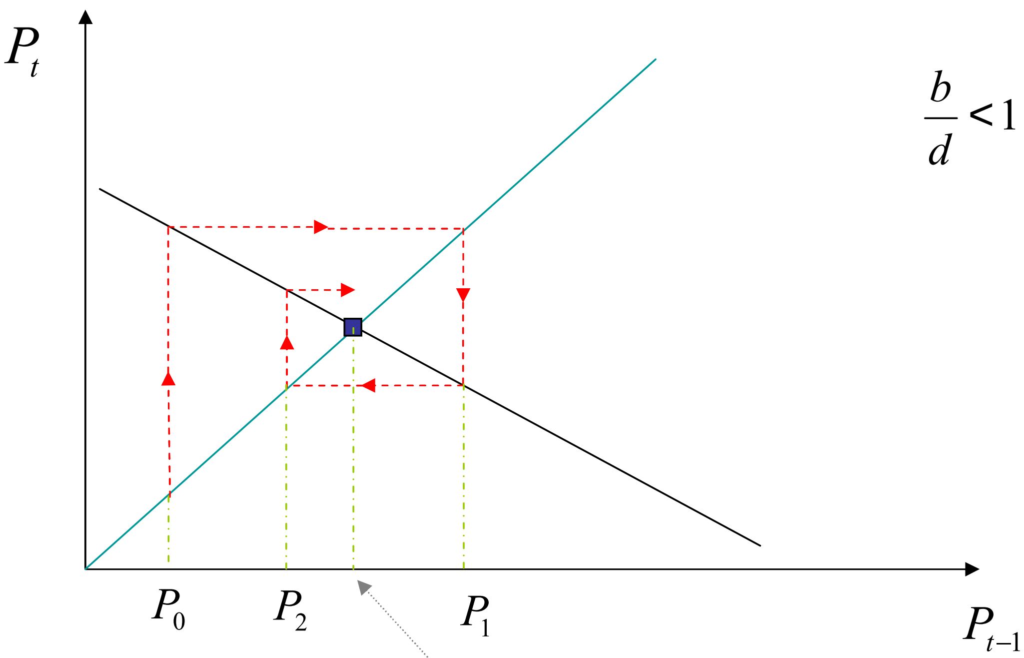
$$D_t = S_t \qquad \text{market clearing}$$

The 1-D map $P_t = a - \frac{b}{d}(P_{t-1} - c)$



$\frac{b}{d} < 1$ (stable case)

$$P^* = \frac{ad + bc}{b + d}$$

$D_t = \dfrac{a - P_t}{b}$ demand

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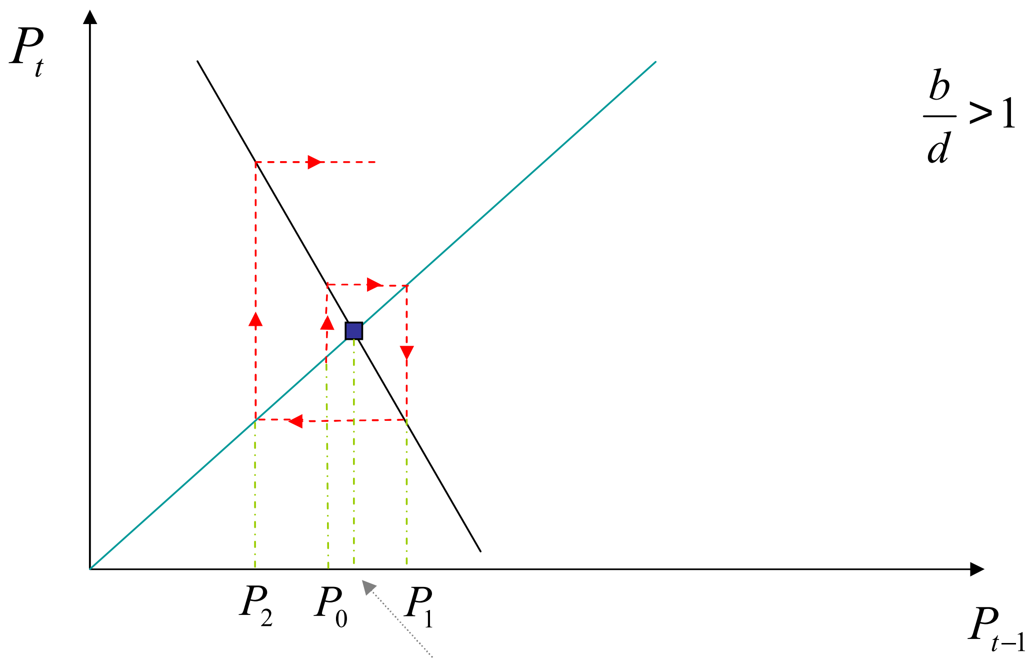


$\dfrac{b}{d} > 1$ (unstable case)

$$P^* = \frac{ad + bc}{b + d}$$

Nonlinear extensions / cyclical and complex price dynamics

Single market

- Chiarella (1988), Puu (1991), Day (1994), Hommes (1994, 1998): nonlinearities in demand and supply curves, *adaptive* expectations schemes (e.g. $P_t^e = (1-\alpha)P_{t-1}^e + \alpha P_{t-1}$)

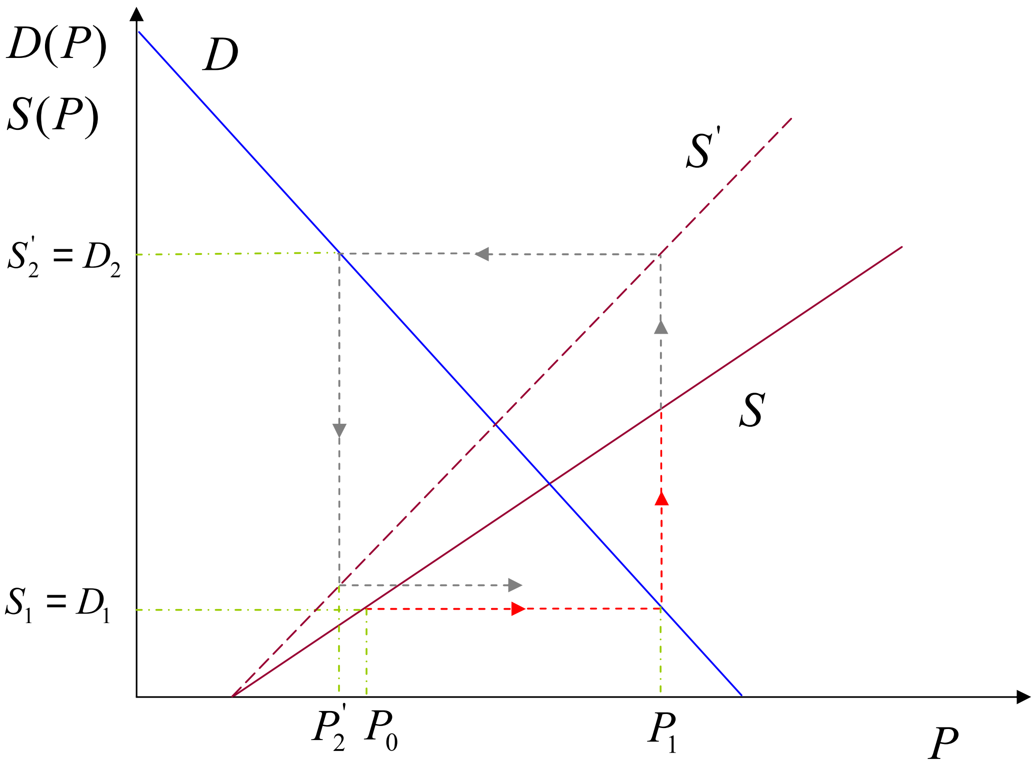
- Brock and Hommes (1997), Goeree and Hommes (2000), Branch (2002), Chiarella and He (2003): linear demand and supply, producers switching among *free naïve* or *costly rational* expectation formation rules

- Boussard (1996), Chiarella et al. (2006): boundedly rational heterogeneous producers, endogenously varying second-moment beliefs, risk aversion

Multiple markets

- Hommes and van Eekelen (1996), Currie and Kubin (1995): interdependent cobweb economies, linked from the side of demand (*complements* or *substitutes* goods).

Effect of entry / exit of additional (identical) producers



$D(P_t) = \frac{a - P_t}{b}$ demand curve $\qquad S(P_{t-1}) = \frac{P_{t-1} - c}{d}$ supply curve (initial number of producers)

$S'(P_{t-1}) = \frac{P_{t-1} - c}{d'}, \quad d' = 0.8d$ supply curve (25% increase in the number of producers)

This model: supply-side nonlinear interactions

- Two markets, X and Z
- N producers, W_X and $W_Z = 1 - W_X$: their proportions in each market
- Supply of a single producer, e.g. in market X: $S_{X,t} = (P_{X,t-1} - c_X)/d_X$
- Supply curve, e.g. in market X becomes

$$S^a_{X,t} = NW_X S_{X,t} = NW_X \frac{P_{X,t-1} - c_X}{d_X}$$

- For fixed N, changes in the distribution of producers across markets result in changes of the slope of S^a_X
- Such changes are produced *endogenously* in this model (which becomes a *non-linear* model).

Interacting ‘cobweb’ markets

- When farmers reduce their production of rye because they expect a lower price, they are likely to increase their production in another market, say, the wheat market. In our framework producers may select between one of two markets
- The choice depends on how profitable the two markets have been in the recent past and is updated over time
- As a consequence, the demand curves remain unchanged, while the position of the supply curves may change over time *endogenously*, because the number of producers in each market is time-varying (depending on realized profits). This introduces nonlinearity into the model
- Impact on the *local stability properties* of the long-run equilibrium prices and on the *global dynamics*

2 The model

- Two markets (goods) X and Z, fixed number of producers (N), $W_{X,t}$, $W_{Z,t} = 1 - W_{X,t}$, their proportions in markets X and Z (in period t)

- Linear demand curves (a_X, a_Z, b_X, $b_Z > 0$)

$$D_{X,t} = (a_X - P_{X,t})/b_X; \quad D_{Z,t} = (a_Z - P_{Z,t})/b_Z$$

- Linear supply of a single producer under naïve expectations, $E_{t-1}[P_t] = P_{t-1}$ (c_X, $c_Z \geq 0$, d_X, $d_Z > 0$)

$$S_{X,t} = (P_{X,t-1} - c_X)/d_X; \quad S_{Z,t} = (P_{Z,t-1} - c_Z)/d_Z$$

- An individual producer either supplies quantity $S_{X,t}$ of good X, or $S_{Z,t}$ of good Z; $NW_{X,t}\ S_{X,t}$, $NW_{Z,t}\ S_{Z,t}$, total supply in the two markets

- Market clearing occurs in every period implying that $D_{X,t} = NW_{X,t}S_{X,t}$, $D_{Z,t} = NW_{Z,t}S_{Z,t}$

- Market clearing conditions yield the laws of motion of the two prices

$$P_{X,t} = a_X - \frac{b_X}{d_X} NW_{X,t}(P_{X,t-1} - c_X), \quad P_{Z,t} = a_Z - \frac{b_Z}{d_Z} NW_{Z,t}(P_{Z,t-1} - c_Z)$$

Remarks

- Formally, time-varying quantities $b_X NW_{X,t}$, $b_Z NW_{Z,t}$, replace 'demand' parameter b used above to illustrate the 'textbook' cobweb.
- Constant proportions $W_{X,t} = \overline{W}_X$, $W_{Z,t} = \overline{W}_Z$ result in two independent first-order linear difference equations. The unique fixed point, e.g. of market X is

$$\overline{P}_X = \frac{a_X d_X + N\overline{W}_X b_X c_X}{d_X + N\overline{W}_X b_X}$$

  and it is Locally Asymptotically Stable (LAS) iff $(N\overline{W}_X b_X)/d_X < 1$

- Time varying fractions, dependent on past relative profitability, result in interdependent markets

Time-varying proportions

- *Bounded rational* producers tend to select the market with the higher realized profit (in the last period).

- $W_{X,t}$ and $W_{Z,t}$ determined via a *discrete-choice* model (see e.g. Brock and Hommes 1997, 1998)

$$W_{X,t} = \frac{\exp(f\ \pi_{X,t-1})}{\exp(f\ \pi_{X,t-1}) + \exp(f\ \pi_{Z,t-1})}; \quad W_{Z,t} = 1 - W_{X,t}$$

- $\pi_{X,t-1}$, $\pi_{Z,t-1}$ most recent realized profits, under quadratic cost functions of the type $C(S_t) = cS_t + eS_t^2$

  $$\pi_{X,t-1} = (P_{X,t-1} - c_X)S_{X,t-1} - e_X S_{X,t-1}^2; \ \ \pi_{Z,t-1} = (P_{Z,t-1} - c_Z)S_{Z,t-1} - e_Z S_{Z,t-1}^2$$

  $f \geq 0$: *intensity of choice* (sensitivity of the mass of producers to selecting the most profitable market)

Cost parameters, 'risk perception' and supply curves

- Producers' expectation of the profit in period t

$$\pi_t^e := P_t^e S_t - C(S_t) = (P_t^e - c)S_t - eS_t^2$$

- The output choice of *profit-maximizing* producers

$$S_t = \frac{P_t^e - c}{2e} = \arg\max \pi_t^e$$

- The output choice of *risk-averse* producers, maximizing *risk-adjusted* expected profit (γ constant absolute risk aversion parameter, $\sigma_{P,t}^2$ price variance)

$$S_t = \frac{P_t^e - c}{2e + \gamma\sigma_{P,t}^2} = \arg\max \left\{ \pi_t^e - \frac{\gamma}{2} S_t^2 \sigma_{P,t}^2 \right\}$$

- In general (under naïve expectations and constant variance σ_P^2)

$$S_t = (P_{t-1} - c)/d$$

where $d := 2e + \gamma\sigma_P^2 > 2e$ ($d = 2e$ for *risk-neutral* producers).

Summary of the general model

Market-clearing prices

$$\begin{aligned} P_{X,t} &= a_X - \frac{b_X}{d_X} N W_{X,t} (P_{X,t-1} - c_X) \\ P_{Z,t} &= a_Z - \frac{b_Z}{d_Z} N W_{Z,t} (P_{Z,t-1} - c_Z) \end{aligned}$$

Proportions of producers in markets X and Z

$$W_{X,t} = \frac{\exp(f\ \pi_{X,t-1})}{\exp(f\ \pi_{X,t-1}) + \exp(f\ \pi_{Z,t-1})}; \quad W_{Z,t} = 1 - W_{X,t}$$

Realized profits

$$\pi_{X,t-1} = (P_{X,t-1} - c_X) S_{X,t-1} - e_X S^2_{X,t-1}; \;\; \pi_{Z,t-1} = (P_{Z,t-1} - c_Z) S_{Z,t-1} - e_Z S^2_{Z,t-1}$$

Quantities supplied

$$S_{X,t-1} = (P_{X,t-2} - c_X)/d_X; \quad S_{Z,t-1} = (P_{Z,t-2} - c_Z)/d_Z$$

A system of two nonlinear second-order difference equations

Dynamical system

Define

$$\Omega_t := W_{X,t} - W_{Z,t} = \tanh\left[\frac{f}{2}(\pi_{X,t-1} - \ \pi_{Z,t-1})\right]$$

with $-1 < \Omega_t < 1$, where $\Omega_t \to 1 \Leftrightarrow W_{X,t} \to 1$ and $\Omega_t \to -1 \Leftrightarrow W_{Z,t} \to 1$.

Note that

$$W_{X,t} = (1 + \Omega_t)/2, \quad W_{Z,t} = (1 - \Omega_t)/2.$$

We rewrite the model as a 4-D dynamical system in P_X, P_Z, S_X, S_Z:

$$P_{X,t} = \frac{a_X d_X - g_X(1 + \Omega_t)(P_{X,t-1} - c_X)}{d_X}, \quad P_{Z,t} = \frac{a_Z d_Z - g_Z(1 - \Omega_t)(P_{Z,t-1} - c_Z)}{d_Z}$$

$$S_{X,t} = \frac{P_{X,t-1} - c_X}{d_X}, \quad S_{Z,t} = \frac{P_{Z,t-1} - c_Z}{d_Z}$$

where $g_X := (Nb_X)/2$, $g_Z := (Nb_Z)/2$, and $\Omega_t := \Omega(P_{X,t-1}, P_{Z,t-1}, S_{X,t-1}, S_{Z,t-1})$, namely:

$$\Omega_t = \tanh\left\{\frac{f}{2}\left[(P_{X,t-1} - c_X)S_{X,t-1} - e_X S_{X,t-1}^2 - (P_{Z,t-1} - c_Z)S_{Z,t-1} + e_Z S_{Z,t-1}^2\right]\right\}$$

3 Steady state

- Steady-state (s.s.) difference of proportions $\overline{\Omega}$ is implicitly defined by

$$\overline{\Omega} = \tanh\left\{\frac{f}{2}\left[\frac{(d_X - e_X)(a_X - c_X)^2}{\left[d_X + g_X(1+\overline{\Omega})\right]^2} - \frac{(d_Z - e_Z)(a_Z - c_Z)^2}{\left[d_Z + g_Z(1-\overline{\Omega})\right]^2}\right]\right\}$$

where s.s. prices, quantities, profits are given, respectively, by

$$\overline{P}_X = \frac{a_X d_X + g_X(1+\overline{\Omega})c_X}{d_X + g_X(1+\overline{\Omega})}, \qquad \overline{P}_Z = \frac{a_Z d_Z + g_Z(1-\overline{\Omega})c_Z}{d_Z + g_Z(1-\overline{\Omega})}$$

$$\overline{S}_X = \frac{\overline{P}_X - c_X}{d_X} = \frac{a_X - c_X}{d_X + g_X(1+\overline{\Omega})}, \qquad \overline{S}_Z = \frac{\overline{P}_Z - c_Z}{d_Z} = \frac{a_Z - c_Z}{d_Z + g_Z(1-\overline{\Omega})}$$

$$\overline{\pi}_X = (d_X - e_X)\overline{S}_X^2 = \frac{(d_X - e_X)(a_X - c_X)^2}{\left[d_X + g_X(1+\overline{\Omega})\right]^2}, \quad \overline{\pi}_Z = (d_Z - e_Z)\overline{S}_Z^2 = \frac{(d_Z - e_Z)(a_Z - c_Z)^2}{\left[d_Z + g_Z(1-\overline{\Omega})\right]^2}$$

- Uniqueness of the s.s. can be easily proven (though s.s. cannot be computed explicitly), s.s. coordinates depend on f

Steady state: two 'benchmark' scenarios

(i) *fixed-proportions* model with the *actual* steady state proportions of the complete model:

$$P_{X,t} = a_X - \frac{b_X}{d_X} N\overline{W}_X (P_{X,t-1} - c_X), \quad P_{Z,t} = a_Z - \frac{b_Z}{d_Z} N\overline{W}_Z (P_{Z,t-1} - c_Z)$$

i.e.

$$P_{X,t} = \frac{a_X d_X - g_X (1 + \overline{\Omega})(P_{X,t-1} - c_X)}{d_X}, \quad P_{Z,t} = \frac{a_Z d_Z - g_Z (1 - \overline{\Omega})(P_{Z,t-1} - c_Z)}{d_Z}$$

The s.s. of the two isolated markets are LAS, respectively, iff

$$\frac{g_X}{d_X}(1 + \overline{\Omega}) < 1, \quad \frac{g_Z}{d_Z}(1 - \overline{\Omega}) < 1$$

(ii) *fixed-proportions* model corresponding to the case $f = 0$ (and therefore $\overline{\Omega} = 0$), s.s. is LAS iff

$$\frac{g_X}{d_X} < 1, \quad \frac{g_Z}{d_Z} < 1$$

Remarkable steady state properties

- For the case of interacting markets ($f > 0$), $\overline{\Omega}$ is positive (negative) iff the s.s. profit differential of the two markets, *considered in isolation*, is positive (negative), i.e. iff $\overline{\pi}_X^0 - \overline{\pi}_Z^0 > 0$ ($\overline{\pi}_X^0 - \overline{\pi}_Z^0 < 0$), where

$$\overline{\pi}_X^0 := \frac{(d_X - e_X)(a_X - c_X)^2}{(d_X + g_X)^2}, \quad \overline{\pi}_Z^0 := \frac{(d_Z - e_Z)(a_Z - c_Z)^2}{(d_Z + g_Z)^2}$$

  The market that attracts the majority of producers, *at the steady state solution*, is the one that would be more profitable in the absence of interaction

- If $\overline{\pi}_X^0 - \overline{\pi}_Z^0 > 0$ ($\overline{\pi}_X^0 - \overline{\pi}_Z^0 < 0$), $\overline{\Omega}$ is a strictly increasing (decreasing) function of the switching intensity f, for f ranging from zero to infinity.

Remarkable steady state properties (continued)
Consider the difference of proportions

$$\begin{aligned}\Omega_t &= \Omega(P_{X,t-1}, P_{Z,t-1}, S_{X,t-1}, S_{Z,t-1}) = \tanh\left(\frac{f}{2}\left(\pi_{X,t-1} - \pi_{Z,t-1}\right)\right) \\ &= \tanh\left\{\frac{f}{2}\left[P_{X,t-1}S_{X,t-1} - C_X(S_{X,t-1}) - P_{Z,t-1}S_{Z,t-1} + C_Z(S_{Z,t-1})\right]\right\}\end{aligned}$$

The partial derivative with respect, e.g. to S_X (and similarly for S_Z)

$$\frac{\partial \Omega}{\partial S_X} = \left[1 - \tanh^2\left(\frac{f}{2}\left(\pi_X - \pi_Z\right)\right)\right]\left[P_X - C'_X(S_X)\right]$$

Two useful *steady state* properties:

- $1 - \tanh^2\left(\frac{f}{2}\left(\overline{\pi}_X - \overline{\pi}_Z\right)\right) = 1 - \overline{\Omega}^2$ (from the definition of Ω_t)
- $\overline{P}_X - C'_X(\overline{S}_X) = (d_X - 2e_X)\overline{S}_X$ (from the f.o.c. of risk-adjusted profit maximization problem), $\overline{P}_X - C'_X(\overline{S}_X) = 0$ for risk-neutral producers

4 Two particular cases

- Further analytical results are hard to obtain and/or interpret in the general case. We then consider two particular cases

- **Case A**: profit-maximizing (*risk neutral*) producers: $e_X = \dfrac{d_X}{2}$, $e_Z = \dfrac{d_Z}{2}$.

- **Case B**: *symmetric* markets, i.e. identical supply and demand parameters in the two markets: $a_X = a_Z := a$, $g_X = g_Z := g$, $c_X = c_Z := c$, $d_X = d_Z := d$, which implies $\overline{\Omega} = 0$.

- Focus on conditions for the s.s. to be LAS, compared with the case of absence of interaction

- Focus on the range of possible long-run out-of-equilibrium dynamics

4.1 Case A: profit-maximizing producers

Model summary

We rewrite the model as a 4-D dynamical system in P_X, P_Z, S_X, S_Z:

$$\begin{aligned} P_{X,t} &= F_X(P_{X,t-1}, P_{Z,t-1}, S_{X,t-1}, S_{Z,t-1}) := \frac{a_X d_X - g_X(1+\Omega_t)(P_{X,t-1} - c_X)}{d_X} \\ P_{Z,t} &= F_Z(P_{X,t-1}, P_{Z,t-1}, S_{X,t-1}, S_{Z,t-1}) := \frac{a_Z d_Z - g_Z(1-\Omega_t)(P_{Z,t-1} - c_Z)}{d_Z} \end{aligned}$$

$$\begin{aligned} S_{X,t} &= G_X(P_{X,t-1}) := \frac{P_{X,t-1} - c_X}{d_X} \\ S_{Z,t} &= G_Z(P_{Z,t-1}) := \frac{P_{Z,t-1} - c_Z}{d_Z} \end{aligned}$$

where $\Omega_t := \Omega(P_{X,t-1}, P_{Z,t-1}, S_{X,t-1}, S_{Z,t-1})$, namely:

$$\Omega_t = \tanh\left\{\frac{f}{2}\left[(P_{X,t-1} - c_X)S_{X,t-1} - \frac{d_X}{2}S_{X,t-1}^2 - (P_{Z,t-1} - c_Z)S_{Z,t-1} + \frac{d_Z}{2}S_{Z,t-1}^2\right]\right\}$$

Summary of analytical results about stability

- The Jacobian at the s.s. (denote it by J) can be rewritten as a function of the steady state distribution of producers across markets, $\overline{\Omega}$

- J has a block triangular structure: $J = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}$, where $\mathbf{A}$, $\mathbf{C}$, are two-dimensional blocks while $\mathbf{0}$ denotes the two-dimensional null matrix.

- A null block occupies the upper right corner because F_X and F_Z depend on S_X and S_Z only via Ω_t, but $\dfrac{\partial \Omega}{\partial S_X}$ and $\dfrac{\partial \Omega}{\partial S_Z}$ include the factors $P_X - C'_X(S_X)$ and $P_Z - C'_Z(S_Z)$, respectively, which vanish at the steady state.

- Two eigenvalues, say λ_3, λ_4, are zero (thus smaller than one in modulus), while the remaining eigenvalues (λ_1, λ_2) are the ones of the $2 - D$ block $\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} -\frac{g_X}{d_X}\left[(1-\overline{\Omega}^2)\frac{f}{2}d_X\overline{S}_X^2 + (1+\overline{\Omega})\right] & (1-\overline{\Omega}^2)\frac{f}{2}g_X\overline{S}_X\overline{S}_Z \\ (1-\overline{\Omega}^2)\frac{f}{2}g_Z\overline{S}_X\overline{S}_Z & -\frac{g_Z}{d_Z}\left[(1-\overline{\Omega}^2)\frac{f}{2}d_Z\overline{S}_Z^2 + (1-\overline{\Omega})\right] \end{bmatrix}$$

- λ_1, λ_2 are the solutions of the characteristic equation

$$P(\lambda) = \lambda^2 - Tr(\mathbf{A})\lambda + Det(\mathbf{A}) = 0$$

- Necessary and sufficient condition for both eigenvalues of $\mathbf{A}$ to be smaller than one in modulus (LAS steady state) is given by (see e.g. Medio and Lines, 2001)

$$\begin{aligned} P(1) &= 1 - Tr(\mathbf{A}) + Det(\mathbf{A}) > 0 \\ P(-1) &= 1 + Tr(\mathbf{A}) + Det(\mathbf{A}) > 0 \\ P(0) &= Det(\mathbf{A}) < 1 \end{aligned}$$

  which can be rewritten as a function of the parameters (and of $\overline{\Omega}$).

- Findings: $[Tr(\mathbf{A})]^2 - 4Det(\mathbf{A}) > 0$, eigenvalues are real (*Neimark-Sacker* bifurcation not possible); moreover $Tr(\mathbf{A}) < 0$, $Det(\mathbf{A}) > 0$, *saddle-node* bifurcation not possible.

- Stability can be lost only via Flip-bifurcation, when condition $1 + Tr(\mathbf{A}) + Det(\mathbf{A}) > 0$ is violated (and this occurs for f large enough)

Markets in isolation vs. interconnected markets

- *Benchmark scenario (ii)*, $f = 0$ (and therefore $\overline{\Omega} = 0$), producers permanently splitting evenly across markets. Stability conditions become

$$g_X/d_X < 1, \quad g_Z/d_Z < 1$$

- *Complete model*, time-varying supply and interacting markets. A *necessary condition* for the steady state to be LAS is

$$\frac{g_X}{d_X}(1+\overline{\Omega}) + \frac{f}{2} g_X \overline{S}_X^2 (1-\overline{\Omega}^2) \leq 1, \quad \frac{g_Z}{d_Z}(1-\overline{\Omega}) + \frac{f}{2} g_Z \overline{S}_Z^2 (1-\overline{\Omega}^2) \leq 1$$

- Compare the *complete model* with *benchmark scenario (ii)*: stability of the (isolated) market with higher s.s. profit, is a necessary condition for the s.s. of the coupled system to be LAS

  Put differently, stability of the whole system of interacting markets requires stability of at least one of the two 'isolated' markets (the one with higher s.s. profit).

Markets in isolation vs. interconnected markets (continued)

A more intuitive and direct comparison is between the *complete model* and

- *Benchmark scenario (i)*: fixed-fraction model *corresponding to the actual s.s. proportions* of producers. Stable s.s. iff

$$\frac{g_X}{d_X}(1+\overline{\Omega}) < 1, \quad \frac{g_Z}{d_Z}(1-\overline{\Omega}) < 1$$

- *Complete model*, the *necessary condition* for the steady state stability

$$\frac{g_X}{d_X}(1+\overline{\Omega}) + \frac{f}{2} g_X \overline{S}_X^2 (1-\overline{\Omega}^2) \leq 1, \quad \frac{g_Z}{d_Z}(1-\overline{\Omega}) + \frac{f}{2} g_Z \overline{S}_Z^2 (1-\overline{\Omega}^2) \leq 1$$

- Compare the *complete model* with *benchmark scenario (i)*. Both markets considered in isolation must be stable in order the steady state of the full system to be LAS.

- More restrictive stability conditions, *destabilizing effect* of interactions

Markets in isolation vs. interconnected markets (continued)

- In the symmetric case, stability condition reduces to

$$\frac{g}{d}\left(1 + df\overline{S}^2\right) < 1$$

where $\overline{S} = (a - c)/(d + g)$ is the supply in equilibrium, or equivalently

$$f < f_{Flip} \equiv \frac{d - g}{gd\overline{S}^2}$$

where f_{Flip} is the Flip-bifurcation value for the intensity of choice

- If agents are allowed to select the most profitable market, this may have a "destabilizing" effect on the equilibrium prices. In particular, in the symmetric case the stability condition is more requiring than for independent symmetric markets ($g/d < 1$)

Numerical example

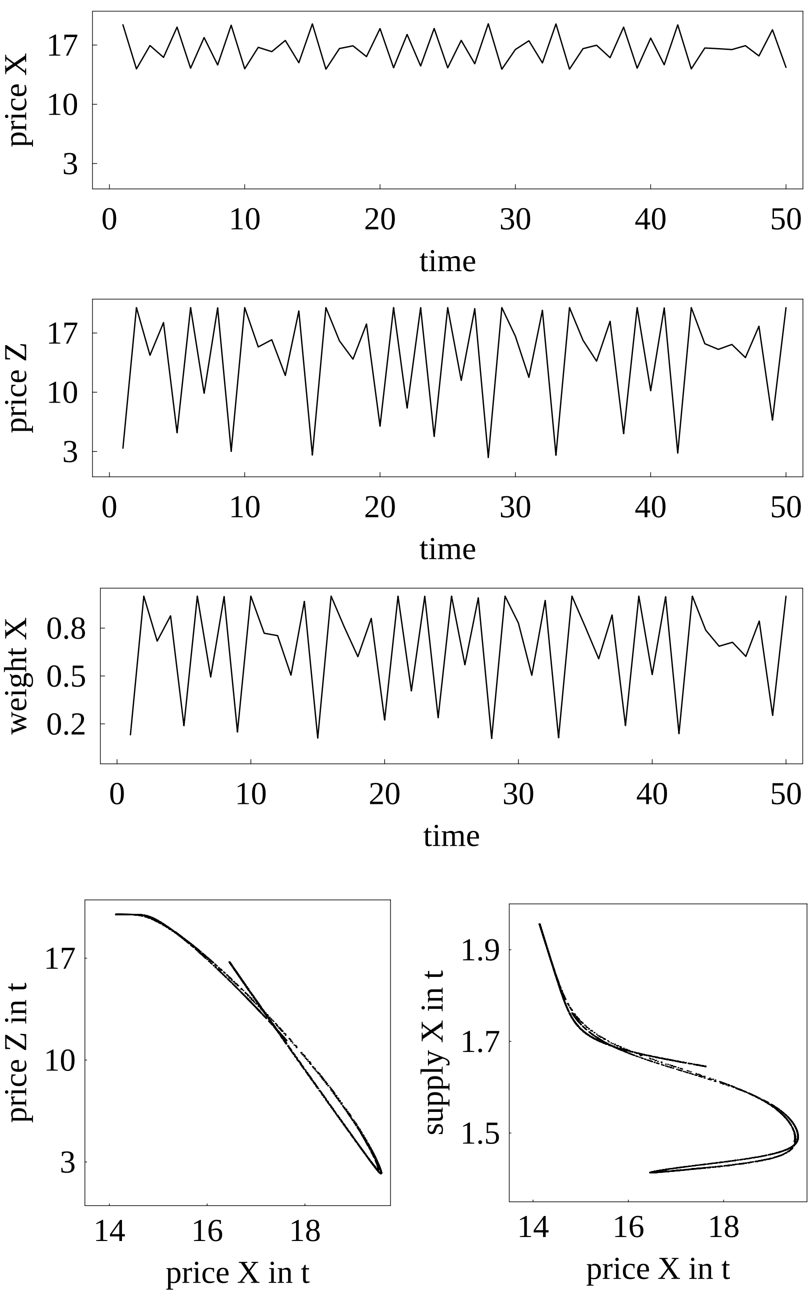
- Parameters: $a_X = a_Z = 20$, $c_X = c_Z = 0$, $d_X = d_Z = 10$, $g_X = 1.5$, $g_Z = 5$, $f = 0.375$.

- In case of no switching / no interactions ($f = 0 \Longrightarrow \overline{\Omega} = 0$), the steady-state profits of the independent markets would be $\overline{\pi}_X^0 \simeq 15.123$, $\overline{\pi}_Z^0 \simeq 8.889$.

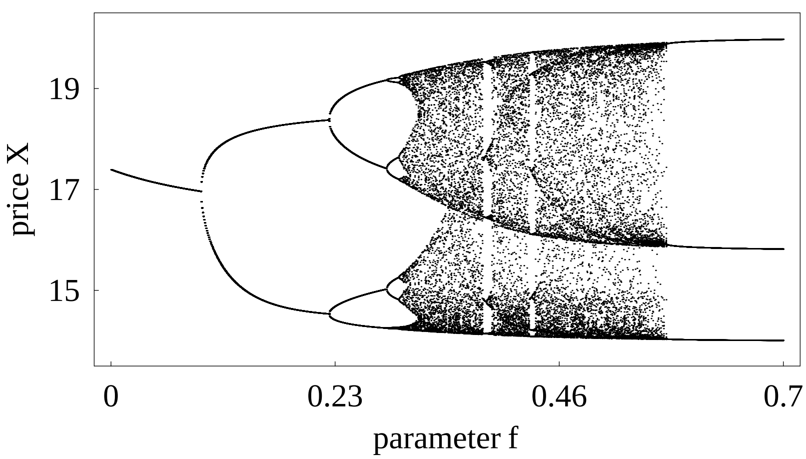
- This implies $\overline{\Omega} > 0$ (i.e. $\overline{W}_X > 50\%$) at the steady state of the complete model for any $f > 0$, and $\overline{\Omega}$ (as well as $\overline{W}_X$) increasing with f.

- Since $g_X/d_X = 0.15$, $g_Z/d_Z = 0.5$: in the case of no switching / no interactions ($f = 0 \Longrightarrow \overline{\Omega} = 0$), the steady states of the two independent markets *would both be globally asymptotically stable.*

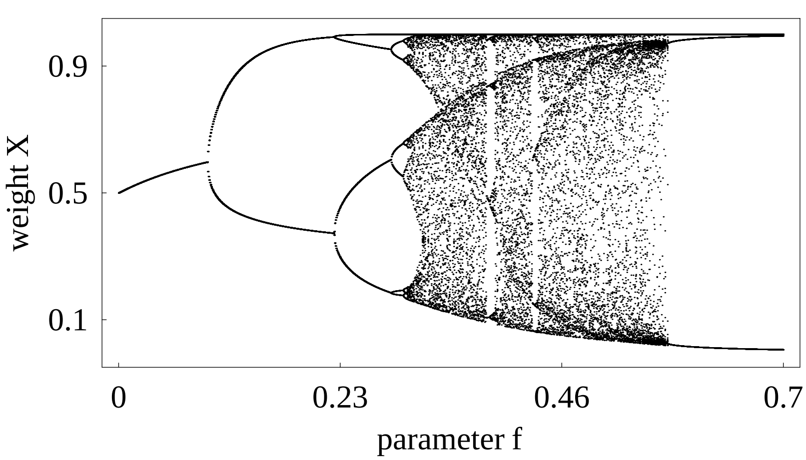
- However, the steady state of the system of interconnected markets is LAS only for $f < f_{Flip} \simeq 0.093311$, larger f brings about the familiar period-doubling bifurcation sequence.



price X
17
10
3
0
10
20
30
40
50
time
price Z
weight X
0.8
0.5
0.2
price Z in t
price X in t
14
16
18
supply X in t
1.9
1.7
1.5



price X
19
17
15
0
0.23
0.46
0.7
parameter f



weight X
0.9
0.5
0.1
0
0.23
0.46
0.7
parameter f

4.2 Case B: symmetric markets

Steady state. In the symmetric case: $a_X = a_Z = a$, $g_X = g_Z = g$, $c_X = c_Z = c$, $d_X = d_Z = d$, $e_X = e_Z = e$, the equilibrium is independent on f; moreover $\Omega = 0$ (producers split uniformly across markets) and

$$P_X = P_Z = \overline{P} = \frac{ad + cg}{d + g}$$

$$S_X = S_Z = \overline{S} = \frac{\overline{P} - c}{d} = \frac{a - c}{d + g}$$

Local stability. The characteristic polynomial of the Jacobian J can be factorized as

$$P(\lambda) = \lambda \left(\frac{g}{d} + \lambda\right) \left[\lambda^2 + \left(gf\ \overline{S}^2 + \frac{g}{d}\right)\lambda + \frac{gf(d - 2e)\overline{S}^2}{d}\right] =$$
$$\lambda \left(\frac{g}{d} + \lambda\right) P_1(\lambda)$$

- Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -g/d$, λ_3 and λ_4 are the roots of $P_1(\lambda)$.
- Condition for the steady state to be LAS is then given by

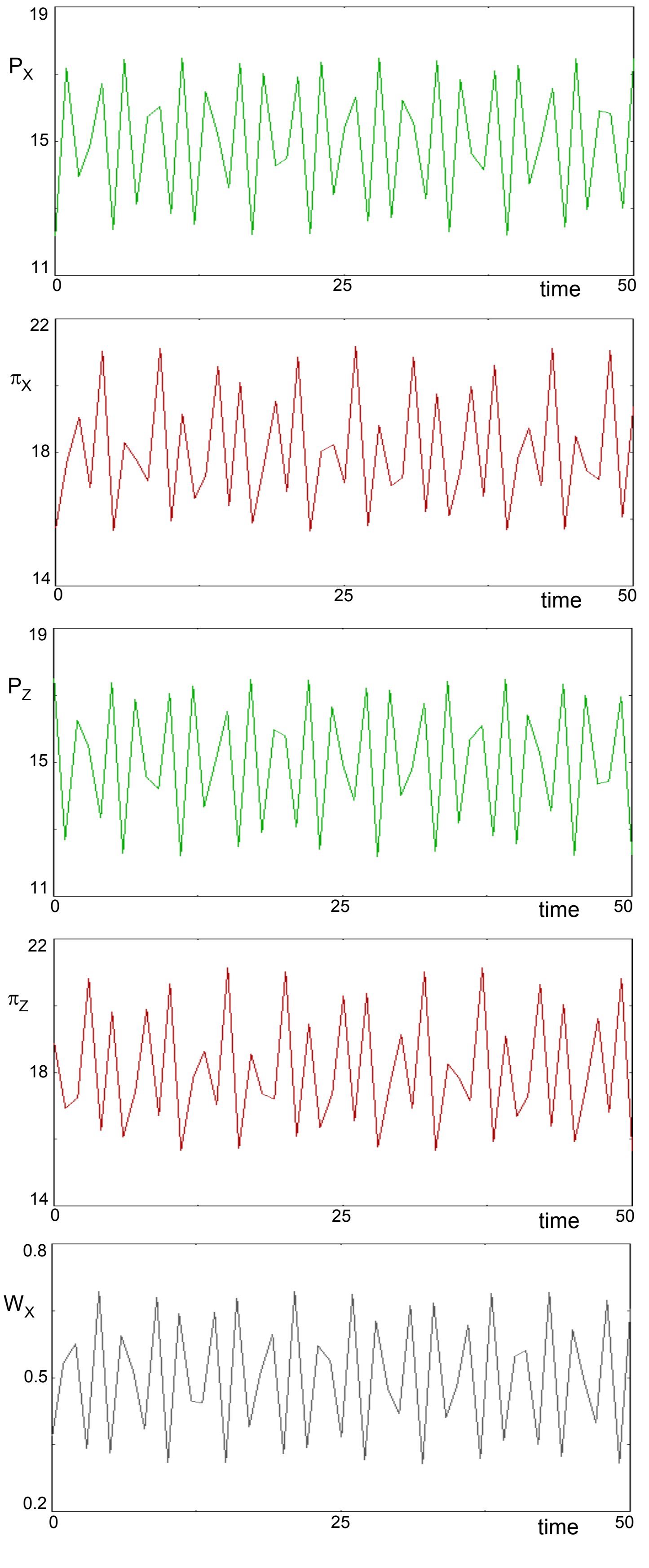
$$\frac{g}{d} < 1$$

$$P_1(1) > 0 \qquad \text{always true}$$

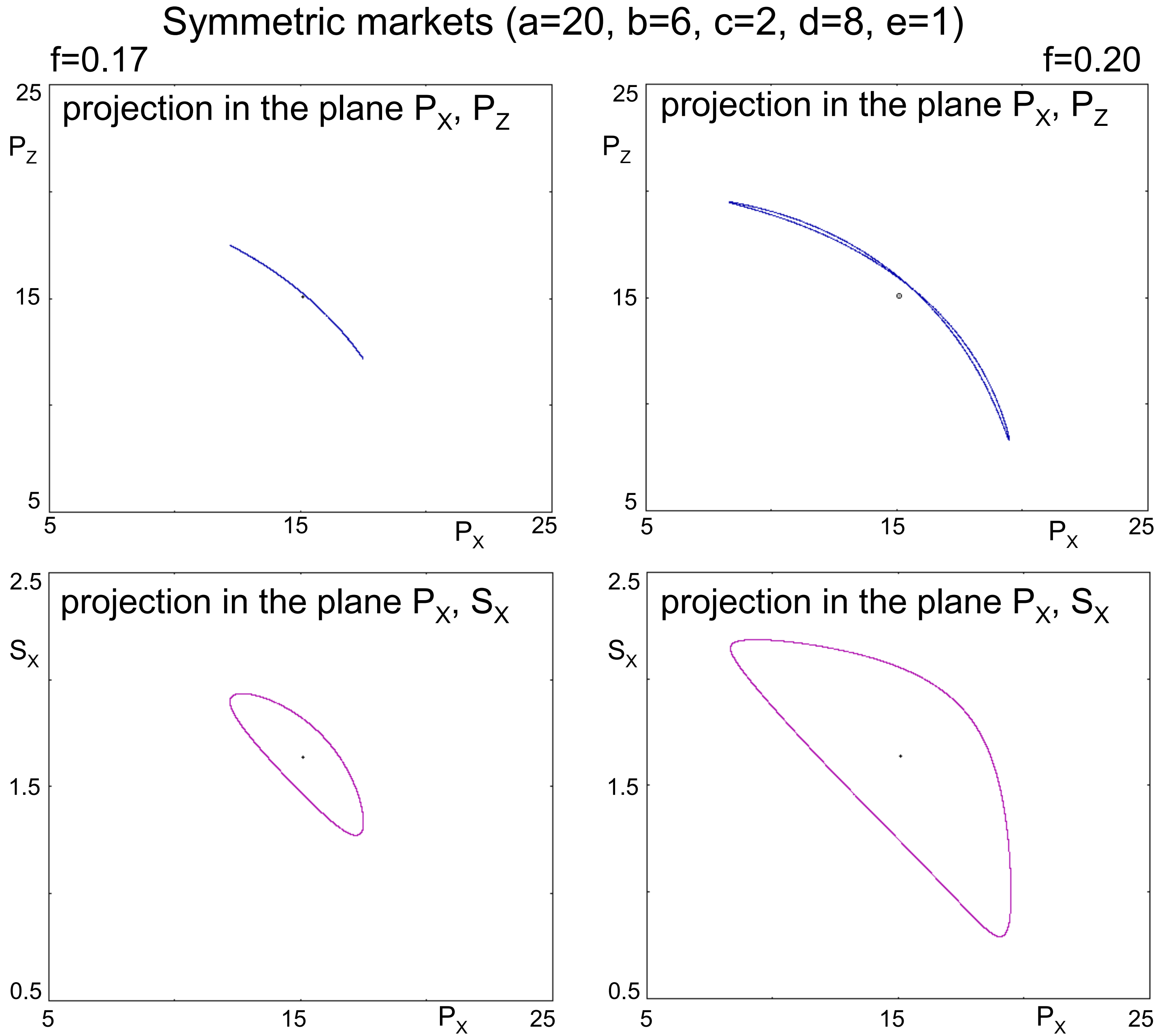
$$P_1(-1) > 0 \qquad \text{i.e.} \qquad \frac{g}{d} < u_F := \frac{1}{1 + 2ef\overline{S}^2}$$

$$P_1(0) < 1 \qquad \text{i.e.} \qquad \frac{g}{d} < u_{NS} := \frac{1}{f(d - 2e)\overline{S}^2}$$

- Stability condition is therefore: $\frac{g}{d} < \min[u_F, u_{NS}]$.
- For $f > 0$, more strict stability condition than the case of a single market
- Stability can be lost either via *Flip-bifurcation* or via *Neimark-Sacker bifurcation*
- Neimark-Sacker scenario is absent under the case of risk-neutral producers ($d \to 2e$)



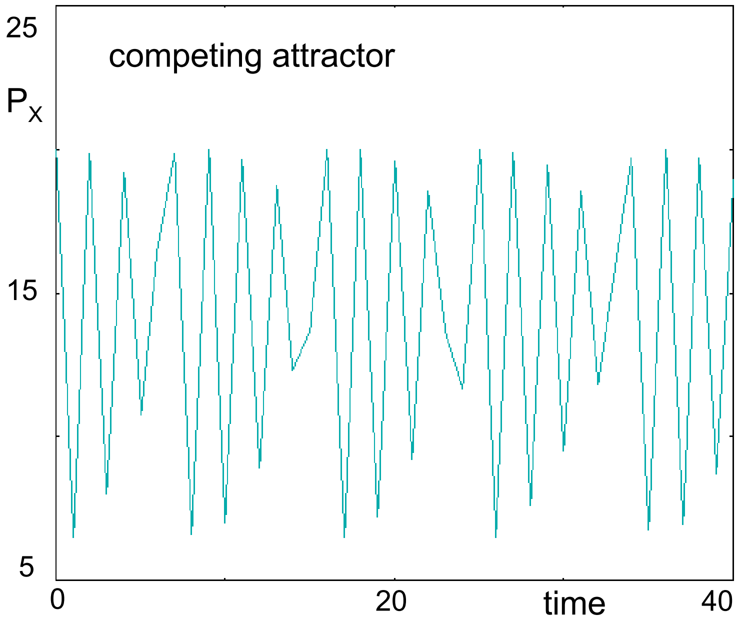
Symmetric markets
a=20, b=6, c=2, d=8,
e=1, f=0.17
g:=Nb/2=3 (with N=1)

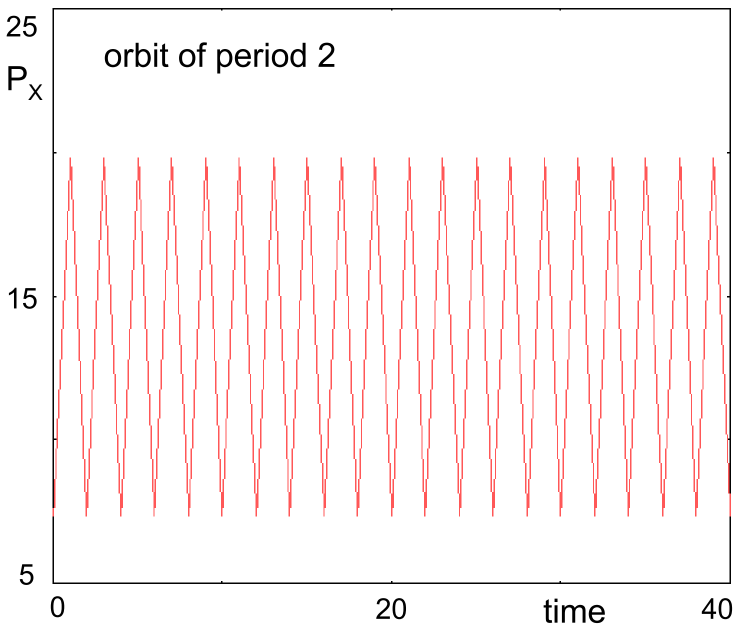


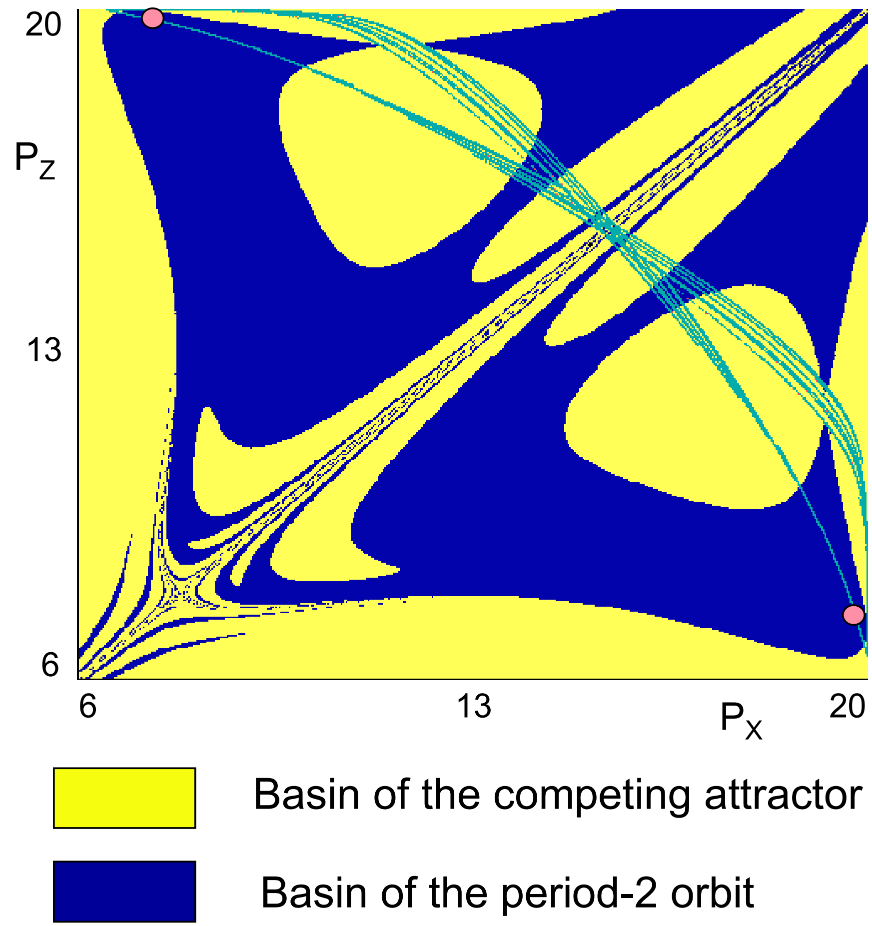
Symmetric markets (a=20, b=6, c=2, d=8, e=1)
f=0.17
f=0.20
projection in the plane P_X, P_Z
P_Z
25
15
5
P_X
5
15
25
projection in the plane P_X, P_Z
P_Z
25
15
5
P_X
5
15
25
projection in the plane P_X, S_X
S_X
2.5
1.5
0.5
P_X
5
15
25
projection in the plane P_X, S_X
S_X
2.5
1.5
0.5
P_X
5
15
25

Coexistence of attractors

f=0.64295

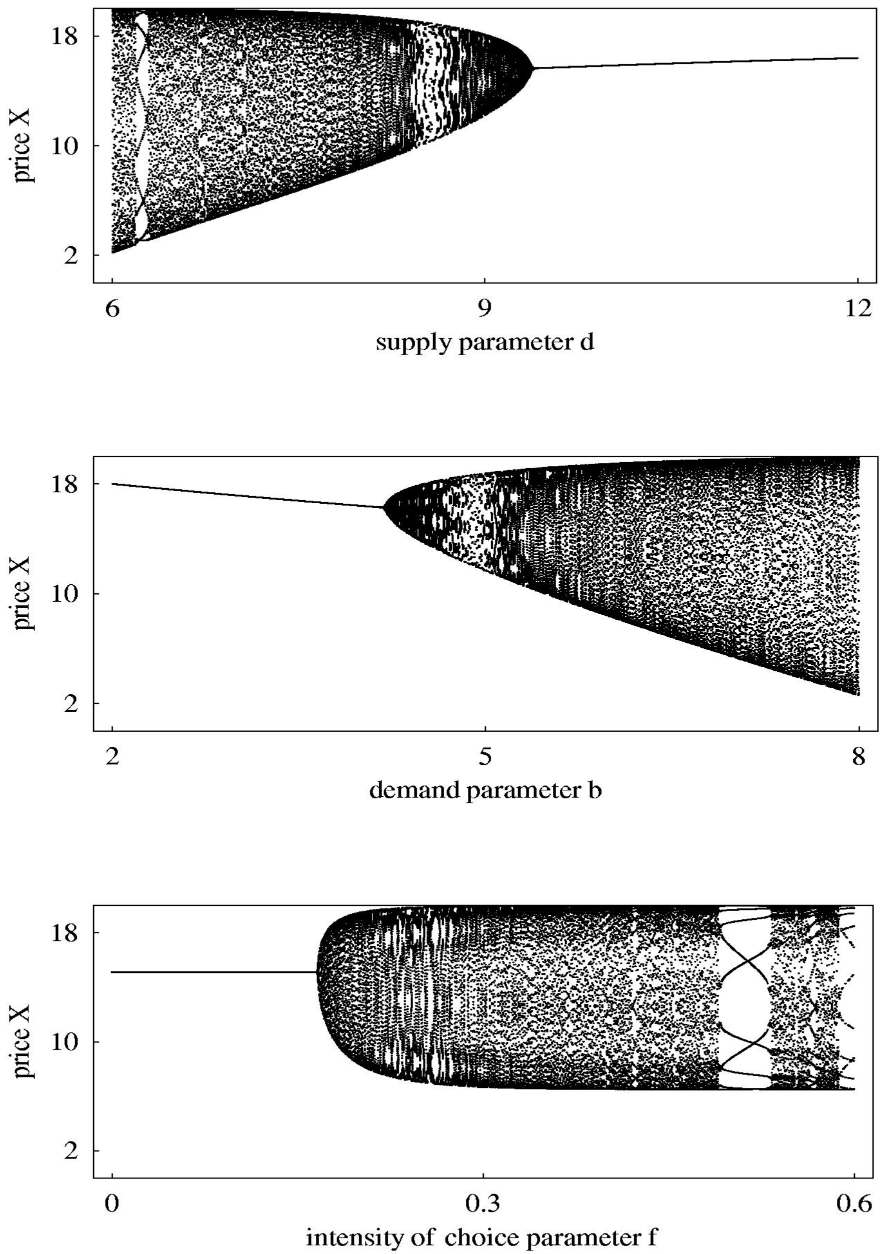






Basins computed along the section of equation:

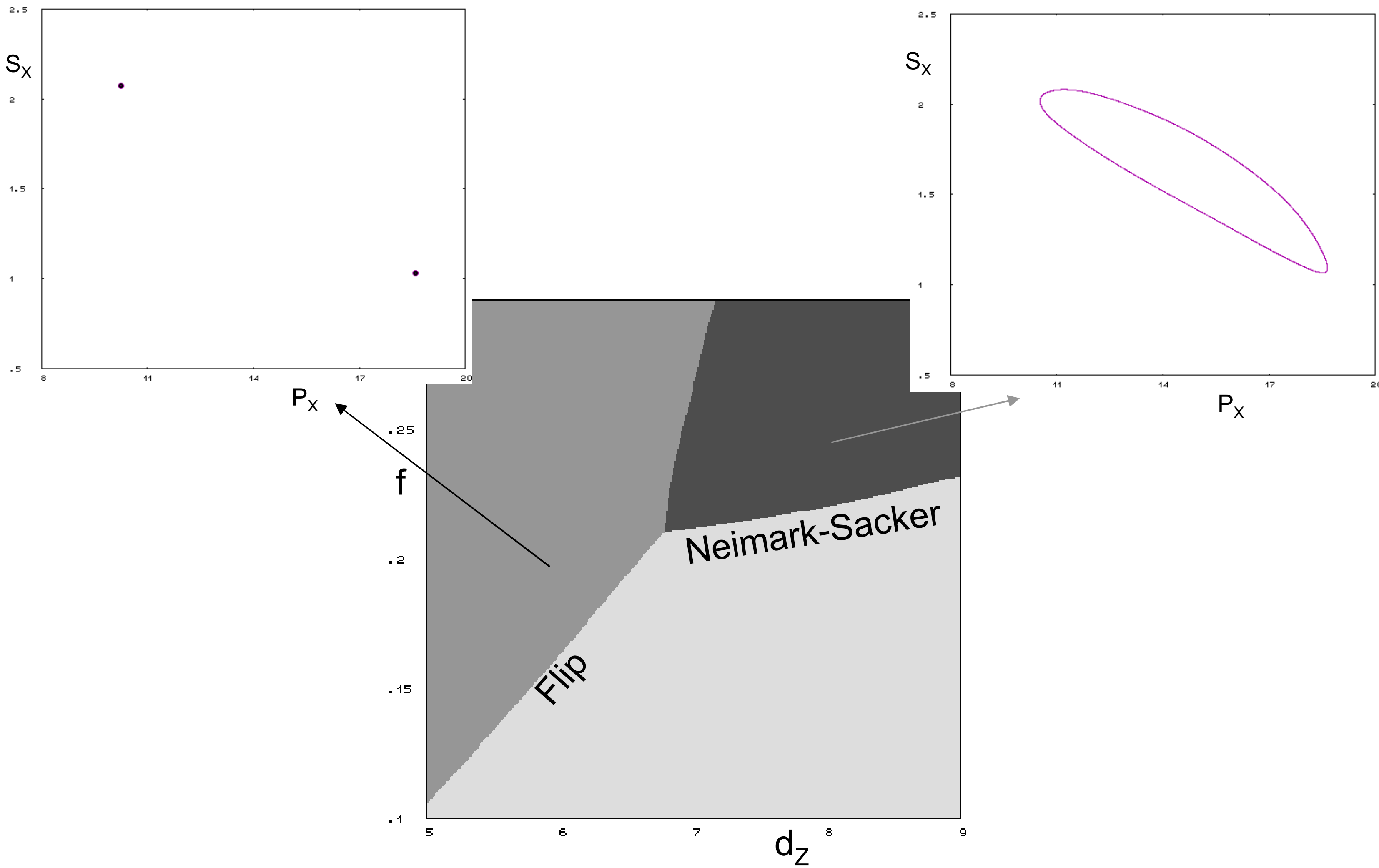
$$S_X = S_Z = \overline{S}$$



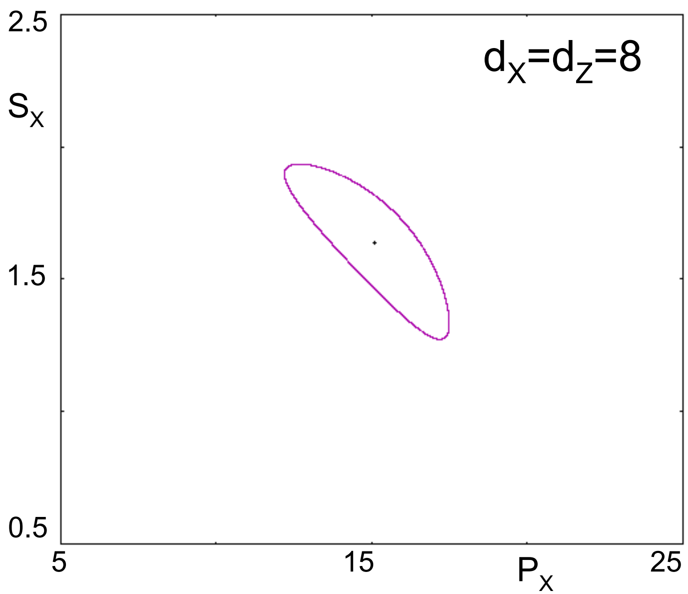
Bifurcation diagrams

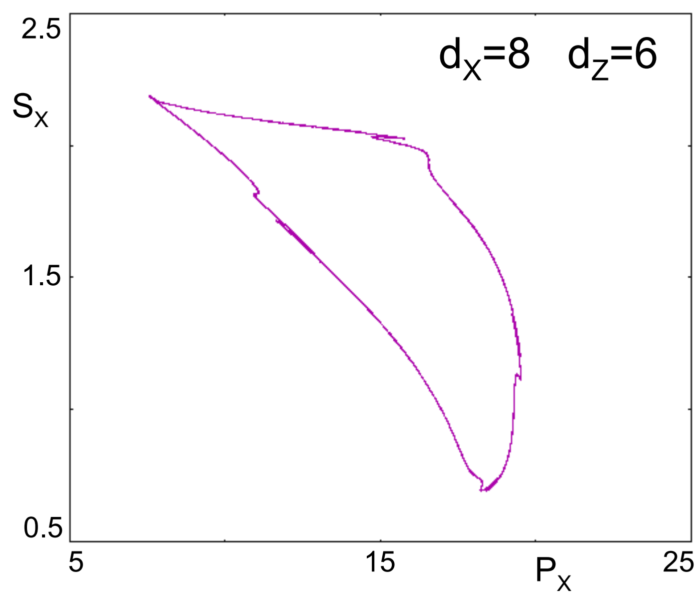
Parameters:
a=20, b=6, c=2, d=8, e=1, f=0.17

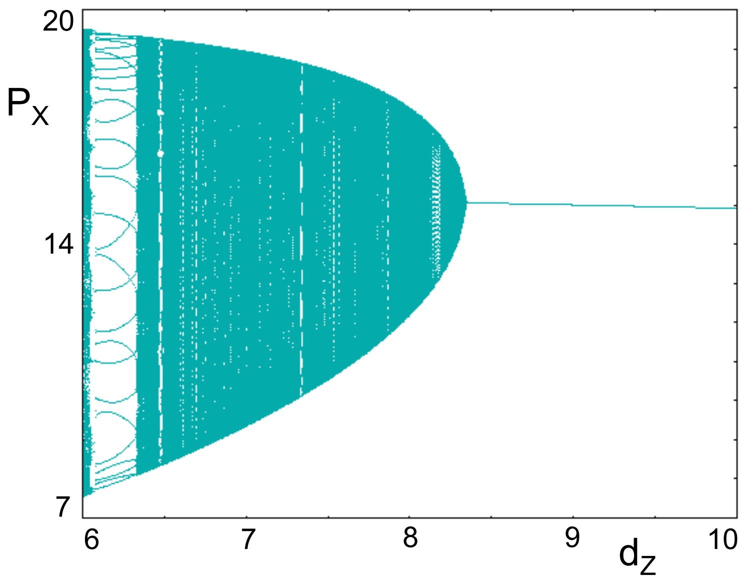
2-parameter bifurcation diagram under symmetry-breaking



Parameters: a_X=20, a_Z=18, b_X=6, b_Z=4, c_X=2, c_Z=1, e_X=1, e_Z=2, d_X=8







Effect of symmetry breaking

Parameters:
$a_X=a_Z=20$, $b_X=b_Z=6$, $c_X=c_Z=2$
$e_X=e_Z=1$, $d_X=8$, $f=0.17$

5 Summary of results

- The model sticks to the classical ‘cobweb’ as far as possible and introduces supply-side interactions between two cobweb markets.
- Interacting markets may add to the cyclical behavior of commodity prices captured by the classical cobweb model, and become a further source of instability and complexity
- Main findings
  - (a) interactions destabilize otherwise locally (and globally) stable equilibrium prices
  - (b) endogenous dynamics and complex behavior may emerge, particularly when agents react sensitive to profit differentials
  - (c) different types of bifurcations associated to different assumptions about risk attitudes / perceptions