Applications to Economics:
Modeling interacting markets with
higher-dimensional discrete dynamical systems

"Bull" and "Bear" dynamics in a 3D model
of interconnected markets

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1 Introduction

- Literature on asset price dynamics with boundedly rational heterogeneous agents has become well-developed (see surveys by Hommes 2006, LeBaron 2006, Lux 2008, Westerhoff 2008, Hens and Shenk-Hoppé (eds.) 2009).

- Most models include nonlinear elements, arising from trading rules / demand functions (e.g. Day and Huang 1990, Chiarella 1992), or switching among available strategies (e.g. Brock and Hommes 1998, Lux 1998).

- Main contribution of this branch of research

  (i) successful replication of "stylized facts" of financial markets

  (ii) financial asset prices at least partially driven by endogenous laws of motion (aggregation of heterogeneous beliefs / demand functions, expectations feedback, evolutionary competition between forecasting rules, ...)

  (iii) possible implications for design and supervision of markets

Recent developments: interacting financial markets with heterogeneous agents

- Multiple risky assets setups (e.g. Böhm and Wenzelburger 2005, Chiarella, Dieci and Gardini 2005, Chiarella, Dieci and He 2007): interactions due to heterogeneous agents' mean-variance portfolio diversification under endogenously changing first- and second-moment beliefs.

- Interactions between speculative asset markets: speculators switch between different markets depending on relative profitability (e.g. Westerhoff 2004, Westerhoff and Dieci 2006)

- Interacting stock and foreign exchange markets (Corona et al. 2008, Tramontana et al. 2009, Dieci and Westerhoff 2010).

Focus on the impact of interactions (whether "stabilizing" or "destabilizing"), effect of transaction taxes, comovements of stock prices, and the way endogenous price movements / price volatility that originate in one market spill over throughout the system of interconnected markets.

Day and Huang (1990): complex ‘bull’ and ’bear’ fluctuations due to nonlinear interactions between chartists and fundamentalists

- Destabilizing behavior of chartists, who believe in the persistence of positive, or negative mispricing. Stabilizing impact of fundamental traders, who expect mean reversion towards the fundamental.

- Key feature: the larger the mispricing, the more ‘aggressive’ fundamentalists become (*chance* function)

- Asset price dynamics driven by a one-dimensional *cubic* map, with three fixed points: an unstable fundamental between two further *non-fundamental* fixed points.

- Cycles of various periods and then chaotic dynamics may emerge within two different *bull* and *bear* market regions, as a consequence of *period-doubling* and *homoclinic* bifurcation sequences involving each of two coexisting non-fundamental equilibria.

- The two chaotic areas may eventually merge via a further *homoclinic* bifurcation, that brings about a scenario of apparently random switches between ‘bull’ and ‘bear’ markets.

This model

- One-asset (one-dimensional) model with technical and fundamental traders, close in spirit to Day and Huang (1990).
- We embed the model in a three-dimensional system of interdependent market, and *explore how the coupling of the markets affects the emergence of 'bull and bear' dynamics*.
- Focus on the *global (homoclinic) bifurcations* that change the model behavior from coexistence of multiple equilibria to chaotic dynamics across bull and bear regions.
- Different 'levels' of interaction between markets (due to restrictions to investors' trading activity) result in lower-dimensional particular cases, embedded in the full 3D model.
- *Similarities and differences of the relevant bifurcation sequences*, across dynamical systems of increasing dimension.

2 One-asset model

- Price dynamics

$$
\begin{aligned}
S_{t+1} &= S_t + d\left(D^S_{C,t} + D^S_{F,t}\right), \quad d > 0 \\
D^S_{C,t} &= e(S_t - F^S), \quad e > 0 \\
D^S_{F,t} &= f(F^S - S_t)^3, \quad f > 0
\end{aligned}
$$

$D^S_{C,t}$, $D^S_{F,t}$, speculative (excess) demand by chartists and fundamentalists, F^S, fundamental price.

- Chartists believe in the persistence of 'bull' markets or 'bear' markets, e.g. they optimistically buy as long as price is high

- Fundamentalists expect mean reversion and seek to exploit misalignments using a nonlinear trading rule.

- A linear price impact function is assumed for simplicity

- A simple $1D$ model in deviations $x := (S - F^S)$:

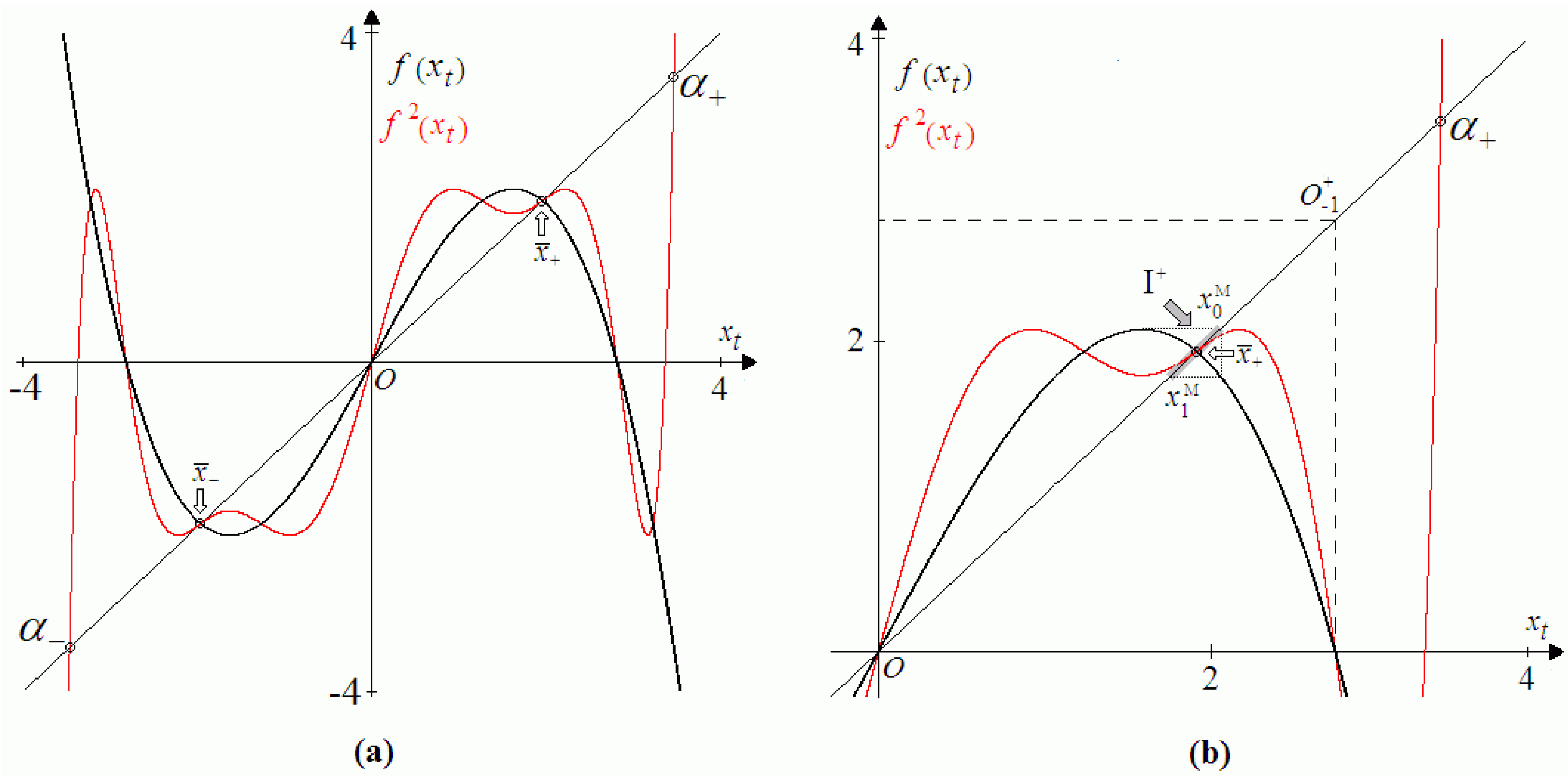
$$x_{t+1} = \varphi(x_t) = x_t(1 + de) - dfx_t^3$$

Stedy state properties

- Symmetric cubic map, three equilibria for any e, $f > 0$
- $\overline{x} := 0$ *fundamental* equilibrium, $\overline{x}_- := -\sqrt{e/f}$ and $\overline{x}_+ := \sqrt{e/f}$ *non-fundamental* equilibria
- $\varphi'(0) = 1 + de > 1$, $\varphi'(\overline{x}_-) = \varphi'(\overline{x}_+) = 1 - 2de < 1$
- Unstable fundamental equilibrium, non-fundamental equilibria locally stable for small e.
- Non-fundamental equilibria become unstable via Flip-bifurcation for $e > 1/d$, followed by the usual period doubling cascade

Stable non-fundamental steady states

parameters: $d=0.35$, $e=2.687$, $f=0.7$



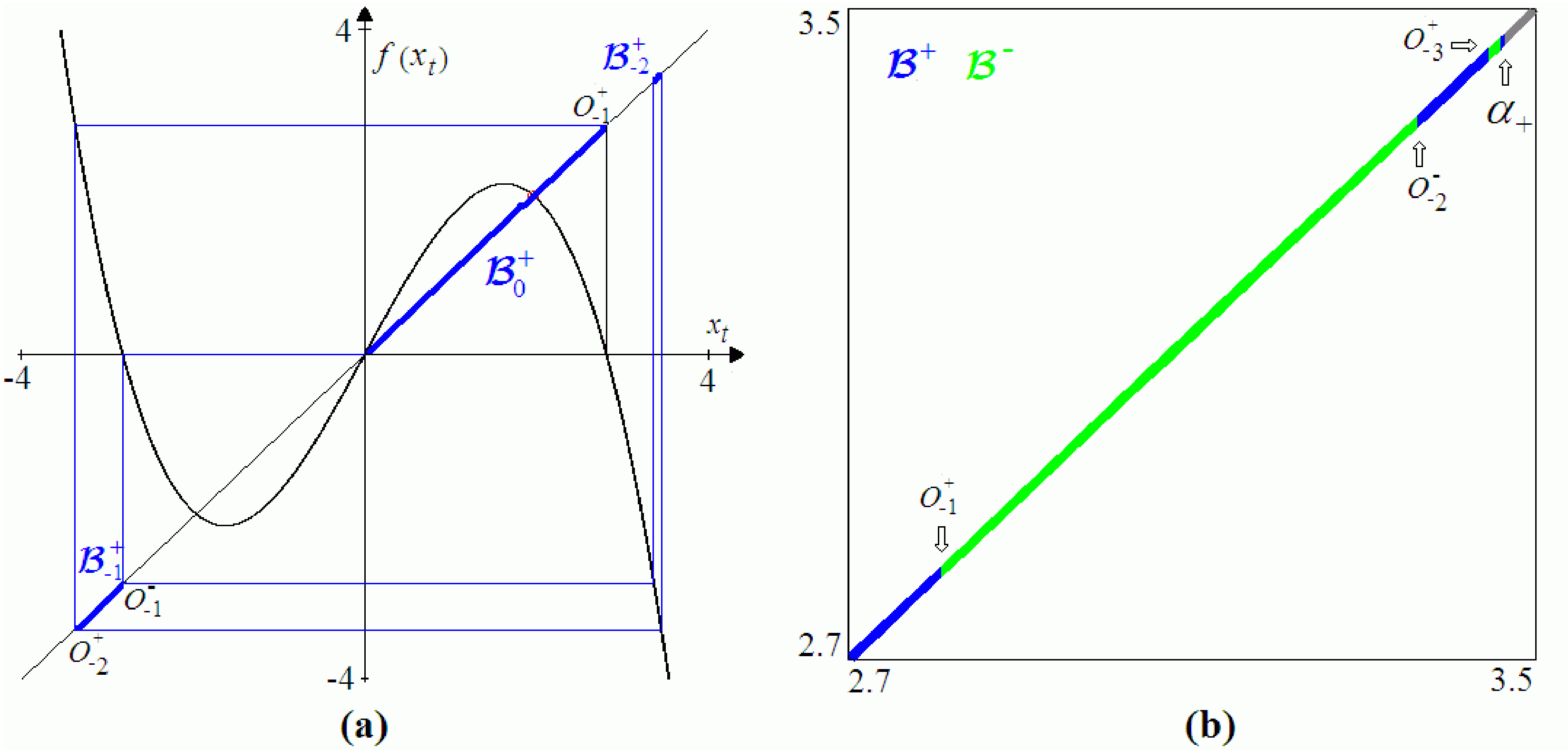
Basins of attraction

(a) the immediate basin of the steady state x_+ and its rank-1 and rank-2 preimages (in blue).

(b) enlargement of the interval between O_{-1}^+ and α_+ .

Intervals belonging to the basins of x_+ and x_- alternate on the real line.

parameters: $d=0.35$, $e=2.687$, $f=0.7$



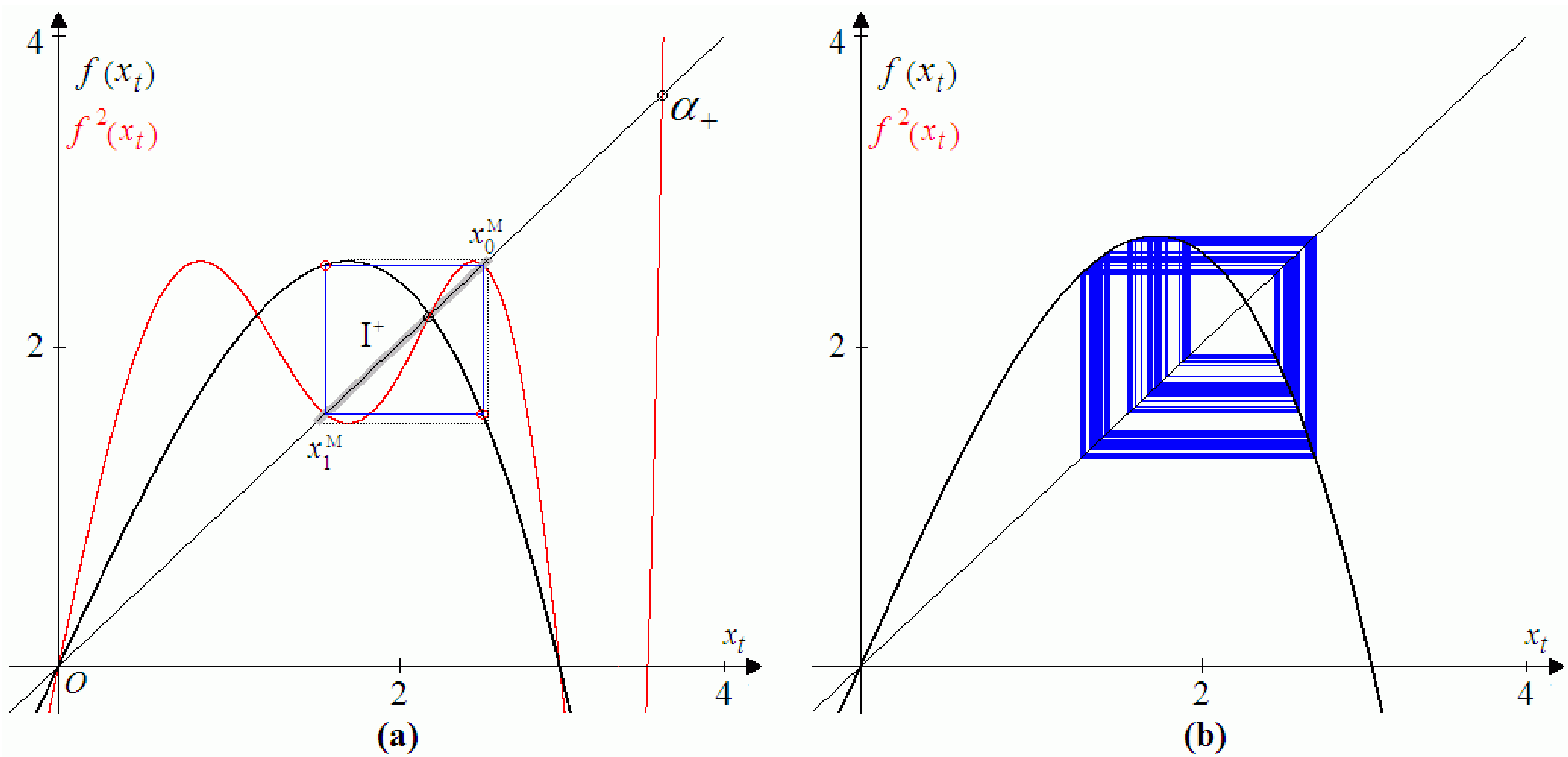
(a) (b)

Periodic and chaotic attractors

(a) a stable 2-cycle for $e=3.483$

(b) a chaotic attractor for $e=3.7436$

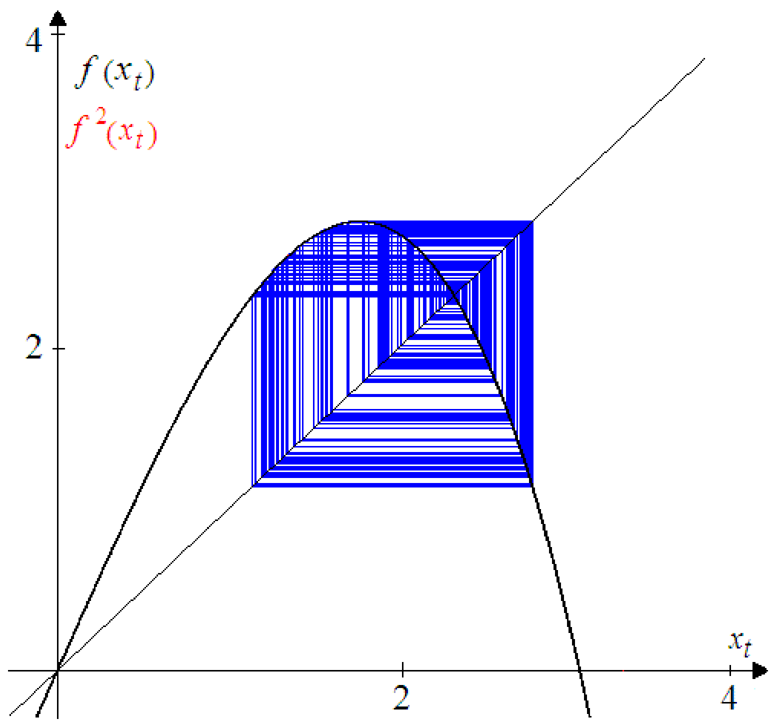
other parameters: $d=0.35$, $f=0.7$

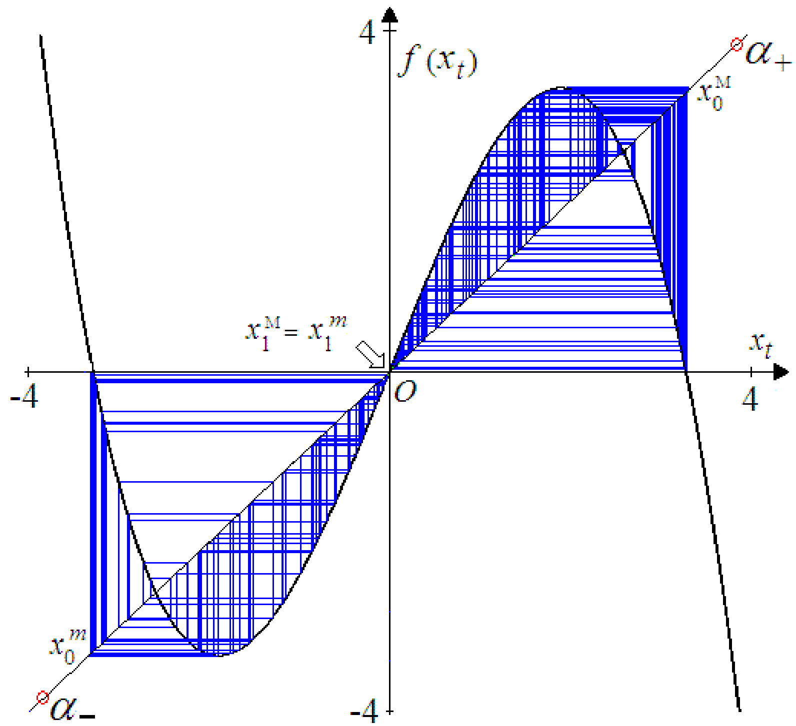


Homoclinic bifurcation of x_+
The intervals on the two sides of x_+ merge into a unique chaotic interval for $e \simeq 3.89$

Homoclinic bifurcation of O
The chaotic intervals around x_+ and x_- merge into a unique interval for $e \simeq 4.5659$

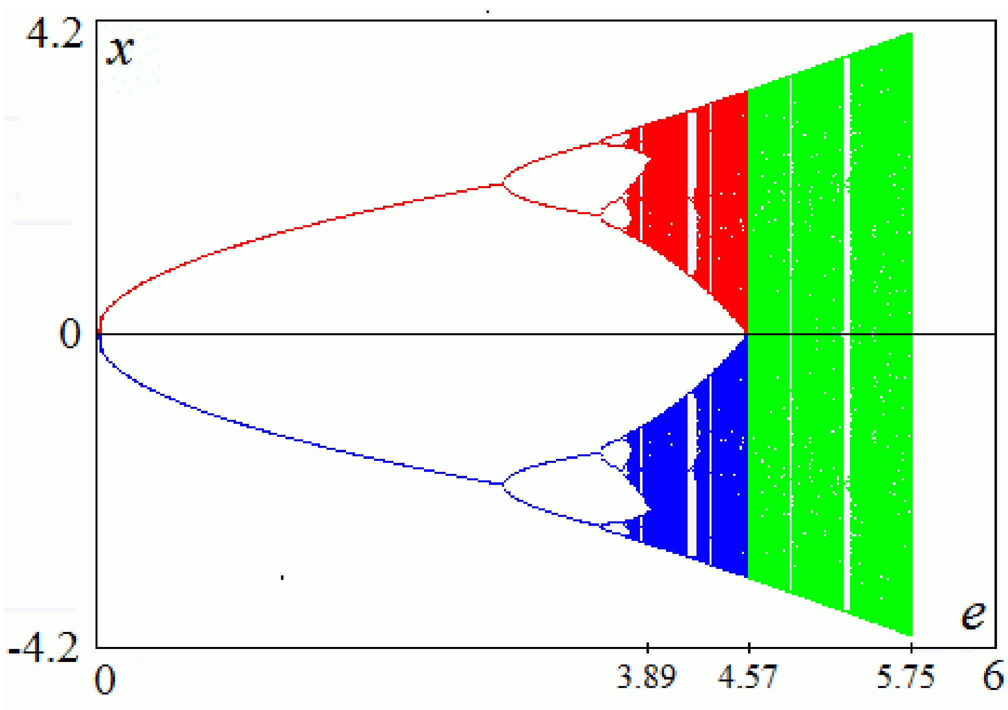
other parameters: $d=0.35$, $f=0.7$



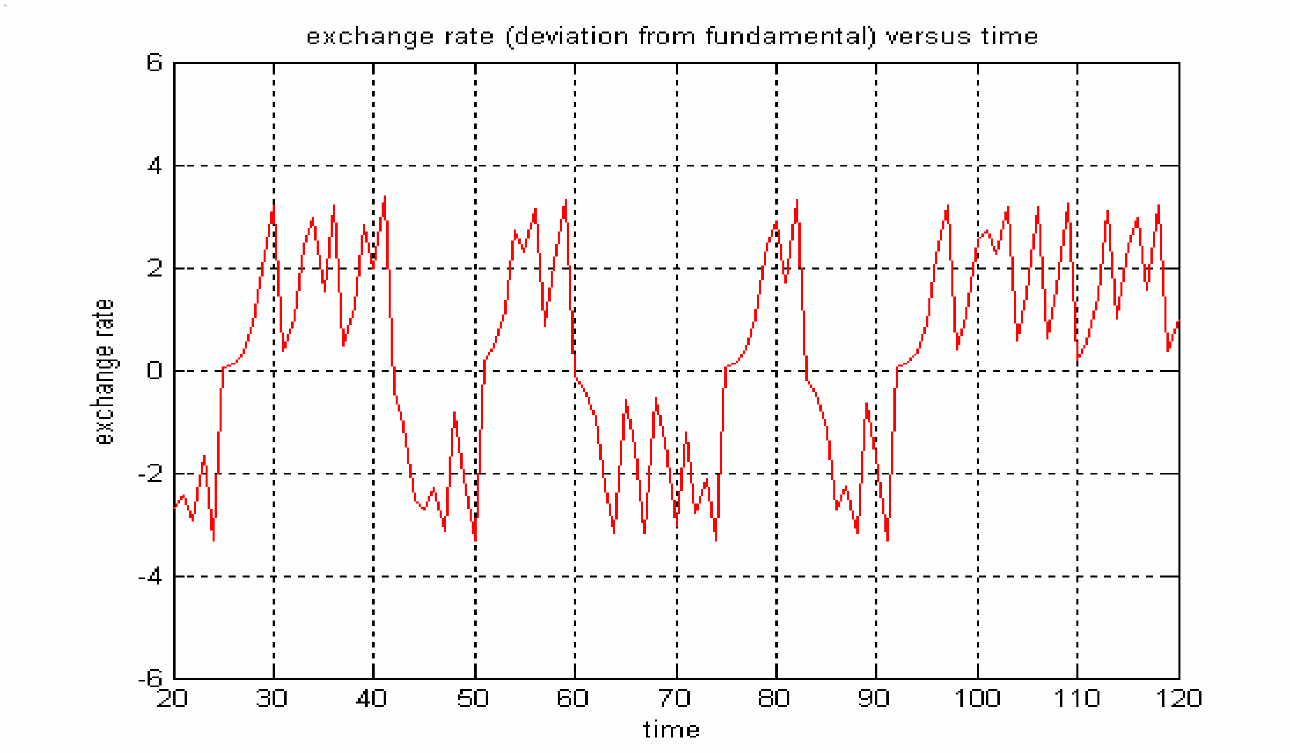


Bifurcation diagram versus parameter e for the 1D model parameters: $d=0.35$, $f=0.7$

Blue: i.c. close to x_- Red: i.c. close to x_+ Green: both



A trajectory of the asset price (deviation) x, switching across ‘bull’ and ‘bear’ regions, after the homoclinic bifurcation of O ($e=4.75$)



3 Interdependent markets

- S_t: exchange rate (price of currency H(ome) in terms of currency A(broad))

- Two stock markets, H and A, where *fundamental traders* from both H and A are active

- Connections between markets: *(i)* stock market traders who invest abroad have to consider potential exchange rate adjustments when they enter a speculative position; *(ii)* these agents buy/sell foreign currency to conduct their transactions, which generates exchange rate adjustments

- A three-dimensional dynamical system, in prices P^H, P^A and exchange rate S, two linear equations and one nonlinear.

- The dynamical system

$$
\begin{aligned}
P^H_{t+1} &= P^H_t + a^H(D^{HH}_{F,t} + D^{HA}_{F,t}), \quad a^H > 0 \\
P^A_{t+1} &= P^A_t + a^A(D^{AA}_{F,t} + D^{AH}_{F,t}), \quad a^A > 0 \\
S_{t+1} &= S_t + d\left(P^H_t D^{HA}_{F,t} - \frac{P^A_t}{S_t} D^{AH}_{F,t} + D^S_{C,t} + D^S_{F,t}\right)
\end{aligned}
$$

- $D^S_{C,t} = e(S_t - F^S)$, $D^S_{F,t} = f(F^S - S_t)^3$ speculative (excess) demand for currency H by chartists and fundamentalists, F^S, F^H, F^A fundamental values;

- $D^{HH}_{F,t} = b^H(F^H - P^H_t)$, $b^H > 0$: demand (in real units) for stock H by fundamentalists from country H;

- $D^{HA}_{F,t} = c^H\left[(F^H - P^H_t) + \gamma^H(F^S - S_t)\right]$, $c^H \geq 0$, $\gamma^H > 0$: demand for stock H by fundamentalists from A;

- $D^{AA}_{F,t} = b^A(F^A - P^A_t)$, $b^A > 0$, demand for stock A by fundamentalists from A;

- $D^{AH}_{F,t} = c^A\left[(F^A - P^A_t) + \gamma^A\left(\frac{1}{F^S} - \frac{1}{S_t}\right)\right]$, $c^A \geq 0$, $\gamma^A > 0$, demand for stock A by fundamentalists from H.

Particular cases (restrictions to foreign traders)

- $c^H = c^A = 0$, three independent dynamic equations

$$P^H_{t+1} = G^H(P^H_t), \quad S_{t+1} = G^S(S_t), \quad P^A_{t+1} = G^A(P^A_t)$$

  Exchange rate S as in the previous $1D$ model, 'fundamental' equilibrium prices of the two stock markets globally stable iff $a^H b^H < 2$, $a^A b^A < 2$.

- $c^H > 0$, $c^A = 0$, and independent $2D$ system for P^H and S

$$\begin{aligned} P^H_{t+1} &= G^H(P^H_t, S_t) \\ S_{t+1} &= G^S(P^H_t, S_t) \end{aligned}$$

  (Multiple) steady states structure analogous to that of the nonlinear $1D$ model

3.1 The two-dimensional case

- The $2D$ model in deviations $x := (P^H - F^H)$ and $y := (S - F^S)$:

$$\begin{array}{l} x_{t+1} = x_t - a^H \left[(b^H + c^H) x_t + c^H \gamma^H y_t \right] \\ y_{t+1} = y_t - d \left[c^H \left(x_t + F^H \right) \left(x_t + \gamma^H y_t \right) - e y_t + f y_t^3 \right] \end{array} .$$

- Fundamental steady state O $(x = 0,\ y = 0)$ is LAS for

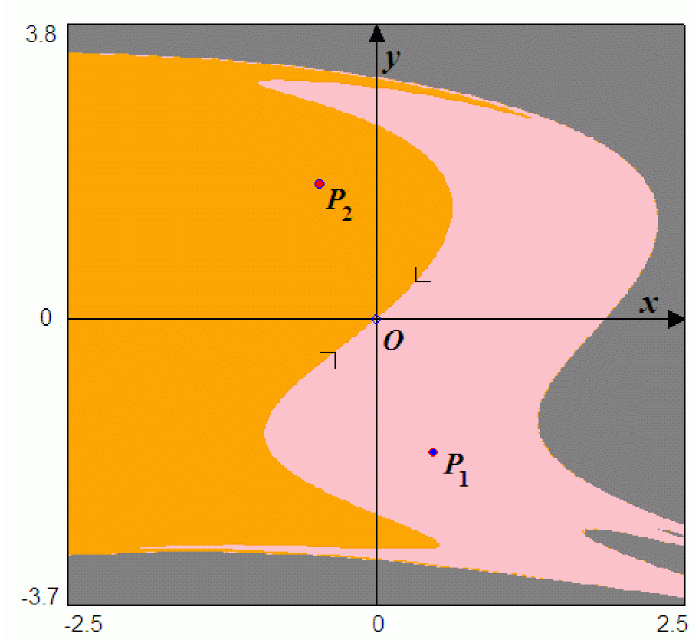
$$e < e_{CS} := b^H F^H c^H \gamma^H / (b^H + c^H)$$

- Steady state structure changes via a *saddle-node* followed by *transcritical* bifurcation.

- Two LAS non-fundamental steady states exist on opposite sides of the unstable (saddle) O for $e > e_{CS}$: $P_1 = (x_1, y_1)$, $x_1 > 0$, $y_1 < 0$, and $P_2 = (x_2, y_2)$, $x_2 < 0$, $y_2 > 0$.

- P_1 and P_2 undergo a sequence of period doubling bifurcations when e is increased

Bi-stability in the 2D case.
The stable manifold of the saddle O is the border between the basins of equilibria P_1 and P_2

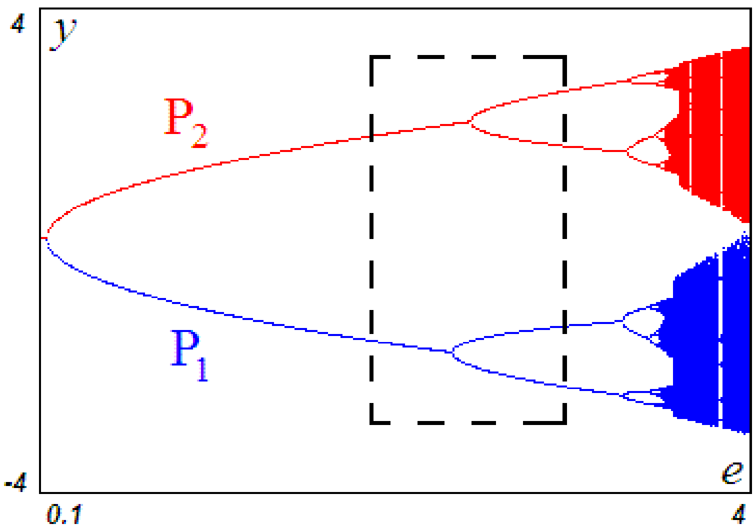


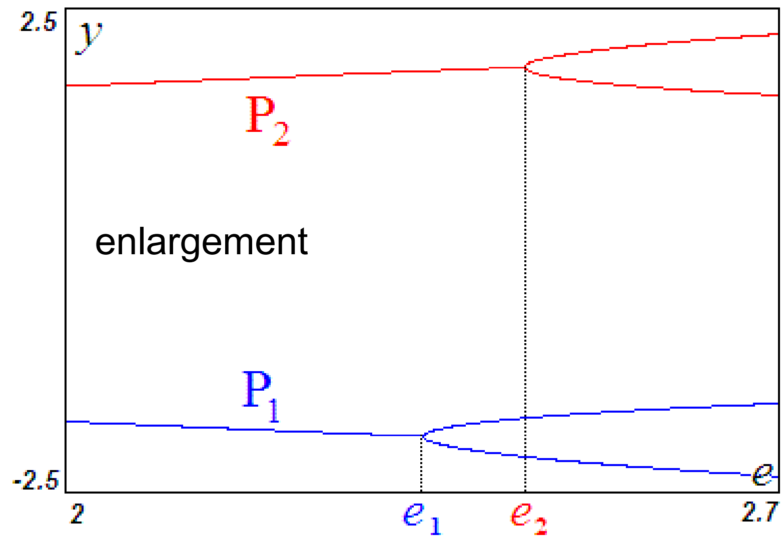
Parameters

$a^H = 0.41 \quad b^H = 0.11 \quad c^H = 0.83 \quad \gamma^H = 0.3 \quad F^H = 4.279$

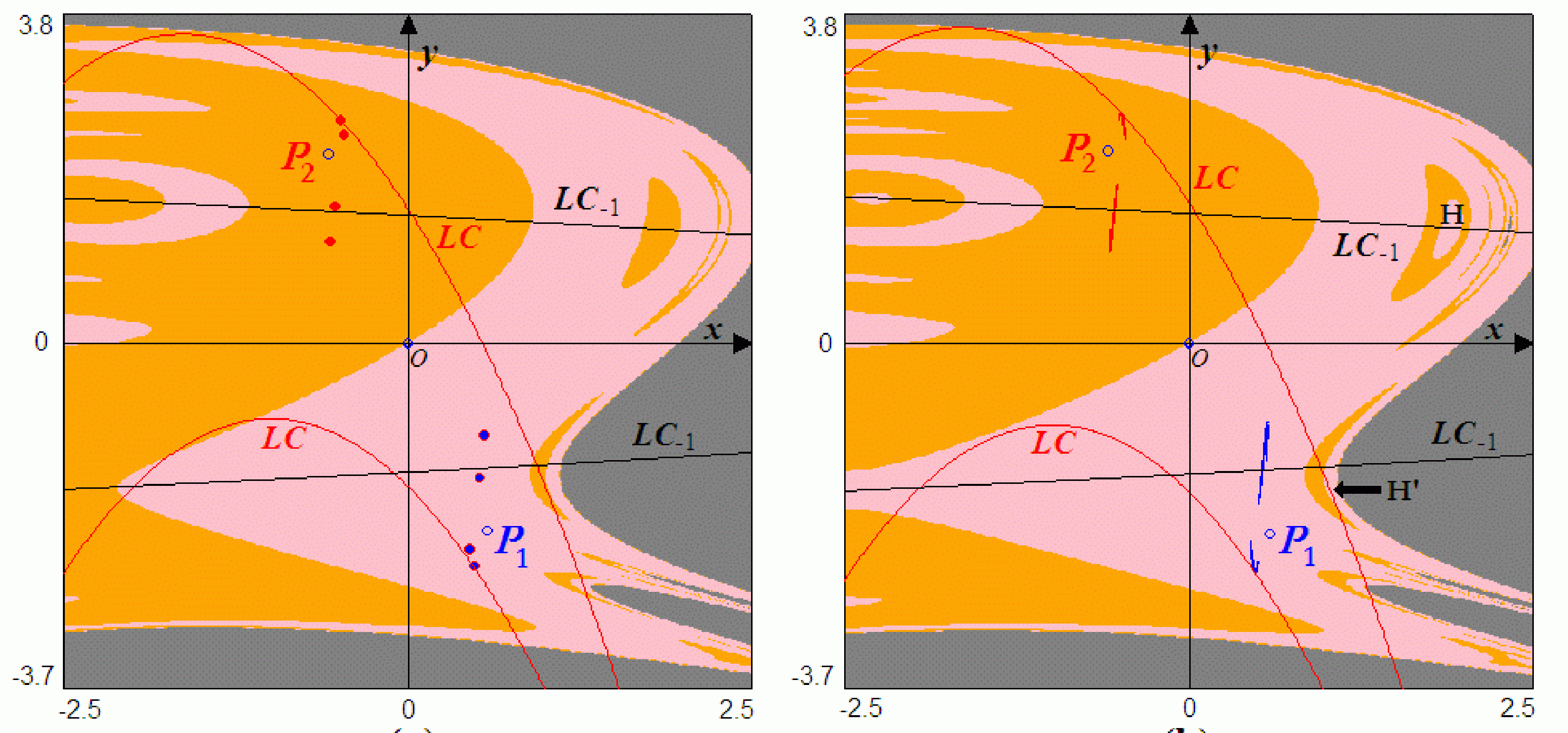
$d = 0.35 \quad e = 2.22 \quad f = 0.7$

Bifurcation diagram of y versus parameter e
Asynchronous period-doubling bifurcations
Blue: i.c. close to P_1
Red: i.c. close to P_2

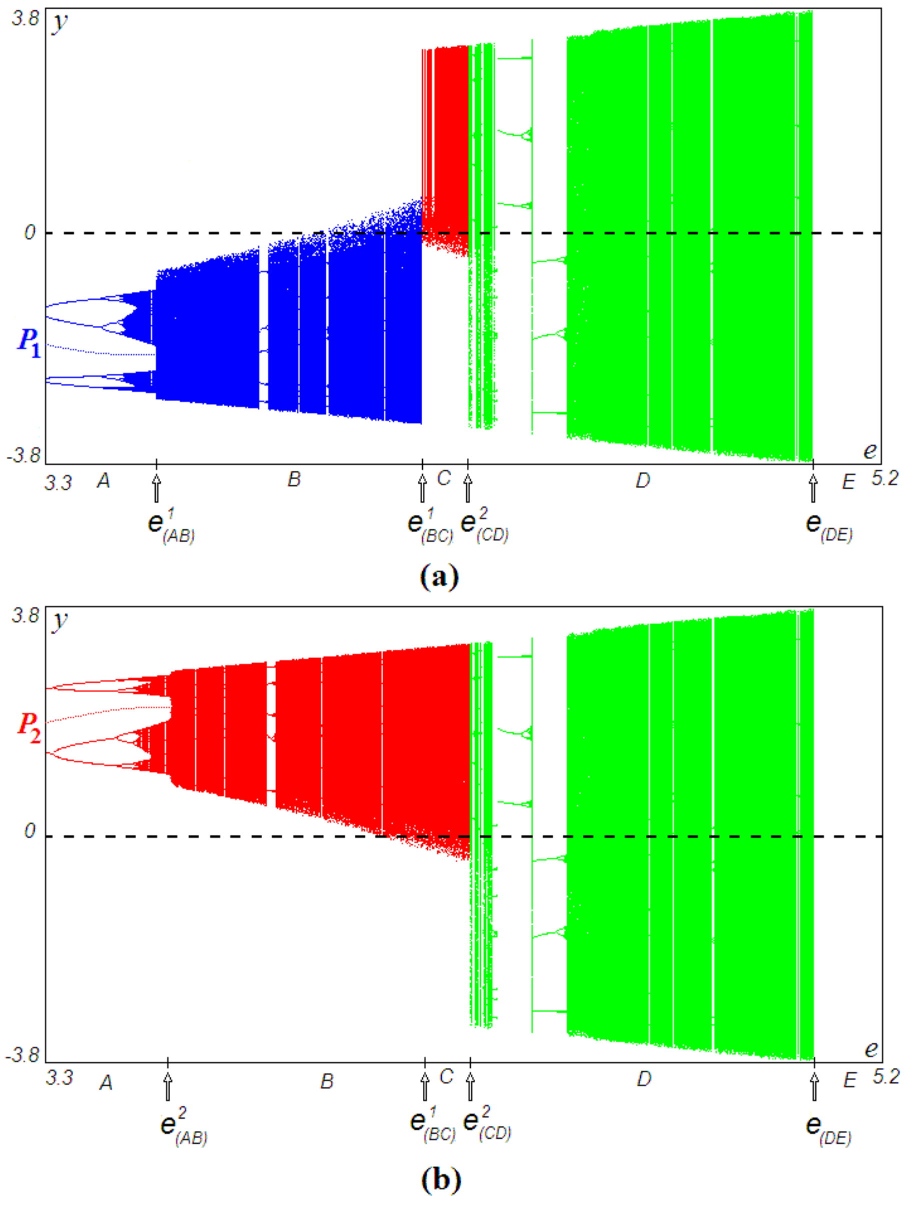




Basins of attraction and critical curves
(a) Basins of coexisting 4-cycles ($e=3.43$)
(b) Basins of coexisting 2-piece chaotic attractors ($e=3.56$)
New disconnected portions of basins appear around LC_{-1}
whenever a basin boundary crosses LC



(a) (b)

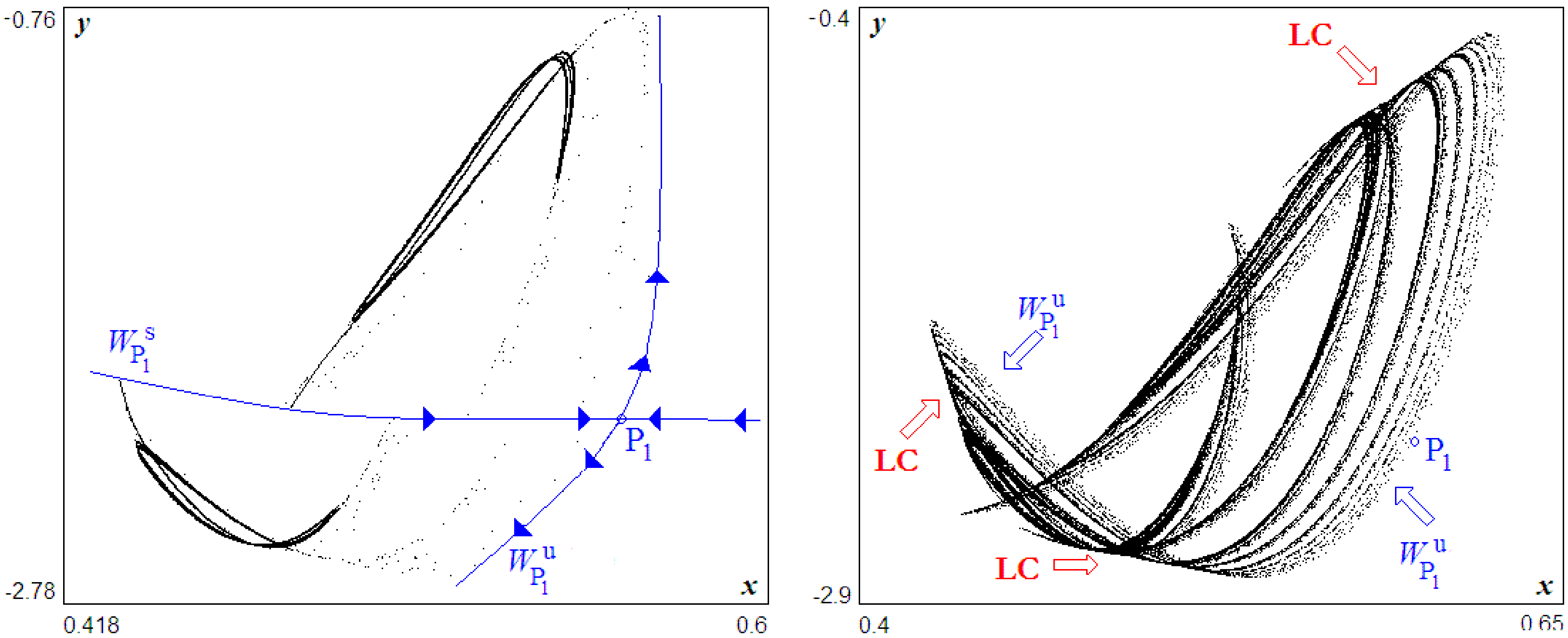


Bifurcation diagram (large *e*)
Asynchronous *homoclinic bifurcations*

Blue: i.c. close to P_1
Red: i.c. close to P_2
Green: both

First homoclinic bifurcation of P_1 (*interior crisis*)

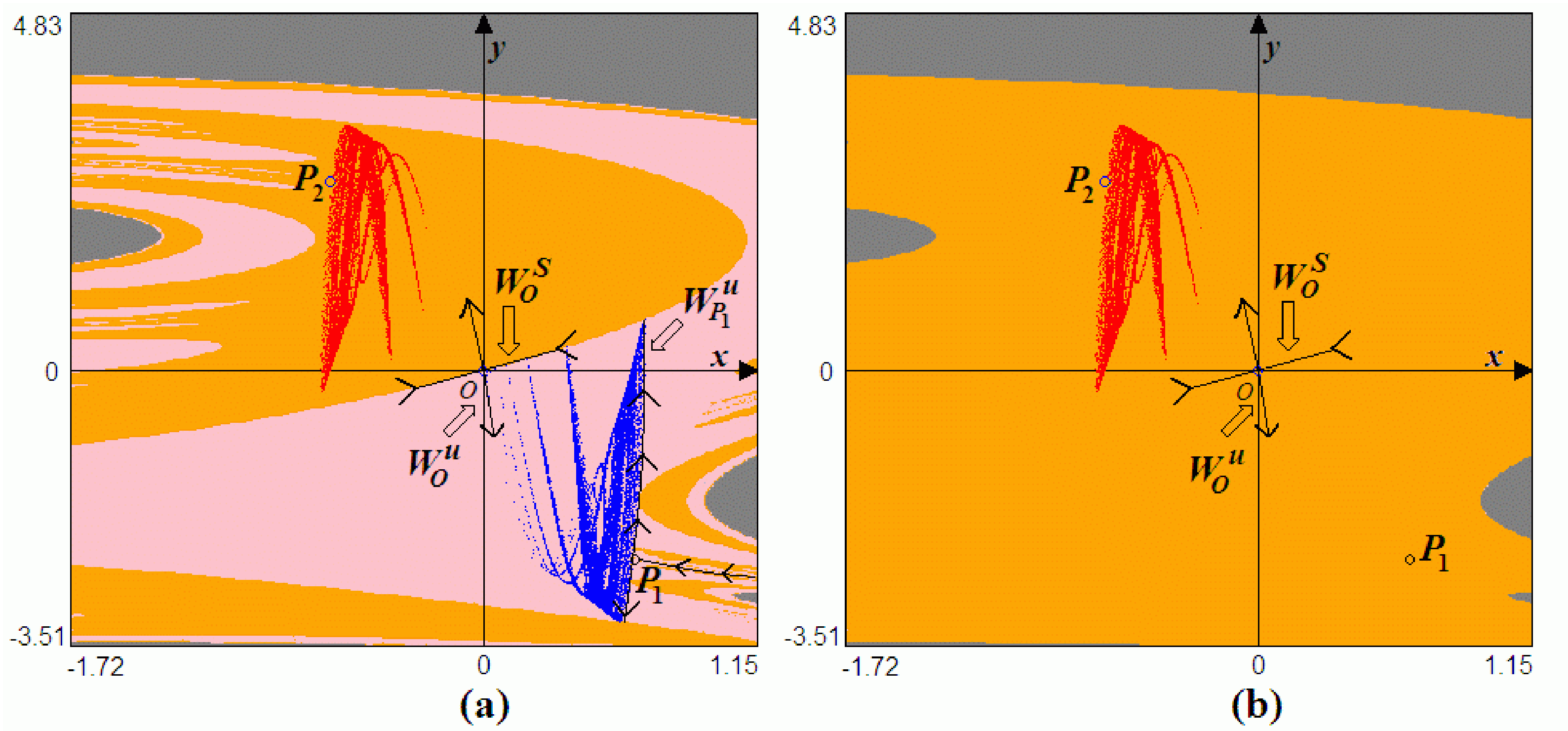
(a) contact between the two pieces of attractor A_1 and the stable set of the saddle P_1 ($e=3.6$)
(b) one-piece chaotic area A_1 after the bifurcation ($e=3.65$)
Boundary is made up by segments of both critical curves and unstable manifold of P_1
Only one branch (the one included in the chaotic area) of the stable manifold has homoclinic points



(a) (b)

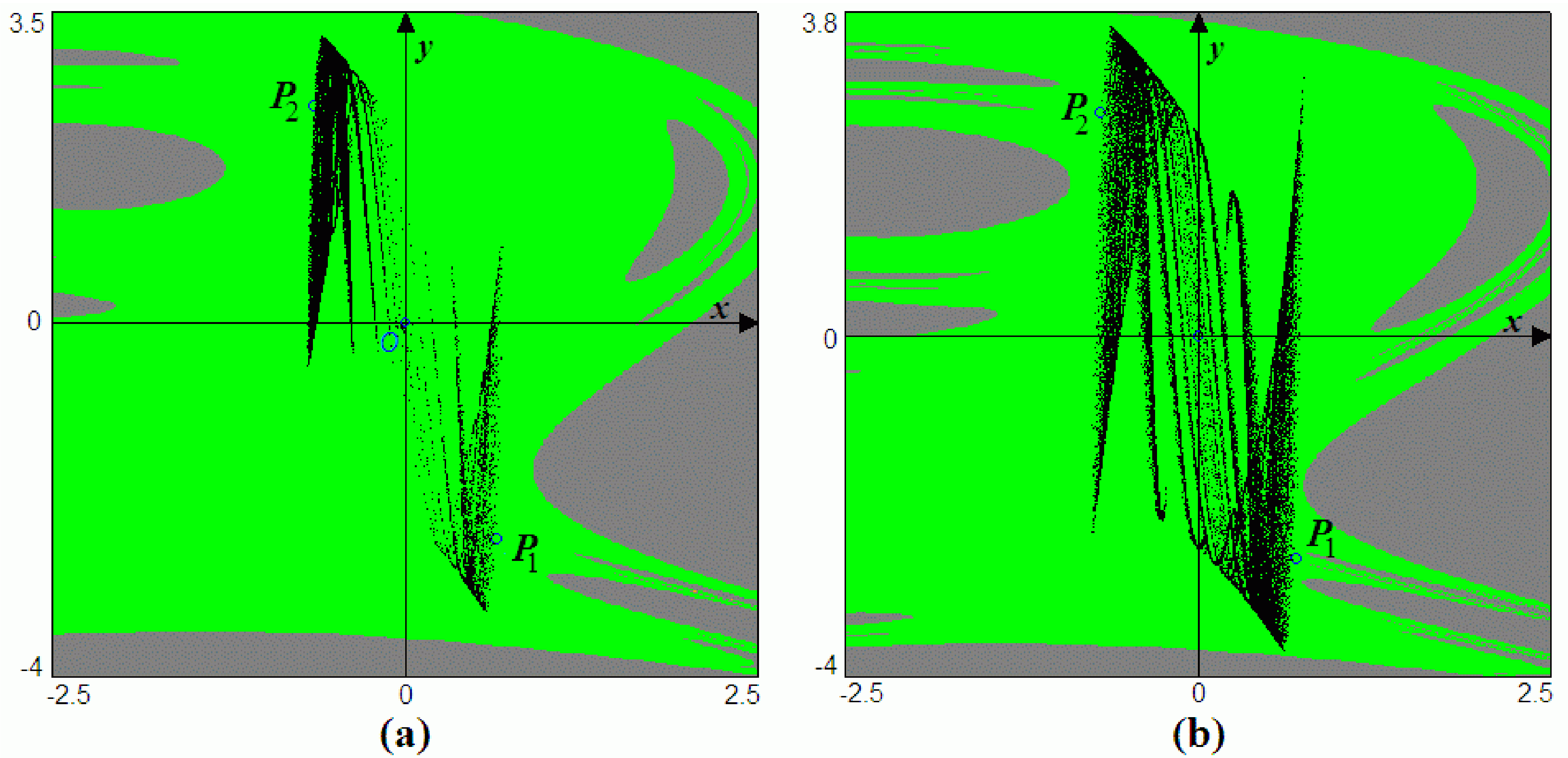
Second homoclinic bifurcation of P_1 and *first* homoclinic bifurcation of O
exterior crisis: disappearance of attractor A_1

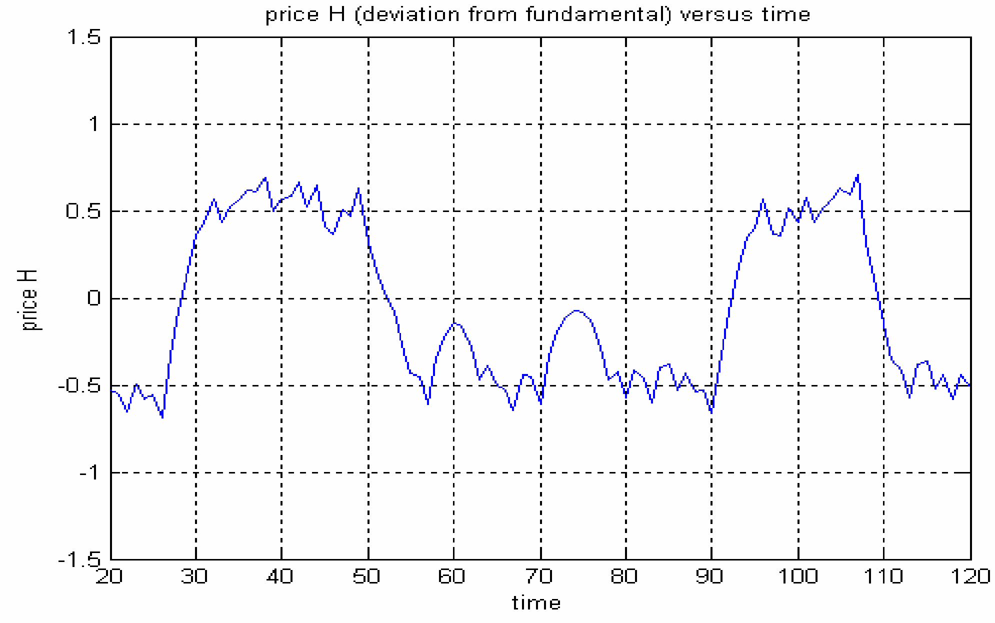
The bifurcation involves the second branch of the stable manifold of P_1
(a) Situation at the bifurcation value ($e \cong 4.198$)
(b) Just after the bifurcation ($e \cong 4.2$)

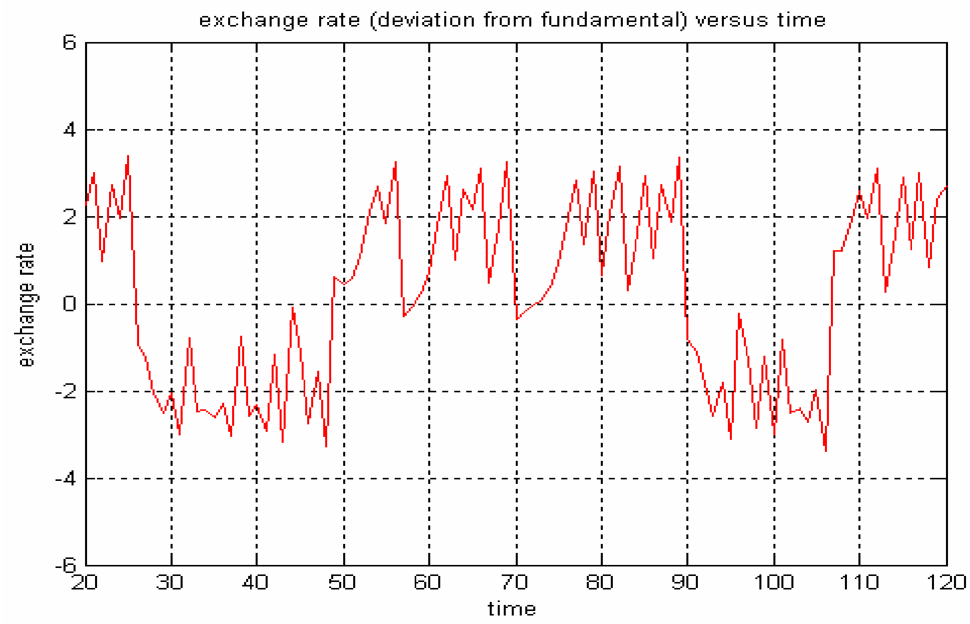


(a) *Second* homoclinic bifurcation of P_2 and *second* homoclinic bifurcation of O
exterior crisis: 'explosion' of attractor A_2
($e \cong 4.3$)

(b) Towards the 'final' bifurcation ($e \cong 4.893$)







Trajectories of *x* and *y*, switching across 'bull' and 'bear' regions, after the second homoclinic bifurcation of *O* ($e=4.75$)

3.2 The complete model

- The full $3D$ model in deviations $x := (P^H - F^H)$, $y := (S - F^S)$ and $z := (P^A - F^A)$

$$x_{t+1} = x_t - a^H \left[(b^H + c^H)x_t + c^H \gamma^H y_t\right]$$

$$y_{t+1} = y_t - d \left[c^H \left(x_t + F^H\right) \left(x_t + \gamma^H y_t\right) + c^A \frac{z_t + F^A}{y_t + F^S} \left(\gamma^A \frac{y_t}{F^S(y_t + F^S)} - z_t \right) \right.$$
$$\left. - e y_t + f y_t^3 \right]$$

$$z_{t+1} = z_t - a^A \left[\left(b^A + c^A\right) z_t - c^A \gamma^A \frac{y_t}{F^S(y_t + F^S)} \right]$$

- Again, for sufficiently large e, the (unstable) fundamental steady state $O = (0, 0, 0)$ is surrounded by two LAS nonfundamental steady states, $P_1 = (x_1, y_1, z_1)$, $x_1 > 0$, $y_1 < 0$, $z_1 < 0$ and $P_2 = (x_2, y_2, z_2)$, $x_2 < 0$, $y_2 > 0$, $z_2 > 0$.

- P_1 and P_2 undergo a sequence of period doubling bifurcations when e is increased

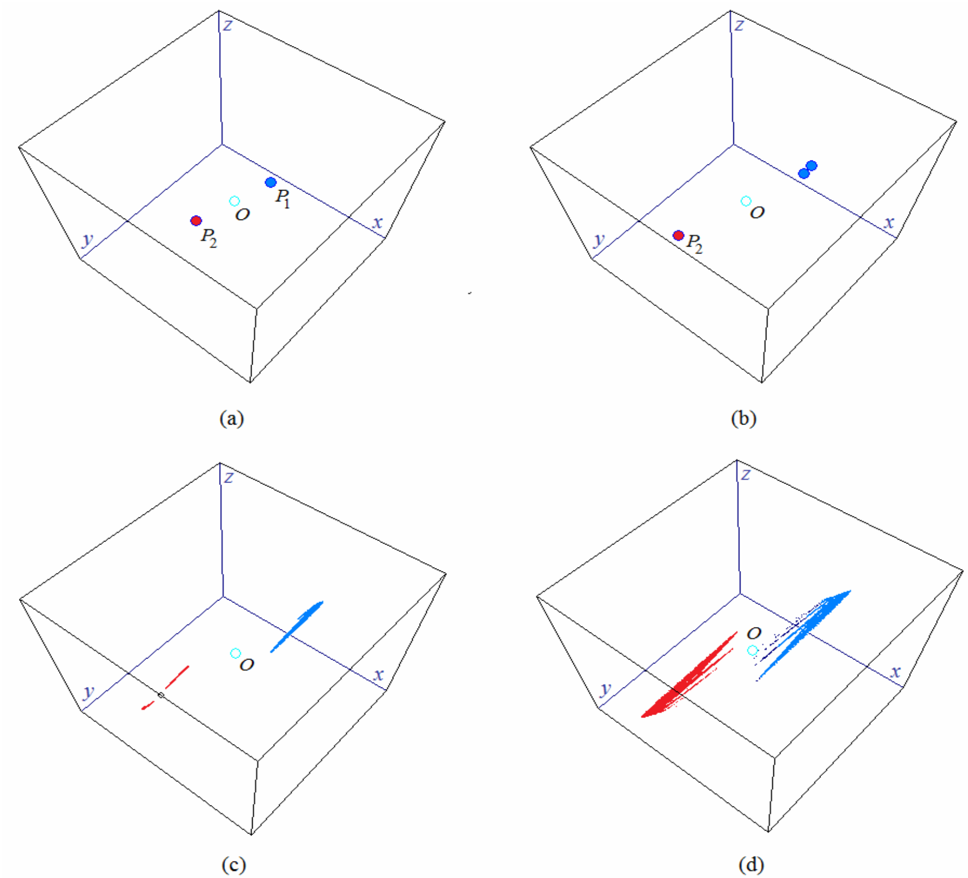
(a) Two stable equilibria P_1 and P_2 (*e=0.89*)

(b) Stable equilibrium P_2 and stable 2-cycle (*e=2.43*)

(c) One-piece chaotic attractor (after the first homoclinic bifurcation of P_1) and a two-piece chaotic attractor (before the first homoclinic bifurcation of P_2) (*e=3.576*)

(d) Two one-piece attractors (*e=4.1841*)

Coexisting attractors for increasing *e*



Parameters

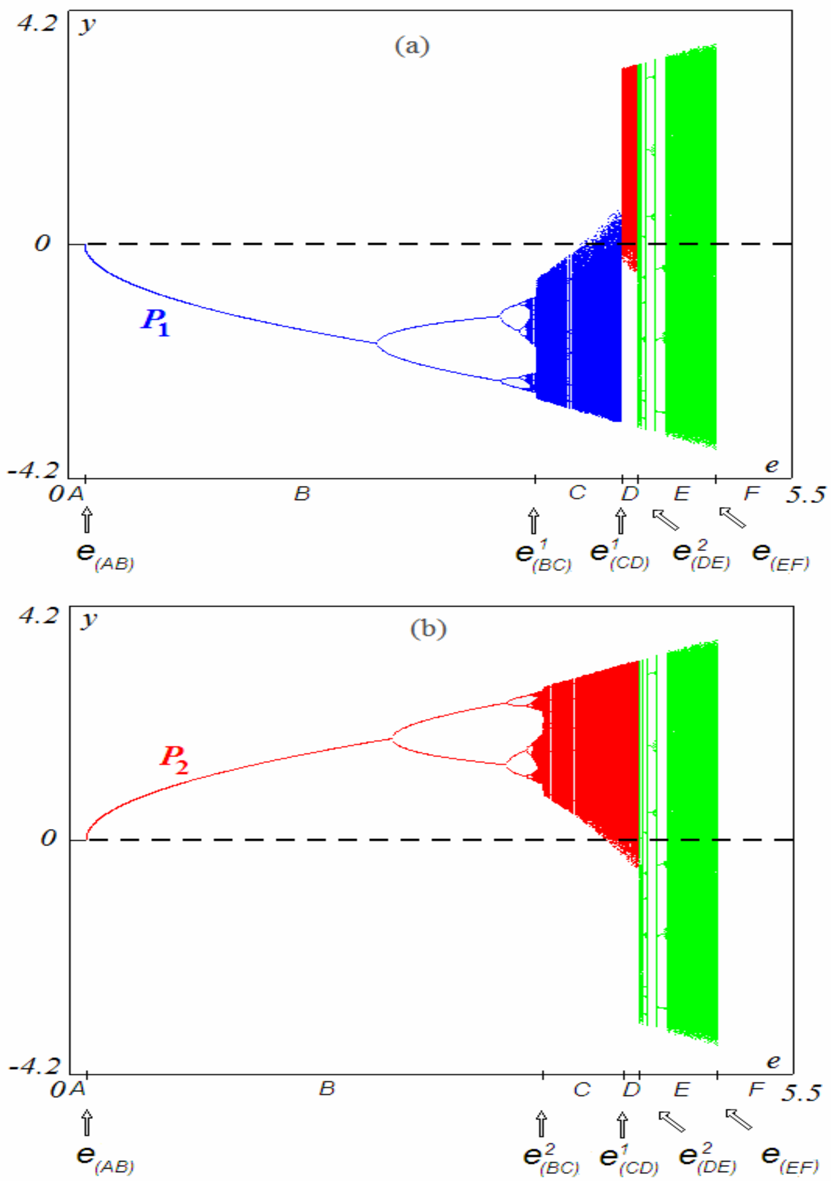
$a^H = 0.41 \quad b^H = 0.11 \quad c^H = 0.83$

$\gamma^H = 0.3 \quad F^H = 4.279$

$a^A = 0.43 \quad b^A = 0.21 \quad c^A = 0.9$

$\gamma^A = 0.36 \quad F^A = 1.1$

$d = 0.35 \quad f = 0.7 \quad F^S = 6.07$



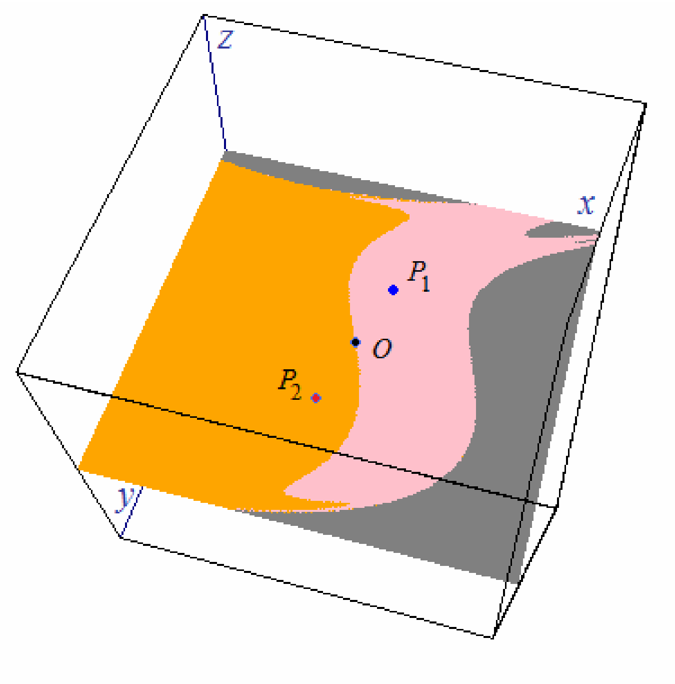
Bifurcation diagram of *y* against parameter *e* and homoclinic bifurcation values of *e*

Blue: i.c. close to P_1

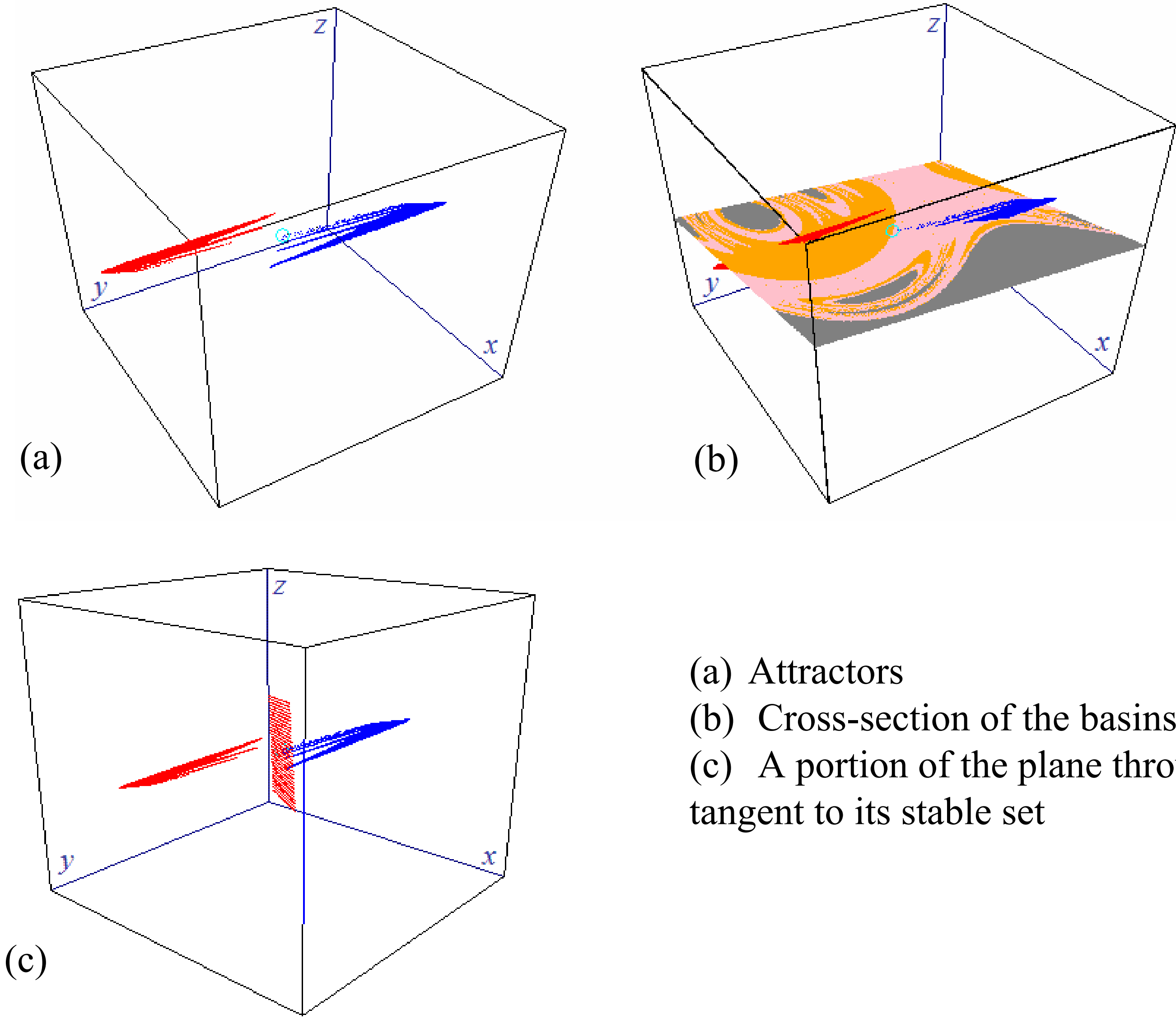
Red: i.c. close to P_2

Green: both

Bi-stability and basins of attraction for *e=0.89*



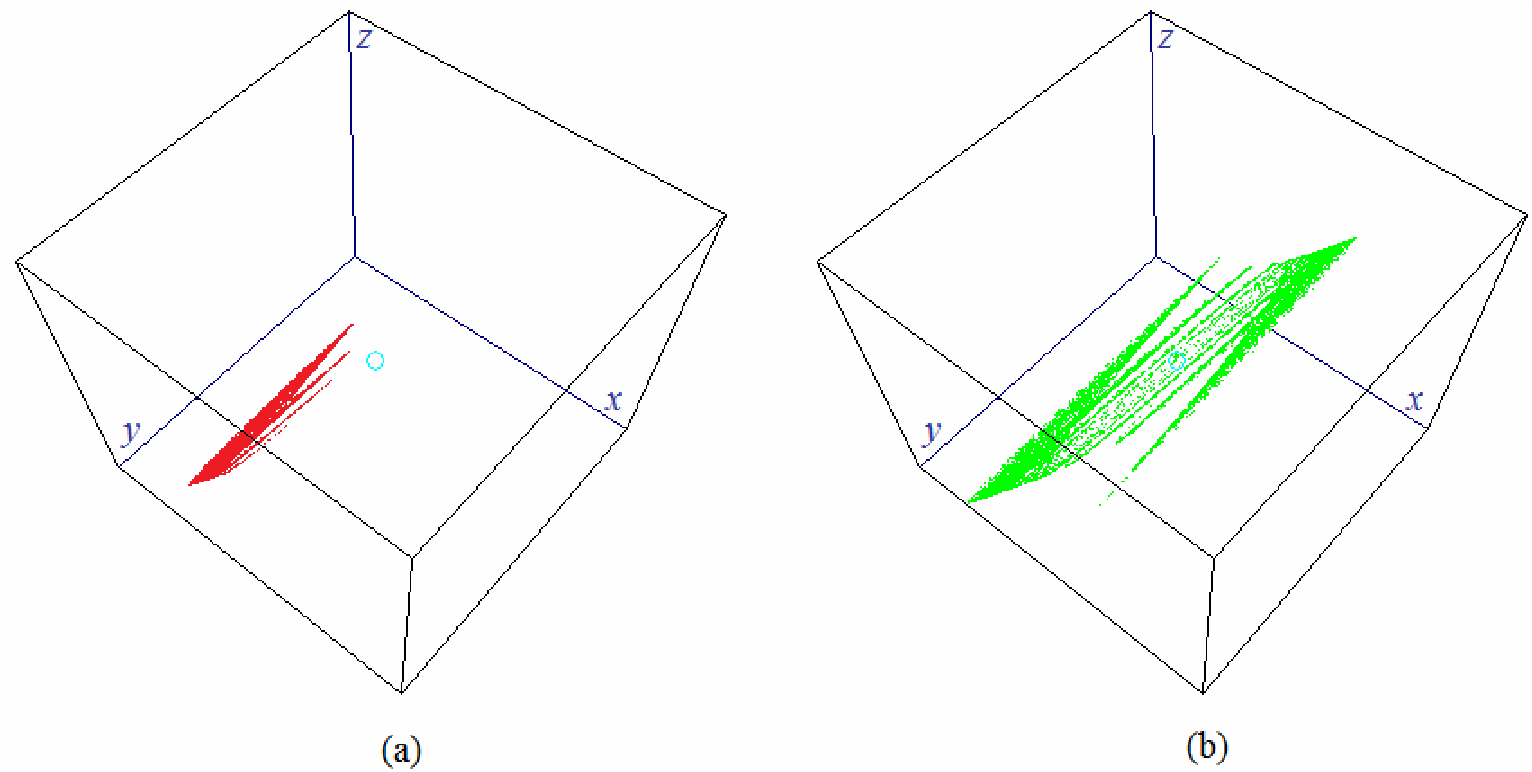
Close to the second homoclinic bifurcation of P_1 (and first homoclinic bifurcation of O) ($e=4.1841$)



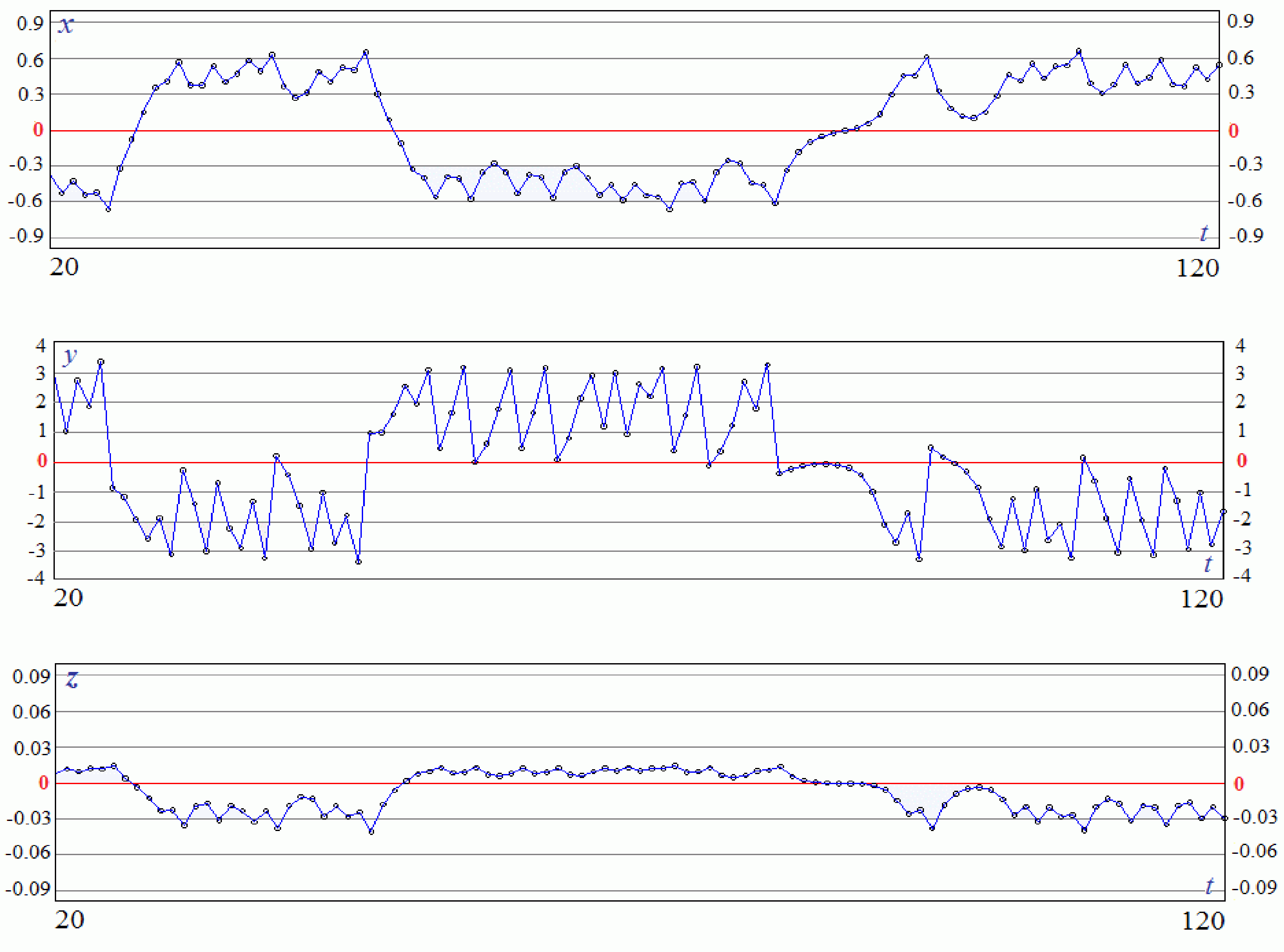
(a) Attractors
(b) Cross-section of the basins
(c) A portion of the plane through the saddle O, tangent to its stable set

Second homoclinic bifurcation of P_2
(and second homoclinic bifurcation of O)

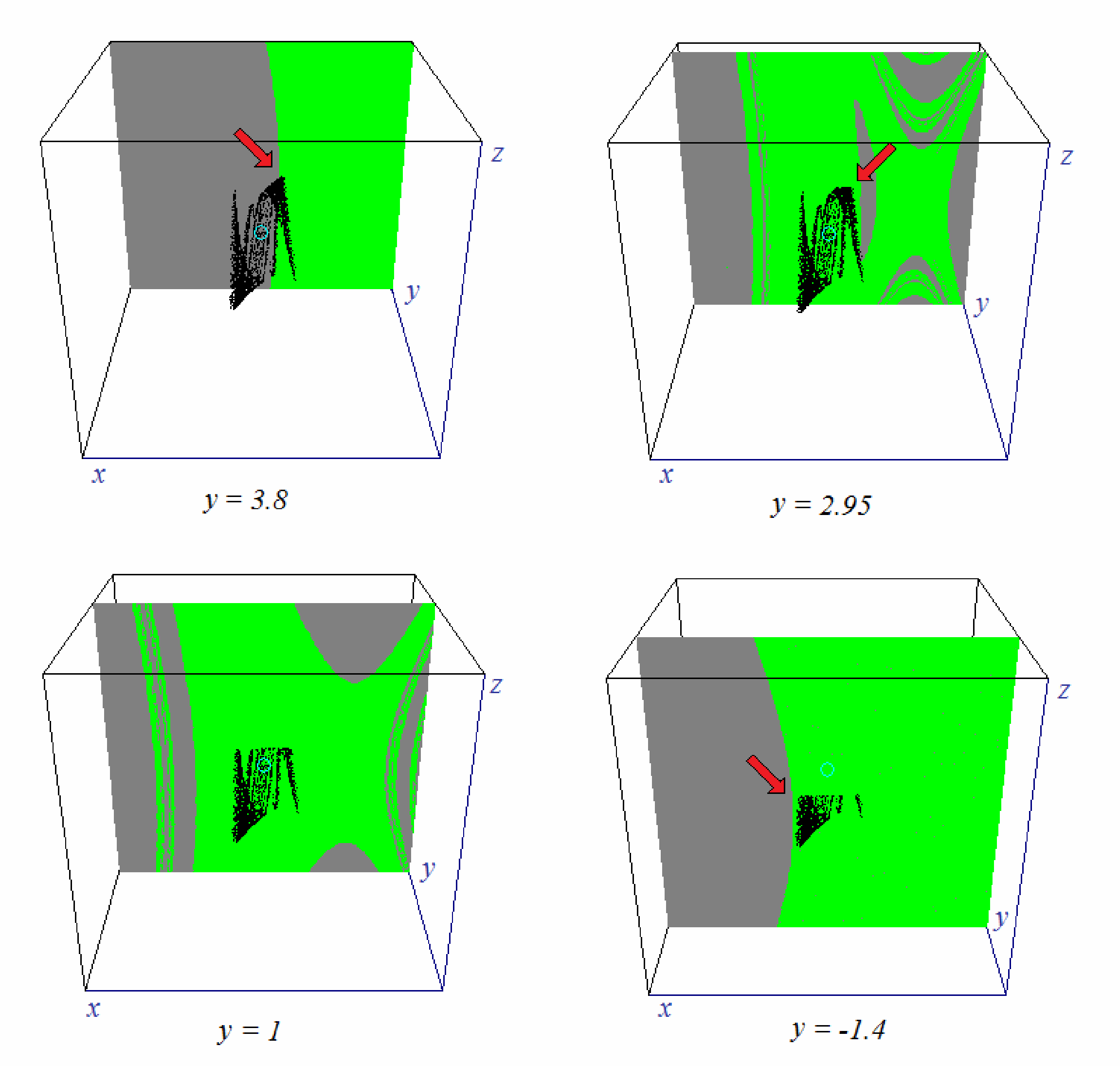
(a) Unique chaotic attractor in the 'bull' region, after the first homoclinic bifurcation of the saddle O ($e=4.208$)
(b) Unique chaotic attractor covering both 'bull' and 'bear' regions after the second homoclinic bifurcation of O ($e=4.761$)



Trajectories of x, y, and z switching across 'bull' and 'bear' regions, after the second homoclinic bifurcation of O ($e=4.75$)



Towards the ‘final’ bifurcation:
attractor and four different sections of the 3D basins



4 Summary of results

- A stylized deterministic model of two stock markets that interact *via* and *with* the foreign exchange market reproduces, in three dimensions, a regime of alternating 'bull' and 'bear' markets, first described by Day and Huang (1990) in a 1D model.

- The possibility to reduce the dimension of the dynamical system, via restrictions imposed on the activity of foreign traders, results in simplified one- and two-dimensional setups.

- The two- and complete three-dimensional models can be studied by properly extending the methods and concepts of the one-dimensional analysis (critical sets and properties of noninvertible maps, homoclinic bifurcations, ...)

- The two- and three-dimensional cases require a suitable mix of analytical, numerical, and graphical techniques.

- A sequence of homoclinic bifurcations, analogous to those of the one-dimensional case, takes the model across increasingly complex scenarios: coexistence of two attractors in two distinct 'bull' and 'bear' areas, sudden disappearance of one of them, chaotic behavior on a unique, larger attractor, with prices unpredictably switching among different regions of the phase space.