Global properties of 1D smooth DDS

Let us consider a dynamic model which is described by iterating some process: the state of the system changes under the action of some function T. The state x is assumed to be a scalar variable, the state space is a set $X \subseteq R$, and $T : X \to X$.

The discrete dynamical system (DDS) is represented by the standard notation

$$x_{n+1} = T(x_n) \quad or \quad x' = T(x)$$

The object of the theory of DDS is to understand

a) which kind of values will be obtained asymptotically, and this depending on the initial value, or initial condition, x_0, thus a description in the state space, and

b) how things change when the parameters of the system are varied, and this is a description in the parameter space, characterizing the bifurcations which may occur.

A *bifurcation* occurs when the changes occurring in the phase space cannot be obtained via a smooth transformations (the phase space before/after are not topologically conjugated).

Local bifurcations are those which can be studied via an approximation of the map (locally, in some fixed point or cycle).

Global bifurcations are those which cannot be studied by a Taylor expansion in some point, and require global properties of the map. Of this kind are the contact bifurcations between invariant sets, which cause qualitative changes in the structure of the basin, or in the structure of the attracting set, or both, and homoclinic bifurcations.

As we have seen, the simplest notion of invarian set is that of "fixed point". We say that x^* is a fixed point (or equilibrium point) of the DDS if it satisfies

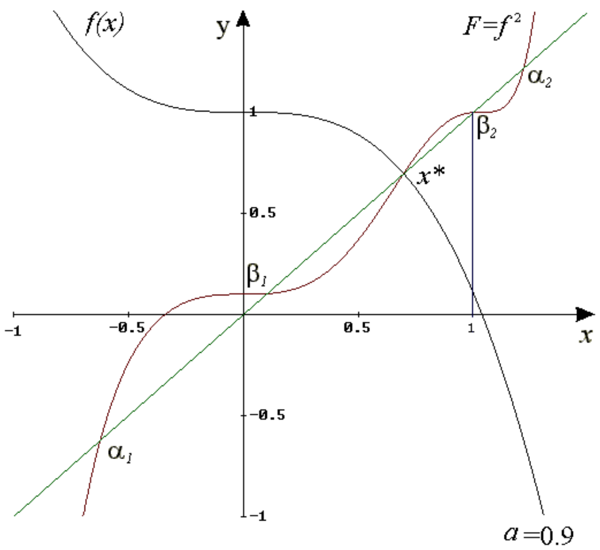
$$x^* = T(x^*)$$

Then for nonlinear functions the stability/instability is a local property, which may be investigated by the first order approximation of the function in the fixed point. We can summarize as follows:

$$\begin{aligned} -1 &< f'(x^*) < 1 \ : \ locally\ stable\ fixed\ point \\ f'(x^*) &= +1\ bifurcation \\ & \quad (fold,\ transcritical\ or\ pitchfork) \\ f'(x^*) &= -1\ flip\ bifurcation \end{aligned}$$

In the case of monotone increasing one dimensional functions the only possible invariant sets are fixed points which may be alternating: one stable, one unstable, the basins of attractions of the stable fixed points are bounded by the unstable fixed points or by infinity.

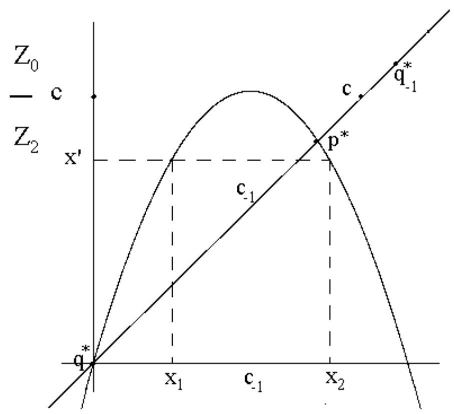
In the generic case of a decreasing one dimensional functions the only possible invariant sets are one fixed point, and 2-cycles, which may be alternating: one stable, one unstable.



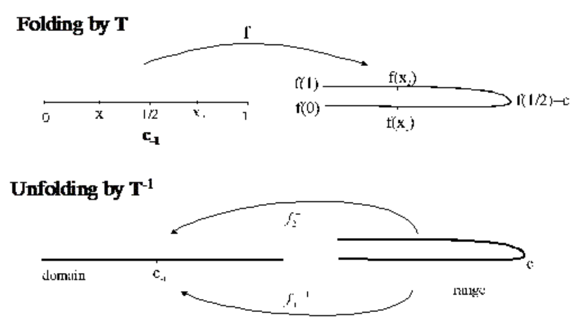
In functions with generic shape cycles may occur. A $k-$cycle is a sequence of k distinc points x_i, $i = 1, 2, ..., k$ visited iteratively by the map, and such that $f^k(x_i) = x_i$ for any point x_i. That is, stated in other words, each of the periodic points is a fixed point of the map $f^k = f \circ f \circ ... \circ f$. The stability/instability of a cycle is determined by the stability/instability condition of a fixed point of the map f^k and from the chain rule we have, for each point x_i of the cycle,

$$\frac{d}{dx}(f^k(x))|_{x_i} = \prod_{j=1}^{k} f'(x_j)$$

In the one-dimensional case we can see that once that the monotonicity (i.e. the invertibility property) is lost, then very complicated paths may occur, which may be predictable or not (although the model is completely deterministic).



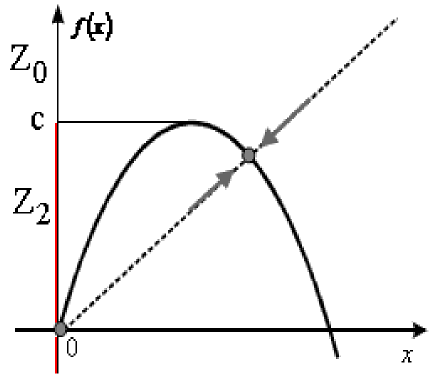
(a)



(b)

As a standard example let us consider the simple logistic map (whose graph is a parabola):

$$f(x) = \mu x(1-x) \quad , \quad \mu \in [3,4]$$



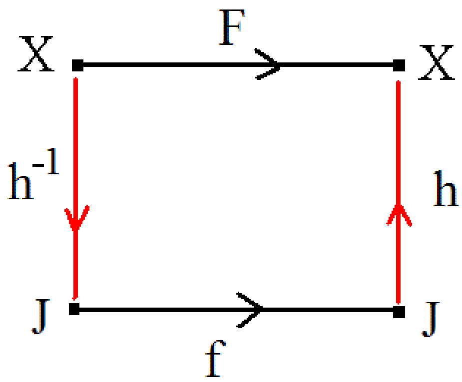
which for $\mu > 3$ has the origin as unstable fixed point and the positive fixed point which may be stable or unstable, depending on the slope (or eigenvalue) in that point.

Or, as an equivalent model, we may consider any function which is obtained by using a change of variable with an homeomorphisms h (a continuous and invertible function).

We are so introducing the concept of *Topological conjugacy*: let

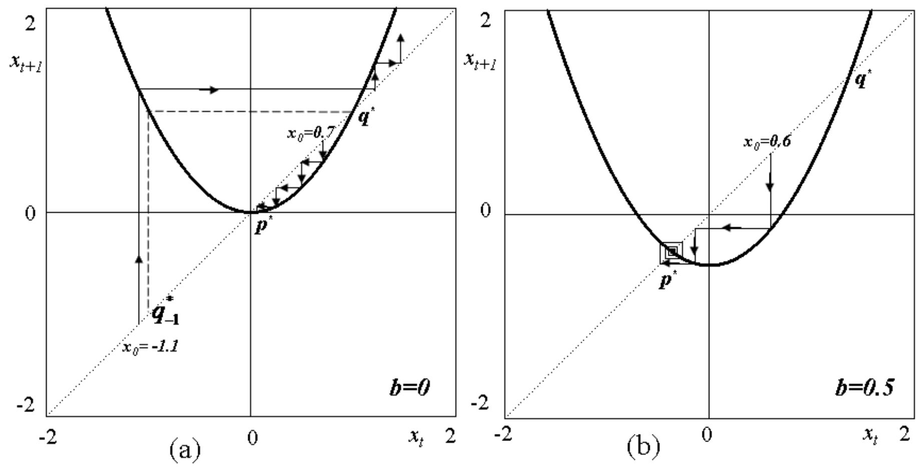
$$F = h \circ f \circ h^{-1}$$

then the maps F and f are called topologically conjugated.

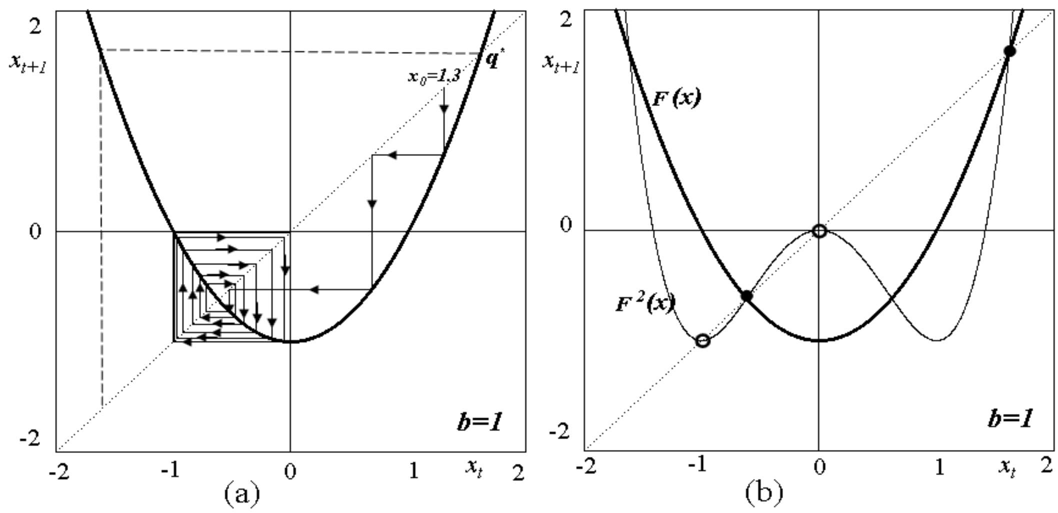


Topologically conjugated maps have the same dynamics: all the trajectories can be put in one-to-one correspondence by the homeomorphisms h. It is easy to see that via a linear homeomorphism we can transform the logistic function into the Myrber's map (from the name of the first author who studied in details the bifurcations of these non-invertible one-dimensional maps, still in 1965):

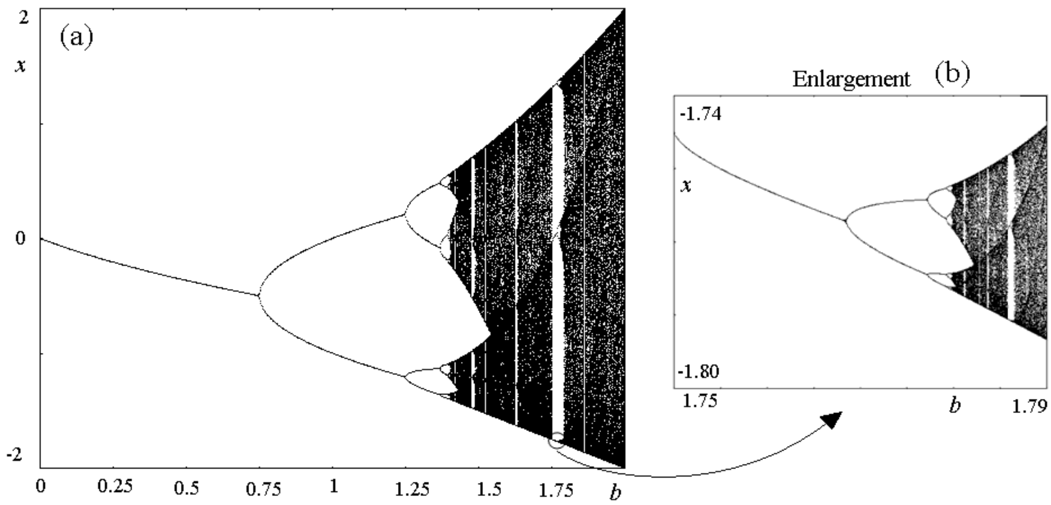
$$x' = F(x) \quad : \quad F(x) = x^2 - b$$

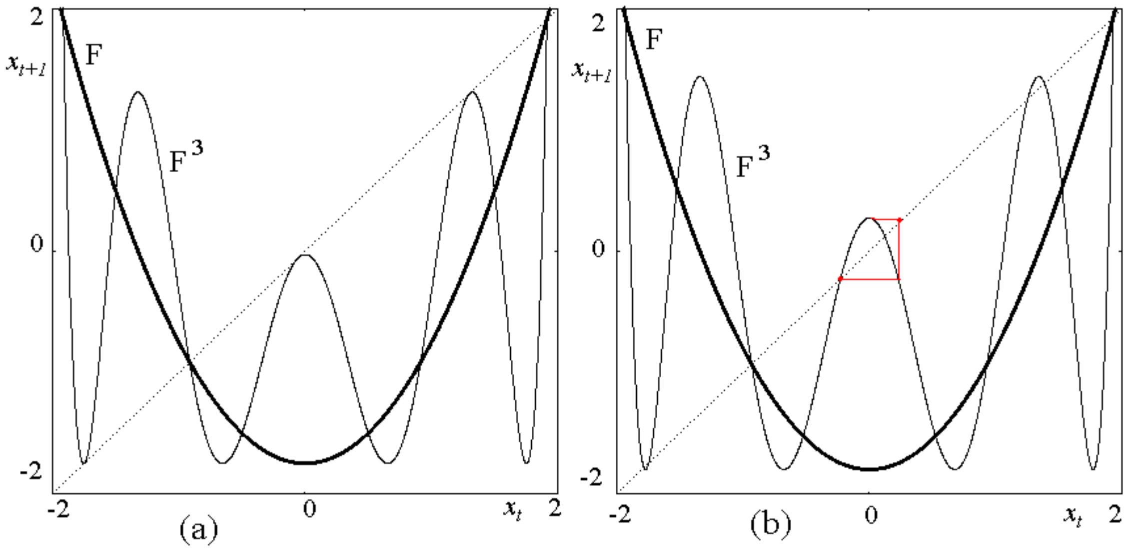


For $b \in [0, 2]$ we have $F : X \longrightarrow X,\ X = [q^*_{-1}, q^*]$ where q^* is the repelling positive fixed point. At $b = 0$ the slope at the stable fixed point p^* is zero in Fig.a (also called superstable), and then, increasing b, the slope from positive will become negative, reaching the value -1 and undergoing a flip bifurcation, leading to the appearance of a stable cycle of period 2.



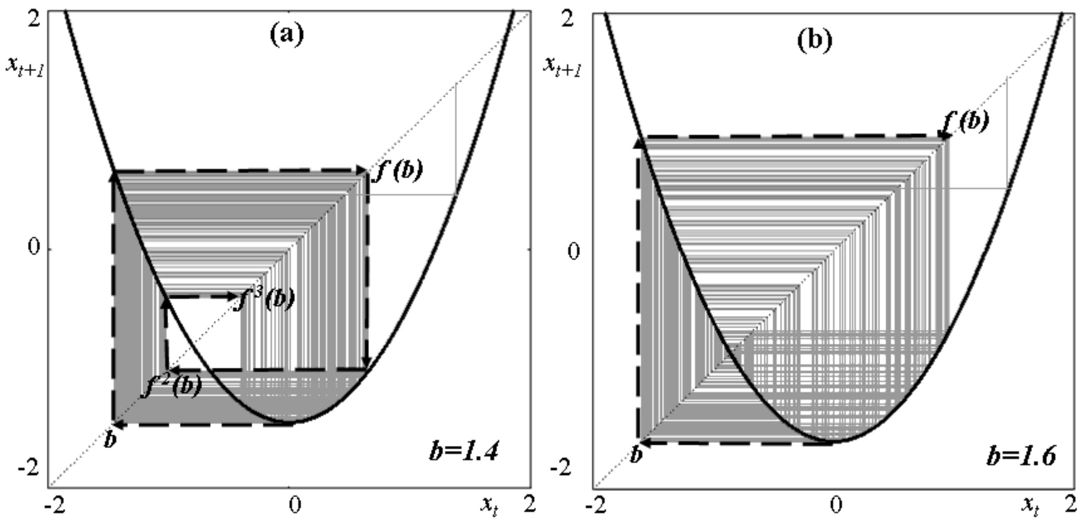
From the shape of the second iterate of the function we can see that locally the fixed point of the map F^2 (2-cycle of F) behaves as previously for the fixed point of the function F: the stable 2-cycle becomes superstable and then the slope becomes negative, reaching the value -1, and so on. By self-similarity all the cycles of period 2^n will be generated and become unstable leading, as n tends to infinity, to a critical bifurcation value $b = b_{1s}$ (Feigenbaum point) after which the map has a so-called chaotic behavior, because a set Λ invariant for the map, i.e. $F(\Lambda) = \Lambda$, always exists on which the restriction is chaotic. This is often represented in a bifurcation diagram which shows the asymptotic behavior of a generic point of the interval $X = [q^*_{-1}, q^*]$ as a function of the parameter b.



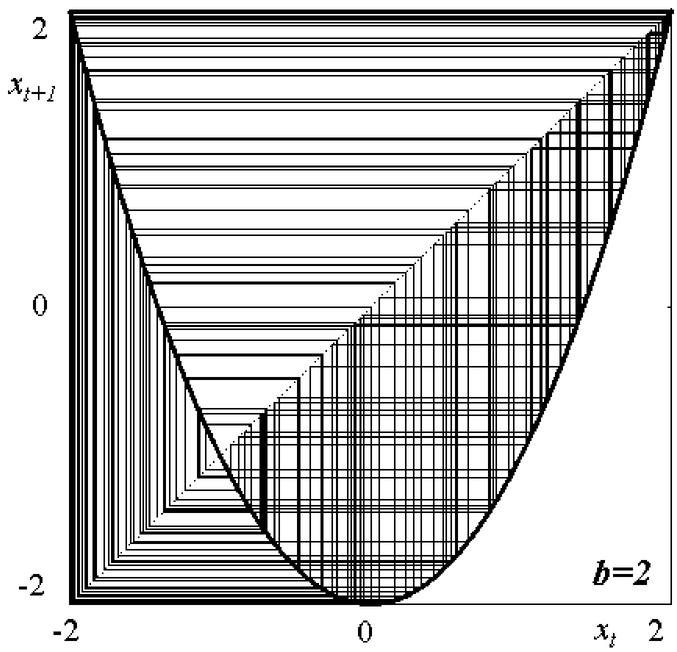


The bifurcation diagram is "self-similar" as for any period (and several boxes exist having the same period) we can repeat the period-doubling route to chaos described above. As an example the enlargement shows the "box" associated with the period-3 cycle: a pair of these cycles appear by saddle node-bifurcation, and the stable one, for the map F^3, will have the same bifurcation structure.

We also note that although the dynamic behavior is unpredictable when we are in a chaotic regime, some global properties can still be very useful. For example the iterates of the critical point determine cyclical intervals or one single interval inside which the trajectories are confined, and such intervals are trapping: starting in a different point of the interval $X = [q^*_{-1}, q^*]$ a trajectory enters such absorbing interval from which it will never escape.

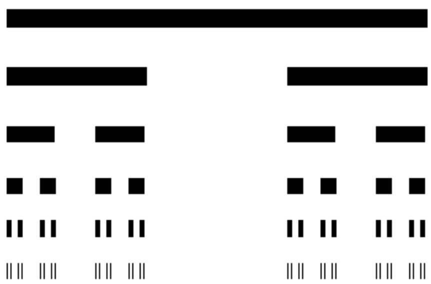


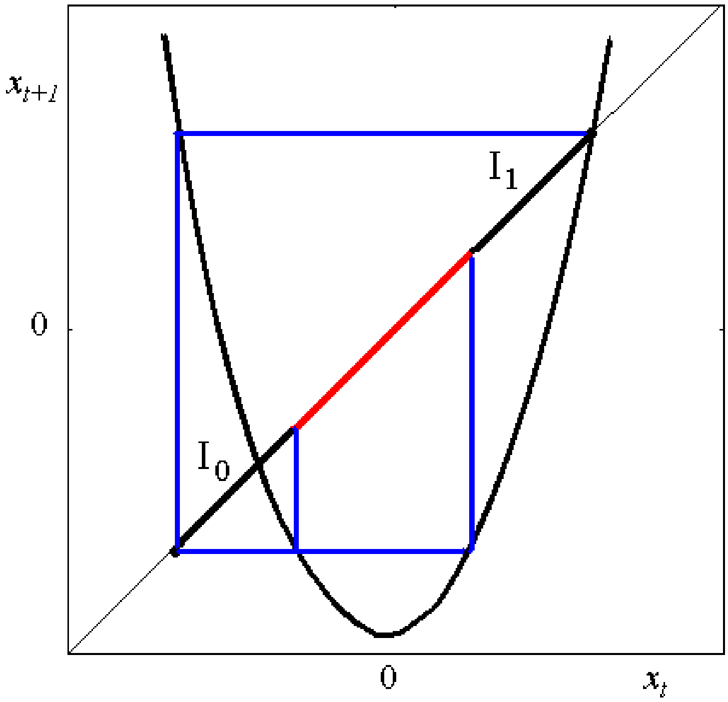
a "final bifurcation" is known to occur at the bifurcation value $b = 2$, when the preimage of the unstable fixed point becomes equal to the critical value, that is: the invariant interval $X = [q^*_{-1}, q^*]$ becomes an invariant chaotic interval, and after, for $b > 2$, the generic trajectory will be divergent. However a set which is invariant inside X still exists. It is a Cantor set inside which the map F is chaotic.



Cantor set

A set Λ is a Cantor set if it is closed, totally disconnected and perfect. The simplest example is the "Middle-third Cantor set": start with a closed interval X and remove the open "middle third" of the interval. Next, from each of the two remaining closed intervals, say I_0 and I_1, remove again the open "middle thirds", and so on. After n iterations, we have 2^n closed intervals inside the two intervals I_0 and I_1.





It is quite clear the similarity of this construction with that of the invariant set for the Myrberg's map for any $b > 2$. Considering our unimodal map, for any point ξ belonging to the interval $X = [q^*_{-1}, q^*]$ there are two distinct inverse functions:

$$F_0^{-1}(\xi) = -(b + \xi)^{1/2} \quad , \quad F_1^{-1}(\xi) = +(b + \xi)^{1/2}$$

The set of points whose dynamics is bounded forever in the interval X can be obtained removing from the interval all the points which exit the interval after n iterations, for $n = 1, 2, \ldots$.. Thus let us start with the two closed disjoint intervals

$$F^{-1}(X) = F_0^{-1}(X) \cup F_1^{-1}(X) = I_0 \cup I_1$$

i.e. we have removed the points leaving X after one iteration. Next we remove the points exiting after two iterations obtaining four closed disjoint intervals

$$F^{-2}(X) = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}$$

defining in a natural way

$$F^{-1}(I_0) = F_0^{-1}(I_0) \cup F_1^{-1}(I_0) = I_{00} \cup I_{10}$$

and

$$F^{-1}(I_1) = F_0^{-1}(I_1) \cup F_1^{-1}(I_1) = I_{01} \cup I_{11}.$$

Note that if a point x belongs to I_{01} (or to I_{11}) then $F(x)$ belongs to I_1 (i.e. one iteration means dropping the first symbol of the index). Continuing the elimination process we have that $F^{-n}(X)$ consists of 2^n disjoint closed intervals (satisfying $F^{-(n+1)}(X) \subset F^{-n}(X)$), and in the limit we get

$$\Lambda = \cap_{n=0}^{\infty} F^{-n}(X) = \lim_{n \to \infty} F^{-n}(X).$$

The set Λ is closed (as intersection of closed intervals), invariant by construction (as $F^{-1}(\Lambda) = F^{-1}(\cap_{n=0}^{\infty} F^{-n}(X)) = \cap_{n=0}^{\infty} F^{-n}(X) = \Lambda$).

Let us consider $b > 2$ and such that $|F'(x)| > 1$ for any $x \in I_0 \cup I_1$ (the property holds for any $b > 2$, but the proof is more complicated), then Λ cannot include any interval (because F is expanding). Thus Λ is totally disconnected, and perfect by construction, so that it is a Cantor set.

Moreover, by construction, to any element $x \in \Lambda$ we can associate a symbolic sequence, called Itinerary, or address, of x : $S_x = (s_0 s_1 s_2 s_3 ...)$ with $s_i \in \{0, 1\}$, i.e. S_x belongs to the set of all one-sided infinite sequences of two symbols Σ_2. S_x comes from the symbols we put as indices to the intervals in the construction process, and there exists a one-to-one correspondence between the points $x \in \Lambda$ and the elements $S_x \in \Sigma_2$. Also, from the construction process we have that if x belongs to the interval $I_{s_0 s_1 ... s_n}$ then $F(x)$ belongs to $I_{s_1 ... s_n}$. Thus the action of the function F on the points of Λ corresponds to the application of the "shift map σ" to the itinerary S_x in the code space Σ_2:

$$\begin{aligned} if \quad x &\in \Lambda \; has \; S_x = (s_0 s_1 s_2 s_3 ...) \\ & \qquad then \\ F(x) &\in \Lambda \; has \; S_{F(x)} = (s_1 s_2 s_3 ...) \\ &= \sigma(s_0 s_1 s_2 s_3 ...) = \sigma(S_x) \end{aligned}$$

Given a point $x \in \Lambda$ how do we construct its itinerary S_x? In the obvious way: we put $s_0 = 0$ if $x \in I_0$ or $s_0 = 1$ if $x \in I_1$, then we consider $F(x)$ and we put $s_1 = 0$ if $F(x) \in I_0$ or $s_1 = 1$ if $F(x) \in I_1$, and so on.

It follows that F is chaotic in Λ, because it is topologically conjugated with the shift map, which is the prototype of the chaotic map.

We recall that, following the definition of chaos given by Devaney, an invariant set is chaotic under the action of a map F if

1) there exist infinitely many periodic orbits,
   dense in the invariant set
2) there exist an aperiodic trajectory
   dense in the invariant set

As a consequence of the above two conditions we have that the sensitivity with respect to the initial conditions also exists (which often is added as a third condition).

The shift map is chaotic in Σ_2. Indeed, it is easy to see that the two properties hold. Notice that each periodic sequence of symbols of period k represents a periodic orbit with k distinct points, and thus a so-called $k-$cycle. Since the elements of Σ_2 can be put in one-to-one correspondence with the real numbers, we have that the periodic sequences are dense in the space, thus (1): the periodic orbits are dense in Λ. Also there are infinitely many aperiodic sequences (i.e. trajectories) which are dense in Λ thus (2) also is satified (it is enough to consider the binary representation of an irrational number).

Note that the main property in the previous construction is the existence of two disjoint intervals, I_0 and I_1, such that

$$\begin{aligned} F^k(I_0) &\supset I_0 \cup I_1 \\ F^k(I_1) &\supset I_0 \cup I_1 \end{aligned}$$

for a suitable k, and indeed this propery is the key feature in any dimension (i.e. for maps in R^m whith $m \geq 1$), to prove the existence of chaos. Also it is the basic feature in the theory of IFS since the pioneering work by Barnsley.

Definition. An Iterated Function System (IFS) $\{D; H_1, ...H_m\}$ is a collection of m mappings H_i of a compact metric space D into itself.

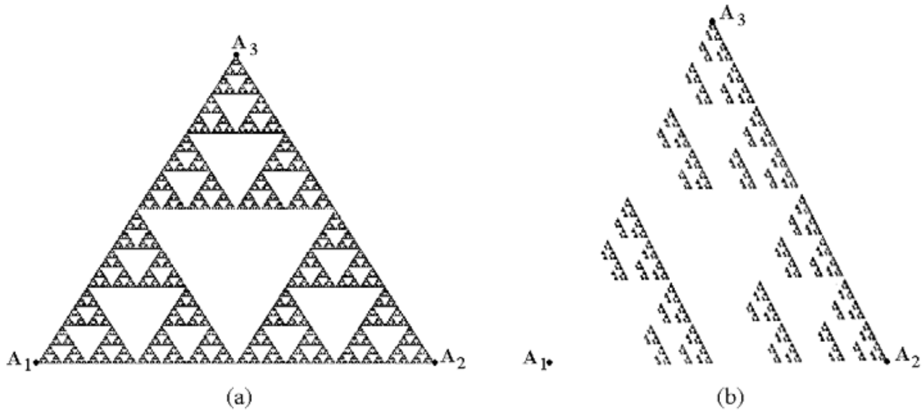
We can so define $W = H_1 \cup ... \cup H_m$. Denoting by s_i the contractivity factor of H_i then the contractivity factor of W is $s = max\{s_1, ...s_m\}$, and for any point or set $X \subseteq D$ we define

$$W(X) = H_1(X) \cup ... \cup H_m(X).$$

The main property of this definition is given in the following theorem:

Theorem (Barnsley 1988, p. 82). Let $\{D; H_1, ...H_m\}$ be an IFS. If the H_i are contraction functions then there exists a "unique attractor" Λ such that $\Lambda = W(\Lambda)$ and $\Lambda = \lim_{n\to\infty} W^n(X)$ for any non-empty set $X \subseteq D$.

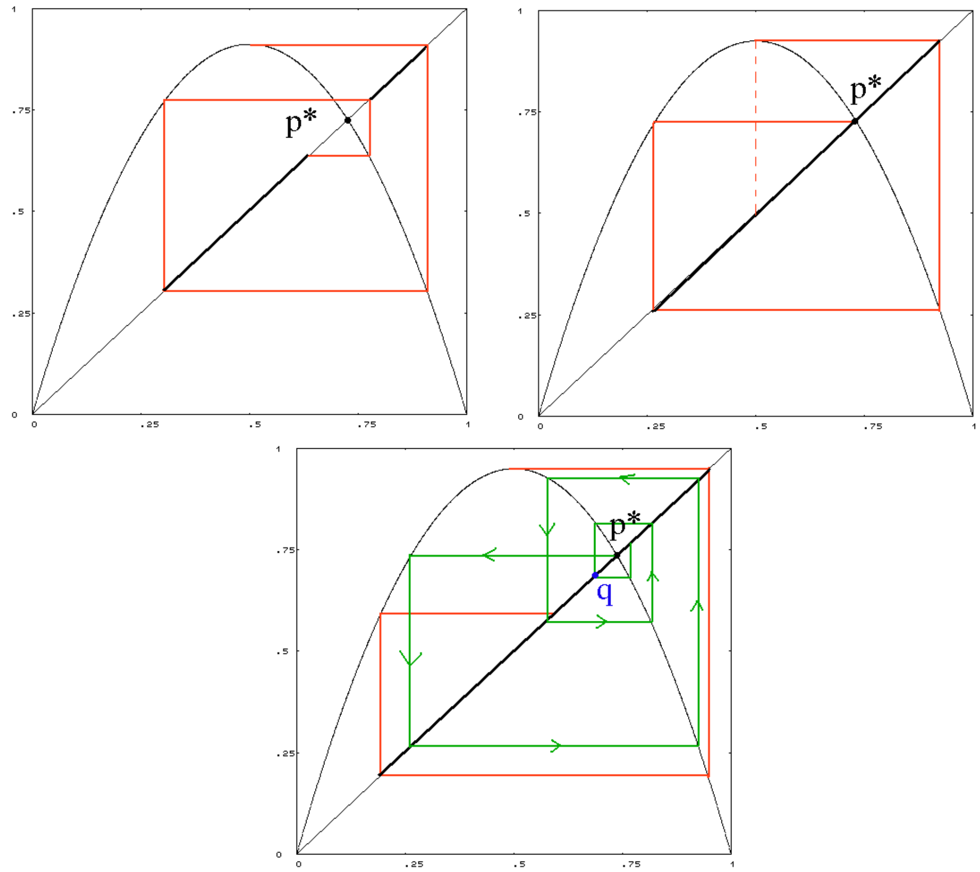
In the case previously described with the Myrberg's map we have $D = X$, $H_1 = F_0^{-1}$, $H_2 = F_1^{-1}$.



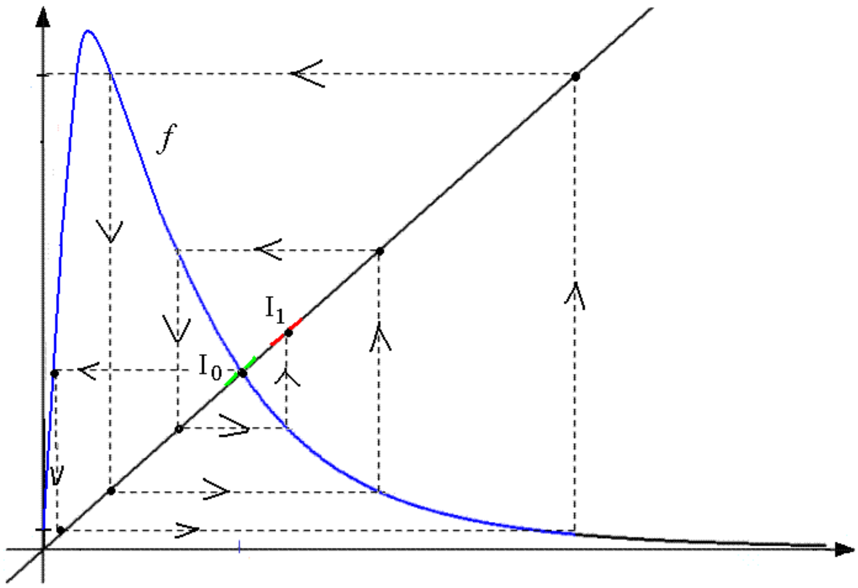
Example: the Sierpinski triangle is the unique attractor Λ of the ITF $\{D; H_1, H_2, H_3\}$. A subset of the Sierpinski triangle Λ^* is the unique attractor of the RIFS $\{D; H_1, H_2, H_3\}$ with the restriction that H_1 is never applied twice consecutively.

Snap back repellers

A similar property (leading to the construction of an invariant set on which the restriction of the map is chaotic) can be repeated whenever we have an homoclinic trajectory to some fixed point or cycle. A homoclinic trajectory is one which tends to some invariant set in the forward process, and in the backward one. For example, in the figures of the Myrberg's map or logistic map we can easily see homoclinic trajectories when the unstable fixed point p^* becomes homoclinic (also called snap back repeller, after Marotto 78).



See also the following unimodal map $f(x)$



When an homoclinc orbit exists (and the fixed point is called SBR) then an invariant chaotic set exists. In fact, it is possible to find two intervals I_0 and I_1 such that $f^k(I_0) \supset I_0 \cup I_1$ and $f^k(I_1) \supset I_0 \cup I_1$ for a suitable k (see also Gardini 1994, Gardini and Tramontana 2009, 2010).

Let U be a neighborhood of the unstable fixed point p^* in which the map is expansive ($|f'(x)| > 1$), and consider f_0^{-1} the local inverse and f_1^{-1} the other one. Take a suitable number of preimages following the homoclinic orbit, such that $I_1 = f_1^{-1} \circ \ldots . \circ f_0^{-1}(U) \subset U$ and take the same number, say k, of local preimages leading to $I_0 = f_0^{-1} \circ .... \circ f_0^{-1}(U) \subset U$. Then I_0 is disjoint from I_1 (as $f_0^{-1}(U)$ is disjoint from $f_1^{-1}(U)$), and such that $f^k(I_0) \supset I_0 \cup I_1$ and $f^k(I_1) \supset I_0 \cup I_1$, so we are done.

A remarkable application of this homoclinic theorem in the economic context occurs in the study of models formulated in the so called "backward dynamics". That is, as discrete models in the form $x_t = F(x_{t+1})$, and the interest is in the behavior of the forward values of the state variable $(x_t, x_{t+1}, x_{t+2}...)$. Two well known examples are the overlapping generations (OLG)-model (Grandmont and others) and the cash-in-advance model (Michener and Ravikumar and others). No problem when the function $F(.)$ is invertible (as $x_{t+1} = F^{-1}(x_t)$ is a standard dynamical system), while difficulties arise when $F(.)$ has not a unique inverse, and difficulties may also arise in the interpretation of the models. Mathematically, this kind of models have been investigated considering the inverse limit approach is rather abstract (as it always considers infinitely many states all together at once, without a real selection of the states step by step), so we prefer to follow a different approach, which is based on the theory of Iterated Function Systems. Whenever we have some homoclinic cycle we can apply the theory of Dynamical Systems and the theory of IFS to describe fractal "attractors" in the forward states of backward models.

Let us summarize some of the properties of the maps which are topologically conjugated to the logistic map or Myrberg's map $T : x' = x^2 - b$, say $T : X \to X$, $X = [q_1^{-1}, q_1]$ where q_1 is the fixed point always repelling for $b \in [-1/4, 2]$. The absorbing interval is $I = [T(x_c), T^2(x_c)]$ where x_c is the critical point.

On the x-axis, the repelling cycles and their preimages and limit points have a fractal organization when $b \geq b_{1s}$ where b_{1s} denotes the first Feigenbaum point, i.e. the limit point of the first flip bifurcation sequence of the $2-$cycle of T. For each value of the parameter b, $b \geq b_{1s}$, the fractal structure of the map singularities is completely identified from the box-within-a-box bifurcation structure described in the years 1975 by Mira (1987). Consider b $(b \geq b_{1s})$ such that the map has an attracting $k-$cycle C, then for the map T^k this cycle gives k attracting fixed points P_i, $i = 1, ..., k$, each of them with an immediate basin $B_0(P_i)$, and a total non connected basin

$B(P_i) = \cup_{n>0} T^{-kn} d_0(P_i)$. The total basins $B(P_i)$ have a fractal structure, and a strange repeller Λ_i belongs to the boundary of $\cup_{n=1}^{k} B(P_i)$. For the map T this is reflected in a cyclical property, so that the basin $B(C)$ is the union of the k basins and its fronties is a strange repeller Λ, i.e. an invariant set, $T(\Lambda) = \Lambda$, such that the restriction $T : \Lambda \to \Lambda$ is chaotic (in the sense of Devaney, i.e. topological chaos with positive topological entropy). This frontier (on which the map is chaotic) if a set of zero measure in the interval X.

When the parameter b varies in the interval $-1/4 \leq b \leq 2$ sequences of "boxes" occur, with the related bifurcations. Each box of the first kind is opened by a fold bifurcation giving rise to a pair of cycles, such a box of first kind closes when the cycle with $\lambda > 1$ becomes critical for the first time (i.e. the first time that a critical point merges in it, at its first homoclinic bifurcation). Inside each box of first kind the cycle with $\lambda < 1$ starts an infinite sequence of flip bifurcations, each of which opens a box of second class which closes when it becomes critical for the first time (i.e. at its first homoclinic bifurcation). Such sequences of boxes have a fractal structure due to the self similar property. All the boundaries of boxes of first or second class are bifurcation values.

For any value of b almost all the points x of the interval $]q_1^{-1}, q_1[$ (i.e. apart from at most a set of points of zero Lebesgue measure) have the same asymptotic behavior, which sometimes is called metric attractor A, due to this property, and independently on its nature. This metric attractor A can only be one of the following three typologies:

(1) a $k-$cycle (of any period $k \geq 1$, either stable ($|\lambda| < 1$), or neutral ($|\lambda| = 1$);

(2) a critical attractor (A_{cr}) with Cantor like structure, of zero Lebesgue measure;

(3) $k-$cyclic chaotic intervals, $k \geq 1$.

In the case (1) the generic omega limit set $\omega(x)$ is equal to the omega limit set of the critical point x_C, and the trajectory of x_C tends to the $k-$cycle, stable or neutral $A_\lambda,\ \omega(x_C) = A_\lambda$. In the case in which $|\lambda| = 1$ the cycle belongs to the frontier of its basin (or better, stable set). In the case in which $|\lambda| < 1$

the cycle is an attractor of T. For $b > b_{1s}$ the frontier of the basin of attraction is a strange repeller Λ, i.e. an invariant set, $T(\Lambda) = \Lambda$ such that the restriction $T : \Lambda \rightarrow \Lambda$ is chaotic (in the sense of Devaney). This frontier (on which the map is chaotic) is a set of zero measure in the interval X, and it is a topological repellor, i.e. a repelling set in the definition given above.

In the case (2) the generic omega limit set $\omega(x)$ is equal to $\omega(x_C) = A_{Cr}$ and $x_C \in A_{Cr}$. In this case $T : A_{Cr} \rightarrow A_{Cr}$ is chaotic, however A_{Cr} is not a topological attractor, that is, an "attractor of T" in the usual definition, but an "attractor in Milnor' sense" and its stable set is the whole interval, so that we can say that it is globally attracting in the interval.

We recall that an invariant set is an "attractor in Milnor' sense" when its stable set has positive Lebesgue measure in the space of the map.

In the case (3) the critical point x_C is either periodic or preperiodic, merging into a repelling cycle ($|\lambda| > 1$), which is called a critical periodic orbit, and at this parameter value a homoclinic bifurcation of this cycle occurs. The critical periodic orbit belongs to the chaotic intervals A. In this case $T : A \rightarrow A$ is chaotic, and A may be a topological attractor or an "attractor in Milnor' sense" depending on the parameter value (for example, at the closure of a box of second kind, it is a topological attractor, while at the closure of a box of first kind it is an attractor in Milnor's sense, but globally attracting in the open interval).

In all the cases (1), (2) and (3), the chaotic set is the closure of all the repelling points in I.

Noticing that in (2) and (3) above the chaotic sets attracts all the points of the interval, we may generically speak of "chaotic attractors", but the chaotic set is of *full measure* only in the case (3).

Let us define as

b_p the set of parameter values in the interval $[-1/4, 2]$ at which the typology (1) occurs,

b_{cr} and b_{ch} respectively the set of parameter values in the same interval $[-1/4, 2]$ at which the typology (2) and (3) respectively occurs.

Then it is important to notice that the set b_p consists of infinitely many nontrivial **intervals** having a fractal structure in the interval $[-1/4, 2]$ and **dense** in it (i.e. Closure(b_p) $=[-1/4, 2]$). The set b_{cr} is a completely disconnected set of zero Lebesgue measure while the set b_{ch} is a completely disconnected set of positive Lebesgue measure (for the proofs we refer to Thunberg [2001] and references therein).

Thus the set of points in the parameter space $[-1/4, 2]$ in which we have chaotic attracting sets of full measure in X is a set of positive Lebesgue measure.

At all the opening values of the boxes, the map is of typology (1), while all the closure values are global (homoclinic) bifurcations (belonging to the set b_{ch}), and the map is of typology (3). Inside each box of first kind there exists a limit value of boxes of second kind at which the the map is of typology (2) (the so called Feigenbaum point). Particular bifurcation values of b are those which are limit points of other bifurcation values (for example boundaries of boxes of first class), such bifurcation values belong to the set b_{ch} and the map is of typology (3). In particular, when the critical point x_c is periodic or preperiodic the map is of typology (3).

Let us also recall the *analytic solution* of the Myrberg map at $b = 2$, $x_{n+1} = x_n^2 - 2$ (MGBC book WS96 pag.35). Given an initial condition x_0 the explicit solution is:

$$x_n = 2\cos(2^n \arccos(\frac{x_0}{2})) \quad , \quad if \quad -2 \le x_0 \le 2$$
$$x_n = 2\cosh(2^n \cosh^{-1}(\frac{x_0}{2})) \quad , \qquad if \quad x_0 > 2$$

it is topologically conjugated with the

$$L\ (Logistic): \quad x_{n+1} = 4x_n(1 - x_n)$$

(via a linear homeomorphisms $h(x) = \alpha x + \beta$ where α and β can be easily found), it is also topologically conjugated to the shift map σ in the space of symbolic sequences Σ_2:

$$\sigma : \ \Sigma_2 \rightarrow \Sigma_2$$

to the

$$Q \ (Quadratic): \quad x_{n+1} = 2x_n^2 - 1$$

and conjugated or semi conjugated to the following maps:

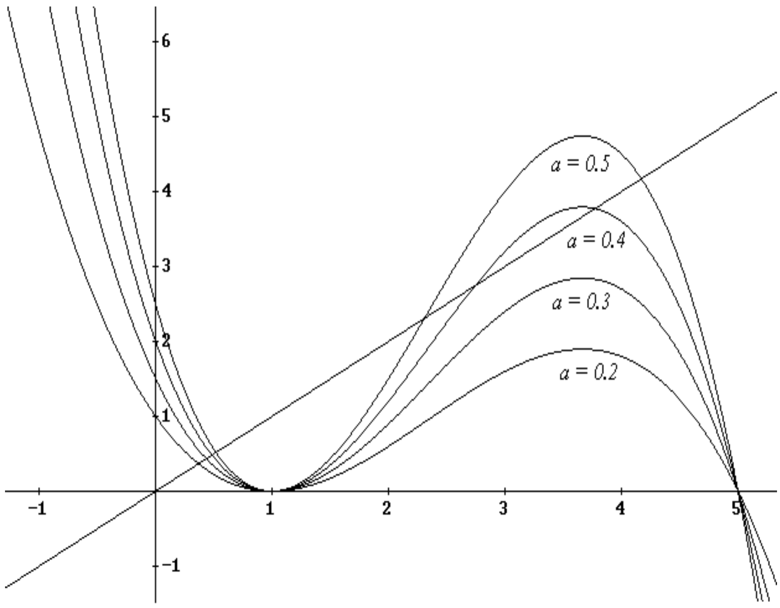
$$T \ (Tent): \quad \begin{cases} x_{n+1} = 2x_n \ \ if \ \ 0 \leq x_n \leq 1/2 \\ x_{n+1} = 2 - 2x_n \ \ if \ \ 1/2 \leq x_n \leq 1 \end{cases}$$

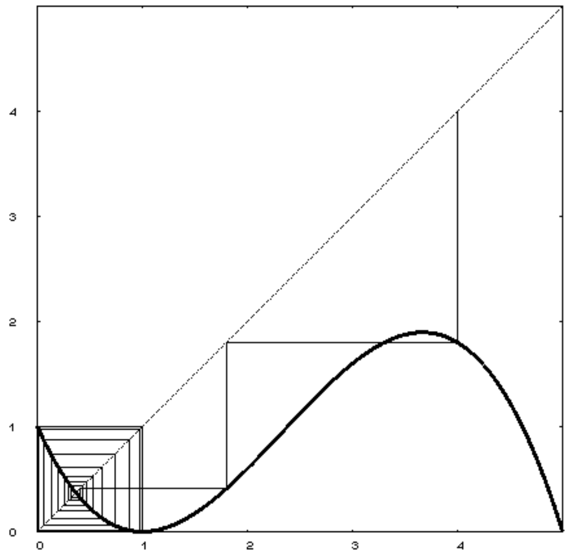
$$B \ (Baker): \quad x_{n+1} = \begin{cases} 2x_n & \text{if} \ \ 0 \leq x_n \leq \frac{1}{2} \\ 2x_n - 1 & \text{if} \ \ \frac{1}{2} < x_n \leq 1 \end{cases}$$

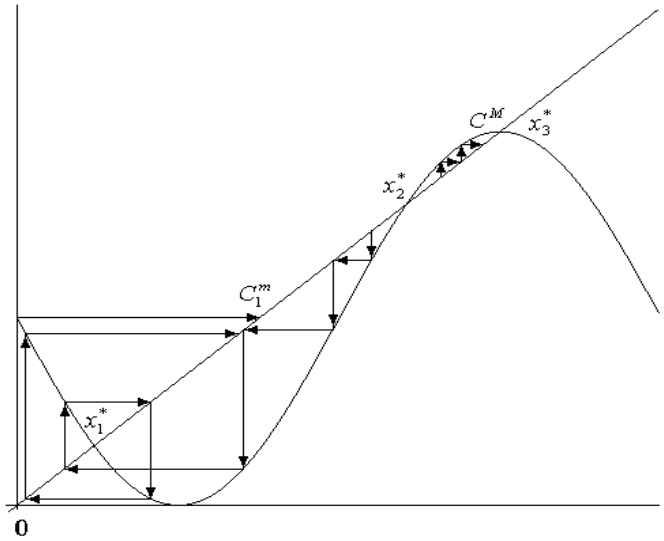
$$\begin{aligned} S \ (square) \quad &: \quad \theta_{n+1} = 2\theta_n \quad \text{mod} \quad 2\pi \\ \theta_{n+1} \quad &= \quad \begin{cases} 2\theta_n & \text{if} \ \ 0 \leq \theta_n \leq \pi \\ 2\theta_n - 2\pi & \text{if} \ \ \pi < \theta_n < 2\pi \end{cases} \end{aligned}$$

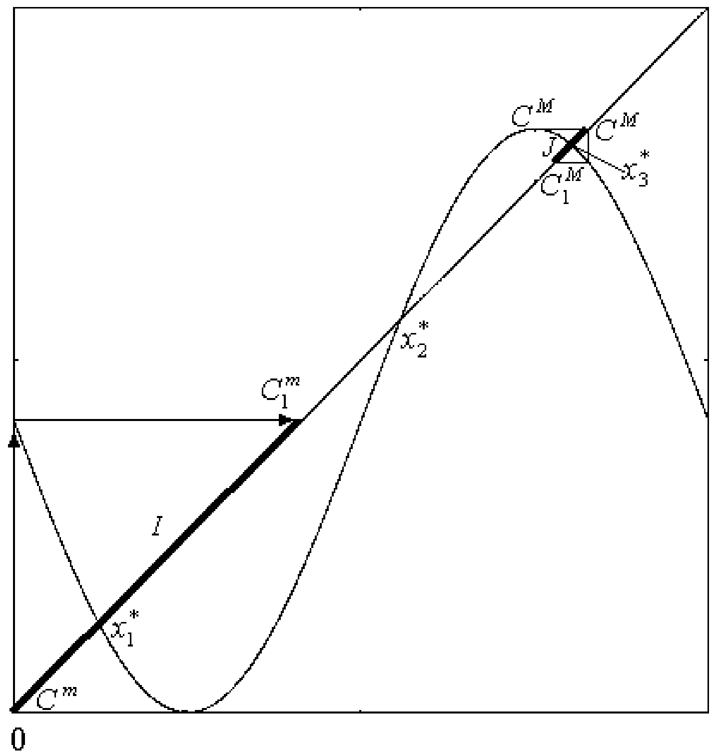
Generalizations

$$x' = a(1 - \sin(x))$$

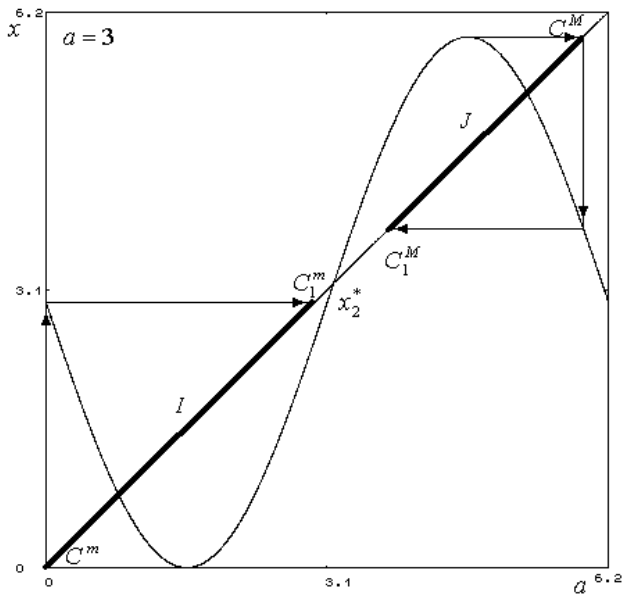



0
1
2
3
4
0
1
2
3
4

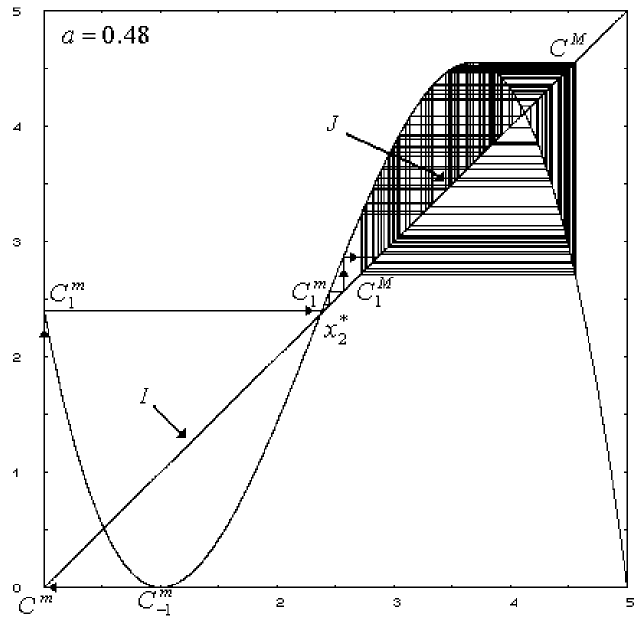

C^M
x_3^*
x_2^*
C_1^m
x_1^*
0



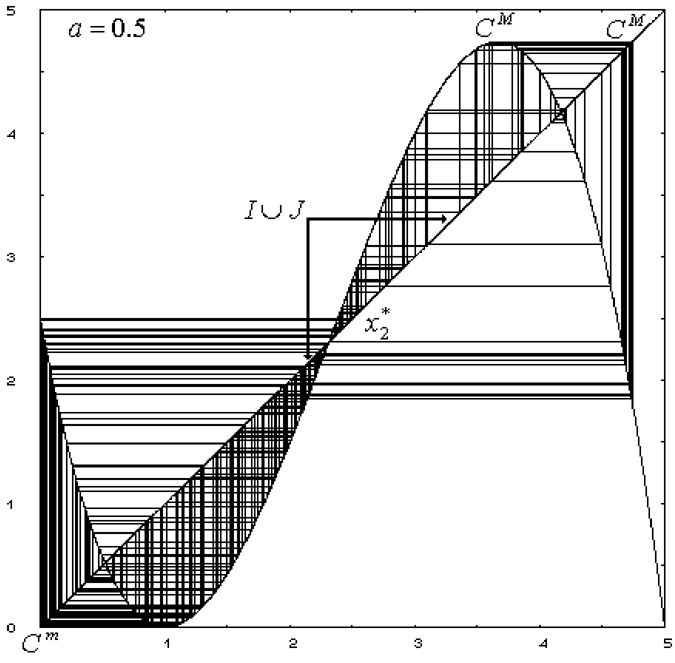
C^M
C^M
J
C_1^M
x_3^*
x_2^*
C_1^m
I
x_1^*
C^m
0



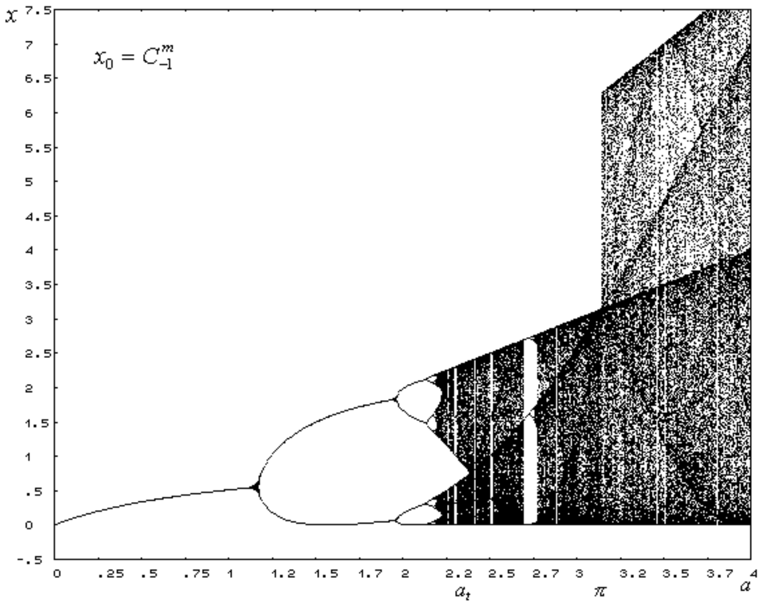
6.2
x
$a = 3$
C^M
J
C_1^M
3.1
C_1^m
x_2^*
I
C^m
0
0
3.1
a 6.2

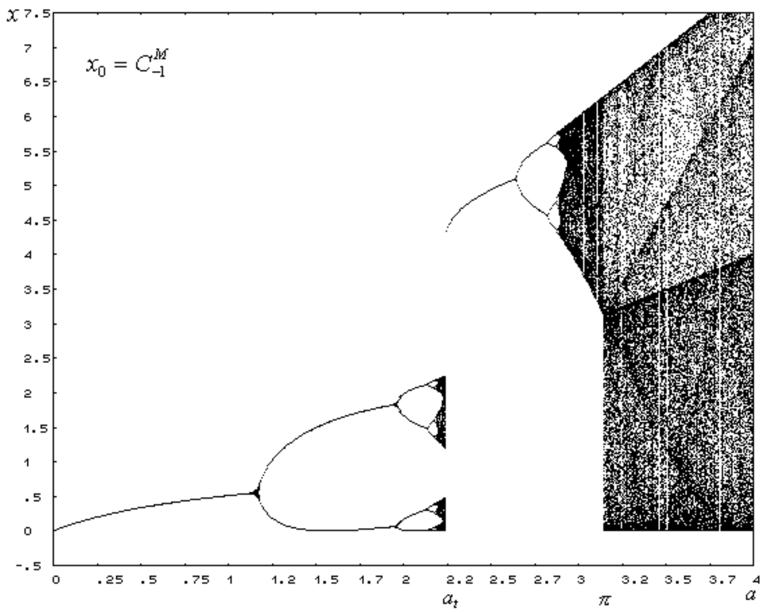


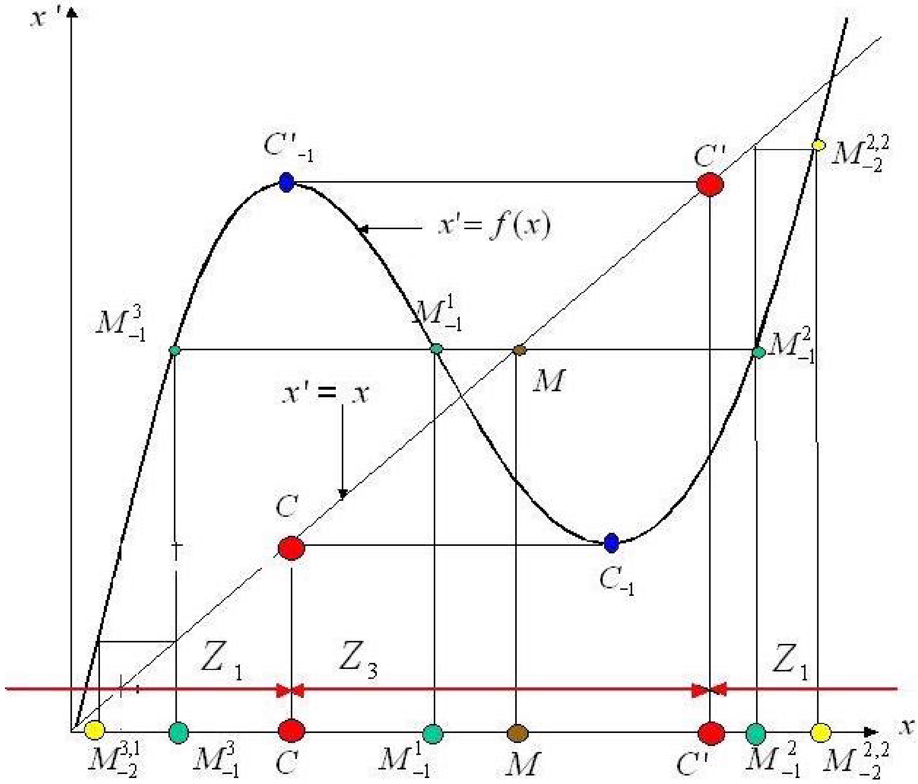
a = 0.48
J
I



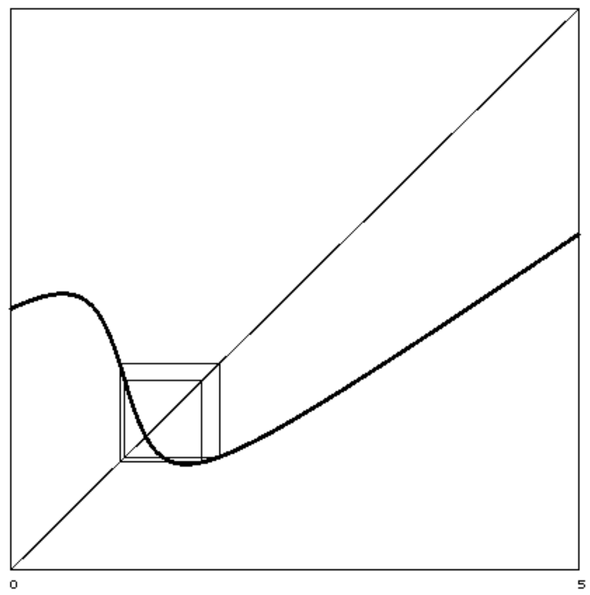
a = 0.5
I ∪ J

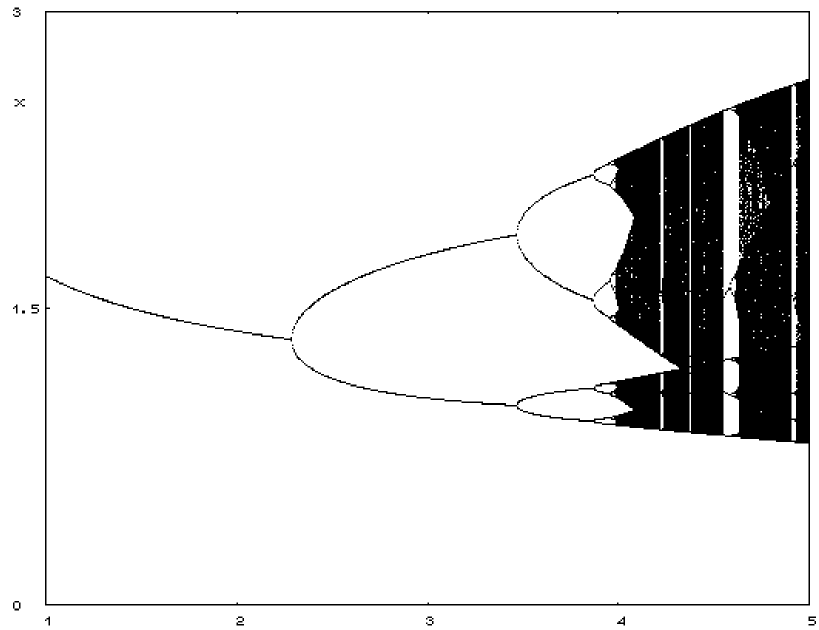

x
7.5
7
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
.5
0
-.5
$x_0 = C_{-1}^m$
0 .25 .5 .75 1 1.2 1.5 1.7 2 2.2 2.5 2.7 3 3.2 3.5 3.7 4
a_t
π
a

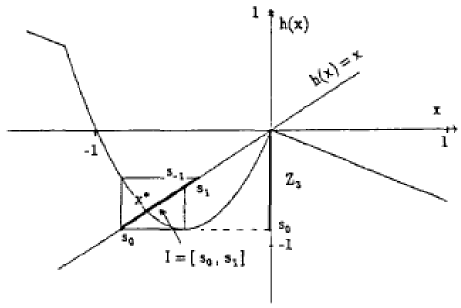

x
7.5
7
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
.5
0
-.5
$x_0 = C_{-1}^M$
0 .25 .5 .75 1 1.2 1.5 1.7 2 2.2 2.5 2.7 3 3.2 3.5 3.7 4
a_t
π
a

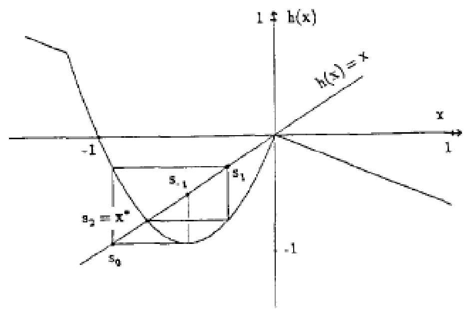


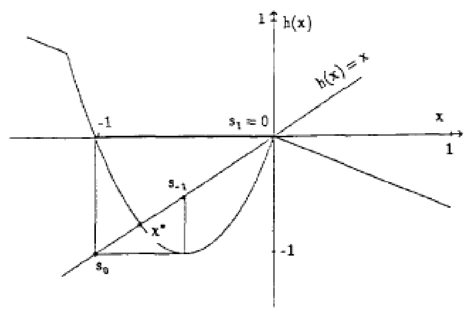
x'
C'_{-1}
C'
$M_{-2}^{2,2}$
$x' = f(x)$
M_{-1}^{3}
M_{-1}^{1}
M_{-1}^{2}
M
$x' = x$
C
C_{-1}
Z_1
Z_3
Z_1
x
$M_{-2}^{3,1}$
M_{-1}^{3}
C
M_{-1}^{1}
M
C'
M_{-1}^{2}
$M_{-2}^{2,2}$

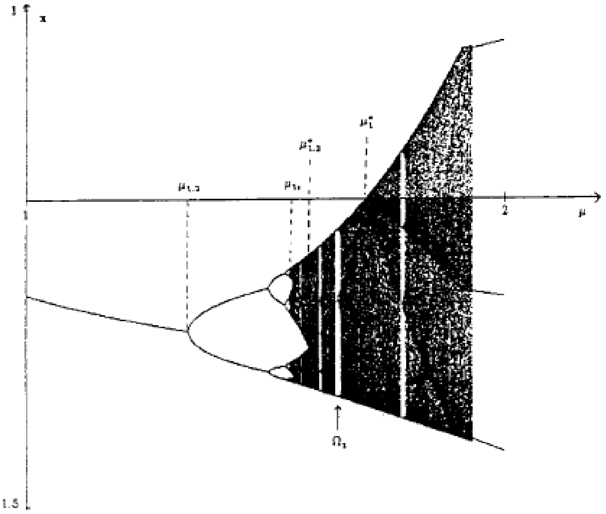

0
5

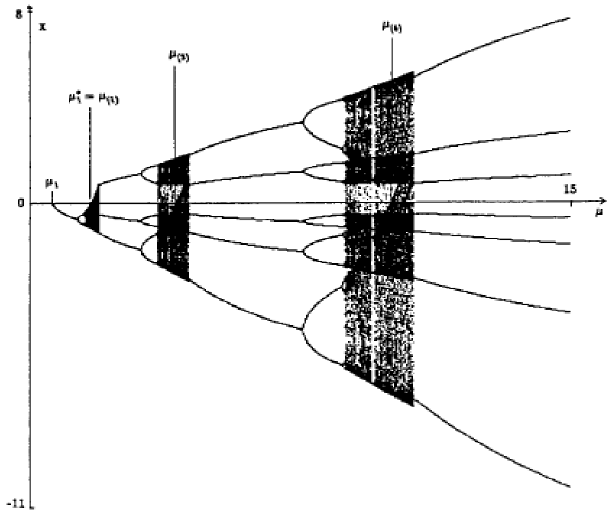

3
x
1.5
0
1
2
3
4
5

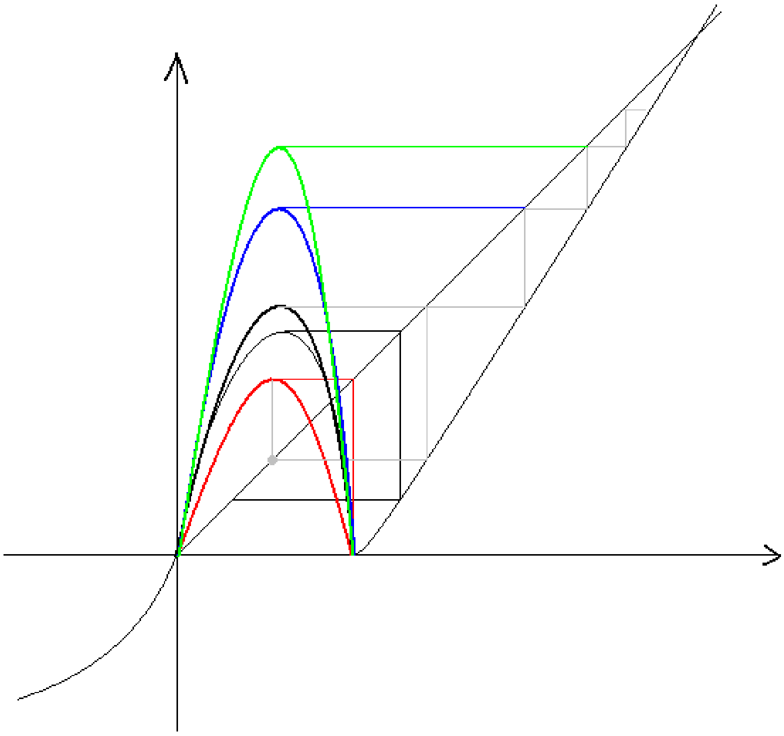

1
h(x)
h(x) = x
x
1
-1
s_{-1}
s_1
x*
Z_3
s_0
-1
s_0
I = [s_0 , s_1]


1
h(x)
h(x) = x
x
1
-1
s_{-1}
s_1
s_2 = x*
s_0
-1


1
h(x)
h(x) = x
x
-1
s_1 = 0
1
s_{-1}
x*
-1
s_0


1
x
$\mu_{1.2}$
$\mu_{1.2}^*$
μ_1^*
2
μ
Ω_1
-1.5


8
x
$\mu_{(6)}$
$\mu_{(3)}$
$\mu_1^* = \mu_{(1)}$
μ_1
0
15
μ
-11



Consider now a very simple map: a linear-fractional f:

$$f : x \mapsto f(x) = \mu + \frac{k}{x}$$

with asymptotes $x = 0$ and $f(x) = \mu$. The map f has two fixed points

$$x_{1,2} = (\mu \pm (\mu^2 + 4k)^{1/2})/2$$

which are real for

$$\mu^2 + 4k > 0$$

We are interested in the parameter range $\mu^2 + 4k < 0$ in which the map f has no real fixed points.

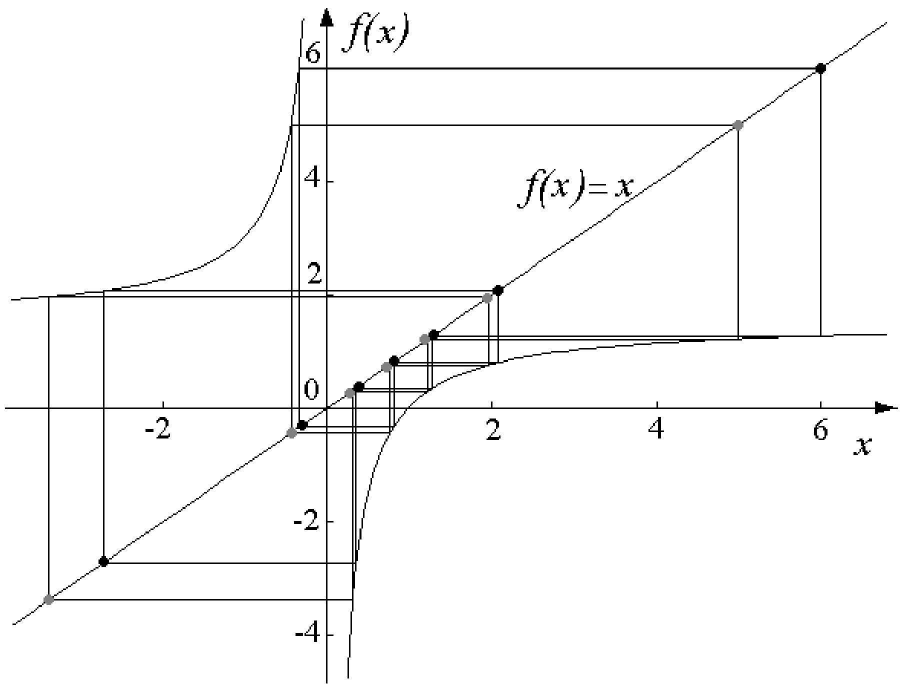
The dynamics of the map f for such parameter range is described in the following (SG09):

Proposition. Let $|\mu| < 2(|k|)^{1/2}$. *Then considering*

$$\frac{\mu}{2(|k|)^{1/2}} = \cos(\pi\rho)$$

when $\rho = m/n$ *is rational, then any trajectory of the map* f *is* n*-periodic with rational rotation number* m/n;

when ρ *is irrational, then any trajectory of* f *is quasiperiodic and dense on the real line.*



$\rho = 2/7$

Notice that the result holds for a generic linear-fractional

$$g : y \mapsto \frac{ay+b}{cy+d}$$

with $bc - ad \neq 0$, $c \neq 0$ in which, without loss of generality, we consider $c > 0$,

in fact it is topologically conjugate to the map

$$f : x \mapsto \mu + \frac{k}{x}$$

used above via the homeomorphism

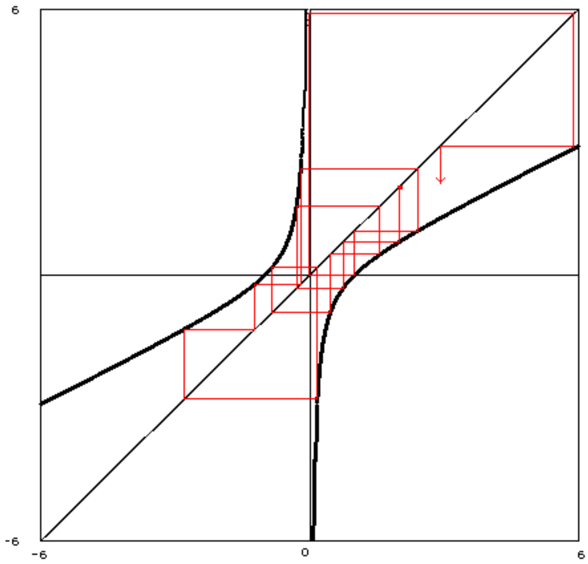
$$h(x) = x/c^{1/2} - d/c$$

and

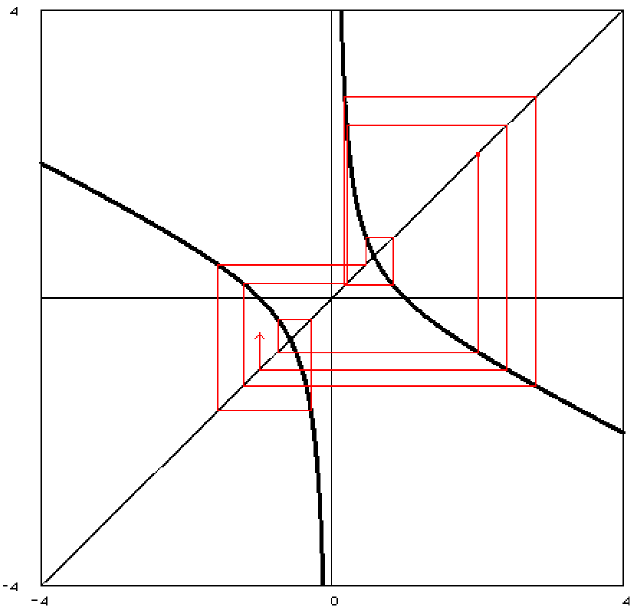
$$k = (bc - ad)/c \ , \ \mu = (a+d)/c^{1/2}$$

Thus the result shown above holds: when the map g has no real fixed points, and the dynamics of the points $y \in R$ either are all periodic, of the same period (which depends on the parameters), or all are quasiperiodic, with a trajectory dense in R.

Clearly this nice result is due to the existence of a unique inverse. As soon as the inverses are more, as we know, things may change.

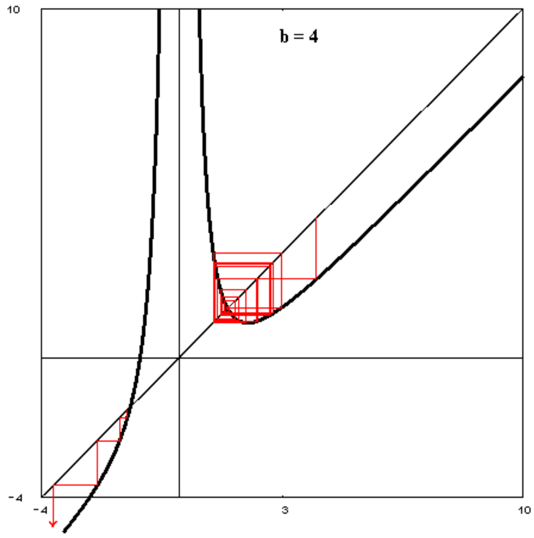


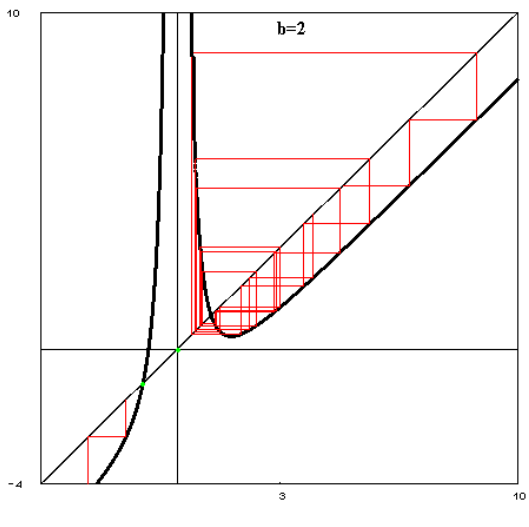
$$x' = (x^2 - a)/x$$

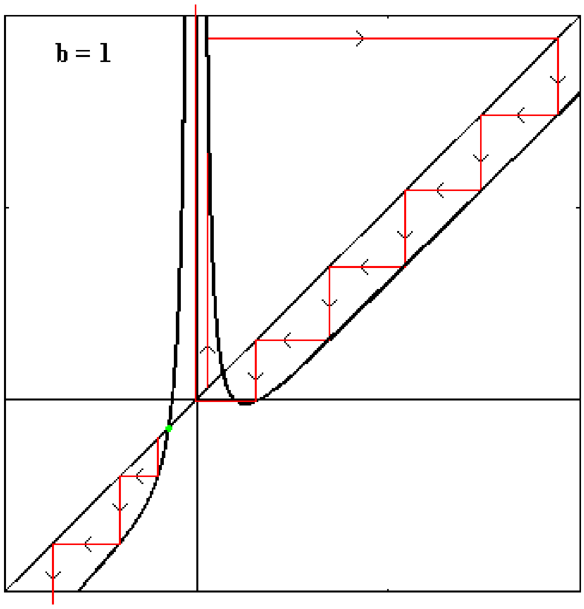


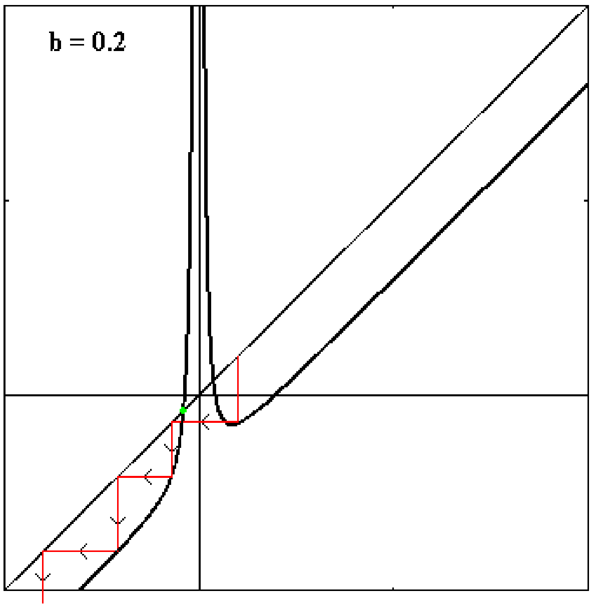
$$x' = (a - x^2)/2x$$

$$x' = x + \frac{b}{x^2} - 2$$






b = 1


b = 0.2

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