Examples and applications of one-dimensional discrete dynamical systems in economic, social and ecological systems

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Overview

- One-Sector Growth (OSG)
  - the basic OSG model
  - One-Sector Optimal Growth
  - OSG with pollution
  - from OSG to natural resources
- Dynamics of exploited renewable resources
  - Constant harvesting
  - Constant effort
  - Constant effort with depensation
- Overlapping Generation Models (OGM)
- Cobwebs
  - adaptive expectations
  - linear and nonlinear cobwebs
- Binary Choice models

Main Assumptions:

1. $Y = F(K, L)$ production function, K capital and L labour
2. F is an homogeneous function of degree 1;
3. $S = sY$, total savings proportional to outputs (s constant marginal savings rate)
4. $L(t+1) = L(t)(1+n)$, population with constant growth rate n
5. $K(t+1) = S(t) + K(t) - \delta K(t)$, δ obsolescence of capital

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The model:

- By 2.: $\frac{F(K,L)}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right) = f(k)$, per capita production of k (per capita capital)
- Rewrite 5.: $K(t+1) = sL(t)f(k(t)) + K(t)(1-\delta)$
- Dividing by $L(t+1)$ in 4. we get a one-dimensional map for state variable k:

$$\left(\frac{K(t+1)}{L(t+1)} =\right) k(t+1) = G(k(t)) = \frac{sf(k(t))}{(1+n)} + \frac{(1-\delta)}{(1+n)}k(t)$$

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One Sector Growth

Basic version (Solow-Swan)

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Problem

Show analytically that k_1^* is a repelling fixed point and k_2^* is asymptotically stable

Problem

Given the capital stock $k(0) = k_0$, find consumption plans $c(0), c(1), \ldots$ such that

$$
\begin{aligned}
v(k_0) &= \sup \sum_{t=0}^{+\infty} \beta^t \left[u(c(t))\right] \\
s.t.\ k(t+1) &= f(k(t)) - c(t)
\end{aligned}
$$

with $\beta \in (0,1)$. By the constraint, we get $c(t) = f(k(t)) - k(t+1)$ and so the problem can be restated as

$$
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v(k_0) &= \sup \sum_{t=0}^{+\infty} \beta^t u\left\{\left[f(k(t)) - k(t+1)\right]\right\} \\
s.t.\ 0 &\leq k(t+1) \leq f(k(t))
\end{aligned}
$$

- Take $u(x) = \log x$ and $f(k) = k^{\alpha}$, $1 > \alpha > 0$ (Cobb-Douglas with $\delta = 1$)

Solution

Bellman equation is

$$v(x) = \sup_{0 \leq y \leq x^{\alpha}} \left[\log\left(x^{\alpha} - y\right) + \beta v(y) \right]$$

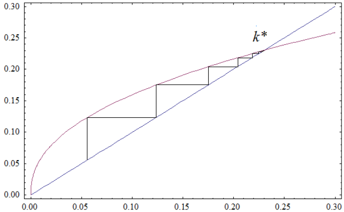
Try a solution of the type $v(x) = A + B \log x$ *and solve for* A *and* B*. By first order condition for maximization, we get* $y = \frac{B\beta}{1+B\beta} x^{\alpha}$*, which satisfies the constraint. After some algebra, we get*

$$A = \frac{1}{1-\beta}\left[\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right]$$

$$B = \frac{\alpha}{1-\alpha\beta}$$

One Sector Optimal Growth

- Substituing the definition of B into the first order condition we have the map
$$k(t+1) = G(k(t+1)) = \alpha\beta k(t)^{\alpha}$$
- *Turnpike* property: optimal capital stock converge to the fixed point k^*
- Mathematically, the map is identical with the one in the Solow-Swan example where it is postulated that saving propensity is a constant part of income.
- Boldrin and Montrucchio, 1986 provide a constructive method to find an optimal growth model with a given optimal policy function!



Example

Assume $f(k) = Ak^{\alpha}(m-k)^{\gamma}, k \leq m; A, \alpha, \gamma \geq 0, .$ Pollution is increased as a consequence of production, and resources have to be invested to reduce pollution, as reflected by the term $(m-k)^{\gamma}$.

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Solution

Take $\alpha = m = \gamma = \delta = 1$, the model reduces to $k(t+1) = \frac{sA}{(1+n)}k(t)(1-k(t))$. By letting $\frac{sA}{(1+n)} = a$ we get the standard logistic equation $k(t+1) = ak(t)(1-k(t))$

- Take the previous example by Day, 1982 and assume $\delta \geq 1$ to obtain $k(t+1) = rk(t)(1-k(t)) - qEk(t)$, where $\frac{(1-\delta)}{(1+n)} = -qE \leq 0$.

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- When $E = 0$, the fish population follows a logistic growth
- When $qE > 0$, the harvesting $H = qEk$ is proportional to the present biomass (Schaefer catch equation, see Clark, 1990)

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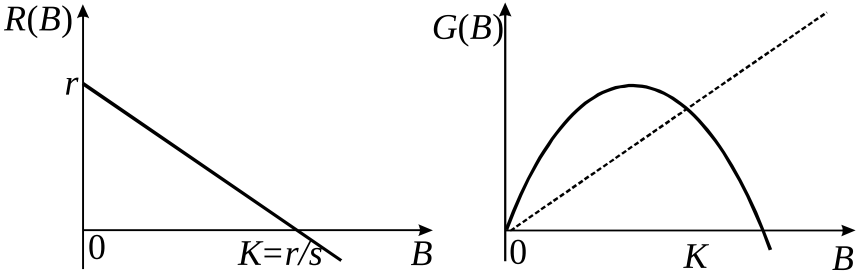

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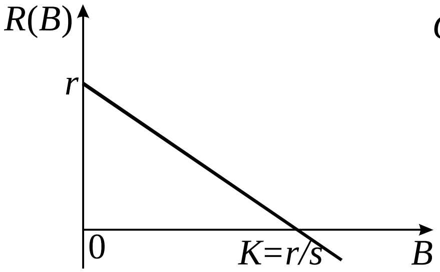

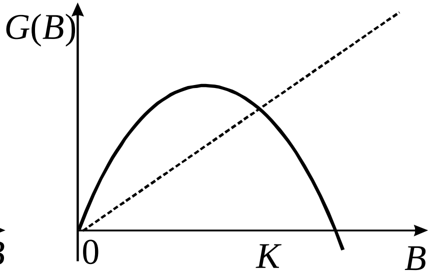

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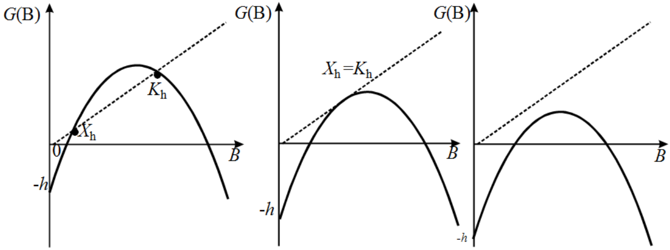
Dynamics of exploited renewable resources

Constant harvesting - Fold Bifurcation



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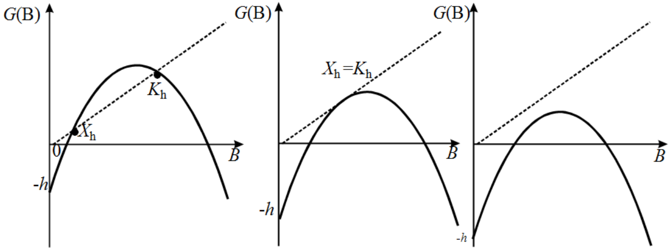

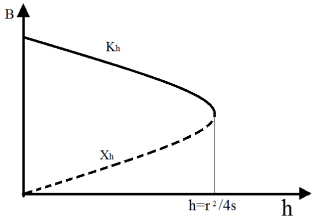

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Schaefer catch equation $H(t) = qEB(t)$, where E is harvesting *effort* and q is the *catchability* coefficient

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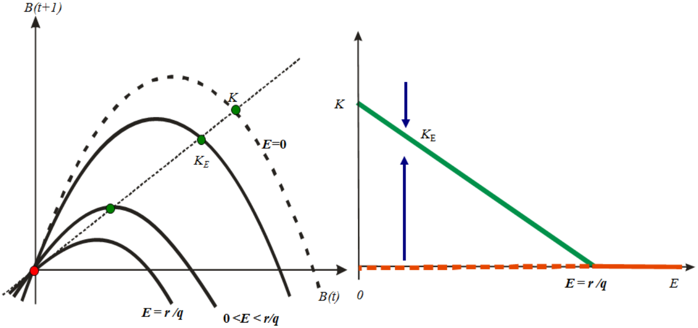
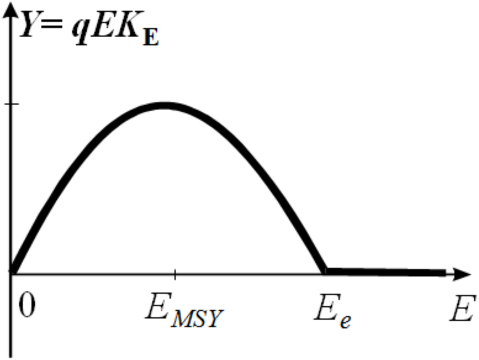
Two equilibria: $B_0 = 0$ (*extinction*) and $K_E = \frac{r-qE}{s}$ (*modified carrying capacity*)

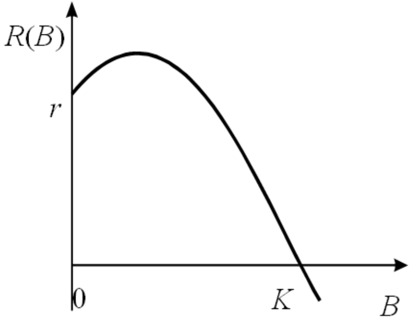
Dynamics of exploited renewable resources
Increasing fishing effort and transcritical bifurcation

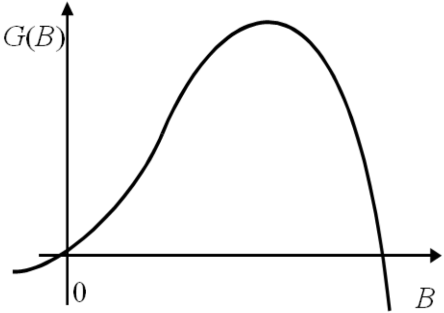




Yield-Effort curve

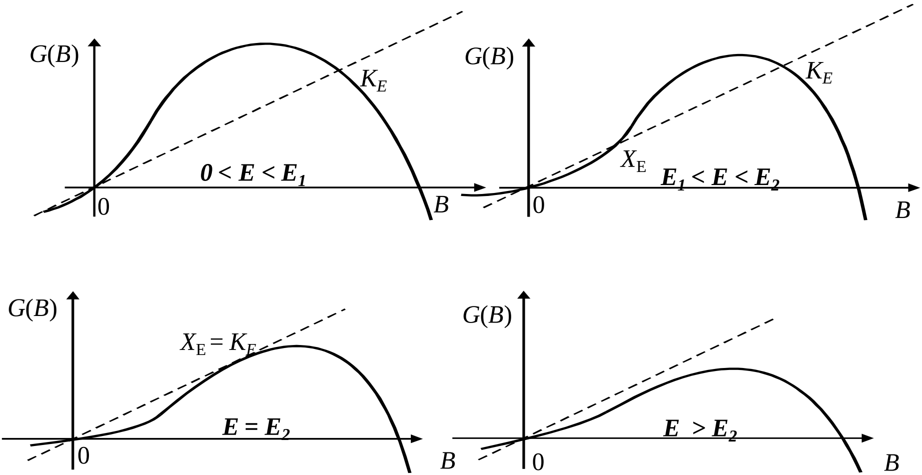
Actual biological populations might exhibit *depensation* (unimodal growth rate)

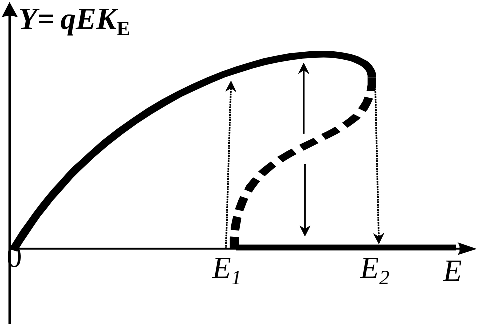




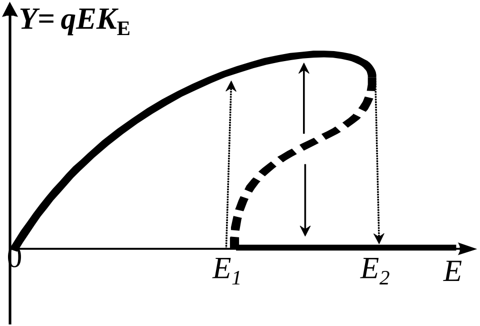
Dynamics of exploited renewable resources

Growth with depensation





Yield-Effort curve



Yield-Effort curve

The system exhibits *hysteresis* effects

Overlapping generation models

Basic Example

At each time period, only two agents operate in the economy, one old and one young. When young (e.g. time t), an agent solves the problem

$$\max_{c_0(t), c_1(t+1)} u\left(c_0(t), c_1(t+1)\right)$$

$$s.t. \begin{cases} c_1(t+1) = w_1 + (1+i_t)\left[w_0 - c_0(t)\right] \quad \text{(budget constraint)} \\ c_0(t) \geq 0; c_1(t+1) \geq 0 \quad \text{(non-negativity constraint)} \end{cases}$$

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- w_0, w_1 wages when, respectively, young and old (constants)
- i_t periodic interest rate at time t

At each time period, only two agents operate in the economy, one old and one young. When young (e.g. time t), an agent solves the problem

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Overlapping generation models
Basic Example

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- Writing the condition of market equilibrium (1) as $c_0(t+1) + c_1(t+1) = w_0 + w_1$, we obtain the unidimensional map

$$c_0(t+1) = w_0 + w_1 - G(c_0(t))$$

Example (see Gandolfo 1997)

$u\left(c_0(t), c_1(t+1)\right) = u\left(c_0, c_1\right) = ac_0 - \frac{b}{2}c_0^2 + c_1$, where $c_0 \in \left[0, \frac{a}{b}\right]$, $w_0 = 0; w_1 = \widehat{w} > \frac{a}{b}$. Condition (2) is

$$a - bc_0 - \frac{\widehat{w} - c_1}{c_0} = 0 \Longleftrightarrow c_1(t+1) - \widehat{w} = -c_0(t)\left[a - bc_0(t)\right]$$

and by (1), $c_0(t+1) + c_1(t+1) = \widehat{w}$,

$$c_0(t+1) = c_0(t)\left[a - bc_0(t)\right]$$

by letting $c_0(t) = \frac{a}{b}x(t)$, we have the standard logistic map

$$x(t+1) = ax(t)\left[1 - x(t)\right]$$

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- Consumption depletes the environment
- Zhang derives an unimodal map, for which chaos a là Li-Yorke emerges (3-cycle).

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Cobwebs

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4. By (3), we can compute actual prices from expected ones.

Example

$D(p) = a - bp$ and $S(\widehat{p}) = -m + s\widehat{p}$, $a, b, m, s > 0$.

$$\widehat{p}\,(t) = w\frac{a+m}{b} + \left[1 - w - \frac{sw}{b}\right]\widehat{p}\,(t-1)$$

is a linear difference equation of the first order. The equilibrium (expected and actual) price is

$$p^* = \frac{a+m}{b+s} > 0$$

Assuming $w > 0$, p^* is stable $\Longleftrightarrow -1 < 1 - w - \frac{sw}{b} < 1$, $\Longleftrightarrow w\left(1 + \frac{s}{b}\right) < 2 \Longleftrightarrow \begin{cases} b > s \\ b \leq s \text{ and } w \in \left[0, \frac{2b}{b+s}\right) \end{cases}$. At $w\left(1+\frac{s}{b}\right) = 2$, fluctuations remain of constant magnitude (2-cycle). For $w\left(1+\frac{s}{b}\right) > 2$, fluctuations increase in magnitude with each period.

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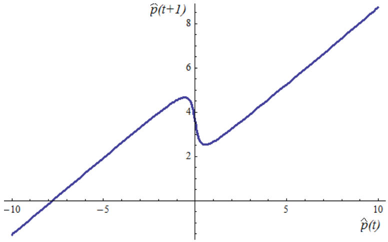

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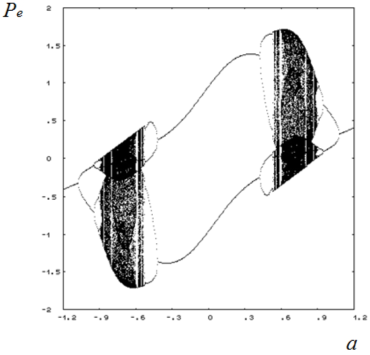

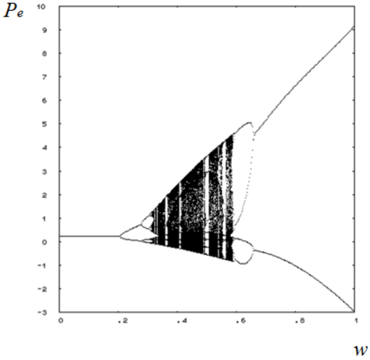

Binary Choices
Basic ideas, Schelling, 1973

- Each agent of a large population makes a binary decision (A or B)
- Denote by $x \in [0, 1]$ the fraction of players that choose strategy A.
- Payoffs are continuous functions of x, $A(x) : [0, 1] \rightarrow \mathbb{R}$, $B(x) : [0, 1] \rightarrow \mathbb{R}$ where $A(x)$ and $B(x)$ represent the payoff associated to strategies A and B, respectively.
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- Should I wear the helmet or not during the hockey match?
  - *It depends if the other guys do or not.*
- Should I carry a weapon or going unarmed?
  - *It depends on what other guys do (apply to nations)*
- Should I take the car or the train ?
- Should I invest in R&D or not? (consider spillover effects)
- Join or not? (switch watches to daylight saving time or stay on standard time)
- Should I dress elegant or not at the annual meeting of my society?
- Should I get annual flu vaccination or not ?
- Should I spray the insecticide in my garden or not?
- Should I go to vote for my favourite party or not?

- Population of N firms, each with two strategies available:
  - S_1: invest in R&D with payoff A
  - S_2: just spillovers with payoff B
- Let $x = n/N \in [0,1]$ be the fraction of players that choose strategy S_1, $(1-x)$ choose S_2 :
  - $x = 0$: all choose S_2 (just spill)
  - $x = 1$: all choose S_1 (invest in R&D)
- Payoffs are functions $A(x)$ and $B(x)$ defined in $[0,1]$

$$A(x) = (a+b)x - c; B(x) = bx$$

- Population of N firms, each with two strategies available:
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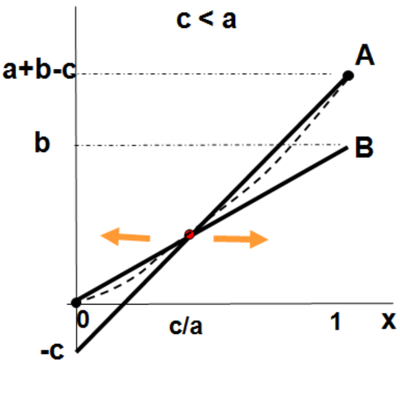
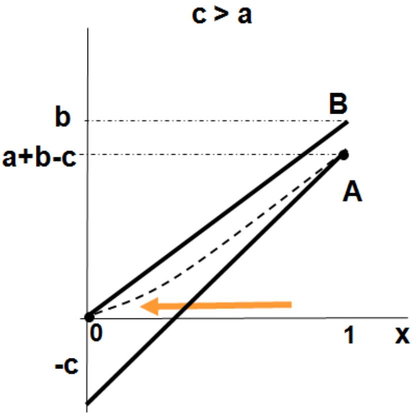
Binary Choices





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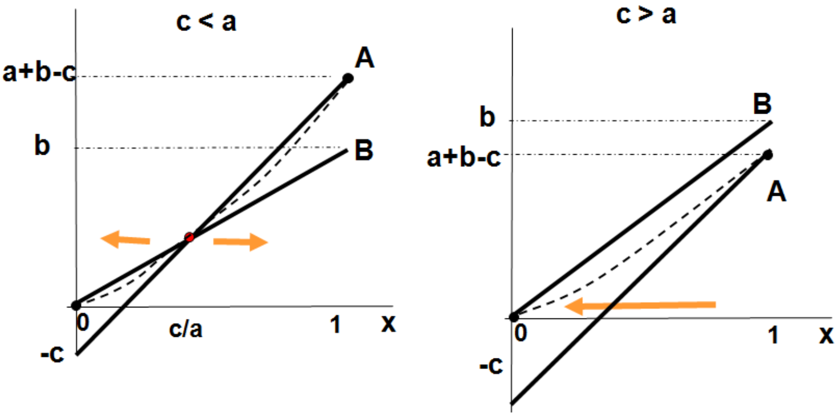
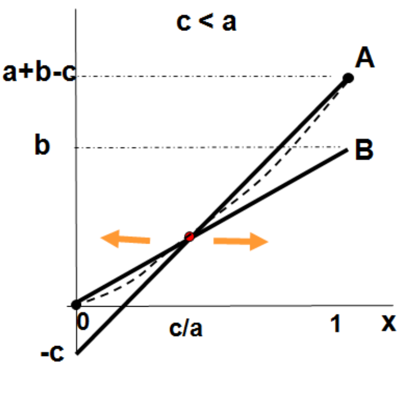


- Collective efficiency:

$$xA(x) + (1-x)B(x) = x(ax + bx - c) + (1-x)bx = ax^2 + (b-c)x$$

Binary Choices



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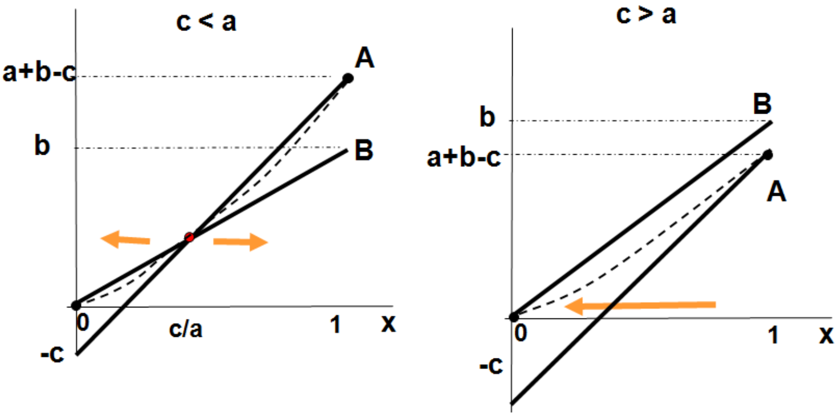
Binary Choices



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- Equilibria are solutions x^* of the equation $A(x^*) = B(x^*)$, or $x = 0$ (if $A(0) < B(0)$) or $x = 1$ (if $A(1) > B(1)$).
- Bischi and Merlone, 2009a, consider a repeated binary choice at discrete time, where $x(t)$ is the fraction of agents playing strategy A at time t.
- Agents at time t observe the choices of the population and try to increase their short-run payoff (myopic agents).
- If $A(x(t)) > B(x(t))$, then a fraction $(1 - x(t))$ of agents playing B will switch to strategy A in the next period and vice-versa.

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Binary Choices

Dynamic formulation

The dynamic of $x(t)$ can be modelled as

$$x(t+1) = f(x(t)) =$$

$$\begin{cases} x(t) + \delta_A g\left[\lambda(A(x(t)) - B(x(t)))\right](1 - x(t)) & \text{if } A(x(t)) \geq B(x(t) \\ x(t) - \delta_B g\left[\lambda(B(x(t)) - A(x(t)))\right] x(t) & \text{if } A(x(t)) < B(x(t) \end{cases}$$

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where

- $\delta_A, \delta_B \in [0,1]$ are propensities to switch to other strategy
- $g : \mathbb{R}^+ \to [0,1]$ is a continuous function such that $g(0) = 0$ and $\lim_{y \to +\infty} g(y) = 1$
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Theorem (case 1)

Assuming that

1. $A(0) < B(0)$
2. $A(1) > B(1)$
3. *there is a unique* $x^* \in (0,1)$ *such that* $A(x^*) = B(x^*)$.

Then $x = 0, x = 1, x = x^*$ *are fixed points.* x^* *is unstable and constitutes the boundary separating the basins of attraction of the stable fixed points* 0 *and* 1. *The dynamics converges monotonically to 0 if* $x(0) < x^*$ *or to 1 if* $x(0) > x^*$.

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This theorem applies to the previous example (R&D vs. spillovers)

- Similarly to Hommes, take $g(x) = \frac{2}{\pi}\arctan(x)$
- $A(x) = x; B(x) = 0.25 + 0.5x; \delta_A = 0.1; \delta_B = 0.9; \lambda = 40;$
- Piecewise smooth map; the previous theorem applies

Binary Choices

Dynamic formulation

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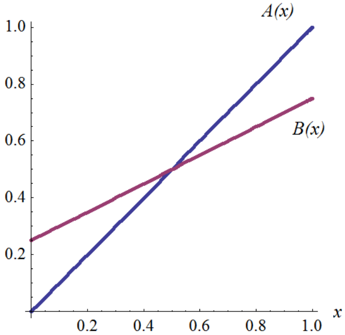

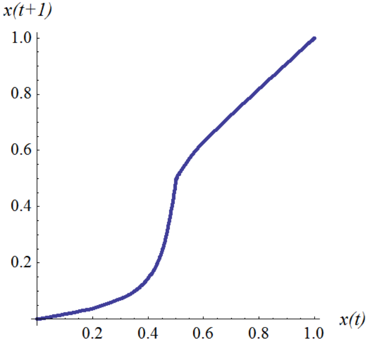

Theorem (case 2)

Assuming that

1. $A(0) > B(0)$
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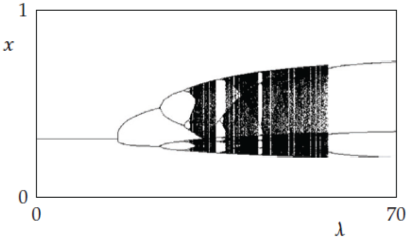
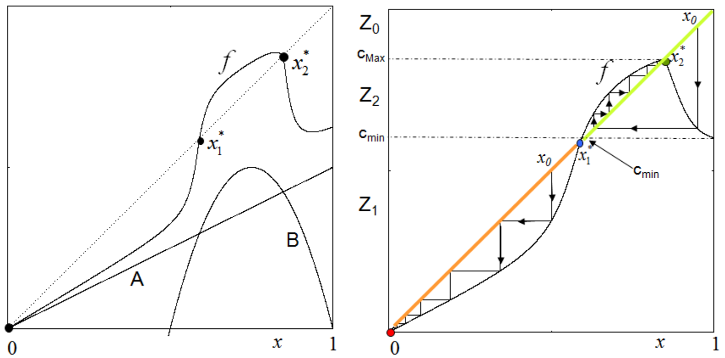
$A(x) = 0.25 + 0.5x;$
$B(x) = 1.5x;$
$\delta_A = \delta_B = 0.5$

- Theorem of case 2 applies
- Period doubling route to chaos as λ is increased

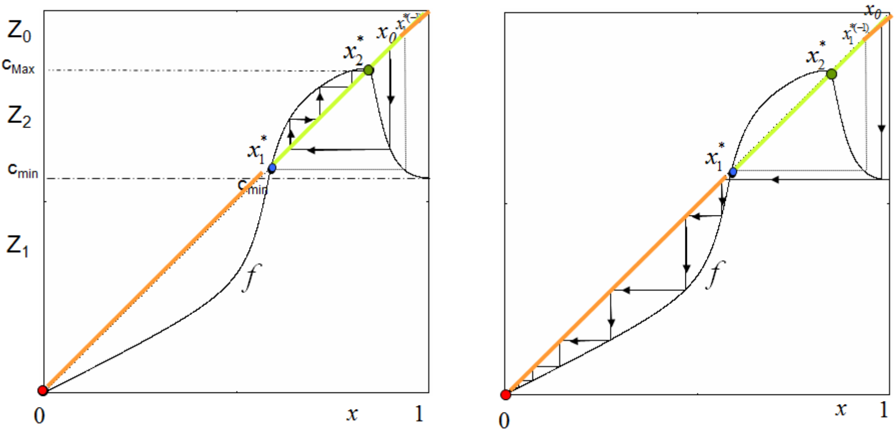


- Schelling provides exemples for unimodal payoff functions
- Bischi and Merlone 2009b carry on an example with $A(x) = 0.5x$;

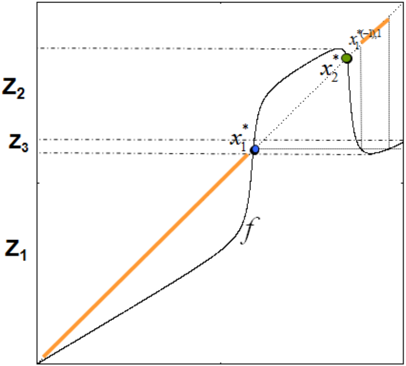
$B(x) = -8x^2 + 12x - 4$; $\delta_A = \delta_B = 0.5$ and $\lambda = 6$



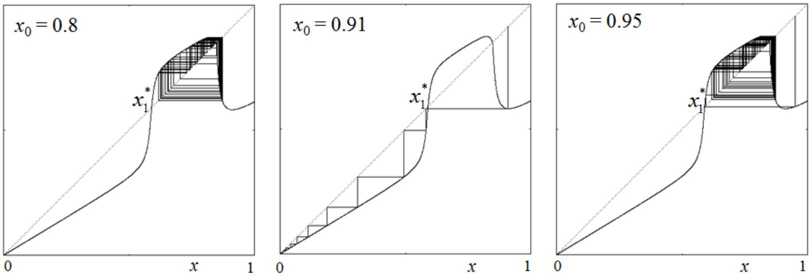
- $\lambda = 10$



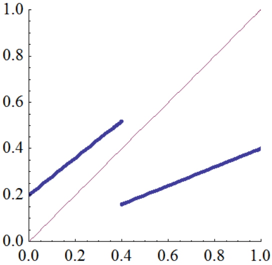
- $\lambda = 60$



- $\lambda = 60$; role of initial conditions



- When $\lambda \to \infty$ agents are impulsive and the map is discontinuous



- Bischi, Gardini, Merlone 2009, show that in some cases stable periodic cycles of any period that appear and disappear through border-collision bifurcations.

Concluding remarks

- The list of models is not exhaustive...
- Maps are often derived by solving static, dynamic or 'myopic' optimization problems, by conditions on stocks and flows, equilibrium equations, ...
- Main mathematical concepts employed so far include:
    - linear and nonlinear maps, stability of equilibria and cycles
    - bifurcations, bifurcation diagrams and hysteresis
    - conjugacy and period doubling route to chaos
    - noninvertible maps and critical points
    - basins of attraction and global bifurcations
    - piecewise and discontinuous maps
    - ...

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