On the Evolution of Market Shares in the Presence of Internal and External Spillover Effects

G.-I. Bischi* H. Dawid† M. Kopel‡

Abstract

In this paper we consider the impact of spillovers occurring within and between two populations of firms on the long run agglomeration patterns in a market. In each period every single firm can either produce for this market or choose some outside option (e.g. a risky asset). Firms switch between the two options based on information about the relative profitability of the market and the outside option. In the market, due to spillovers, the production costs are influenced by the total number of firms from both populations in the market. The resulting model describes the evolution of the size of the two firm clusters and their market shares over time. We provide a global analysis of the existence and basins of attraction of equilibria to address the question what impact different constellations of spillover effects have on the growth of dominant respectively incoming clusters. We demonstrate that the basins of attraction of coexisting long run equilibria do not depend continuously on the size of the spillover effects. Furthermore, an increase in the initial cluster size is not necessarily beneficial if the switching behavior of firms is fast.

Keywords: spillover effects, evolutionary dynamics, equilibrium selection, basins of attraction, critical curves

1 Introduction

In many industries a well-observed phenomenon is the clustering of firms related to this industry in relatively small geographical regions. Industries

*Istituto di Scienze Economiche, University of Urbino, Italy

†Department of Economics, University of Southern California

‡Department of Managerial Economics and Industrial Organization, Vienna University of Technology, email: kopel@mail.ibab.tuwien.ac.at

where such agglomeration patterns have been observed include car manufacturing, computer manufacturing, fashion designing or the chemical industry (see Audretsch and Feldman (1996) and, in particular, Ellison and Glaeser (1997) for an extensive analysis of the geographic concentration of several different US industries). Two major reasons for this agglomeration have been identified in the literature: first, endowment effects suggesting that certain regions are especially well suited for the production of certain goods and, second, positive externalities due to technological and intellectual spillovers from other similar companies in a region. Beginning with Marshall (1920) it has been argued in numerous places that the flow of information on production techniques and product design as well as the pooling of skilled labor in a certain region may lead to considerable reduction of the production costs and make it attractive for firms to produce similar or related products. It has also been pointed out that the size of these spillover effects between firms does not only depend on local proximity of the production facilities but also on the ownership of the company. For example, Head et al. (1995) show that internal spillovers within each groups of Japanese and American car manufacturers in the US are larger than external spillovers between these groups[1]. Nevertheless, there is evidence for positive spillovers from companies from different groups and also between regions (see e.g. Mansfield (1988) for an empirical study). The size of the internal and external spillovers has been shown to differ between different regions and industries (Mansfield (1988)) which is due to differing management and production techniques but also due to differences in the infrastructure (means of transportation, etc.) or incentives for foreign direct investment and human capital exchange programs (see e.g. Chuang and Lin (1999) or Neven and Siotis (1996) on the connection between foreign direct investment and spillovers). The ability of Japanese car manufacturers to yield a sustainable market share on western markets is generally attributed to their *keiretsu* organization which facilitates internal spillovers and their superior ability for technological sourcing which allows them to use externally based technologies much faster and more efficient than their U.S. competitors (Mansfield (1988)).

This raises the question whether we can gain some general insights on the conditions which allow for the evolution of a viable cluster of firms producing for a certain market in a region. In particular, it is interesting to examine which differences in internal and external spillover effects allow an initially marginal cluster to coexist with another cluster of firms in the mar-

[1] Relatedly, Ellison and Glaeser (1997) find in their empirical study that within county spillovers are stronger than nearby-county spillovers.

ket and under which circumstances internal and/or external spillover effects may lead to a market takeover by one cluster. Furthermore, the question whether an increase of internal or external spillover effects is more advantageous from the long run perspective of a local industry group arises. These questions are particularly relevant for regional planners who are interested in attracting or building up a certain industry cluster in their region by providing appropriate infrastructure and incentives for the emergence of spillovers and by attracting initial investments. A few authors have provided policy recommendations on these issues based on empirical studies (Chuang and Lin (1999)) or static models (Carlisle (1992)), however without explicitly taking into account the interaction of distinct clusters in the market. In this paper we carry out a dynamic evolutionary analysis of the competition of two industry groups (say Japanese and US car manufacturers) in a market where internal and external spillover effects are present. The size of the two industry groups depends on the attractivity of the market for both groups, which is expressed by the profits achieved by firms who are producing for the market in relation to the profit of some outside option. The market entry and exit decision of firms is made on the basis of information about the relative profitability of the market which has been collected via direct communication within the group. In order to address the questions posed above, we analyze the evolution of the size and market shares respectively for both firm clusters. In particular, we will investigate to which steady state the process converges depending on the sizes of the spillover effects, their relation to each other and the initial market shares of both groups. We will characterize the set of initial market shares which lead to convergence to a certain equilibrium – the basin of attraction of this equilibrium – and will use the size of the different basins as a measure of the efficacy of policies leading to increases or advantages in internal or external spillovers.

Besides dealing with these economic questions this paper also addresses a general, more technical, point related to dynamic evolutionary analysis. For the last two decades dynamic evolutionary models based on local interaction of boundedly rational agents have been used frequently to gain a new understanding of several economic problems[2]. Whereas this type of research initially focused on the notion of evolutionary stability (ESS) and deterministic evolutionary dynamics, the larger part of recent contributions has used stochastic models including 'experimentation' or 'error' terms. These mod-

[2] The fields of application include bargaining models (Young (1993), Ellingsen (1997)), auctions (Dawid (1999a), Lu and McAfee (1996)), the formation of social norms (Young (1998) or various models of market interaction (Quin and Stuart (1997), Vega-Redondo (1997)).

els are usually analyzed using the so called 'Freidlin-Wentzell method' which characterizes the states which stay in the limit distribution of the process as the probability for experiments goes to zero. An appealing feature of this technique is that it generates, for most of the models, a unique long run prediction and, therefore, is a useful tool for equilibrium selection based on evolutionary arguments. However, it has to be pointed out that these predictions often hold only in the very long run and there may be a large probability that the process stays for a very long time close to some state different from the stochastically stable one[3]. One the other hand, deterministic models normally do not provide unique predictions about the long run outcome. It is well-known that there might exist several (non-interior) evolutionary stable states which are locally stable for many prominent evolutionary dynamics. Thus, the long run prediction in general depends on the initial conditions of the process, and the long run in this framework involves a much shorter time horizon than for the stochastic models.

Considering our model of two competing groups of firms it seems that the initial conditions (i.e. the initial market shares of the two groups) indeed has a crucial influence on the outcome of the competition in a reasonable time frame. Thus, for the purpose of this paper, we find it more suitable to look at the intermediate run using a deterministic model of the evolution of the size of the two industry groups. We will follow Friedman (1998) who points out that "*[I]n most applied work it suffices to identify the evolutionary equilibria and their basins of attraction*", (p. 34; see also Friedman (1991) p. 639) and will characterize the equilibria in our model and their basins for different constellations of internal and external spillovers. Obtaining such a characterization for higher dimensional non-linear models by analytic means (and not only numerically) is, however, an intricate task. So far few results other than for standard normal form games are available (see e.g. Friedman (1998)). This might be seen as a reason that very few such dynamic analyses have been carried out so far, but in this paper we demonstrate how the theory of critical curves can be used to understand and predict the changes of the basins of attraction in highly non-linear evolutionary models with multiple coexisting steady states. Although this technique has been sporadically applied in economic modeling in the past (see e.g. Bischi et al. (2000) and Bischi and Kopel (1999)), this is to our knowledge the first study where the merits of this approach for evolutionary analysis are explored. The use of

[3] For example Ellison (2000) points out that "*..., it is inherently limited in scope to a characterization of the very long run limit. This can be problematic because evolution in these models is at times so slow as to be of limited practical importance.*"

this technique should facilitate the applicability of deterministic evolutionary analysis in many other models as well, and could in some cases generate predictions for a more realistic time horizon than the stochastic approach. Thus, the analysis provided here should be of general interest for researchers working in the field of evolutionary modeling as well.

The paper is organized as follows. In section 2 we introduce the model. Section 3 deals with the case where only internal spillovers exist. Section 4 then studies the changes of the long run outcomes when spillovers exist not only within a population but also between populations. Section 5 briefly summarizes the main findings. All proofs are given the Appendix.

2 Markets and Spillovers

Consider two groups of firms $i = 1, 2$ which have to decide whether to produce for a certain market or not. Alternatively, you might think of two groups of investors who have to decide whether to invest in a certain local industry branch or not. As laid out in the introduction, the assignment into one of the groups may be due to several reasons like the location or the national origin of the firms. The only crucial observation here is that there is more flow of information within a group than between the two groups. To keep matters as simple as possible we assume that the two groups are of the same size and denote by x_{it} the fraction of firms in population $i, i = 1, 2$ which are in the market at time t. This fraction might as well be interpreted as the fraction of available capital in each country which is invested in firms in the market. Assuming constant returns to scale, it does not make a difference for any of our arguments whether output within one country is produced by several large or many small firms as long as no firm has relevant market power.

For reasons of simplicity it is assumed that every firm in the market produces one unit of a homogeneous good per period. Aggregate output in the market is then given by (x_1+x_2) times the number of firms. The market clearing price is determined by an inverse demand function

$$p = p(x_1 + x_2).$$

Due to internal spillovers within the group, costs reductions are higher the more other firms in the population produce the same good. Additionally, if both, internal and external spillovers exist, costs reductions also arise due to production activities in the other population. We include such cost externalities in our model by assuming that the unit costs for a firm depend on the number of firms from both populations which produce the good in the market. Hence, we express unit costs of a firm in population i by $c_i(x_1, x_2)$, where $\frac{\partial c_i}{\partial x_j} \leq 0,\ i,j = 1,2,\ \frac{\partial c_i}{\partial x_i} \leq \frac{\partial c_i}{\partial x_j},\ i,j = 1,2, i \neq j$. This gives a per period profit of

$$\pi_i(x_1, x_2) = p(x_1 + x_2) - c_i(x_1, x_2).$$

The profit of a firm which stays out of the market (i.e. chooses the outside option) is modeled as a stochastic variable. Outside profit of firm f in population i at time t is $u^i_{f,t}$ with expected value U_i. Outside profits are independent across individual firms in a population and time. We write $u^i_{f,t} = U_i + \epsilon^i_{f,t}$ where the density of $\epsilon^i_{f,t}$ is independent of f and t, has full support $\mathbb{R}$ and is unimodal and symmetric with respect to 0. The distribution function of $\epsilon^i_{f,t}$ is denoted by Θ_i.

Each period $t = 0, \ldots, \infty$ every firm decides whether to enter or to exit the market[4]. If a firm samples another firm (in the same population) which has chosen a different action in the previous period, it switches (i.e. exits or enters) whenever the profit of the other firm has been larger than its own profit. The sampling procedure is assumed to be stochastically independent from the outside profit $u^i_{f,t}$ and this implies that the probability that an arbitrary firm in population i which is now in the market exits after period t is given by:

$$\begin{aligned} p^i_{out}(x_1, x_2) &= (1 - x_i)\mathbb{P}(\pi_i(x_1, x_2) < U_i + \epsilon^i_{f,t}) \\ &= (1 - x_i)(1 - \Theta_i(\pi_i(x_1, x_2) - U_i)), \end{aligned}$$

On the other hand, a firm currently outside the market enters the market with probability

$$\begin{aligned} p^i_{in}(x_1, x_2) &= x_i\mathbb{P}(\pi_i(x_1, x_2) > U_i + \epsilon^i_{f,t}) \\ &= x_i\Theta_i(\pi_i(x_1, x_2) - U_i), \end{aligned}$$

The expected fraction of population i firms ($i = 1, 2$) in the market is therefore:

$$\begin{aligned} x_{i,t+1} &= x_{i,t} + (1 - x_{i,t})p^i_{in}(x_{1,t}, x_{2,t}) - x_{i,t}p^i_{out}(x_{1,t}x_{2,t}) \\ &= x_{i,t} + x_{i,t}(1 - x_{i,t})(\Theta_i(\pi_i(x_{1,t}, x_{2,t}) - U_i) - (1 - \Theta(\pi_i(x_{1,t}, x_{2,t}) - U_i)) \\ &= x_{i,t} + x_{i,t}(1 - x_{i,t})G_i(\pi_i(x_{1,t}, x_{2,t}) - U_i), \qquad (1) \end{aligned}$$

[4] As indicated above, this can alternatively be interpreted as the decision of an investor in the population to invest in this market by founding a production firm or to withdraw capital respectively.

where $G_i(x) := 2\Theta_i(x) - 1$. The evolution of the fractions x_1 and x_2 are hence described by a nonlinear deterministic system in discrete time. Word of mouth dynamics similar to this one have been analyzed by Fudenberg and Ellison (1993, 1995) and Dawid (1999b). It is obvious that the shape of the function G_i depends on the distribution function Θ_i. However, from the fact that Θ_i is a distribution function and the properties of the corresponding density (unimodality and symmetry) it is easy to derive the following statements for $i = 1, 2$:

$$G_i(0) = 0, \quad \lim_{x\to\infty} G_i(x) = 1, \quad \lim_{x\to-\infty} G_i(x) = -1.$$

Furthermore, $G_i(x)$ is symmetric with respect to 0, convex on $(-\infty, 0]$ and concave on $[0, \infty)$. The slope of G_i at 0 is twice the altitude of the hump of the unimodal density. It will turn out that the qualitative properties of the long run behavior of the dynamics in many cases crucially depend on the 'speed' of the flow towards the action with the higher expected profit. Hence, we will use $G_i'(0)$ as a measure of this speed and denote it by λ_i[5].

Studying the nonlinear two-dimensional dynamical system (1) allows us to derive qualitative features of the evolution of the fraction of firms of the two populations which are in the considered market. In particular, we are interested in the question how initial market shares of firms of the two populations, $x_{1,0}$ and $x_{2,0}$, and differences in (internal and external) spillovers influence the convergence properties of the evolutionary process to some long run equilibrium (the agglomeration pattern). To answer this question, we will provide an extensive analysis of the equilibria and their basins of attraction. However, before we proceed we need to be more specific about the functions involved.

We will assume that the demand curve is linear

$$p(x_1 + x_2) = P_0 - B(x_1 + x_2).$$

Furthermore, we use the following rational expression for the unit costs of a firm in population i:

$$c_i(x_1, x_2) = \frac{C_i}{1 + \beta_i x_i + \gamma_i x_j} \quad i, j \in \{1, 2\}, \ i \neq j,$$

The parameter β_i incorporates the effect of internal spillovers, whereas γ_i characterizes in how far spillovers occur externally between the two populations. As explained in the introduction, the qualitative properties of the

[5] For many classes of distribution functions, like the normal distribution, a large slope of G_i at zero corresponds to a small variance of the outside profit; for example for the normal distribution λ_i is inversely proportional to σ.

cost function are inspired by existing theoretical and empirical work. To take account of the fact that internal spillovers are stronger than those between populations (see Ellison and Glaeser (1997), Head et al. (1995)), we assume that $\beta_i \geq \gamma_i$. The profit of a firm in population i who is in the market is then given by

$$\pi_i(x_1, x_2) = P_0 - B(x_1 + x_2) - \frac{C_i}{1 + \beta_i x_i + \gamma_i x_j} \quad i, j = 1, 2, \ i \neq j. \tag{2}$$

We will always assume that if all firms from both populations are in the market, the payoff for firms in the market is smaller than the expected outside profit. This assumption rules out the rather unrealistic and uninteresting case where the market is so much more attractive than the outside option that all firms from both populations want to enter or stay under all circumstances. On the other hand, the monopoly profit of the firm entering the market should be larger than the expected outside profit. These conditions are represented by the following inequalities:

$$\begin{array}{rcll} P_0 - 2B - \frac{C_i}{1+\beta_i+\gamma_i} & < & U_i & i = 1, 2 \\ P_0 - C_i & > & U_i & i = 1, 2 \end{array} \tag{3}$$

Additionally, we make the more technical assumption that, if there were no spillovers, the expected profit of the outside option would be higher than the profit in the market if half of the firms are in the market

$$P_0 - B - C_i < U_i \quad i = 1, 2. \tag{4}$$

One of the main points we will make in this paper is to show that in the presence of large spillovers a population of firms can invade a market despite the fact that their initial market share is small. Accordingly, it is sensible to assume that these spillovers are the reason why the market may become, on average, more attractive than the outside option. By making this assumption we avoid the discussion of several cases. However, it would be straightforward to extend the analysis to cases where this assumption does not hold.

Using the expressions given above, we obtain the following evolutionary model which describes the dynamics of the fraction of firms from both populations in the market:

$$\begin{array}{rcl} x_{1,t+1} & = & x_{1,t} + x_{1,t}(1 - x_{1,t})G_1\left(A_1 - B(x_{1,t} + x_{2,t}) - \frac{C_1}{1+\beta_1 x_1+\gamma_1 x_2}\right) \\ x_{2,t+1} & = & x_{2,t} + x_{2,t}(1 - x_{2,t})G_2\left(A_2 - B(x_{1,t} + x_{2,t}) - \frac{C_2}{1+\beta_2 x_2+\gamma_2 x_1}\right), \end{array} \tag{5}$$

where $A_i := P_0 - U_i$. We define $T : [0,1]^2 \mapsto [0,1]^2$ as the right hand side of (5) and using this notation the system reads $x_{t+1} = T(x_t)$. In general terms we have derived a two-population evolutionary model with non-linear payoff functions and inter- and intra population interaction. In the remainder of the paper, we will proceed as follows: First, we will examine the set of possible equilibria of this dynamical system, which gives the potential (long run) agglomeration patterns of firms of the two populations in the market. Second, we will study their local stability properties. Third, we will turn to the equilibrium selection problem. We will focus on a global analysis and characterize the basins of attraction of the different stable equilibria for different parameter constellations.

3 Fixed Points and Local Stability

Obviously, the state space $\mathcal{S} := [0,1] \times [0,1]$ is invariant under the dynamics (5) and all four corners are fixed points. A standard local stability analysis further shows that under assumptions (3) the two corners $(0,0)$ and $(1,1)$ are unstable[6]. The corner $(1,0)$ is locally asymptotically stable if

$$P_0 - B - \frac{C_1}{1+\beta_1} \quad > \quad U_1 \tag{6}$$

$$P_0 - B - \frac{C_2}{1+\gamma_2} \quad < \quad U_2 \tag{7}$$

and $(0,1)$ is locally asymptotically stable under the symmetric conditions

$$P_0 - B - \frac{C_1}{1+\gamma_1} \quad < \quad U_1 \tag{8}$$

$$P_0 - B - \frac{C_2}{1+\beta_2} \quad > \quad U_2. \tag{9}$$

Besides these four fixed points the system may also have additional stationary points. Every point in the interior of $\mathcal{S}$ where $\pi_1(x_1,x_2) = U_1$ and $\pi_2(x_1,x_2) = U_2$ is a fixed point of (5). Furthermore, fixed points exist on the upper and lower boundary of $\mathcal{S}$ where $\pi_1(x_1,x_2) = U_1$ and on the left and right boundary where $\pi_2(x_1,x_2) = U_2$. To facilitate the analysis we define the curves $F_i,\ i = 1,2$ as the set of all points (x_1,x_2) where the profit in the market equals the expected outside profit for a firm of population i, i.e.

$$F_i = \{(x_1,x_2) \in [0,1]^2 | \pi_i(x_1,x_2) = U_i\}. \tag{10}$$

[6] The details of the local stability analysis can be found in a technical Appendix available upon request from the authors.

Interior equilibria exist at all intersections of the curves F_1 and F_2. Fixed points at the boundary occur either at the intersection of F_1 with $x_2 = 0$ or $x_2 = 1$, or at the intersection of F_2 with $x_1 = 0$ or $x_1 = 1$. Note however, that fixed points on the boundary might not correspond to Nash equilibria of the model.

In the following proposition we characterize the fixed points (other than the vertices) of the dynamics under our assumptions.

Proposition 1 *For $\beta_i > 0$ the dynamical system (5) can have at most one fixed point in the interior of $[0,1]^2$ and if it exists it is always unstable. Additionally, there can be either at most two fixed points on the boundary $x_1 = 1$ or one fixed point on $x_1 = 0$ and either at most two fixed points on the boundary $x_2 = 1$ or one fixed point on $x_2 = 0$. There can never be fixed points on $x_i = 0$ and on $x_i = 1,\ i = 1, 2$ simultaneously for the same values of the parameters.*

Taking into account proposition 1 we conclude that our model can have up to nine coexisting fixed points. There are always four at the vertices of $[0,1]^2$, at most four on the boundary and at most one in the interior of the unit square. Short introspection further establishes that there can be at most one stable equilibrium in the interior of a boundary line of the unit square. Therefore, all together there can be at most four locally stable fixed points. Any of these stable fixed points is the potential long run outcome of the evolutionary process driven by the switching behavior of the firms. Standard arguments used in the evolutionary game theory literature (e.g. Weibull (1995)) imply that every locally stable fixed point corresponds to a Nash equilibrium of the underlying two population game. Which locally stable equilibrium is actually chosen depends on the initial market share firms in population i have. In order to obtain a thorough understanding of the interplay between initial market shares, long run outcomes and their dependence on parameters, for each equilibrium we need a characterization of the set of initial conditions for which the process converges to it. In other words, what is needed is not only an identification of the set of locally stable fixed points, but also a characterization of their basins of attraction and the changes these basins undergo as parameters are varied.

To make matters simple, we introduce a coherent notation for all fixed points on the boundary which we will use throughout the analysis. We denote the vertices as: $0 = (0,0), V_I = (1,0), V_{II} = (1,1), V_{III} = (0,1)$, and the unique interior fixed point as $S = (s_1, s_2)$. Obviously, V_I characterizes an agglomeration pattern where all firms of population 1 are in the market and all firms of population 2 choose the outside option. Conversely, in

V_{III} population 2 firms are in the market and population 1 firms choose the outside option. We denote by $P_I = (p_{I1}, 0)$ the interior fixed point on the boundary line $x_2 = 0$, by $Q_{II} = (1, q_{II2})$ and $P_{II} = (1, p_{II2})$ the two fixed points on $x_1 = 1$, where $q_{II2} < p_{II2}$, by $Q_{III} = (q_{III1}, 1)$ and $P_{III} = (p_{III1}, 1)$ the two fixed points on $x_2 = 1$, where $q_{III1} < p_{III1}$ and by $P_{IV} = (0, p_{IV2})$ the interior fixed point on $x_1 = 0$. For example, P_{III} characterizes a situation where in the long run firms of population 2 dominate the market. However, not all firms of population 1 are driven out of the market: a fraction of firms of population 1 coexists.

To keep our exposition as clear and simple as possible we assume that the constant unit costs of a single firm from either population are identical: $C_1 = C_2 = C$. In other words, the profit a single firm can achieve when entering a market where no other firm of the same population is in, is independent of the population the firm belongs to. Furthermore, we assume that the distribution of the outside profit is identical in both populations, i.e. $\Theta_1 = \Theta_2$, which in particular implies $G_1 = G_2 := G$ and $U_1 = U_2$. Therefore, we have $A_1 = A_2 = A$. The populations might differ, however, with regard to their infrastructure facilitating spillovers and cost externalities between their members (i.e. with respect to β_i and γ_i).

4 Internal Spillovers

We start our analysis by considering a scenario where internal, but no external spillovers exist. That is, we assume $\beta_i > 0$ and $\gamma_i = 0$ for $i = 1, 2$. Discrete time dynamics have the generic property that they might 'overshoot' equilibria if the step-size is too large. Throughout this first part of the analysis we will avoid this by assuming that the dynamics is "sufficiently slow" (i.e. $\lambda_1 = \lambda_2 := \lambda$ is sufficiently small), such that no local overshooting occurs at any fixed point. This corresponds to a situation where the variance of the outside profit is large. Later on in our analysis we will also deal with the effect of an increase of the speed of the flow in and out of the market.

4.1 Symmetric Spillovers and Slow Dynamics

Let us start with the symmetric case where the internal spillovers in both populations are equal, i.e. $\beta_1 = \beta_2 := \beta$. Initially, we will assume very small spillover effects and characterize how the set of fixed points and the set of initial conditions for the fraction of firms ($x_{1,0}$ and $x_{2,0}$) which converge to these equilibria change as β is increased. In the limit case $\beta = 0$, the curves F_1 and F_2 are identical straight lines with slope -1. It follows from assumption (4) that they are both below the line $x_1+x_2=1$. Note that this implies that we have a continuum of interior fixed points. Generically, in such a case different initial conditions lead to different long run states. If we slightly increase β, the curve F_1 bends upwards; the intersection point with $x_1=0$ is fixed and the intersection point with $x_2=0$ moves to the right. The curve F_2 changes in a symmetric way. We can therefore conclude that for positive, but very small values of β, additionally to the four vertices, there are three fixed points: the single interior equilibrium, S, P_I on the line $x_1=0$ and P_{IV} on the line $x_2=0$. Due to (4), the local stability conditions (6) and (9) for V_I and V_{III} are violated for sufficiently small β. Thus, the two equilibria on the boundary, $P_I=(p_{I1},0)$ and $P_{IV}=(0,p_{IV2})$, where p_{I1} and p_{IV2} are identical and satisfy $A-Bp-\frac{C}{1+\beta p}=0$, are the only two stable equilibria. We know that the interior fixed point, S, is a saddle point and the complete symmetry of the dynamics with respect to x_1 and x_2 implies that the stable set of the saddle is the diagonal of the unit square. Figure 1 depicts the curves F_1, F_2, all coexisting fixed points and the diagonal. In all our numerical illustrations we use the parameter values $P_0=300$, $B=100$, $C=190$, $U=32$ for the market environment and expected outside profits. These values satisfy assumptions (3) and (4) for all $\beta_i,\gamma_i\in[0,1]$. Other values of the parameters which satisfy these assumptions would yield qualitatively similar results.

Figure 1: Insert here!

Due to the symmetry properties, the diagonal is invariant with respect to the dynamics and this suggests, that it separates the basins of attraction of the two stable fixed points P_I and P_{IV}. In order to rigorously establish this fact, we have to show that the dynamics never maps a point from one side of the diagonal to the other side. One way to show this, is to prove that the diagonal is not only forwards, but also backwards invariant with respect to the dynamics. A simple continuity argument then establishes that if there is a backwards invariant curve either all points are mapped from one side to the other or none. Since we know that, for example, $T(V_I)=V_I$, showing that the diagonal is backwards invariant is sufficient to show that it separates the basins of attraction of P_I and P_{IV}. If the inverse of the generating map of the dynamics is single-valued, it is trivial that the forwards invariant diagonal is also backwards invariant. In order to see whether the inverse is unique, it is useful to consider the so-called critical curves LC of the map T. Critical curves separate areas where the number of (rank-1) preimages of points coincide. Whenever points have different numbers of (rank-1) preimages, there has to be at least one critical curve between these points (see Appendix C for a short introduction into the critical curve technique and Mira et al. (1996), Bischi and Kopel (1999) or Bischi et al. (2000) for more details on critical curves). If we denote the set of all points where the determinant of the Jacobian of the map T vanishes by LC_{-1}, then the critical curve LC can be determined by applying the map T to all points of this set, i.e. $LC = T(LC_{-1})$. In Appendix B it is shown that every intersection of a critical curve and the diagonal is a critical point of the restriction of T to the diagonal. Accordingly, the number of preimages of T and its restriction to the diagonal coincides for elements of this set. This implies that if the number of rank-1 preimages of points on the diagonal is greater than 1, all additional rank-1 preimages have to be on the diagonal. Therefore, the diagonal is backwards invariant and indeed separates the basins of attraction of P_I and P_{IV}.

An intuitive interpretation of this result can be given easily. If internal spillovers are very small and symmetric, the population of firms which initially has the smaller fraction of firms in the market completely leaves the market in the long run and the population with the larger initial market share completely takes over the market. However, since spillovers (and hence cost externalities) are small, it only pays to be in the market if the price is rather high. Consequently, if the number of firms from the own population in the market is too large, using the outside option is, on average, more advantageous, even if the other population has completely left the market. As a consequence, only a certain fraction of firms stays in the market in the long run, whereas some members of the population (together with all firms of the other population) end up choosing the outside option.

Looking at the transient part of a path where the initial number of firms in the market is small in both populations with slight advantages for population 1, it can be observed that during the early periods the market is attractive for both populations and both x_1 and x_2 increase (see Figure 1). At some point, however, the number of firms in the market becomes so large that firms in population 2 (which enjoy only smaller cost reductions due to spillovers) start leaving the market. Since the market is still attractive for population 1, these firms are replaced by members from population 1 and the path converges to P_I. So, the strong effect of the small initial advantages in market share for population 1 becomes apparent only with a certain delay, but in the long run only the industry cluster with the initial advantage in market share will survive.

If the size of internal spillover effects is increased, the equilibrium P_I moves to the right and eventually collides with the corner V_I. At the point where

$$\beta = \hat{\beta} := \frac{C}{A-B} - 1 > 0,$$

the fixed point P_I leaves the unit square and thus becomes irrelevant for our study. The fixed point V_I becomes locally asymptotically stable[7]. For the same parameter value of β also P_{IV} moves through V_{III} and V_{III} becomes locally asymptotically stable. Thus, now the equilibria $(1,0)$ and $(0,1)$ are the only two stable equilibria. The stable manifold of the interior saddle point still separates the basins of attraction. We depict the equilibria and the corresponding curves F_1, F_2 in figure 2.

Figure 2: Insert here!

From an economic point of view, we now have a situation where the cost savings in the market due to internal spillovers are sufficiently large to make the market option always attractive if there are no firms from the other population in the market. However, spillovers are still not large enough to make the market in the long run attractive for the population with the smaller market share. Trajectories here look very similar to that observed in figure 1. If both groups initially are small, then the number of firms from both populations increases. However, at some point firms from the population with the smaller market share start leaving the market and, eventually, this group vanishes from the market. On the other hand, all firms in the population which initially has the larger market share eventually enter the market. Thus, for such values of the parameters the long run result is a complete market takeover. Since the distribution of the outside profit – which is the profit earned by all firms who left the market – is identical in

[7] Mathematically speaking we have a transcritical bifurcation (see Lorenz (1993) p. 111. Note that this bifurcation occurs only for $A - B > 0$. Otherwise, for all values of β and γ only P_I and P_{IV} are stable. We ignore this rather uninteresting case in the further analysis.

both populations, it follows that the firms in the population which initially has the larger fraction of firms in the market end up with higher profits.

As internal spillovers in both populations become even larger, the curve F_1 moves up further and, analogously, F_2 moves to the right. Again, we will only describe the effects of the changes of F_1 in detail, since the effects on F_2 follow from the symmetry of the map T. Remember that the intersection point of F_1 with the border line $x_1 = 0$ is not affected by the size of $\beta_1 = \beta$. However, the remaining part of the curve moves upwards when β is increased and eventually touches the horizontal border line $x_2 = 1$. The exact value of β where this tangency occurs is given in the following Lemma.

Lemma 1 *For $\gamma = 0$ and $\beta > \beta^*$, where $\beta^* > 0$ is the larger of the two roots of*

$$((A - B)\beta^* - B)^2 - 4\beta^* B(C - (A - B)) = 0 \tag{11}$$

the curve F_1 has two positive intersection points with $x_2 = 1$ and F_2 has two positive intersection points with $x_1 = 1$.

The pair of intersection points of F_1 and $x_2 = 1$ might either appear left or right of the corner $(1, 1)$ of the unit square. In cases where the curve F_1 touches the line $x_2 = 1$ right of $(1, 1)$, the behavior of the system in $[0, 1]^2$ does not change. Thus we focus on the case where F_1 touches $x_2 = 1$ inside the unit square. We will see that in this case the number of locally stable equilibria and the qualitative properties of the process suddenly change. But before we go on to discuss this transition, we summarize the findings for small β in the following proposition.

Proposition 2 *If internal spillover effects are symmetric between the two populations with $\beta < \beta^*$ and there exist no external spillovers, all firms from the population with the smaller initial market share eventually leave the market and the market is completely taken over by firms from the population with the larger initial market share. Depending on the size of β only a fraction or all of these firms stay in the market in the long run.*

To understand the changes in the number of fixed points and their local and global stability properties as the size of the internal spillovers cross the level $\beta = \beta^*$, it is useful to consider the restriction of the map T to the invariant line $x_2 = 1$. Many properties of the two-dimensional dynamical system on $[0, 1]^2$ can be inferred from this one-dimensional restriction. For $x_2 = 1$, the time evolution of x_1 is given by the system $x_{1,t+1} = h(x_{1,t})$ where

$$h(x_{1,t}) = x_{1,t} + x_{1,t}(1 - x_{1,t})G\left(A - B\left(x_{1,t} + 1\right) - \frac{C}{1 + \beta x_{1,t}}\right) \tag{12}$$

Figure 3: Insert here!

If we look at the graph of h, it is obvious (see the analysis above) that for small values of β the graph lies below the diagonal on the entire interval $(0, 1)$. This implies that the map h has only two fixed points: 0 is locally stable and 1 is unstable. If β is increased, at the value β^* the graph of h touches the diagonal from below and, for even larger β, a pair of additional fixed points appears[8] (depending on the parameter values this pair is in $[0, 1]$ or not– see the arguments given in the previous paragraph). The stability properties of the corner fixed points remain unchanged, whereas the left of the two new additional fixed points, q_{III1}, is unstable and the right, p_{III1} is stable. This can be easily seen in figure 3 where we show a schematic representation of the restricted map h for $\beta > \beta^*$. The map h is monotone if λ is sufficiently small and thus any trajectory starting right of q_{III1} stays right of q_{III1} and converges to p_{III1}. On the other hand, any trajectory with initial condition $x_{1,0} < q_{III1}$ converges to 0. We can see that the unstable fixed point q_{III1} has only one rank-1 preimage, namely itself. The local stability properties of the additional fixed points q_{III1} and p_{III1} of h determine the stability of the fixed points Q_{III} and P_{III} of the map T along the invariant manifold $x_2 = 1$. Standard local stability analysis shows that the eigenvalue for the transversal eigenvector is given by $1-G(\pi_2(x_1, 1)-U)$, where $x_1 = q_{III1}$ or $x_1 = p_{III1}$ depending on the fixed point we consider. We treat the case of Q_{III}, but all arguments also apply to P_{III}. Since Q_{III} is a fixed point, we have

$$\pi_1(q_{III1}, 1) - U = A - B(1 + q_{III1}) - \frac{1}{1 + \beta q_{III1}} = 0,$$

which implies

$$\pi_2(q_{III1}, 1) - U = A - B(1 + q_{III1}) - \frac{1}{1 + \beta} > 0.$$

Thus, $G(\pi_2(q_{III1}, 1) - U) > 0$ and the transversal eigenvalue is in $(0, 1)$. This implies that Q_{III} is a saddle point and P_{III} is locally asymptotically stable.

[8] In mathematical terms the map h undergoes a tangent bifurcation, see Lorenz (1993).

Thus, we now have four coexisting locally stable equilibria, two with market takeovers in the long run and two where clusters of firms from both populations stay in the market. This makes the characterization of the different basins substantially more difficult than in the cases we have looked at so far. We will illustrate the following results by figures depicting the different basins of attraction and – primarily in the section on fast dynamics – use the actual shape of the critical curves of the dynamics. This, however, cannot be done without further specification of the switching function G. In the numerical illustrations we provide below we always use the specification $G(x) = \frac{2}{\pi}\arctan\left(\frac{\lambda\pi}{2}x\right)$, where $\lambda = \lambda_1 = \lambda_2$. This function satisfies all the assumptions for the switching function G and $\lambda = G'(0)$[9].

The same line of arguments as used to show that the backwards invariant diagonal cannot be crossed by a trajectory imply that the stable set of the saddle point Q_{III} cannot be crossed if all points in this set have only one rank-1 preimage. The properties of G (in particular its S-shape) imply that the map of the dynamics (5) is invertible on $[0,1]^2$ for sufficiently small λ. This implies that the considerations for $x_2 = 1$ can be extended to the whole unit square and for $\beta > \beta^*$ and sufficiently small λ the stable set of the saddle point Q_{III} is a smooth curve connecting Q_{III} and 0. The stable set of Q_{III} splits the former basin of attraction of V_{III} into a smaller basin of V_{III} and a basin of attraction of the new stable equilibrium P_{III}. The triangular basin below the diagonal undergoes exactly the same transition. Thus, we have the following result

Proposition 3 *For $\gamma = 0$, $\beta > \beta^*$ and sufficiently small λ there are four locally stable equilibria – $V_I, P_{II}, V_{III}, P_{III}$ – with simply connected basins of attraction. The boundaries of the basins are given by the diagonal and by the stable manifold of Q_{II} between the basins of V_I and P_{II} and the stable manifold of Q_{III} between the basins of V_{III} and P_{III}.*

We depict the equilibria and the basins of attraction for such a case in figure 4.

In economic terms this means that, if internal spillovers increase symmetrically in both populations – for example because the exchange of information within firms of a population (e.g. a country) is made easier due

[9] We decided to use this specification rather than, for example, the switching function resulting from a normal distribution for pragmatic reasons: a closed form representation makes the calculation of critical curves easier. With the proper parameterization, a very close match with the switching function stemming from a normal distribution can be achieved, and it is quite obvious that the qualitative properties of the dynamics do not depend on the exact specification.

Figure 4: Insert here!

to improvements in information technology – at some point the long term properties of the system abruptly change. Although it remains true that the population which initially has the larger market share will keep a larger market share also in the long run, after the additional pair of equilibria has emerged it needs a quite large initial advantage to be able to drive firms from the other population completely out of the market. For most initial conditions the system will end up in a state where all firms from the population with the larger initial market share are in the market, but at the same time a large fraction of firms of the other population is able to stay in the market as well (e.g. the equilibrium P_{III} if $x_{1,0} < x_{2,0}$). It is obvious that the market price is much lower at the state P_{III} than at the state V_{III}. Consequently, the payoff for firms of population 2 is smaller at the boundary equilibrium than at the vertex V_{III}. On the other hand, since P_{III} is an equilibrium where a fraction of the firms in population 1 is in the market and a fraction chooses the outside option, it is obvious that the average profit of a population 1 firm in this equilibrium is identical to the average outside profit. Therefore, the average profit in P_{III} is identical to the average profit of population 1 firms in V_{III}. Thus, if we look at the long run outcomes for initial conditions in the basin of attraction of P_{III}, we can conclude that the difference in profits between firms in the population with the larger and with the smaller initial market share respectively suddenly shrinks if β becomes larger than β^* and the new equilibria appear. This confirms our observation for the case $\beta < \beta^*$, namely that a symmetric increase of spillover effects in both populations is harmful for the long run profit of the firms in the population with the initial advantage in market share. Note also that this implies that the equilibria V_I and V_{III} Pareto dominate the equilibria P_{II} and P_{III}.

If we increase the parameter β even further, and again look at the one-dimensional map h, we realize that the fixed point q_{III1} moves to the left whereas the fixed point p_{III1} moves to the right. Note however, that, since the slope of the map at the point 0 is given by

$$h'(0) = 1 + G(A - B - C) < 1,$$

which is independent of β, we have $q_{III1} > 0$ for all β. On the other hand, if we would increase β so much that $q_{III1} = 1$ this would imply

$$A - 2B - \frac{C}{1+\beta} = 0,$$

which contradicts assumption (3). Thus, given our assumptions about the parameters, no further qualitative changes of the properties of the equilibria and the structure of the basins occur for increasing values of β.

4.2 Asymmetric Spillovers and Slow Dynamics

Up to now we have only considered a scenario where the cost savings due to internal spillovers were symmetric in both populations. We have seen that in such cases the population which initially has the larger market share will keep this advantage also in the long run. Additionally, it has been demonstrated that the size of this advantage depends on the value of the spillover parameter β. Now we will turn to the case where firms from one population operate in an environment which provides superior possibilities for knowledge exchange and information flow. We will analyze how such an advantage modifies the relationship between initial market shares of the populations and long run outcome of the evolutionary process with respect to the scenario analyzed above.

We start with the symmetric situation depicted in figure 2, where internal spillovers are below β^* and the two fixed points V_I and V_{III} are the only attractors. We recall that in this situation for each of the two populations in the long run either all firms are in the market or out of the market depending on the initial market share of the population. Now, let us consider a scenario where β_2 stays constant, but for population 1 the conditions of the environment are changed such that the effect of spillovers between firms is more significant, i.e. β_1 is increased . We know that the curve F_i depends only on β_i but not on $\beta_j,\ j \neq i$ and, therefore, curve F_1 rotates upwards whereas curve F_2 remains unchanged. Initially, the only effect of an increase in β_1 is that the interior saddle point moves up along the unchanged curve F_2. The basins of attraction of the equilibria V_I and V_{III} are separated by the stable manifold of the interior fixed point which now lies above the diagonal (note that the diagonal is no longer invariant with asymmetric parameter values). Hence, given that spillovers in population 1 are higher than in population 2, the former is able to take over the market even if the initial number of firms from the other population on this market is slightly larger (see figure 5a). So, increasing spillover effects in population 1, yields 'continuous' effects on the long run market shares if the increase is only small.

Figure 5: Insert here!

However, like in the symmetric case, if β_1 crosses the threshold value β^*, the curve F_1 has a contact with the line $x_2 = 1$ and, if β_1 is further increased, a pair of new equilibria Q_{III} and P_{III} appears. In contrast to the case of a symmetric increase of β_1 and β_2, we can now not state generally whether this pair of additional equilibria appears right or left of the intersection of the curve F_2 with $x_2 = 1$. If the pair of equilibria appears to the left – like in the symmetric case – the fixed point Q_{III} is a saddle point and P_{III} is locally stable (see figure 5b). In this case the transition of the basins of attraction is very similar to the symmetric case. All of a sudden, for all initial conditions between the stable manifold of Q_{III} and the stable manifold of the interior equilibrium S, population 2 no longer controls the whole market in the long run, but a fraction p_{III1} of firms of population 1 will stay in the market. Accordingly, now for more than half of all possible initial market conditions population 1 eventually controls the whole market. Furthermore, for an additional set of initial conditions with positive measure, at least a certain fraction of firms from population 1 stays in the market. Note that this huge competitive advantage for population 1 can be gained by a rather small advantage in the parameter β (in the example depicted in figure 5b it is about 20%). It is worth pointing out that these effects caused by slightly higher internal spillovers in population 1 do not happen continuously, but abruptly. As soon as β_1 is increased above the value β^*, the additional basin of the mixed agglomeration equilibrium P_{III} emerges.

If β_1 is further increased, the effect is, for some time, continuous again. The interior equilibrium and, therefore, also its stable manifold (which is the boundary between the basins of attraction of P_{III} and V_I) moves up and to the left. Hence, the basin of V_I continuously expands as β_1 is increased. In figure 5c we show the basins of the three coexisting stable equilibria, where the difference in internal spillovers in the two populations is almost twice as large as in figure 5b. We see that the extents of the three basins have not significantly changed. The only noticeable changes are a slight increase in the extent of the basin of V_I and a small reduction of the basin of V_{III}. However, starting from this situation, if β_1 is only slightly increased, another abrupt structural change in the basins can be observed (see figure 5d). The basin of attraction of the equilibrium P_{III} suddenly disappears and is replaced by a part of the basin of attraction of the equilibrium V_I. Accordingly, in such a situation for a rather large set of initial conditions, despite the fact that population 2 has a larger initial market share, population 1 is not only able to keep a certain number of firms in the market, but can eventually take over the entire market and drive all firms from population 2 out. Mathematically, the reason for this is that the interior equilibrium S moves through the equilibrium P_{III} on the boundary and an exchange of stability occurs. The former stable equilibrium P_{III} becomes a saddle point and the interior saddle point S leaves the unit square. The stable set of the saddle point P_{III} is the invariant boundary $x_2 = 1$, but this line now is repelling in the transversal direction in a neighborhood of P_{III}. It is easy to see that this transition occurs for a value of β_1 where

$$\beta_1 p_{III1} = \beta_2.$$

This condition means that at P_{III} the cost reduction effects due to internal spillovers are identical in both populations. The value of p_{III1} increases monotonically with increasing β_1. Thus, there exists a unique $\tilde{\beta}_1 > \beta_2$ such that for $\beta_1 = \tilde{\beta}_1$ the equality above is satisfied and P_{III} becomes unstable. Again, it should be pointed out that this kind of transition occurs only if the two fixed points Q_{III} and P_{III} appear to the left of the intersection of F_2 with $x_2 = 1$. In the following proposition we show that this holds only if the spillovers in population 2 larger than a threshold $\bar{\beta}_2$. Also, we provide the exact value of $\tilde{\beta}_1$.

Proposition 4 *Assume that* $\gamma = 0$ *and* $\beta_2 \in (\bar{\beta}_2, \beta^*)$*, where*

$$\bar{\beta}_2 := \frac{B + C - A + \sqrt{C(B + C - A)}}{A - B}. \tag{13}$$

Then for $\beta_1 > \beta^*$ *defined in Lemma 1 there appears a pair of fixed points* P_{III} *and* Q_{III} *on* $x_2 = 1$*, where* Q_{III} *is a saddle point and* P_{III} *is locally asymptotically stable. For all values* $(x_{1,0}, x_{2,0})$ *between the stable manifolds of* Q_{III} *and* S *in the long run all firms of population 2 and a fraction* p_{III1} *of firms in population 1 are in the market. As* β_1 *becomes larger than* $\tilde{\beta}_1 = \frac{\beta_2(1+\beta_2)B}{(1+\beta_2)(A-B)-C} > \beta^*$*,* P_{III} *becomes unstable, its basin of attraction suddenly disappears, and for all initial values* $(x_{1,0}, x_{2,0})$ *below the stable manifold of* Q_{III} *all firms from population 2 eventually leave the market which is completely taken over by population 1 (convergence to* V_I*).*

This analysis shows that for values of β_2 which are not too small, advantages of population 1 in terms of the size of spillover effects do not have continuous effects on the success of this firm cluster in the market. Rather, it is important to cross the two thresholds β^* and $\tilde{\beta}_1$ to be able to stay in the market, respectively take over the market, even if the other population is initially dominant in the market.

If β_1 is further increased, the basin of $(1, 0)$ expands, but we know from our analysis of the symmetric case that the saddle point Q_{III} always stays to the right of V_{III}. Accordingly, the basin of $(1, 0)$ can never cover the whole unit square. In other words, in situations with internal spillovers in the populations, there are always initial market conditions such that the population with the smaller spillovers is able to eventually drive firms from the other population out of the market. Of course, the extent of the set containing these initial market conditions becomes very small as the difference in internal spillovers (i.e. the difference in the values of the parameters β_i) becomes large.

If we start with a small symmetric level of internal spillovers ($\beta_1 = \beta_2 < \bar{\beta}_2$) and increase the size of internal spillovers in population 1, the pair of new equilibria Q_{III}, P_{III} which appears for $\beta_1 = \beta^*$ are initially right of the intersection of F_2 with $x_2 = 1$. This means that both are unstable in the direction transversal to $x_2 = 1$. Accordingly, Q_{III} is unstable and P_{III} is a saddle point where the stable manifold is the line $x_2 = 1$. Thus, also for $\beta_1 > \beta^*$ there exist only two stable equilibria, V_I, V_{III} and the basin of attraction of V_I increases continuously with increasing β_1, where the basin boundary is still given by the stable manifold of S. For $\beta_1 = \tilde{\beta}_1$, the interior equilibrium S wanders through Q_{III} and the stable manifold of S becomes the stable manifold of Q_{III} which is now a saddle point. Again, no discontinuous changes in the basins of attraction occur and for $\beta_1 > \tilde{\beta}_1$ the basin boundary is the stable manifold of Q_{III}. Thus, we see that if the spillovers in population 2 are small, the effects of an increase of the size of internal spillovers in population 1 are continuous and in the long run the market is always entirely taken over by one of the two populations. Summarizing, we have

Proposition 5 *Assume $\gamma = 0$ and $\beta_2 \in \left(\frac{C}{A-B} - 1, \bar{\beta}_2\right)$. Then, the only stable equilibria for $\beta_1 > \beta_2$ are V_I and V_{III}. The basin of attraction of V_I always increases continuously in β_1.*

Finally, let us look at the case where the spillovers are large and asymmetric. Given the results above, it is very easy to understand the effect of an increase of β_1 if initially large symmetric spillover effects with $\beta_1 = \beta_2 > \beta^*$ exist in both populations. Remember, that there are four coexisting stable equilibria in the symmetric case. We know that the curve F_1 rotates upwards as β_1 is increased. Since F_2 does not change and P_{II} and Q_{II} are already below F_1, it is obvious that the position and stability properties of P_{II} and Q_{II} do not change for increasing β_1. On the other hand, the transition on $x_2 = 1$ is exactly as the one described in proposition 4. At the point where β_1 becomes larger than $\tilde{\beta}_1$ and the equilibrium P_{III} becomes unstable, the basin of attraction of P_{III} is added to the basin of P_{II}. This means that now for a large set of initial conditions population 1 gains a larger market share in the long run. However, there always exists a smaller cluster of population 2 firms in the market. As before, further increases in the size of internal spillovers in population 1 do not yield any qualitative changes in the long run behavior of the process.

4.3 Asymmetric Spillovers and Fast Dynamics

Up to now we have assumed that the dynamics of the switching behavior of firms in the two populations is rather slow, i.e. that the parameter λ is very small. This means that the probability that a firm changes to the option which is, on average, more attractive is slowly increasing in the difference of the expected profits. What changes can be observed if the expected net flow towards the better option becomes larger and the dynamics faster? We will now investigate how faster dynamics influence the effect of differences in internal spillovers. Let us again consider the scenario depicted in figure 5b where the internal spillovers in population 1 are larger than the threshold value β^*, whereas $\beta_2 \in (\bar{\beta}_2, \beta^*)$. In this situation three locally stable equilibria coexist. Note that figure 5b depicts a scenario where switching is slow ($\lambda = \frac{0.2}{\pi}$). If the speed of switching is increased ($\lambda = \frac{2.2}{\pi}$), the basin boundaries change; see figure 6a. Whereas the boundary between the basins of V_I and P_{III} is no longer smooth[10], the change of the basins is continuous in λ_i and we still have three simply connected basins.

However, if λ is increased a little bit further, a quite remarkable change of the basins of attraction can be observed (see figure 6b). The basin of attraction of V_{III} has become non-connected, i.e. disjoint portions of it (so-

[10]The non-smoothness of the basin boundary is due to the fact that the interior equilibrium was transformed from a saddle to an unstable node by a sequence of local bifurcations. For a parameter setting like the one in figure 6a the boundary is formed by the stable set of a cycle, and the closure of such a stable set also includes many repelling nodes, whose presence yields the non-smoothness of the boundary.

Figure 6: Insert here!

called "islands") are nested inside the basin of another equilibrium. The basin of the boundary equilibrium P_{III} is now a multiply connected set (i.e. connected with "holes" inside it). This has quite interesting and surprising implications. For a given number of firms in population 1 in the market, an increase in the initial number of firms from population 2 in the market does not necessarily imply a higher long run market share for this population. On the contrary, a higher initial fraction of firms in the market may lead to a long run market share of zero whereas a lower initial fraction leads to convergence to P_{III} and the long run survival of a firm cluster from population 2 in the market.

Note that the constellation of fixed points and their local stability properties have not changed in this transition. Accordingly, and this is important to realize, local analysis cannot be used to explain this change in the long run properties of the process. In order to understand the occurrence of such a global bifurcation from a mathematical point of view we can, however, employ the theory of critical curves.

In figure 6c it can be observed that both LC_{-1} (which is the locus where the determinant of the Jacobian vanishes, $\det DT(x_1, x_2) = 0$) and $LC = T(LC_{-1})$ are closed curves. The region outside LC is the region Z_1 of points with only one rank-1 preimage, and inside LC there are points with three rank-1 preimages, i.e. the region Z_3. Note that the region Z_3 is entirely included in the basins of P_{III} and V_I for this value of λ. As λ is increased, the critical curve LC and the stable set of the boundary fixed point Q_{III}, which constitutes the boundary between the basins of P_{III} and V_{III}, have a contact (in fact, numerical evidence reveals that the first contact of LC and the boundary which separates the basins occurs along the boundary $x_2 = 1$). After this contact occurred, a small portion of Z_3 enters the basin of V_{III} (compare figures 6c and d). This means that suddenly a small portion of the basin of V_{III} has a larger number of preimages, namely three instead of one. The two new rank-1 preimages of this portion merge along LC_{-1}; see figure 6d. Since they are inside Z_3 these preimages again have three (rank-1) preimages (which are rank-2 preimages of the small region which has been created when LC crossed the basin boundary). This leads to an arborescent sequence of preimages. All these preimages belong to the basin of attraction of V_{III}, since they are mapped into the immediate basin of V_{III} after a finite number of iterations.

Since the first contact of LC and the basin boundary between P_{III} and V_{III} occurs along $x_2 = 1$, the occurrence of the global bifurcation which changes the structure of the basins can be understood by looking at the one-dimensional restriction of the map T to the invariant line $x_2 = 1$[11]. The critical points (local maxima and minima) of these restrictions are the intersections of LC and $x_2 = 1$. In figures 7a-c the graph of $h(x)$ is shown for the parameter values corresponding to figures 6a and 6b respectively.

Figure 7: Insert here!

In figure 7a the two stable fixed points have connected basins bounded by unstable fixed points. The local minimum is inside the basin of the positive stable fixed point (see the close-up in figure 7b). The change of λ first causes a contact of the local minimum and the unstable fixed point. After this contact a "hole" of the basin of $x = 0$ is created around the minimum. This can be clearly seen in figure 7c, where H_1, H_2 and H_3 indicate points right of p_{III1} which are mapped to the left of q_{III1} in 1, 2 respectively 3 iterations. All the points in these intervals therefore belong to the basin of attraction of 0. It should be pointed out that such a 'basin bifurcation' has to occur for any switching function G, where λ is sufficiently large and β sufficiently close to β^*. So, the transition which is responsible for non-connected basins in this framework does not depend on the exact specification of G.

As λ is further increased, the portion of the immediate basin of V_{III} inside Z_3 becomes larger and, consequently, the holes enlarge leading to even more fragmented and intermingled basins. It is evident that although the attractors continue to be simple equilibria (stable fixed points) the structure of the basins is getting more and more complex. This causes a greater uncertainty about the long-run evolution of the system starting from a given initial condition in the following sense: A small change in the starting condition has a high probability to cause a crossing of a basin boundary and, as a consequence, the convergence to a different equilibrium. Hence, the

[11]It should be emphasized that this is a special characteristic of this model, and no general property of this kind of basin bifurcations.

long run outcome of the process now depends very sensitively on the initial number of firms from both populations in the market.

We summarize the findings of the analysis in the following proposition.

Proposition 6 *For $\gamma = 0, \beta_2 \in \left(\bar{\beta}_2, \beta^*\right)$ and $\beta_1 > \beta^*$, where β_1 is sufficiently close to β^* and λ sufficiently large, the basin of attraction of V_{III} is non-connected and has islands in the basin of attraction of P_{III}. The number of population 2 firms which stay in the market in the long run does not increase monotonic in the initial number of population 2 firms in the market.*

Having provided an extensive technical analysis of the model for a situation where internal spillovers exist in both populations, we would like to discuss the major economic insights we have obtained. First, starting from a situation where spillovers between firms in both populations are symmetric, all equilibria and basins are symmetric with respect to the diagonal. According to this result, the population which initially has the larger fraction of firms in the market will keep a larger market share in all periods. For small internal spillovers, all firms from the population with the smaller initial market share eventually leave the market and choose the outside option. The market is completely taken over by the population with the larger initial market share. Larger internal spillovers make it possible that firms from both populations coexist in the market. The population with the larger initial market share is able to keep this advantage in the long run. However, for a set of initial conditions, an equilibrium is reached in the long run where also a fraction of firms of the other population might stay in the market. This set of initial conditions captures a setting where the market share of the other population is not too much smaller. It expands in size as internal spillovers become larger, but there remains a set of initial conditions where firms from either population (1 or 2) is driven out of the market. For slow dynamics, which corresponds to a large variance of the outside option, all basins of attraction are simple connected sets. Hence, for initial conditions inside either of the basins, a small change in the initial fraction of firms in the market does not change the long run outcome of the evolutionary process. This does not necessarily hold true if the flow in and out of the market is large (this corresponds to a low variance of the outside profit). In the latter situation the basins of the coexisting equilibrium patterns may have an intermingled structure. Consequently, it is hard to predict whether firms from the population with the initially smaller fraction of firms stay in the market or if the other population takes over the market completely. The long run outcome now depends quite sensitively on the exact values of the initial market shares of the two populations. Thus, our model suggests that, even if the circumstances in two different industries are almost identical, the adaptation process of the firms may lead to qualitatively very different long run result. In this sense the process of industrial evolution is path-dependent. It should be noted, that this insight is not due to some complex or chaotic long run dynamics of the model, but simply caused by the complex topological structure of the basins of attraction of the coexisting stable equilibria.

If the spillover effects in one of the two populations is larger, the basins become asymmetric. Surprisingly, if firms from one population can attain a situation where spillovers between them are higher (in comparison to spillovers in the other population), the generated cost reductions does not have continuous long run effects. Small differences in spillovers, in general, only lead to a small expansion of the set of initial conditions where the market is taken over by the population with larger spillovers. However, as the difference in spillovers (the parameter values of β_i) reaches a certain critical level, there is an abrupt structural change in the basins. Suddenly the set of initial market shares where firms from the population with larger spillovers stay in the market expands due to the instantaneous creation of another equilibrium pattern with its corresponding basin. This suggests that there is something like a minimal advantage in the extent of internal spillovers which is crucial for determining if a population of firms can stay in a market in the long run even if this market is dominated by firms from another population. Our analysis further has revealed that there is a second critical level for the difference of internal spillovers. If the spillovers between the firms of one population are large enough such that the corresponding difference in spillovers is larger than this level, firms of this population can not only coexist in the market with firms from the other population (as before), but are suddenly able to drive firms from the other population out of the market despite the fact that the latter initially dominated the market. Thus, one of our major results is that the changes resulting from investments which increase spillovers between firms of the same population are not continuous. These changes occur suddenly when certain levels of spillover differences are achieved.

Finally, our results indicate that it is advantageous for the population with larger internal spillovers if exit and entry in the market is slow. As the rate at which firms enter and exit the market becomes larger, the set of initial market shares where the population with smaller internal spillovers takes over the entire market in the long run becomes larger since more and more islands of the basin of V_{III} are created in the basin of P_{III}. Although we did not provide a proof that this observation holds true in general, all our numerical analyses suggest that this is in fact a general phenomenon.

5 Internal and External Spillovers

In the previous section we have restricted ourselves to analyzing the long run agglomeration patterns in the market in the presence of internal spillovers between firms of a population only. In this section we focus on the impact of external spillovers. More precisely, we will assume that internal spillovers exist and are equal in both populations ($\beta_1 = \beta_2$) and that, additionally, knowledge also spills over from one population to the other. We will consider only the case where $\gamma_1 > 0, \gamma_2 = 0$. However, the results can be easily generalized to any case where knowledge spillovers are asymmetric, i.e. where the two populations are heterogeneous with respect to these external spillovers. Our assumptions of asymmetric external spillovers are motivated by empirical evidence. Mansfield (1988) found in his study that countries differ with regard to their ability to adopt and use foreign technology. Whereas US and Japanese firms were comparable in exploiting internally developed technologies, the empirical evidence revealed a big difference in the use of externally-based technologies. Japanese firms pursue foreign technology more aggressively and efficiently than their rivals. Accordingly, there is evidence for an asymmetry in spillovers. Several sources of external spillovers can be named, including transfers of knowledge between two countries (due to e.g. an exchange of engineers, managers and workers) and direct foreign investment. See Chuang and Lin (1999), who identify foreign direct investment as the major channel of technology transfer from multinational enterprises to domestic firms.

We begin the analysis with a scenario like the one depicted in figure 2, with weak internal and no external spillovers. Recall that the only two stable equilibria are the points V_I and V_{III}. We know that in this case the basins of attraction of the two equilibria are simply the triangular regions below and above the diagonal, respectively. The curve F_1 has one intersection point with the line $x_1 = 0$ and one with $x_1 = 1$, but does not intersect any of the other two boundaries of the unit square. A symmetric statement holds for F_2. Let us now assume that, due to some of the reasons mentioned above, knowledge generated in population 2 spills over to firms in population 1, or, in other words, the parameter γ_1 is positive. Now the symmetry between the two populations is broken. If γ_1 is increased, the intersection point of

F_1 with $x_1 = 0$ – denoted by $\hat{x}_2$ – which satisfies

$$A - B\hat{x}_2 - \frac{C}{1 + \gamma_1 \hat{x}_2} = 0$$

obviously moves up on the line $x_1 = 0$. The same happens to the intersection point with $x_1 = 1$ – denoted by $\tilde{x}_2$ – which satisfies

$$A - B(1 + \tilde{x}_2) - \frac{C}{1 + \beta + \gamma_1 \tilde{x}_2} = 0.$$

Depending on the parameter setting, two possible scenarios can occur for increasing values of γ_1 which satisfy assumption (3). First – similar to the case discussed in the previous section – a pair of new equilibria might emerge on the line $x_2 = 1$. Second, the curve F_1 may never cross $x_2 = 1$, until γ_1 has been increased so much that $\hat{x}_2 = 1$. In the latter case only one new fixed point appears at V_{III} and moves to the right on the line $x_2 = 1$ as γ_1 is further increased. This fixed point is locally stable, whereas the vertex V_{III} becomes unstable. This second scenario can be developed from the first scenario and we, therefore, will not treat it separately.

If we restrict our attention to the case where a pair of new equilibria on the line $x_2 = 1$ emerges, considering the qualitative dynamics we encounter a similar situation as in the case of internal spillovers with increasing values of β_1. However, here it can be guaranteed under fairly mild assumptions that one of the two emerging fixed points is locally asymptotically stable (proof in Appendix A).

Proposition 7 *Assume that $\beta_1 = \beta_2 = \beta \in (\hat{\beta}, \beta^*)$, $\gamma_2 = 0$ and $C < 4B$. Then for*

$$\gamma_1 = \gamma^*(\beta) := \frac{2\sqrt{\beta BC} - \beta(A - B) - B}{B}$$

there emerges a pair of fixed points P_{III}, Q_{III} on the line $x_2 = 1$, where P_{III} is locally asymptotically stable and Q_{III} is a saddle point. Furthermore, we always have $\gamma^*(\beta) < \beta^* - \beta$.

In the present scenario, population 1 benefits from the external spillovers from population 2. These spillovers enable population 1 firms to stay in the market for a certain set of initial conditions despite the fact that population 2 initially has a larger fraction of firms in the market. As in the case discussed in the previous section, the basin of attraction of P_{III} is created abruptly as a set of positive measure when γ_1 becomes larger than γ^*. The proposition also establishes that the increase in γ_1, which is necessary to produce such a transition for every current level of β, is smaller than the increase in β_1 which is needed for such a structural change.

Thus, given a scenario where the only stable equilibria are V_I and V_{III}, this result suggests the following policy formulation. A policy maker, who is in charge of setting the conditions of the environment for firms of population 1, i.e. who can influence the values of the spillover parameters β_1 and γ_1, is advised to increase γ_1 rather than β_1. This leads to the long run existence of a cluster of firms from population 1 in the market for a larger set of initial market shares than in the case of an increase of β_1 of the same size.

Figure 8: Insert here!

In figure 8a we illustrate this by depicting the basins of attraction, where we used the same parameter setting as in section 4 with $\beta_1 = \beta_2 = 1$ and $\gamma_1 = 0.1$, $\gamma_2 = 0$. Recall that an increase of more than 0.2 in the value of internal spillovers, β_1, was necessary to create a stable equilibrium on the upper boundary of $[0,1]^2$. In contrast to this, for an increase in the value of external spillovers of 0.1, this additional equilibrium not only exists, but already has a rather large basin of attraction. If the value of γ_1 is further increased, the saddle point Q_{III} moves through the vertex V_{III} and the vertex becomes unstable. Note that such a transition is not possible if β_1 is increased, but $\gamma_1 = 0$. After this bifurcation there are only two stable equilibria namely P_{III} and V_I. In other words, regardless of the initial market shares, firms from population 1 are never completely driven out of the market. This situation is illustrated in figure 8b.

A further increase of γ_1 finally leads to a collision of the interior fixed point S with the boundary fixed point P_{III} and after that the only stable fixed point is V_I. Thus, population 1 takes over the entire market regardless of the initial market conditions. Note that, again, we have an instantaneous change of the long run behavior of the system. If γ_1 is only slightly smaller than the bifurcation value, there is still a set of initial market constellations with positive and often significant measure which lead to long run market participation of all population 2 firms. As soon as γ_1 crosses this value and is slightly larger, all firms from population 2 leave the market and choose the outside option for all those initial states. For different parameter settings these two transitions might occur in reversed order, however with the same final result.

If the (symmetric) level of internal spillover effects in both populations is larger than β^* (remember that in such a situation for small γ_1 there are 4 locally stable equilibria; see figure 4), and γ_1 is sufficiently large such that S collides with P_{III}, this yields a transformation of the basin of attraction of P_{III} into a part of the basin of P_{II}. In other words, if the level of internal spillovers is large in both populations and firms from population 1 have a higher ability to import information from population 2, a large set of initial conditions leads to a long run state where all firms in population 1 and a positive fraction of population 2 firms are in the market. If internal spillovers are sufficiently large in population 2, a cluster from this population stays in the market even if external spillovers from population 2 to population 1 is much higher than vice versa.

In the description of the effects of an increase of γ_1, so far we have considered slow dynamics. In particular, in our discussion we have implicitly assumed that the critical curve LC has no intersection point with the stable set of Q_{III} and, therefore, no basin bifurcations occurs. However, as has been demonstrated in the previous section, for large values of λ_1 such basin bifurcations, which change the topological structure of the basins, cannot be ruled out. Thus, in order to deepen our understanding of the possible dynamical patterns and how they depend on the existence and magnitude of internal and external spillovers, we study the effect of fast switching. As an illustration of the following discussion we show an example of the possible transitions of the basins for fast dynamics and increasing γ_1 in figure 9. In this chosen scenario, the critical curve spreads out and a large part of the region close to the boundary lines $x_1 = 1$ and $x_2 = 1$ lies inside the closed critical curve. In particular, for sufficiently large λ, the pair of new fixed points emerging on the boundary line $x_2 = 1$ are located in a region which is surrounded by the critical curve LC. In this case, from similar arguments as in the in the previous section it follows that the basin of attraction of the stable equilibrium P_{III} contains non-connected parts (islands) which belong to the basin of attraction of V_{III} (see figure 9b). In the previous section it has been pointed out that these intermingled basins can be best understood by looking at the map $h(x)$ on the line $x_2 = 1$. Here it can be observed that some of the trajectories with initial condition above q_{III1} are eventually mapped below q_{III1} and thus converge to zero. Accordingly, the basins of 0 and p_{III1} are intermingled. If γ_1 is further increased and the dynamics is sufficiently fast, the slope of $h(x)$ at p_{III1} becomes larger until this fixed point becomes unstable. For even larger γ_1 all points are eventually mapped below q_{III1} and thus the only attractor of the map $h(x)$ is 0[12]. Consequently, the only attractors of the adaptation dynamics generated by the map T on the unit square are again V_I and V_{III} (figure 9c). Interestingly, we get a counterintuitive result stating that the further increase of γ_1 has weakened the position of population 1. The mixed equilibrium has disappeared, and the situation is somehow reminiscent of the scenario before any basin bifurcation has occurred; compare figures 9a and c. Firms of population 1 are now driven out of the market for initial market shares, whereas before at least a fraction of population 1 firms could stay in the market. Note, however, that the fixed point Q_{III} moves towards V_{III} as γ_1 is increased. So eventually, this fixed point crosses the critical curve LC and wanders from the region Z_3 into the region Z_1. After this has happened, the stable manifold of the saddle point is forwards and backwards invariant and no trajectory with initial conditions to the right of this boundary can converge to V_{III}. All the trajectories between the stable manifold of Q_{III} and the stable manifold of the interior equilibrium S converge to some attractor on $x_2 = 1$, which might be either a cycle or a chaotic attractor (see figure 9d; remember that p_{III1} is unstable with respect to $h(x)$).

Figure 9: Insert here!

Figure 9 makes it obvious that after the transition from a) to b), population 2 takes over the market for less initial market shares than before the increase of γ_1. A further increase of γ_1 seems to revert this change, at least, as far as the basins of attraction are concerned. Note, however, that although the basins in a) and c) look similar, the transient behavior of a trajectory $(x_1, x_2)_t$ for these two parameter settings may differ significantly. In particular, in case c) trajectories close to the line $x_2 = 1$ might oscillate for some time before converging to $(0, 1)$. So, if we just look at the transient behavior, increasing external spillovers from population 2 to population 1 have the effect of keeping firms from population 1 in the market much longer, although in both cases they eventually leave the market. This transient effect of a viable population 1 cluster only becomes a long term effect if γ_1 is further increased and scenario d) is obtained. Here for a large

[12]After the p_{III1} becomes unstable increasing values of γ_1 create stable cycles of increasing period before the attractor collides with q_{III1}.

set of initial conditions the number of firms from population 1 in the market does not converge at all, but keeps oscillating around the fixed point P_{III}. It is worth pointing out that figures 9 b) and d) help us to realize that there is no connection between the complexity of the attractor of the dynamics and the complexity of the basins. In figure 9 b) the dynamics of the trajectories is simple, all attractors are single points, but the basins are intermingled. Hence, it is rather difficult for a certain set of initial market conditions to predict which fixed point will be reached. On the other hand, in figures 9 d) all basins are simply connected sets. However, the attractor on the line $x_2 = 1$ is a cycle of period four and the typical trajectory in this case is cycling.

It should be clear by now that a similar transition along the border line $x_1 = 1$ occurs if external spillovers exist from population 1 to population 2, i.e. γ_2 is positive. So, similar to the symmetric cases with large and identical internal spillovers, there will be four coexisting locally stable fixed points if external spillovers in both populations are identical and sufficiently large. In contrast to the case where only internal spillovers exist, a further increase of $\gamma_1 = \gamma_2$ leads to the disappearance of the two stable equilibria on the vertices. Hence, only two stable equilibria are left, one in the interior of the upper edge of the unit square and one in the interior of the right edge of the unit square. Accordingly, our analysis yields a very intuitive result: large transfers of knowledge between the two populations (due to large external spillovers) result in the long-run participation of firms from both populations in the market.

Finally, there is one main point which we would like to emphasize. If one compares the effect of an increase in internal spillovers (due to increased communication and worker rotation within a population) and external spillovers (due to increased transfer of knowledge from one population to the other), our analysis reveals that the long run effect of an increase in external spillovers (γ_i) is always larger than the effect of an increase of the same size of internal spillovers (β_i). Also, if external spillovers do not exist, a population is always able to drive the other population out of the market given that the advantage in the initial market share is sufficiently large. This is valid regardless of the difference in internal spillovers. In contrast to this, a population can guarantee to take over the market – regardless of the initial market shares – if the advantages stemming from external spillovers are sufficiently large. In short, the suggestion for policy makers which follows from our analysis is that enabling import of knowledge from another source might be a more effective means than trying to create and exchange knowledge within a population. Interestingly, in their empirical study Chuang and

Lin (1999) find that foreign direct investment is a substitute to R&D. Their policy implication is the following: *"During the early development stage, technology transfer, especially through direct foreign investment of MNEs, can facilitate industry-wide technological learning and diffusion, and thus may be the most effective way for the developing country to strengthen its technical capability and to absorb appropriate technologies."* (p. 133).

6 Discussion

In this paper we have used an evolutionary industry model to study the effect of local and across-border spillovers in two countries competing on a single good market. The decision of a firm whether to produce a good for the market or to choose some outside option is made using a simple stochastic decision rule. We have focused our analysis on the long run outcome of the dynamic adaptation process in both populations and in particular have characterized the basins of attraction of the stable market constellations for different parameter values. We have demonstrated that a dynamic analysis of the evolution of market shares yields very precise insights into the relationship between the size of internal spillover effects, the inertia in the process of market exit and entry, initial market shares and the long run success of a firm cluster in the market. The main findings of this analysis may be summarized as follows:

- for symmetric spillovers initial advantages in market share lead to long-run dominance in the market
- increasing symmetric spillover effects facilitate the growth of the initially smaller firm cluster
- increasing spillovers have discontinuous effects on the basins of long run agglomeration patterns
- increasing spillovers across country borders is more effective than increasing local spillovers
- slow switching behavior is advantageous for the population with the larger spillover effects
- the sets of initial market shares yielding long run dominance of different populations may be intermingled, hence long run market shares may depend sensitively on initial market shares and are not always

monotonously increasing in the initial number of firms of a population in the market.

More generally, we have shown that the analysis of dynamic evolutionary models with non-linear payoff structure may be challenging due to the presence of numerous coexisting locally stable equilibria with complicated basins of attraction. For a sound understanding of the long run properties of the process a characterization of these basins is necessary and we have demonstrated how the theory of critical curves can be used in addition to local bifurcation theory to explain the structural changes in the long run properties of the process. We hope that the merits of this combination of techniques will in the future be further exploited by scholars interested in evolutionary analysis.

Acknowledgments. We thank Laura Gardini for helpful comments and suggestions. This work has been performed under the under the auspices of CNR, Italy, and under the activity of the national research project "Nonlinear Dynamics and Stochastic Models in Economics and Finance", MURST, Italy.

References

Audretsch, D.B. and Feldman, M.P. (1996): R & D Spillovers and the Geography of Innovation and Production. American Economic Review, 86, 630 - 640.

Bischi, G.I. and Kopel, M. (1999): Equilibrium Selection in a Nonlinear Duopoly Game with Adaptive Expectations, Working Paper, University of Urbino.

Bischi, G.I., Gardini, L. and Kopel, M. (2000): Analysis of global bifurcations in a market share attraction model. Journal of Economic Dynamics and Control, 24, 855 - 879.

Carlisle, E.R. (1992): Spillover Asymmetries and a Comparative Technological Advantage, The American Economist, **36**, 13 - 17.

Chuang, Y.-C. and Lin, C.-M. (1999): Foreign Direct Investment, R & D and Spillover Efficiency: Evidence from Taiwan's Manufacturing Firms, Journal of Development Studies, **35**, 117 - 137.

Dawid, H. (1999a): On the Convergence of Genetic Learning in a Double Auction Market. Journal of Economic Dynamics and Control, **23**, 1545 - 1567.

Dawid, H. (1999b): On the Dynamics of Word of Mouth Learning with and without Anticipations. Annals of Operations Research, **89**, 273 - 295.

Ellison, G. (2000): Basins of Attraction, Long Run Stochastic Stability and the Speed of Step-by-step Evolution. Review of Economic Studies **67**, 17-45.

Ellison, G. and Fudenberg, D. (1993): Rules of Thumb for Social Learning, Journal of Political Economy, **101**, 612 - 643.

Ellison, G. and Fudenberg, D. (1995): Word-of-Mouth Communication and Social Learning. Quarterly Journal of Economics **CX**, 93-125.

Ellison, G. and Glaeser, E.L. (1997): Geographic Concentration in U.S. Manufacturing Industries: A Dartboard Approach. Journal of Political Economy **105**, 889 - 927.

Ellingsen, T. (1997): The Evolution of Bargaining Behavior. The Quarterly Journal of Economics. 581 - 602.

Friedman, D. (1991): Evolutionary Games in Economics. Econometrica **59**, 637-666.

Friedman, D. (1998): On Economic Applications of Evolutionary Game Theory. Journal of Evolutionary Economics **8**, 15 -43.

Head, K., J. Ries and D. Swenson (1995): Agglomeration Benefits and Location Choice: Evidence from Japanese Manufacturing Investments in the United States, Journal of International Economics **38**, 223-247.

Lorenz H.-W. (1993): Nonlinear Dynamical Economics and Chaotic Motion, 2d ed., Springer, Berlin.

Lu, X. and McAfee, R.P. (1996): The Evolutionary Stability of Auctions over Bargaining. Games and Economic Behavior **15**, 228 - 254.

Mansfield, E. (1988): The Speed and Cost of Industrial Innovation in Japan and the United States: External vs. Internal Technology, Management Science **34**, 1157-1168.

Marshall, A. (1920): Principles of Economics: An Introductory Volume. 8th ed. London: Macmillan.

Mira, C., Gardini, L., Barugola, A. and Cathala, J.-C. (1996): Chaotic Dynamics in Two-Dimensional Noninvertible Maps. World Scientific, Singapore.

Neven, D. and Siotis, G. (1996): Technology Sourcing and FDI in the EC: An Empirical Evaluation. International Journal of Industrial Organization **14**, 543 - 560.

Qin, C.-Z. and Stuart, C. (1997): Are Cournot and Bertrand Equilibria Evolutionary Stable Strategies?. Journal of Evolutionary Economics **7**, 41 - 47.

Vega-Redondo, F. (1997): The Evolution of Walrasian Behavior. Econometrica, **65**, 375 - 384.

Weibull, J.W. (1995): Evolutionary Game Theory. MIT Press, Cambridge, MA.

Young, P. (1993): An Evolutionary Model of Bargaining. Journal of Economic Theory **59**, 145 - 168.

Young, P. (1998): Individual Strategy and Social Structure: An Evolutionary Theory of Institutions. Princeton University Press: Princeton.

Appendix A

Proof of Proposition 1.

We start the proof by giving a characterization of the equal profit curves F_1 and F_2. The curve of equal profit F_1 is given by

$$B\beta_1 x_1^2 + B(\beta_1+\gamma_1)x_1x_2 + B\gamma_1 x_2^2 + (B - A_1\beta_1)x_1 + (B - A_1\gamma_1)x_2 + C_1 - A_1 = 0, \tag{14}$$

and the curve F_2 satisfies

$$B\beta_2 x_2^2 + B(\beta_2+\gamma_2)x_1x_2 + B\gamma_2 x_1^2 + (B - A_2\beta_2)x_2 + (B - A_2\gamma_2)x_1 + C_2 - A_2 = 0 \tag{15}$$

Thus, these curves are hyperbolae with centers $K_1 = \left(-\frac{B+A_1\gamma_1}{B(\beta_1-\gamma_1)}, \frac{B+A_1\beta_1}{B(\beta_1-\gamma_1)}\right)$ for F_1 and $K_2 = \left(\frac{B+A_2\beta_2}{B(\beta_2-\gamma_2)}, -\frac{B+A_2\gamma_2}{B(\beta_2-\gamma_2)}\right)$ for F_2. The slopes of the asymptotes are -1 and $-\frac{\beta_1}{\gamma_1} < -1$ for F_1 and -1 and $-\frac{\gamma_2}{\beta_2} > -1$ for F_2. We concentrate on the properties of F_1, the corresponding properties of F_2 follow by symmetry. Since the center of F_1 is left of $[0,1]^2$, the curve F_1 in this area is upward bending (it has positive curvature). Analogously, the curve F_2 is downward bending (it has negative curvature). It is further easy to see from (14) that there has to be exactly one intersection of F_1 with the line segment $\{0\} \times [0,\infty)$. Note further that along any straight line in $[0,1]^2$ with slope -1 the overall number of firms in the market and, therefore, also the market price stays constant. Since we assume $\beta_i \geq \gamma_i$, the profit difference $\pi_i(x_1,x_2) - U_i$ increases along any such line. In particular, this means that if we draw a straight line L with slope -1 through an arbitrary point (x_1,x_2) of F_1 we have $\pi_1(\tilde{x}_1,\tilde{x}_2) > U_1$ for every point $(\tilde{x}_1,\tilde{x}_2) \in L$, such that $\tilde{x}_1 > x_1$ and $\pi_1(\tilde{x}_1,\tilde{x}_2) < U_1$ for every point $(\tilde{x}_1,\tilde{x}_2) \in L$, such that $\tilde{x}_1 < x_1$. The same argument shows that if L is a straight line with slope -1 through a point (x_1,x_2) on F_2 $\pi_2(\tilde{x}_1,\tilde{x}_2) < U_2$ for every point $(\tilde{x}_1,\tilde{x}_2) \in L$, such that $\tilde{x}_1 > x_1$, and $\pi_2(\tilde{x}_1,\tilde{x}_2) > U_2$ for every point $(\tilde{x}_1,\tilde{x}_2) \in L$, such that $\tilde{x}_1 < x_1$.

We will now characterize the fixed points on the boundary. To minimize notation let us denote the boundary line of $[0,1]^2$ with $x_2 = 0$ by BL_1 the one with $x_1 = 1$ by BL_2, the one with $x_2 = 1$ by BL_3 and the one with $x_1 = 0$ by BL_4. Due to our assumptions (3) and (4) there has to be an odd number of intersections of F_1 respectively F_2 with $BL_1 \cup BL_2$ and an odd number of intersections in $BL_3 \cup BL_4$. Note further that due to our assumption that $\beta_i > \gamma_i$, F_1 cannot intersect with BL_1 without intersecting with BL_4 and F_2 cannot intersect with BL_4 without intersecting with BL_1. Furthermore, the arguments in the last paragraph imply that if F_1 intersects with BL_1 the whole curve must lie below the straight line with slope -1 through this intersection point, which in particular shows that F_1 cannot also intersect with BL_3. This leaves us with 5 different cases: a) the equal profit curve never enters $[0,1]^2$; b) F_1 has one intersection point with BL_1, then there has to be exactly one intersection point with BL_4 but no intersection points with BL_2 and BL_3. If F_1 has no intersection point with BL_1 then there has to be exactly one intersection point with BL_2 (if there were more than one intersection points there would have to be at least three which is ruled out by the hyperbolic shape of F_1). This gives three more cases: c) there is one intersection point with BL_2 and one with BL_4; d) there is one intersection point with BL_2 and one with BL_3; e) there is one intersection point with BL_2, two intersection points with BL_3 and one intersection point with BL_4. The claim of the proposition concerning the boundary equilibria follows directly.

To show that there can be at most one interior fixed point, we simply have to observe that if we draw a line with slope -1 through an intersection point (x_1^*, x_2^*) of F_1 and F_2 all points on F_1 with $x_1 > x_1^*$ have to lie above this line, all points on F_1 with $x_1 < x_1^*$ have to lie below this line. On the other hand, every point on F_2 with $x_2 > x_2^*$ has to lie below the line and any point on F_2 with $x_2 < x_2^*$ has to lie above this line. Therefore there cannot be a second point of intersection of F_1 and F_2.

Finally, we show that the interior equilibrium is always unstable. Since we know that the dynamics along the straight line with slope -1 points away from the equilibrium, obviously the interior equilibrium always has to have at least one unstable manifold. Furthermore, it is easy to realize that a straight line between $(0,0)$ and the equilibrium never intersects either F_1 or F_2. Given our assumptions about the direction of the dynamics at $(0,0)$ it follows that the dynamics points towards the interior fixed point along this line. Accordingly, the interior fixed point either has to be a saddle point or a repelling node with one positive and one negative eigenvalue, where both have absolute values larger than one. □

Proof of Lemma 1

The condition for F_1 to have a tangency with $x_2 = 1$ is that the equation

$$A - B(1 + x_1) - \frac{C}{1 + \beta x_1} = 0. \tag{16}$$

which is equivalent to the quadratic equation

$$\beta B x_1^2 - ((A-B)\beta - B)x_1 - (A-B-C) = 0 \tag{17}$$

has a positive real solution. Assumption (4) guarantees that the left hand side is positive for $x_1 = 0$. Hence, if there exist real roots, either both of them are negative or both of them are positive. Real roots exist if

$$((A-B)\beta - B)^2 - 4\beta B(C - (A-B)) \geq 0. \tag{18}$$

To show that there exists a pair of positive real numbers where the inequality is binding, we observe that the left hand side goes to infinity for $\beta \to \infty$, is positive for $\beta = 0$, and, given assumption (4) is negative for $\beta = \frac{B}{A-B}$. We denote the two real roots of (18) by $0 < \hat{\beta} < \beta^*$. It is now straight forward to see that (17) has two negative roots for $\beta < \hat{\beta}$, no real roots for $\hat{\beta} < \beta < \beta^*$ and two positive roots for $\beta > \beta^*$. □

Proof of Proposition 4

First, we have to show that under the conditions given in the proposition the point where the curve F_1 touches $x_2 = 1$ for $\beta_1 = \beta^*$ is left to the intersection of F_2 and $x_2 = 1$. This intersection $(\tilde{x}_1, 1)$ is defined by the equality $A - B(1 + x_1) - \frac{C}{1+\beta_2} = 0$ and therefore we have

$$\tilde{x}_1 = \frac{1}{B}\left(A - B - \frac{C}{1+\beta_2}\right). \tag{19}$$

If β_1 is increased above β^*, either Q_{III} or P_{III} eventually collides with $(\tilde{x}_1, 1)$. To prove our claim we have to show that P_{III}, which is right of Q_{III}, collides with $(\tilde{x}_1, 1)$. Both at Q_{III} and P_{III} we have $A - B(1+x_1) - \frac{C}{1+\beta_1 x_1} = 0$ and thus when one of the two points collides with $(\tilde{x}_1, 1)$ we must have $\beta_1 \tilde{x}_1 = \beta_2$, which implies

$$\beta_1 = \tilde{\beta}_1 := \frac{\beta_2(1+\beta_2)B}{(A-B)(1+\beta_2) - C}.$$

Note that the denominator is positive for $\beta_2 > \bar{\beta}_2$. Considering the derivative of population 1 payoffs with respect to x_1 at this point for $\beta_1 = \tilde{\beta}_1$, we get

$$\begin{aligned}
\frac{\partial \pi_1(\tilde{x}_1, 1)}{\partial x_1} &= -B + \frac{C\tilde{\beta}_1}{(1+\tilde{\beta}_1\tilde{x}_1)^2} \\
&= -B + \frac{C\tilde{\beta}_1}{(1+\beta_2)^2} \\
&= -B + \frac{CB\beta_2}{(1+\beta_2)((A-B)(1+\beta_2) - C)}.
\end{aligned}$$

The last expression is negative if and only if $\frac{C\beta_2}{(1+\beta_2)((A-B)(1+\beta_2)-C)} < 1$. Straightforward calculations show that this is true if and only if (13) holds. It follows from $\pi_1(1,1) < 0$ that, the inequality $\frac{\partial \pi_1(x_1,1)}{\partial x_1} < 0$ can only hold at P_{III} and we have shown the proposition. Note that S moves through P_{III} when P_{III} collides with $(\tilde{x}_1, 1)$. $\square$

Proof of Proposition 7:
The minimal value of γ_1 such that the equilibria P_{III} and Q_{III} are created is given by the minimal value such that there is an $x_1 \in (0,1)$ with

$$A - B(1+x_1) - \frac{C}{1+\beta_1 x_1 + \gamma_1} = 0.$$

For every x_1 the marginal increase of the left hand side if γ_1 is increased is always larger than the marginal increase if β_1 is increased. This implies that for any given value of $\beta_1 = \beta_2 = \beta < \beta^*$ a smaller increase of γ_1 than of β_1 is required to create the stable equilibrium P_{III} and to enable firms from population 1 to stay in the market for an additional set of initial conditions. Thus, $\gamma^*(\beta) < \beta^* - \beta$. Determining the value of γ_1 such that the two solutions of the equation above coincide, yields the value $\gamma^*(\beta)$ given in the proposition.

To show that the emerging fixed point P_{III} is locally asymptotically stable we have to establish that the pair of fixed points P_{III}, Q_{III} appear to the left of the intersection point $(\tilde{x}_1, 1)$ of F_2 with $x_2 = 1$. As in the proof of proposition 4, we show that as γ_1 is further increased, the fixed point P_{III} collides with $(\tilde{x}_1, 1)$. When Q_{III} or P_{III} collide with $(\tilde{x}_1, 1)$ we must have Looking again at the derivative of population 1 payoffs with respect to x_1, the assumption $C < 4B$ gives

$$\begin{aligned} \frac{\partial \pi_1(\tilde{x}_1, 1)}{\partial x_1} &= -B + \frac{C\beta}{(1+\tilde{\gamma}+\beta\tilde{x}_1)^2} \\ &= -B + \frac{C\beta}{(1+\beta)^2} \\ &< 0. \end{aligned}$$

Hence, we conclude that P_{III} collides with $(\tilde{x}_1, 1)$ and we have shown the proposition. $\square$

Appendix B

Since we consider a case with symmetric parameter constellations we have $T_1(x,y) = T_2(y,x)\ \forall (x,y) \in [0,1]^2$. Let us denote the restriction of the map to the diagonal by $f(x)$: $f(x) := T_1(x,x) = T_2(x,x)$. Thus, $f'(x) = T_{1x_1}(x,x) + T_{1x_2}(x,x)$. We show that any intersection of the critical curve separating a region with 3 preimages, Z_3, from the region with only one preimage, Z_1, with the diagonal has to be a critical point of the map f. This implies that the number of preimages of f on the diagonal switches from 1 to 3. Let us denote this intersection by (z,z). Any point on the critical curve has two rank-1 preimages, two merging in a point of LC_{-1} and an extra preimage. Since the diagonal is invariant, at least one of the two rank-1 preimages of (z,z) has to be on the diagonal. However, since the dynamical system is perfectly symmetric with respect to the diagonal this implies that also the second rank-1 preimage is on the diagonal. At the point $(v,v) \in LC_{-1}$, where the two merging rank-1 preimages are located, the determinant of the Jacobian has to vanish (see Appendix C). Due to the symmetry of the map T we have $T_{1x_1}(v,v) = T_{2x_2}(v,v)$, $T_{1x_2}(v,v) = T_{2x_1}(v,v)$ and the characteristic polynomial of the Jacobian at (v,v) is given by $(T_{1x_1} - \nu)^2 - T_{1x_2}^2$. Thus, the eigenvalues of the Jacobian at (v,v) are given by $\nu_1 = T_{1x_1}(v,v) + T_{1x_2}(v,v)$ and $\nu_2 = T_{1x_1}(v,v) - T_{1x_2}(v,v)$. Simple calculations show that

$$\begin{aligned} T_{1x_1}(v,v) - T_{1x_2}(v,v) &= 1 + (1-2v)G\left(A - 2Bv - \tfrac{C}{1+\beta v+\gamma v}\right) \\ &\quad + v(1-v)G'\left(A - 2Bv - \tfrac{C}{1+\beta v+\gamma v}\right)\tfrac{C(\beta-\gamma)}{(1+\beta v+\gamma v)^2}. \end{aligned}$$

From our assumption that $\beta \geq \gamma$ it follows from that this expression is positive. Since the Jacobian has to vanish at (v,v), $T_{1x_1}(v,v)+T_{1x_2}(v,v) = 0$ has to hold. We know that $f'(v) = T_{1x_1}(v) + T_{1x_2}(v)$ and this implies that z has to be a critical point of f. Accordingly, the number of additional preimages of points on the diagonal across the critical curve with respect to T coincide with the number of additional rank-1 preimages with respect to f. In other words, all additional rank-1 preimages have to be on the diagonal. Observing that the point $(0,0)$ has one rank-1 preimage with respect to T and 0 has one rank-1 preimage with respect to f now establishes that for all points on the diagonal the number of preimages with respect to T and with respect to f coincide. Therefore, the diagonal is for all symmetric parameter constellation forward and backward invariant with respect to T. Note that this would not necessarily hold if external spillovers would be larger than internal spillovers (which of course is quite counterintuitive).

Having done this we show that for sufficiently fast switching there is local overshooting around the interior fixed point on the diagonal, which implies that there is a region Z_3 around S in the unit square where T has three preimages. The dynamics along the invariant diagonal reads

$$x_{t+1} = f(x_t) := x_t + x_t(1-x_t)G\left(A - 2Bx_t - \frac{C}{1+\beta x_t}\right),$$

where $x_{1,t} = x_{2,t} = x_t$. The derivative of f is given by

$$\begin{aligned} f'(x) &= 1 + (1-2x)G\left(A - 2Bx - \frac{C}{1+\beta x}\right) \\ &\quad + x(1-x)G'\left(A - 2Bx - \frac{C}{1+\beta x}\right)\left(-2B + \frac{\beta C}{(1+\beta x)^2}\right). \end{aligned}$$

We have $f(0) = 0$, $f(1) = 1$ and, due to assumption (3), $f'(0) > 1$. Furthermore, we know that there is at most one interior fixed point of the map. If such an interior fixed point s exists, we therefore always have $f'(s) < 1$. Since the second term in the expression for f' has to be zero at s, the third one has to be negative which implies $-2B + \frac{\beta C}{(1+\beta s)^2} < 0$. Accordingly, we have

$$f'(s) = 1 + G'(0)\left(-2B + \frac{\beta C}{(1+\beta v)^2}\right),$$

which is negative for sufficiently large $G'(0) = \lambda$. At the value of λ where $f'(s) = 0$ a critical curve appears which surrounds the interior fixed point for larger values of λ.

Appendix C

Noninvertible maps and critical curves.

In this appendix we give some basic definitions, properties and a minimal vocabulary concerning the theory of noninvertible maps of the plane and the method of critical curves. We also describe some properties of the critical curves of the map T defined by (5).

A two-dimensional map $T : (x, y) \to (x', y')$ can be written in the form

$$(x', y') = T(x, y) = (f(x, y), g(x, y)) \tag{20}$$

where $(x, y) \in \mathbb{R}^2$ and f, g are assumed to be real-valued continuous functions. The point $(x', y') \in \mathbb{R}^2$ is called rank-1 image of the point (x, y) under

T. The point $(x(t), y(t)) = T^t(x, y)$, $t \in \mathbb{N}$, is called image (or forward iterate) of rank-t of the point (x, y), where T^0 is identified with the identity map and $T^t(\cdot) = T(T^{t-1}(\cdot))$. The fact that the map T is single-valued does not imply the existence and the uniqueness of its inverse T^{-1}. Indeed, for a given (x', y'), several rank-1 preimages (or backward iterates) $(x, y) = T^{-1}(x', y')$ may exist, i.e. the inverse relation T^{-1} may be multivalued. In this case T is said to be a *noninvertible map.* As the point (x', y') varies in the plane $\mathbb{R}^2$ the number of its rank-1 preimages can change. According to the number of distinct rank-1 preimages associated with each point of $\mathbb{R}^2$, the plane can be subdivided into regions, denoted by Z_k, whose points have k distinct preimages. Generally, pairs of real preimages appear or disappear as the point (x', y') crosses the boundary separating regions characterized by a different number of rank-1 preimages. Accordingly, such boundaries are generally characterized by the presence of two coincident (merging) preimages. This leads us to the definition of *critical curves*, one of the distinguishing features of noninvertible maps. The critical curve of rank-1, denoted by LC (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages. These preimages are located in a set called critical curve of rank-0, denoted by LC_{-1}. The curve LC is the two-dimensional generalization of the notion of critical value (local minimum or maximum value) of a one-dimensional map, and LC_{-1} is the generalization of the notion of critical point (local extremum point). As in the case of differentiable one-dimensional maps, where the derivative necessarily vanishes at the local extremum points, for a two-dimensional continuously differentiable map the set LC_{-1} belongs to the set of points in which the Jacobian determinant vanishes:

$$LC_{-1} \subseteq \left\{(x, y) \in \mathbb{R}^2 | \det J = 0\right\} \tag{21}$$

In fact, as LC_{-1} is defined as the locus of coincident rank-1 preimages of the points of LC, in any neighborhood of a point of LC_{-1} there are at least two distinct points mapped by T in the same point near LC. This means that the map T is not locally invertible in the points of LC_{-1} and, if the map T is continuously differentiable, it follows that $\det J$ necessarily vanishes along LC_{-1}. If the set LC_{-1} is determined by (21), then LC is simply obtained as the image of LC_{-1}, i.e. $LC = T(LC_{-1})$.

Considering the map T for the specifications of G_i used in our numerical illustrations, we realize that it is not always invertible. In fact, given a point $(x'_1, x'_2) \in [0, 1] \times [0, 1]$ its preimages are computed by solving the following system with respect to x_1 and x_2:

$$\begin{cases} x_1 + x_1(1-x_1)\frac{2}{\pi}\arctan\left[\lambda_1\left(A_1 - B\left(x_1+x_2\right) - \frac{C_1}{1+\beta_1 x_1+\gamma_1 x_2}\right)\right] = x_1' \\ x_2 + x_2(1-x_2)\frac{2}{\pi}\arctan\left[\lambda_2\left(A_2 - B\left(x_1+x_2\right) - \frac{C_2}{1+\beta_2 x_2+\gamma_2 x_1}\right)\right] = x_2' \end{cases} \tag{22}$$

For some sets of parameters, e.g. for small λ_i, this system has just one solution, i.e. the map T is invertible, but ranges of the parameters exist such that several solutions can be obtained, so that T is a noninvertible map. For sufficiently high values of λ_i a point (x_1', x_2') may have up to five rank-1 preimages.

In order to find the set LC we start from (21). In our case LC_{-1} coincides with the set of points at which the Jacobian $\det J = 0$, and its numerical computation gives a closed curve. It follows that also $LC = T(LC_{-1})$ is a closed curve surrounding the interior equilibrium S. All points in the area surrounded by the curve have three preimages (Z_3 region), those outside the curve have one preimage (Z_1 region). As λ_i increases the curve expands, and may generate self-intersections (leading to Z_5 regions) and non smooth points (i.e. cusp points).

In order to give a geometrical interpretation of the action of the multivalued inverse relation T^{-1}, it is useful to consider a region Z_k as the superposition of k sheets, each associated with a different inverse. Such a representation is known as *Riemann foliation* of the plane (see e.g. Mira et al. 1996). Different sheets are connected by folds joining two sheets, and the projections of such folds on the phase plane are arcs of LC. The foliations associated with the Z_3 region surrounding the interior fixed point S in our model is called "lip structure" in Mira et al. (1996).

Figure Captions:

Figure 1: The curves $F_i,\ i=1,2$ for $\beta<\hat{\beta}$ and a trajectory of the process for $(x_{1,0},x_{2,0})=(0.2,0.18)$ $(P_0=300,\ B=100,\ C=190, U=32,\ \beta=0.05,\ \hat{\beta}=0.131,\ \gamma=0)$.

Figure 2: The curves $F_i,\ i=1,2$ for $\beta\in(\hat{\beta},\beta^*)$ and a trajectory of the process for $(x_{1,0},x_{2,0})=(0.2,0.18)$ $(P_0=300,\ B=100,\ C=190, U=32,\ \beta=1, \hat{\beta}=0.131,\ \beta^*=1.209,\ \gamma=0)$.

Figure 3: The restriction of the map T to the line $x_2=1$ for $\beta>\beta^*$ and slow dynamics $(P_0=300,\ B=100,\ C=190, U=32,\ \beta=1.5, \beta^*=1.21,\ \gamma=0)$.

Figure 4: The basins of attraction of the four stable equilibria for $\beta>\beta^*$ and slow dynamics $(P_0=300,\ B=100,\ C=190, U=32,\ \beta=1.5,\ \beta^*=1.209,\ \gamma=0)$.

Figure 5: The basins of attraction of the locally stable equilibria for $\beta_2\in(\bar{\beta}_2,\beta^*),\ \beta_1>\beta_2$, no external spillovers and slow dynamics: a) $\beta_1=1.2<\beta^*$; b) $\beta_1=1.21\in(\beta^*,\tilde{\beta}_1)$; c) $\beta_1=1.35\in(\beta^*,\tilde{\beta}_1)$; d) $\beta_1=1.4>\tilde{\beta}_1$ $(P_0=300,\ B=100,\ C=190, U=32,\ \beta_2=1,\ \bar{\beta}_2=0.542,\ \beta^*=1.209,\ \tilde{\beta}_1=1.37)$.

Figure 6: The transition of the basins of attraction of the three locally stable equilibria as the dynamics becomes faster and the critical curve crosses a basin boundary: a) no intersection of the critical curve with the boundary between the basins of V_{III} and P_{III} $(\lambda=\frac{2.2}{\pi})$; b) the critical curve has crossed the boundary between the basins of V_{III} and P_{III} $(\lambda=\frac{2.6}{\pi}$; the shape of the curves LC and LC_{-1} for $\lambda=\frac{2.6}{\pi}$; d) enlargement of the area where the intersection between LC and the basin boundary occurs $(P_0=300,\ B=100,\ C=190, U=32,\ \beta_1=1.21,\ \beta_2=1,\ \gamma=0)$.

Figure 7: Illustration of the effect of the crossing of a basin boundary by a critical curve for the one-dimensional restriction of T to the line $x_2=1$: a,b) $\lambda=\frac{2.2}{\pi}$; c) $\lambda=\frac{2.6}{\pi}$ $(P_0=300,\ B=100,\ C=190, U=32,\ \beta_1=1.21, \gamma=0)$.

Figure 8: The basins of attraction of the locally stable equilibria for slow dynamics, symmetric internal spillovers and positive external spillovers only from population 2 to population 1 firms: a) $\gamma_1 = 0.1$; b) $\gamma_1 = 0.2$ ($P_0 = 300,\ B = 100,\ C = 190, U = 32,\ \beta = 1,\ \gamma_2 = 0,\ \gamma^* = 0.0768$).

Figure 9: The transition of the basins of attraction of the locally stable equilibria for fast dynamics, symmetric internal spillovers and increasing positive external spillovers from population 2 to population 1 firms: a) $\gamma_1 = 0.076$; b) $\gamma_1 = 0.077$; c) $\gamma_1 = 0.08$; d) $\gamma_1 = 0.1$ ($P_0 = 300,\ B = 100,\ C = 190, U = 32,\ \beta = 1,\ \gamma_2 = 0,\ \gamma^* = 0.0768$).

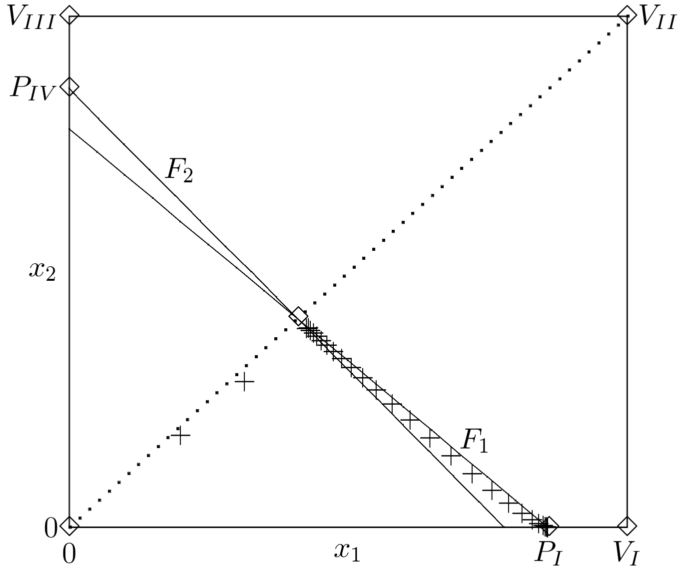


Figure 1

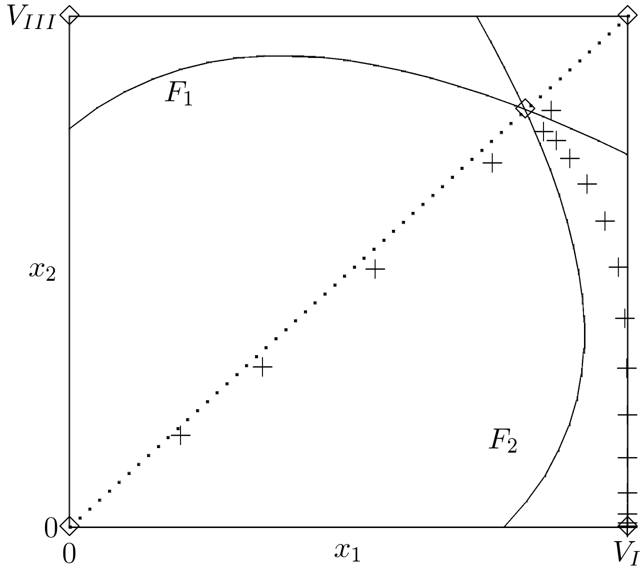


Figure 2

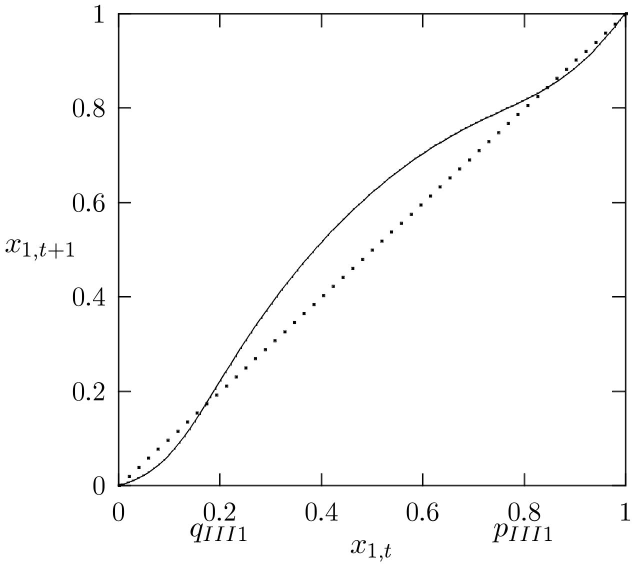
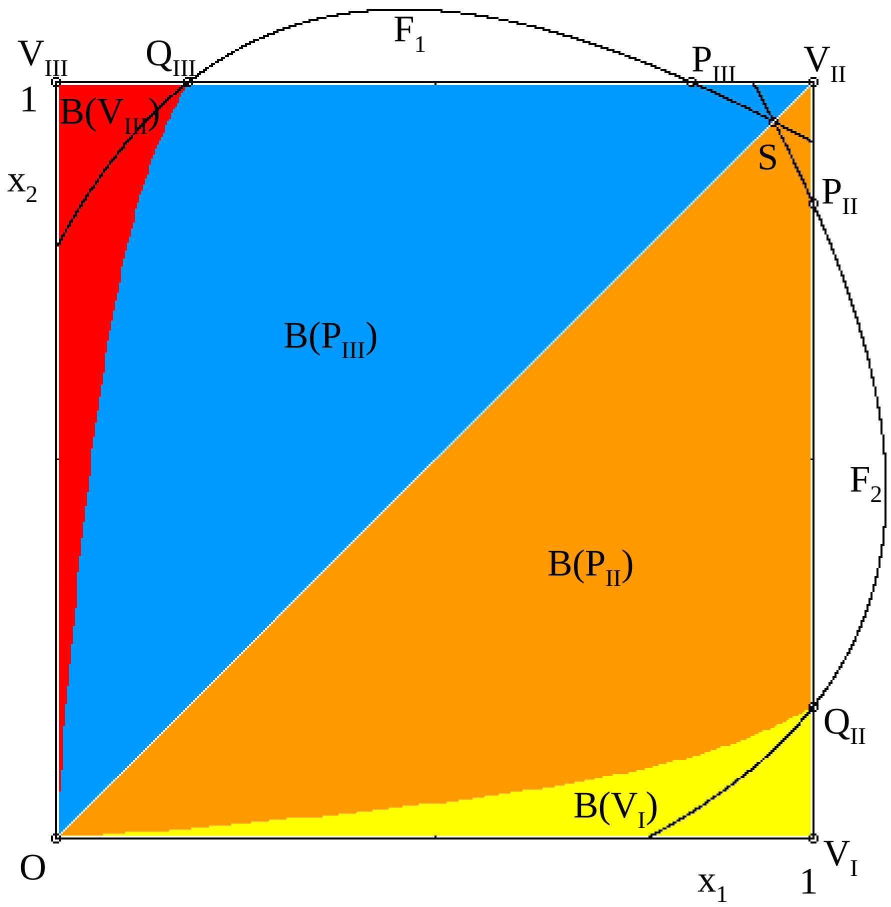
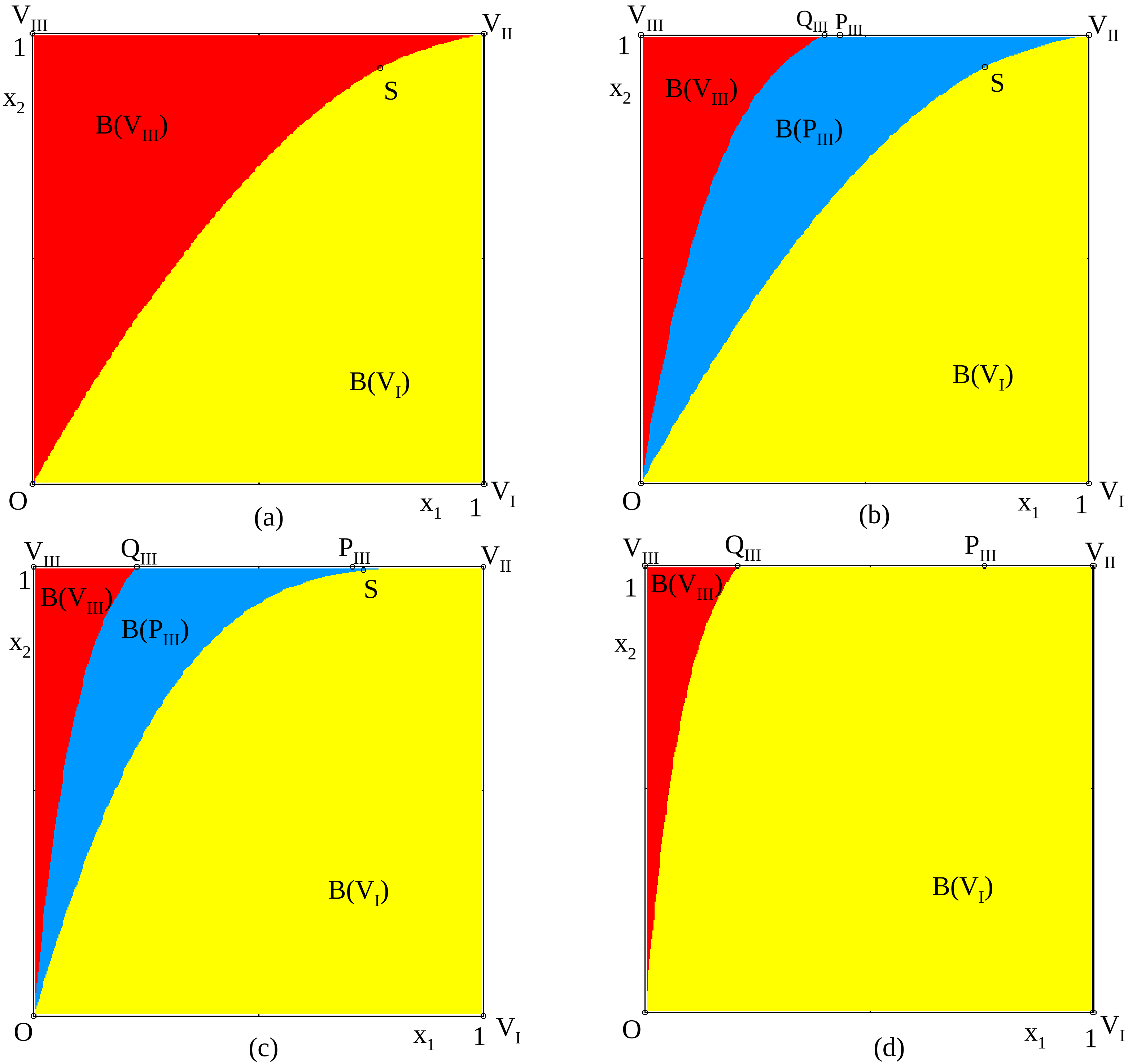


Figure 3



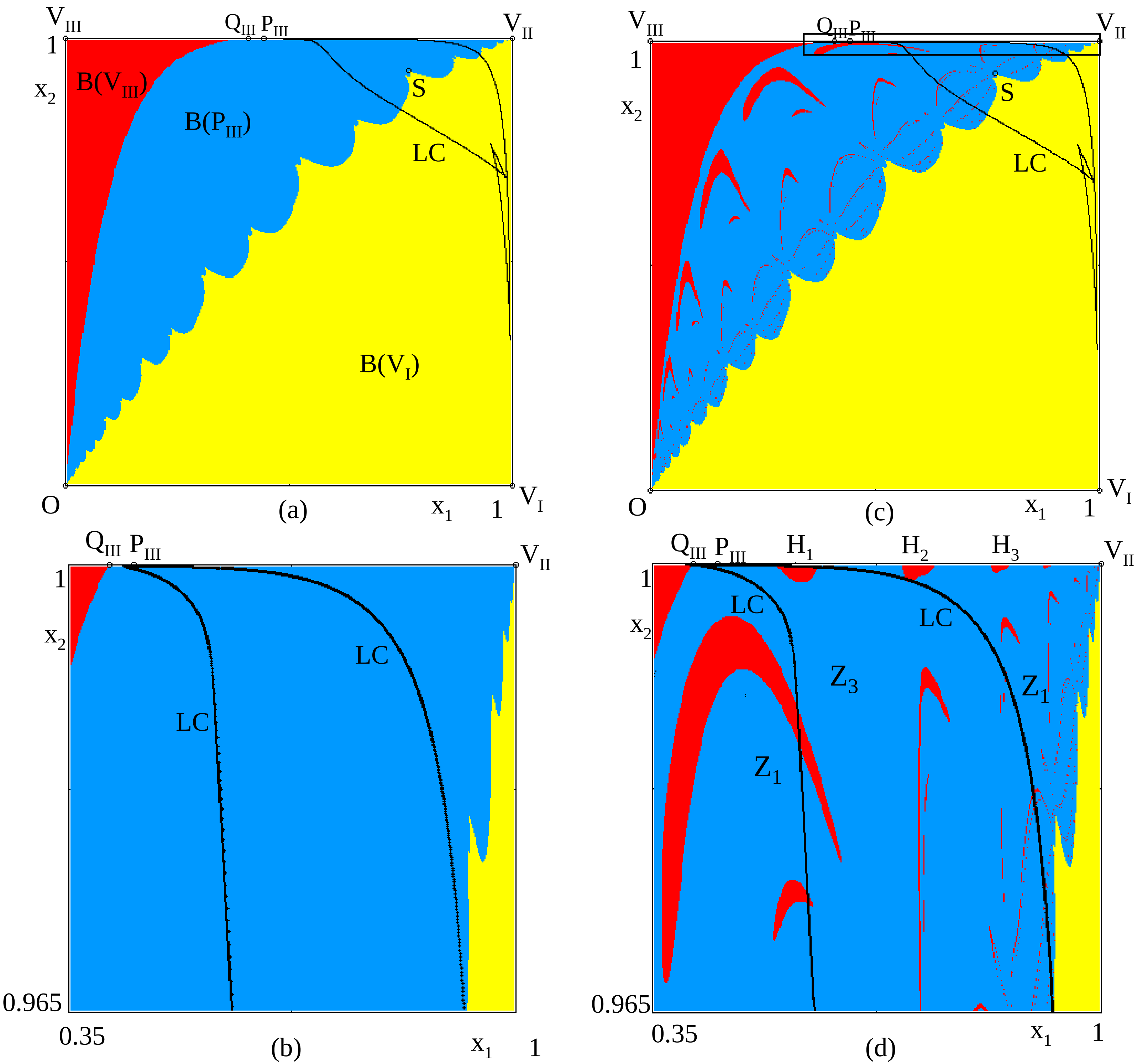
V_III
Q_III
F_1
P_III
V_II
1
B(V_III)
x_2
S
P_II
B(P_III)
F_2
B(P_II)
Q_II
B(V_I)
O
V_I
x_1
1

Fig. 4



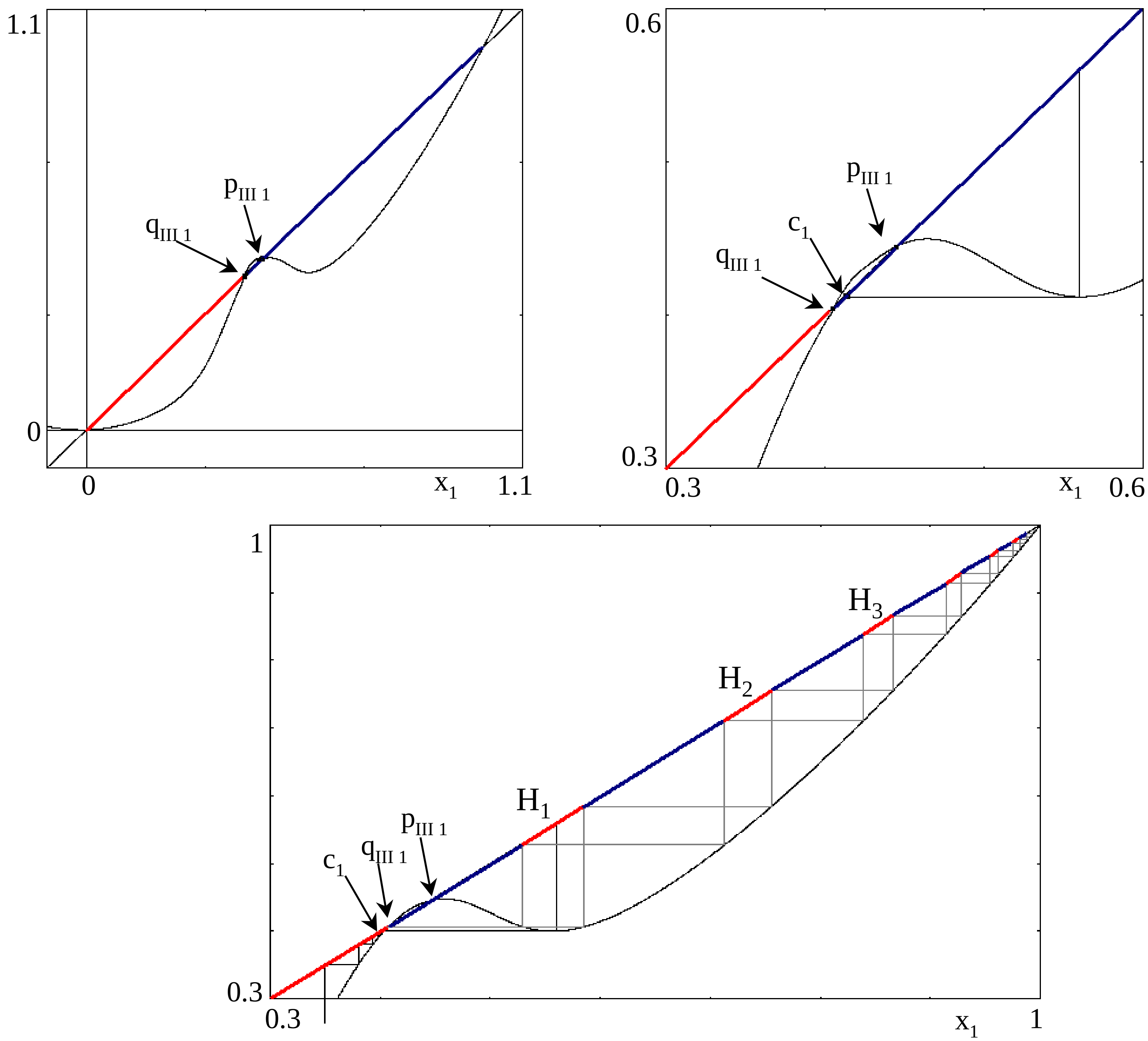
V_III
V_II
1
x_2
S
B(V_III)
B(V_I)
O
x_1
1
V_I
(a)
V_III
Q_III
P_III
V_II
1
x_2
B(V_III)
S
B(P_III)
B(V_I)
O
x_1
1
V_I
(b)
V_III
Q_III
P_III
V_II
1
B(V_III)
S
x_2
B(P_III)
B(V_I)
O
x_1
1
V_I
(c)
V_III
Q_III
P_III
V_II
1
B(V_III)
x_2
B(V_I)
O
x_1
1
V_I
(d)

Fig. 5



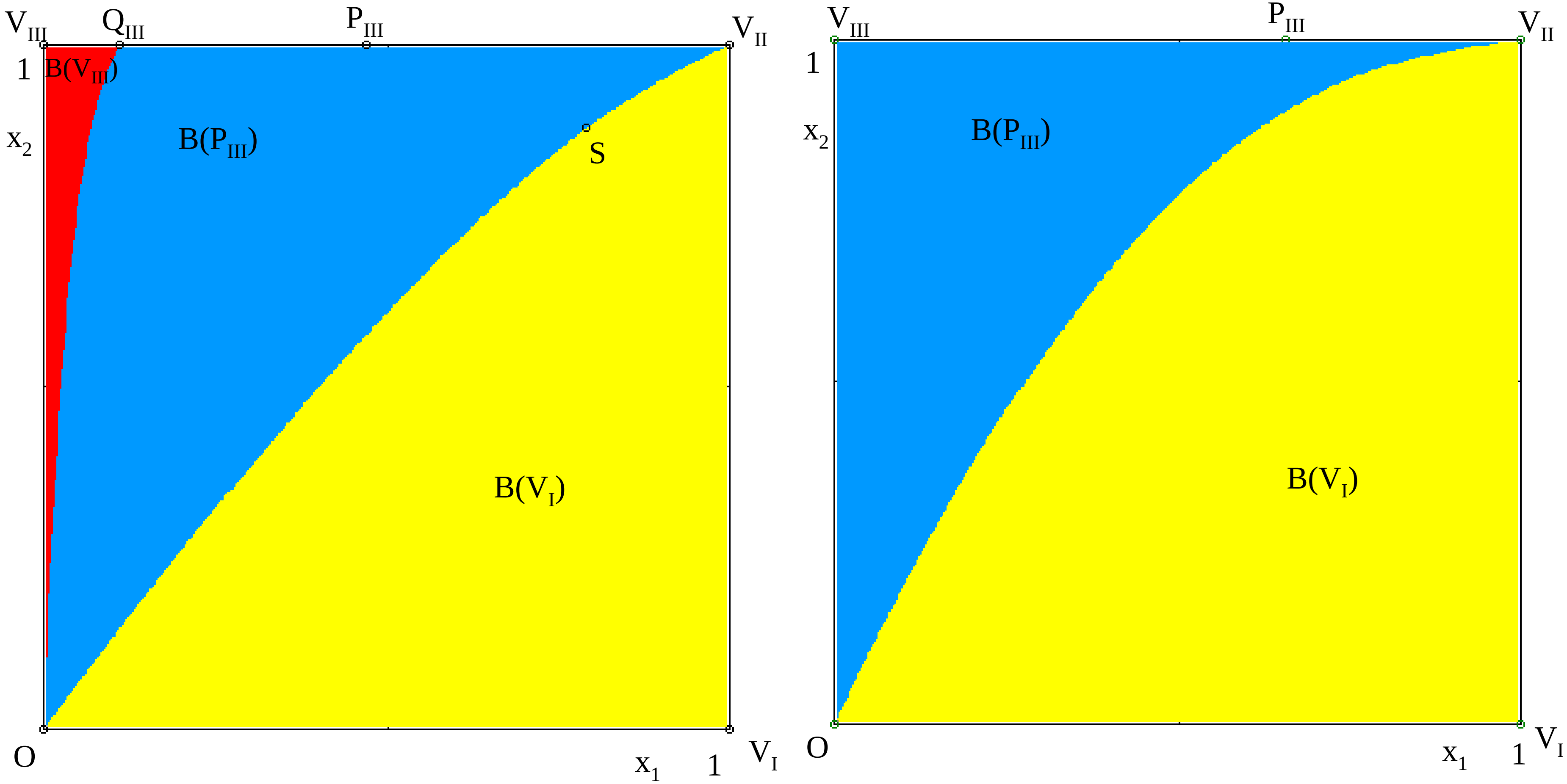
V_III
Q_III P_III
V_II
1
x_2
B(V_III)
B(P_III)
S
LC
B(V_I)
O
(a)
x_1
1
V_I
V_III
Q_III P_III
V_II
1
x_2
S
LC
O
(c)
x_1
1
V_I
Q_III
P_III
V_II
1
x_2
LC
LC
0.965
0.35
(b)
x_1
1
Q_III
P_III
H_1
H_2
H_3
V_II
1
x_2
LC
LC
Z_3
Z_1
Z_1
0.965
0.35
(d)
x_1
1

Fig. 6



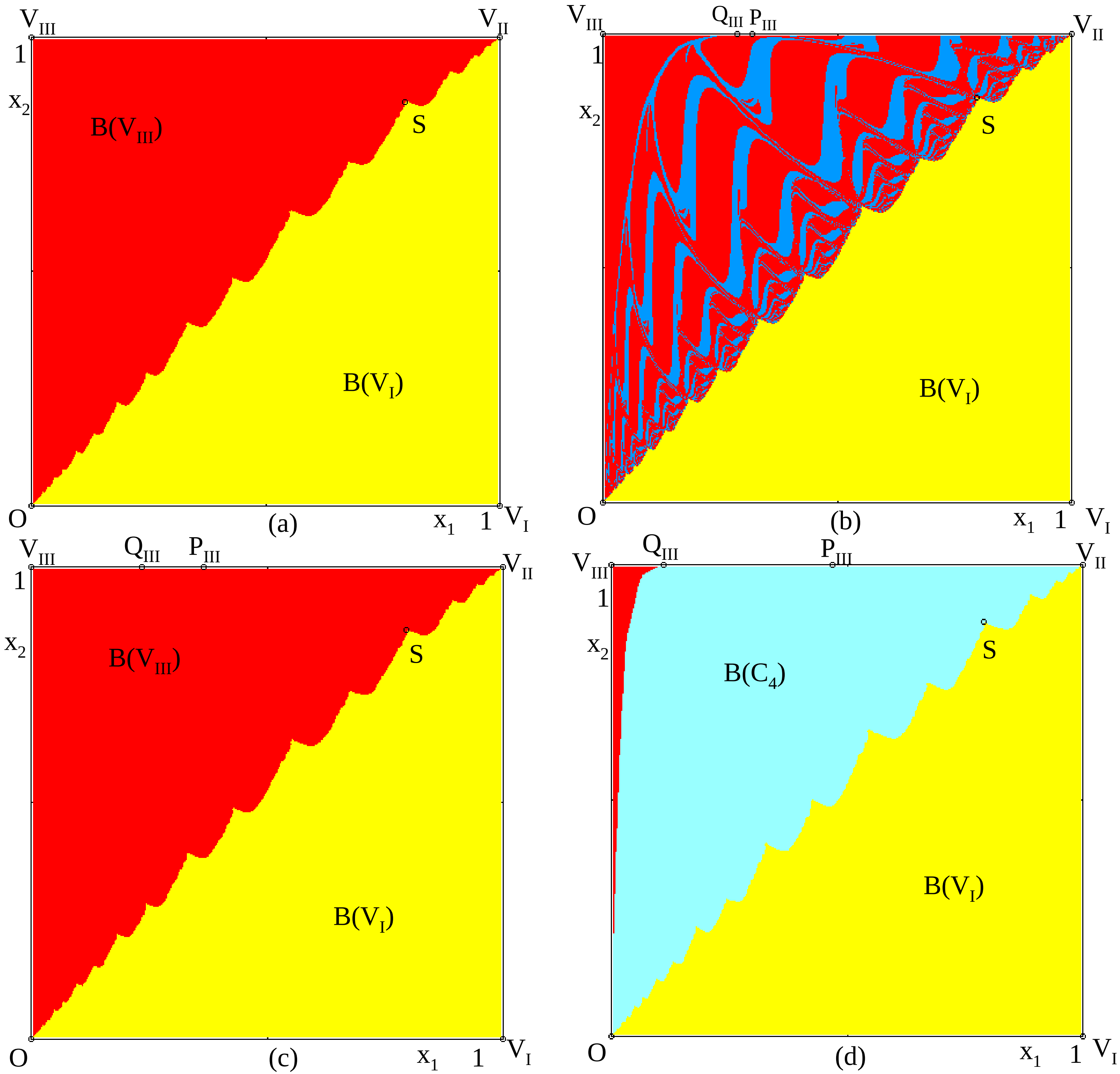
1.1
0
0
x_1
1.1
$p_{III\ 1}$
$q_{III\ 1}$
0.6
0.3
0.3
x_1
0.6
$p_{III\ 1}$
c_1
$q_{III\ 1}$
1
0.3
0.3
x_1
1
H_1
H_2
H_3
c_1
$q_{III\ 1}$
$p_{III\ 1}$

Fig. 7



V_III
Q_III
P_III
V_II
1
B(V_III)
x_2
B(P_III)
S
B(V_I)
O
x_1
1
V_I
V_III
P_III
V_II
1
x_2
B(P_III)
B(V_I)
O
x_1
1
V_I

Fig. 8



V_III
V_II
1
x_2
B(V_III)
S
B(V_I)
O
(a)
x_1
1
V_I
V_III
Q_III
P_III
V_II
1
x_2
S
B(V_I)
O
(b)
x_1
1
V_I
V_III
Q_III
P_III
V_II
1
x_2
B(V_III)
S
B(V_I)
O
(c)
x_1
1
V_I
V_III
Q_III
P_III
V_II
1
x_2
B(C_4)
S
B(V_I)
O
(d)
x_1
1
V_I

Fig. 9