The Dynamics of

Random Economic Models

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Handout for MDEF2000 –
Workshop Modelli Dinamici in Economia e Finanza
Urbino
September 28–30, 2000

Contents

1 Introduction

- **Random Difference Equations**
  - $F : \mathcal{X} \times \mathbb{R}^m \longrightarrow \mathcal{X}$
  - $x_{t+1} = F(x_t, w_t)$
  - x_t state at time t
  - w_t realization of disturbance at time t

$\Longrightarrow$ **Law of motion *plus* a model of exogenous noise**

- recent development of new mathematical techniques
  $\Longrightarrow$ Arnold (1998)
- combining dynamical systems theory with stochastic processes
- allows *dynamic* analysis of stochastic phenomena
- stability analysis of sample paths
- bifurcation theory
- new view of time series analysis in economics

Economic examples

- Linear rational expectations models
- Stochastic Multiplier-accelerator model
- stochastic growth models: – Solow – OLG – RBC
- sequential CAPM model
- dynamic disequilibrium macro model

2 Random Dynamical Systems

Law of motion *plus* a model of exogenous noise

werden für alle Zeitpunkte t beschrieben durch

- $x_{t+1} = F(x_t, w_t)$
- x_t Zustand zum Zeitpunkt t
- w_t Störung zum Zeitpunkt t

Topological dynamical system– modeling the (deterministic) dynamics

Definition 2.1
A (topological) dynamical system in discrete time $\mathbb{N} = \{0, 1, 2, \dots\}$ *on a set* $\mathcal{X} \subset \mathbb{R}^d$ *with parameter space* $\mathbb{R}^m$ *is given by the time–one map*

$$F : \mathcal{X} \times \mathbb{R}^m \longrightarrow \mathcal{X}.$$

For given w *and any initial state* $x \in \mathcal{X}$*, orbits are generated by*

1.

$$x_t = F(x_{t-1}, w) \qquad \forall t \in \mathbb{N}$$

2.

$$x_t = \underbrace{F(., w) \circ \dots \circ F(x, w)}_{t-mal} \qquad \forall t \in \mathbb{N}$$

Ergodic dynamical systems– modeling the noise process

Definition 2.2

A metric dynamical system in discrete time consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *and a measurable invertible time–one map*

$$\theta : \Omega \longrightarrow \Omega,$$

which induces the family of iterated maps (θ^t) *given by*

$$\theta^t = \underbrace{\theta \circ \cdots \circ \theta}_{t-mal}$$

$(\Omega, \mathcal{F}, \mathbb{P}, (\theta^t))$ *satisfies:*

- $\mathbb{P}$ *is ergodic with respect to* θ,
  i.e. invariant sets have $\mathbb{P}$ *measure zero or one*
- θ *is measure preserving, i. e.* $\theta\mathbb{P} = \mathbb{P}$

Examples

- sequences of i.i.d. random variables
- irreducible finite state Markov chains
- "almost" deterministic processes with periodic motion
- the tent map with appropriate measure
- the logistic map $\vartheta\omega = 4\omega(1-\omega)$ with appropriate distribution $\mathbb{P}$ of $\omega \in [0, 1]$

The real noise process

Definition 2.3

A real noise process for the parameter w associated with a metric dynamical system is a stochastic process

$$\{u_t\}, \quad u_t : \Omega \to \mathbb{R}^m$$

,

generated by the metric dynamical system, i.e.

$$u_t = u \circ \theta^t, \quad w_t = u_t(\omega) = u(\theta^t \omega)$$

Definition 2.4 (Arnold 1998)

A random dynamical system consists of a parametrised topological dynamical system, i.e.

$$F : \mathcal{X} \times \mathbb{R}^m \longrightarrow \mathcal{X}$$

and a metric dynamical system with real noise process

$$(\Omega, \mathcal{F}, \mathbb{P}, (\theta^t)), \qquad u_t = u \circ \theta^t$$

Random fixed points and stability

Definition 2.5

A **random fixed point** *of F is a random variable $x_\star : \Omega \longrightarrow \mathbb{R}^m$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that almost surely:*

$$x_\star(\theta\omega) = F(x_\star(\omega), u(\omega))$$

The definition implies that

- If F independent of the perturbation ω, then $x_\star$ is a deterministic fixed point,
- $x_*(\vartheta^{t+1}\omega) = F(x_*(\vartheta^t\omega), u(\vartheta^t\omega))$ for all times t,
- the orbit $\{x_*(\vartheta^t\omega)\}_{t\in\mathbb{N}}$, $\omega \in \Omega$ generated by x_* solves the random difference equation

$$x_{t+1} = F(x_t, u_t(\omega)).$$

- $\{x_*(\vartheta^t)\}_{t\in\mathbb{N}}$ is stationary and ergodic, since ϑ is stationary and ergodic,
- If, in addition, $\mathbb{E}||x_*|| < \infty$, then

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} 1_B(x_*(\vartheta^t\omega)) = x_*\mathbb{P}(B) := \mathbb{P}\{\omega \in \Omega | x_*(\omega) \in B\}$$

  for every $B \in \mathcal{B}(X)$.

- the empirical law of an orbit is well defined and is equal to the distribution $x_*\mathbb{P}$ of x_*.
- if the noise process is i. i. d. $\Longrightarrow$ the orbit is an ergodic Markov equilibrium in the sense of Duffie, Geanakoplos, Mas-Colell & McLennan (1994).

The concept of stability to be used in the context of a random dynamical system is as follows:

Definition 2.6 *A random fixed point x_* is called **attracting** on some set $\mathcal{U} \subset \Omega \times X$ if*

$$\lim_{t\to\infty} ||x_t(\omega) - x_*(\vartheta^t\omega)|| = 0 \quad \textit{for all } (\omega, x_0(\omega)) \in \mathcal{U}.$$

Lra a random fixed point is attracting if nearby orbits converge to the orbit of the random fixed point.

3 Affine Random Dynamical Systems

Iterated Function Systems (IFS)

Affine models with discrete – i. i. d. noise

$$x_{t+1} = A_i x_t + b_i \qquad i \in I$$

Consider a finite family $\{G_i\}$ of affine maps

$$G_i : \mathbb{R}^K \to \mathbb{R}^K$$

with associated probabilities $\{\pi_i\}$, $\pi_i > 0$, $i = 1, \ldots, r$, und $\sum \pi_i = 1$.
If all mappings G_i are contractions, then $\{(G_i); (\pi), i = 1, \ldots, r\}$ is called an **iterated function system**

Lemma 3.1 (Arnold (1998), Barnsley (1988))
Let $\{(G_i); (\pi), i = 1, \ldots, r\}$ be an iterated function system.

1. *$\{(G_i); (\pi), i = 1, \ldots, r\}$ has a unique compact attractor $A \subset [\overline{k}_1, \overline{k}_r]$,*
2. *A is independent of the probabilities $\{\pi_i\}$.*
3. *A is the limit of a decreasing sequence of finite unions of compact intervals (cubes),*
4. *there exists a unique invariant measure μ on A,*
5. *there exists a unique globally attracting random fixed point $x_\star$ of the associated random dynamical system, whose empirical distribution is μ*

$\Longrightarrow$ A is often a fractal set $\to$ Cantor set

$\Longrightarrow$ the measure μ may have a very complex density

Theorem 3.1 (Arnold (1998))
Consider $G : X \times \mathbb{R}^{3m} \longrightarrow X$ *with real noise process* $(A, b)_t := (A, b) \circ \vartheta^t$, $A : \Omega \longrightarrow \mathbb{R}^m \times \mathbb{R}^m$ *and* $b : \Omega \longrightarrow \mathbb{R}^m$ *measurable, over the ergodic dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta^t))$ *where* $G \equiv (A, b)$ *is the family of invertible affine difference equations*

$$x_{t+1} = A(\theta^t\omega)x_t + b(\theta^t\omega).$$

With some ("mild and natural") additional assumptions (hyperbolicity, integrability, contractivity) there exists a unique globally attracting random fixed point.

Example 1: Sequential CAPM

Consider the sequential CAPM model (cif. Böhm & Chiarella 2000) under unbiased prediction with price and expectations process

$$(3.1) \qquad p_t = \frac{1}{R}\left[D_t(\cdot) + R\mu_{t-1} - \mathbb{E}_{t-1}D_t(\cdot)\right]$$

$$(3.2) \qquad \mu_t = \left(\sum_a \tau^a\right)^{-1} \bar{x} + R\mu_{t-1} - \mathbb{E}_{t-1}D_t(\cdot).$$

and an AR(1) dividend process modeled as

$$(3.3) \qquad D_{t+1} = \alpha D_t + \zeta_t, \quad \text{where}$$

with $0 < \alpha < 1$ and $\zeta_t \sim$ uniform i.i.d. over $[a, b]$

Theorem 3.2 *The random dynamical system given by equations* $(3.2) - 3.3$ *has a unique random fixed point* μ_* *if and only if* $R \neq 1$*.* μ_* *is globally attracting if and only if* $0 < R < 1$*.*

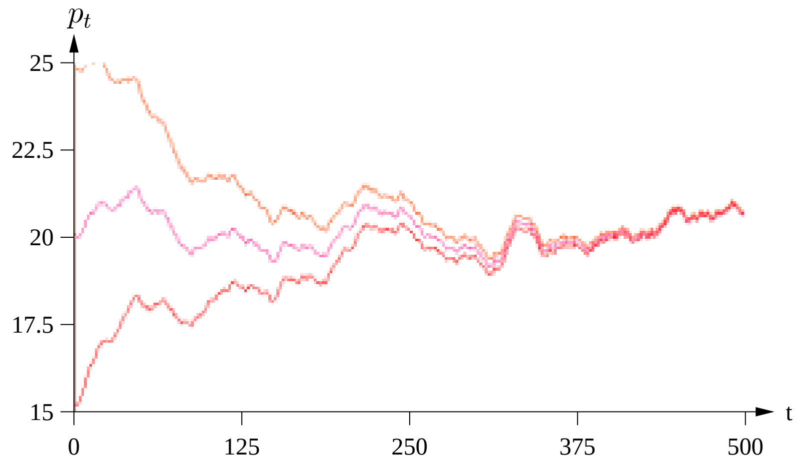

Figure 3.1: Convergence of prices to random fixed point

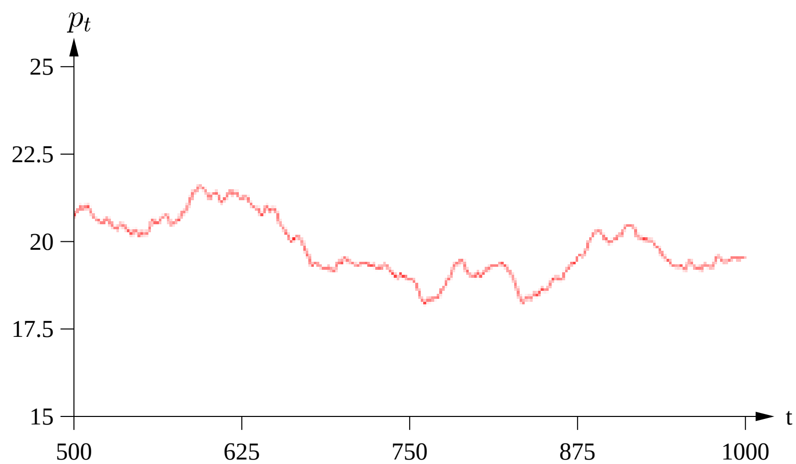

Figure 3.2: Prices with AR(1) dividends

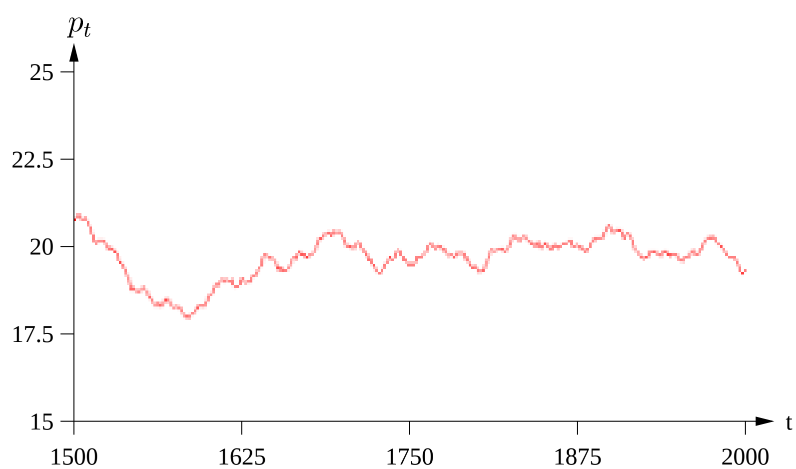

Figure 3.3: prices with AR(1) dividends

Example 2: Random Multiplier accelerator model

(cif. Böhm & Jungeilges 2000)

(3.4) $$C = m^0 + mY_1$$

(3.5) $$I = v^0 + v(Y_{-1} - Y_{-2})$$

(3.6) $$Y = C + I + G$$

(3.7) $$Y = (m^0 + v^0) + (m + v)Y_{-1} - vY_{-2}$$

$\Longrightarrow$

(3.8) $$Y = (m_0 + v_0) + (m + v)Y_{-1} - vY_{-2}$$

$\Longrightarrow$

(3.9) $$(m_i^0, v_i^0) \geq 0, \quad 0 \leq (m_i, v_i) \leq 1$$

(3.10) $$\sum \pi_i = 1 \quad \pi_i \gg 0 \quad i \in I$$

Example 3: Productivity shocks in a linear growth model

Consider

- $0 < a_1 < \ldots < a_i < \cdots < a_r$
- mit Wahrscheinlichkeiten $\{\pi_i\}$, $\pi_i > 0$, $i = 1, \ldots, r$, und $\sum \pi_i = 1$,
- parametrised family of affine maps $\{G_i\}$, $G_i : \mathbb{R} \to \mathbb{R}$

$$G_i(k_t) = \frac{(1 - \delta + sb)k_t + sa_i}{(1 + n)},$$

- mit zugehörigen Fixpunkten

$$\overline{k}_i := \frac{sa_i}{n + \delta - sb} \qquad i = 1, \ldots, r.$$

Beispiel

$r = 2.$

$a_1 = 0.5$, $a_2 = 0.7$,

$\pi_2 = .02$

$s = b = 0.5$

$\delta = 0.5$ bzw. $\delta = 0.85$

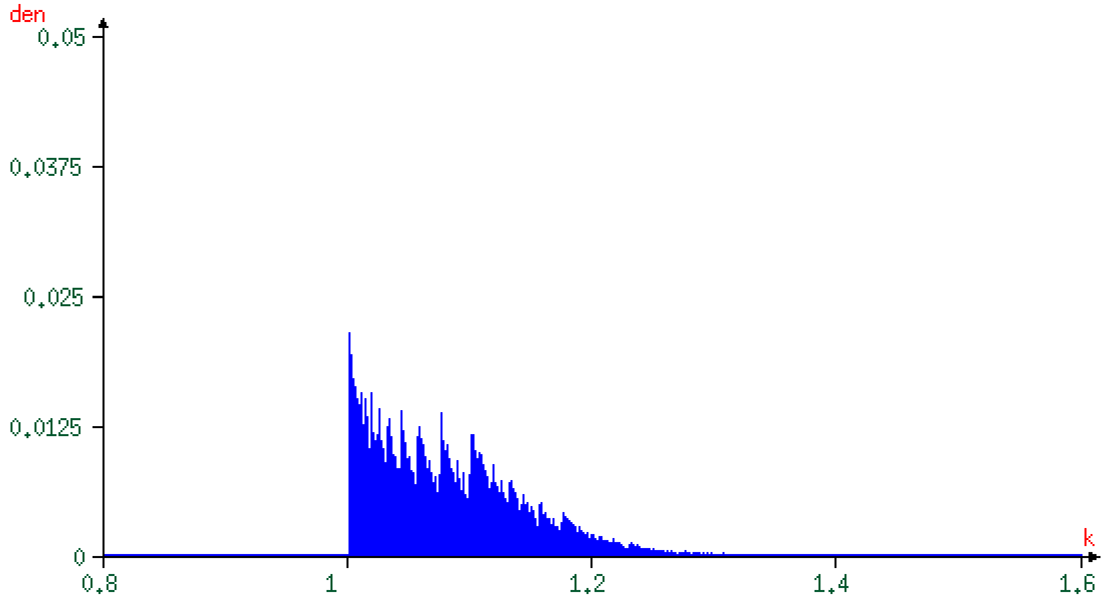

Figure 3.4: Invariante Verteilung im linearen Modell auf einem Intervall

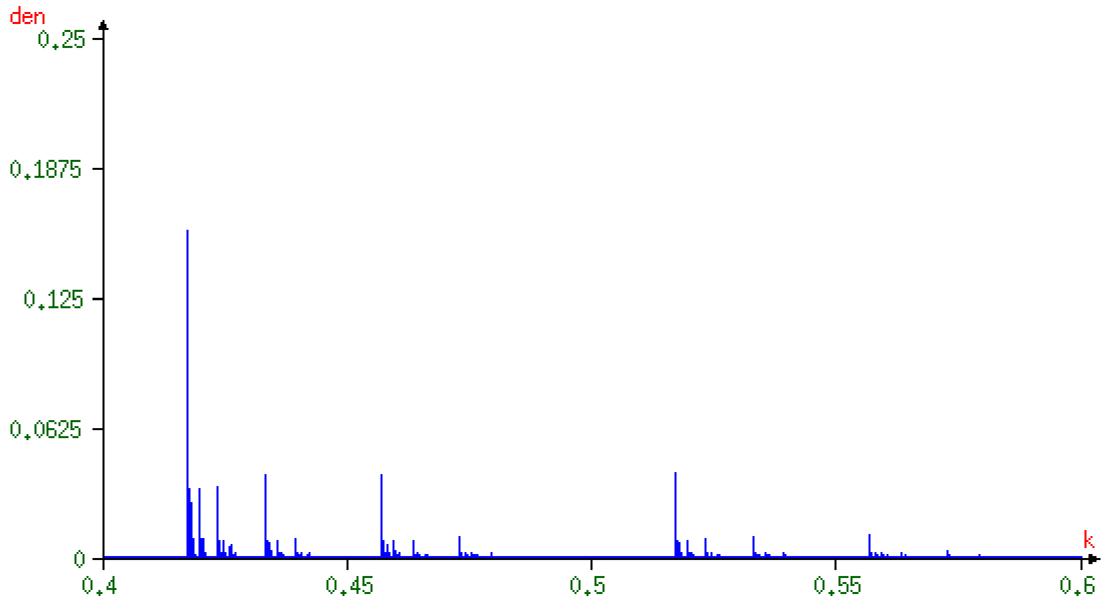

Figure 3.5: Invariante Verteilung im linearen Modell auf einer Cantormenge

4 Aggregate Stochastic Growth Models

Literatur: Mirman (1972, 1973), , Ramsey (1928)

⇒ RBC–Models

• 'local' linearized statistical description near "noisy" steady state

• no stability analysis

• general statistical description of random behavior as
Markov equilibria Becker & Zilcha (1997)

• no stability of sample paths

• no global dynamic analysis

Stability in the stochastic Solow model

- production function $f : \mathbb{R}_+ \to \mathbb{R}_+$, with parameters
- $A > 0$ multiplicative scale factor of f,
- $n > -1$ rate of population growth
- $0 \leq \delta \leq 1$ rate of depreciation
- $0 \leq s \leq 1$ aggregate propensity to save.

$\Longrightarrow$

difference equation parametrized in (δ, n, s, A)

$$k_{t+1} = \frac{(1-\delta)k_t + sAf(k_t)}{(1+n)},$$

$$k_{t+1} = \frac{(1-\delta(\theta^t\omega))k_t + \xi(\theta^t\omega)f(k_t)}{1+n(\theta^t\omega)}$$

with

- $$\delta(\omega) \in [\delta_{min}, \delta_{max}] \subset [0,1]$$
- $$n(\omega) \in [n_{min}, n_{max}] \subset]-1, \infty]$$
- $$\xi(\omega) \in [\xi_{min}, \infty[\subset]0, \infty[\text{ mit } \mathbb{E}\,\xi < \infty$$
- if the Inada conditions hold

$\Longrightarrow$ for every given vector of parameters (δ, n, ξ) there exist a unique nontrivial positive fixed point,

$\Longrightarrow$ strictly decreasing in $n+\delta$ and strictly increasing in ξ.

$\Longrightarrow$

Let

- $\underline{k} := \underline{k}(\delta_{max}, n_{max}, \xi_{min})$ denote the smallest and
- $\overline{k} := \overline{k}(\delta_{min}, n_{min}, \xi_{max})$ the largest possible fixed points of the deterministic model.

$\Longrightarrow$ eventually all sample paths stay in the compact interval $[\underline{k}, \overline{k}]$

$\Longrightarrow$ the long run behavior occurs in the interval which is a forward invariant set of the random dynamical system.

Theorem 4.1 (Schenk-Hoppé & Schmalfuß (1998))
Let f be strictly monotonically increasing, strictly concave, and continuously differentiable. If

i)

$$\delta_{max} + n_{max} > 0$$

ii)

$$0 \leq \lim_{k \to \infty} f(k)/k < (\delta_{max} + n_{max})/\xi_{min} < \lim_{k \to 0} f(k)/k \leq \infty$$

iii)

$$\mathbb{E}\, log \frac{1 - \delta(\omega) + \xi(\omega) f'(\underline{k})}{1 + n(\omega)} < 0,$$

there exists a unique nontrivial globally attracting random fixed point $k_\star$.

Corollary 4.1
If the perturbations $(\delta(\omega), n(\omega), \xi(\omega))$ are i.i.d. , the random fixed point is given by a unique ergodic Markov equilibrium.

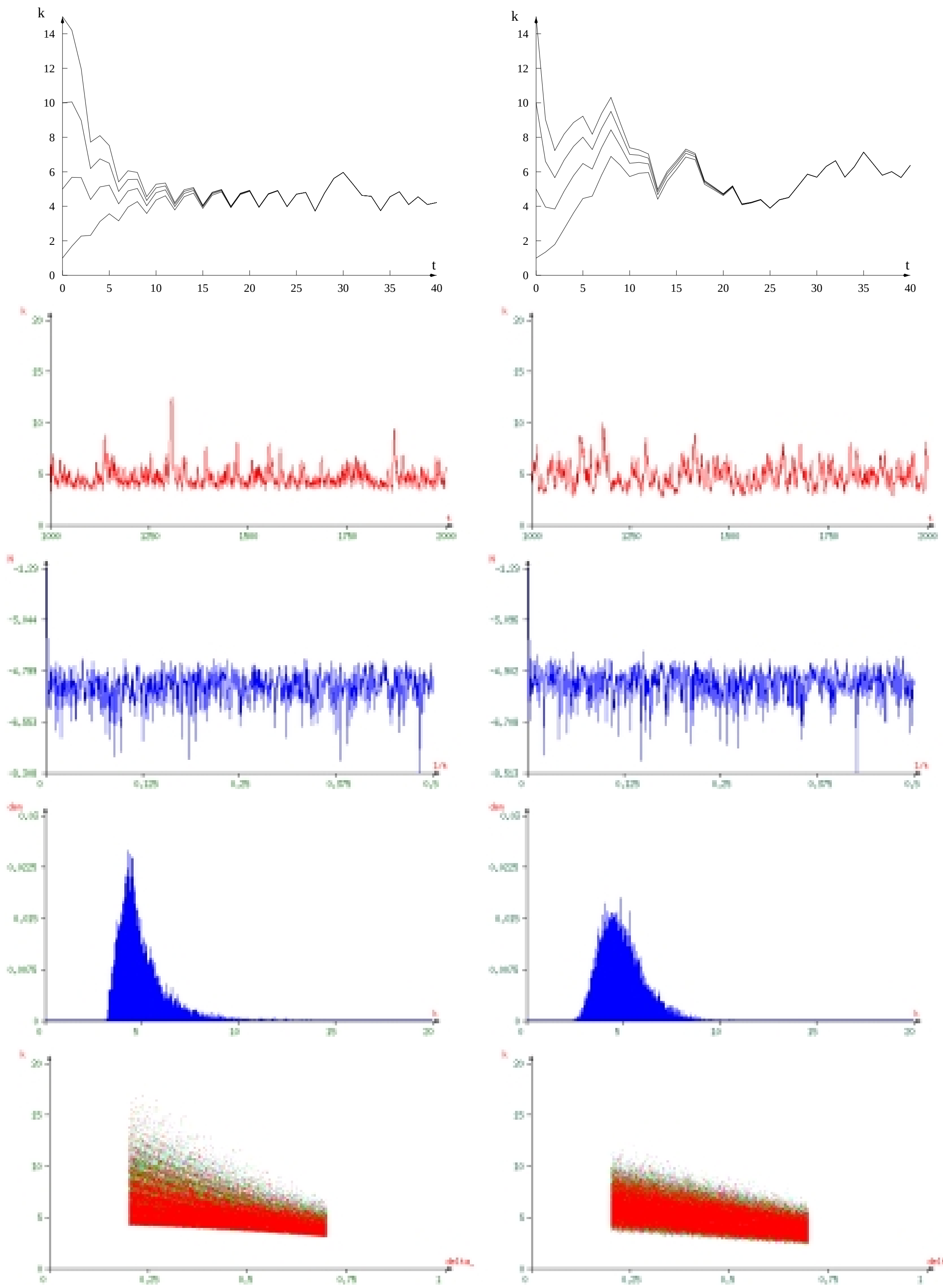

Figure 4.1: Stable random fixed points in the Solow–Swan model. Left coloumn: tent map. Right coloumn: i.i.d. shock.

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