Mean Variance Preferences, Expectations Formation, and the Dynamics of Random Asset Prices

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Abstract

This paper analyzes the dynamics of a general explicit random price process of finitely many assets in an economy with overlapping generations of heterogeneous consumers forming optimal portfolios, extending the one dimensional investigation of Böhm, Deutscher & Wenzelburger (2000). Consumers maximize expected utility with respect to subjective transition probabilities defined by Markov kernels. Given a forecasting rule (predictor) and an exogenous stochastic process of producer dividends, the dynamics of the economy is described as a random dynamical system in the sense of Arnold (1998). The paper investigates existence and stability of random fixed points (invariant measures) for mean–variance preferences under various forecasting schemes, including unbiased predictions as well as OLS forecasting. Numerical simulations show the stability and the performance of the different predictors for linear mean–variance preferences. alternative random dividend processes are provided.

Key words: Random dynamical systems, expectations, learning, random asset pricing, mean variance preferences.
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1 Introduction

One of the most intriguing theoretical problems for a dynamic analysis of asset prices is to model and understand the causes and forces which determine the kind of evolution typically observed in such markets. It is commonly assumed that part of the fluctuations are generated by exogenous unobservable factors which are modeled as stochastic processes. In addition, however, empirical observations as well as straightforward theoretical insight into the nature of the asset price process indicate that traders' expectations about the future development of asset prices have a significant influence on the actual determination of current prices. On a micro based theoretical level the way in which agents form their expectations is well understood. Apart from the usual microeconomic characteristics (like preferences, endowments, and other individual consumption features) these expectations play a fundamental role. The main structural reason for this fundamental influence in intertemporal models is the existence of a so called *expectations feedback* which appears naturally in the demand and supply behavior of interacting agents with multiperiod lives.

The first important implication from this insight is that market clearing prices at each point in time (whether in a deterministic or stochastic environment, in real or asset markets) will be an *endogenous outcome* of the interaction of agents with their expectations. Second, the evolution of prices will depend on the way in which agents *change* their expectations, i. e. which updating rule they use. Therefore, in general, the behavioral features of such rules will have an impact on the *evolution* of prices. From this perspective one would not expect that asset prices can be described by a mere exogenous process.

For deterministic systems numerous contributions investigate different forms of forecasting rules which induce conditions under which complex dynamic behavior may arise (cf. for example Brock & Hommes 1997, 1998), or Chiarella & He (2000). In such cases the theory of (deterministic) dynamical systems and bifurcation theory provide a wide array of results and methods to study the long run behavior of the system and its complexity, as well as the role of different rules. In this way an understanding of the nature of the expectations feedback can lead to an evaluation of the performance of different rules.

For dynamical systems which are simultaneously subject to random perturbations published results characterizing price processes offer much less insight into the properties of the process and their relationship to expectations formation. Most strikingly, the fundamental assumptions of market clearing, of no arbitrage conditions, and of rational expectations dominant in the financial markets literature preclude a standard *dynamic* analysis of the interaction of different forecasting *rules* with other determinants of the market mechanism. Only the outcome of the intertemporal processes of pricing and expectations formation are described as a solution of implicitly given equilibrium conditions and not as orbits of a forward recursive dynamical system (cf. Lucas (1978), Cox, Ingersoll, Jr. & Ross (1985), and Duffie (1996)). This leaves no explanatory room for non rational beliefs, adaptation, or learning. It is evident that explicit forward recursive solutions in such models can be obtained only in very special cases (as for example in the CAPM Sharpe (1964), Lintner (1965), Mossin (1966), and Stapleton & Subrahmanyam (1978)).

The implicit approach describes asset prices as a fixed point in the appropriate space of random variables and provides proofs of existence of a particular equilibrium. Essentially, this means that, excluding exceptional cases, the solutions describing the actual stochastic process cannot be derived explicitly. As a consequence, very little qualitative structure of the price process can be derived. In general nonlinear and multivariate situations it seems difficult to determine global qualitative properties of such fixed points. More important, the implicit solution approach under rational expectations does not describe the process of rational expectations formation; that is, it does not characterize those forecasting rules which, if applied, generate rational expectations orbits. Moreover, no information on nonrational forecasting rules and their impact can be deduced, providing no information on the nature of the expectations feedback.

One way to overcome the implicit non–sequential equilibrium approach is to model the explicit mechanisms determining price formation in markets along with the expectations formation procedures agents use. In this case the methods of dynamical systems theory become available, in particular also those of the so–called random dynamical systems (cf. Arnold 1998). This not only improves the scope of the analysis from a theoretical point of view, but also provides a much more convincing descriptive theory of economic random dynamics. The experience with the fully explicit modeling strategy in other deterministic economic models, (cf. Böhm & Wenzelburger 1999) as well as stochastic models (cf. Böhm & Wenzelburger 1997b) suggests, that these can be applied to financial markets as well. Fortunately, the newly developed theory of random dynamical systems (cf. Arnold 1998) provides applicable and powerful new results.

The asset pricing literature treats the questions of the dynamic forces of markets and of the expectations formation procedures in different ways and different degrees of generality. As already pointed out above, most contributions do not supply an explicit sequential model within a micro–based intertemporal model. Böhm, Deutscher & Wenzelburger (2000) presented the first fully explicit and sequential model with an overlapping generations structure of consumers, where heterogeneous agents can hold arbitrary expectations of future asset prices. The general purpose there was 1) to model the effect of preferences and of expectations as well as of market interaction of agents on asset prices, showing explicitly that market clearing asset prices are determined endogenously, 2) to show that with the description of specific but arbitrary expectations formation rules the forward dynamics of the asset price process was well defined. The model uses an overlapping generations structure for consumers who can trade in two assets, one with a safe and one with a risky return. Such a structure captures the basic features of standard asset market models while providing a general framework to study the impact of heterogeneity of agents on price formation. It provides a reference model to other explicit sequential structures, for example electronic mechanisms like Xetra on the German stock exchange (cf. Deutsche Börse AG 1998). In particular, it sets the stage for a systematic investigation of the influence of alternative adaptive expectation formation and learning on the stability and on the long run properties of endogenous asset price processes.

The current paper extends this model to the general case with an arbitrary finite number of assets. It embodies a fully general analysis of situations with heterogeneous consumers with arbitrary beliefs, preferences, and forecasting rules, providing an explicit sequential modeling of the endogenous price process as a random dynamical system in the sense of Arnold (1998). This makes an application of the new techniques for such systems available, including the analysis of the long run behavior, its stability, and other qualitative properties. In this set up the equilibrium process of the traditional CAPM model becomes a particular stochastic orbit (a random fixed point) of the random dynamical system.

Section 2 of the paper presents the general dynamic model with an arbitrary finite number of assets and a random dividend process and it discusses the dynamic structure of the price process under general forecasting rules modeled as Markov kernels. The notion of perfect (unbiased) Markov kernels introduced in Böhm & Wenzelburger (1997b) is used to show that the expectations feedback for the asset pricing model induces a specific timing and memory structure of unbiased forecasting rules, necessary if rational expectations are guaranteed along all orbits. For the dynamic analysis of the price process the section introduces the concepts of a random fixed point and its stability which are needed to describe the long run behavior of all sample paths of a random dynamical system. Using these notions section 3 analyzes the dynamics of the random asset price process for the class of mean–variance preferences and shows existence of the price process under rational expectations and gives conditions under which it is stable. Section 4 analyzes the dynamics of the asset price process under different forms of "nonrational" forecasting rules including OLS forecasting. Numerical simulations for an AR(1) dividend process are provided. The results exhibit the strong impact of different predictors on stability as well as on prices and returns. Section 5 draws some conclusions.

2 The Model

Consider an economy with one real consumable commodity which is available in each period of time, but which cannot be stored by consumers directly. There exist $k = 1, \dots, K$ (nominal) retradeable assets corresponding to stocks/shares of firms. The production activity of each firm is assumed to induce a stochastic process of dividends over time which are distributed to the share holders (the owners of the assets in each period).

The set of agents participating in the asset markets consists of overlapping generations of finitely many consumers. Let $A = \{1, \dots, N\}$ denote the set of consumers in each generation. Each consumer $a \in A$ lives for two periods. He receives an initial endowment $e^a > 0$ of the consumable commodity in the first period of his life, when he does not consume. To transfer wealth to the second period, consumer a can save part of the endowment to receive a fixed non random rate of return $R > 0$ and purchase any of the K assets. He will choose a portfolio whose proceeds he will consume in period two of his life. Since he receives no additional endowment in the second period of his life, his total consumption is equal to the real wealth accumulated.

For every young consumer $a \in A$, let $u^a : \mathbb{R}_+ \to \mathbb{R}$ denote his von–Neumann-Morgenstern utility function for future consumption. If a purchases a vector of assets $x \in \mathbb{R}^K$ at prices $p \in \mathbb{R}^K_+$ when young, his portfolio[1] $(y, x) := (e^a - p \cdot x, x) \in \mathbb{R}^{K+1}$ implies wealth/consumption when old given by $Re^a + x \cdot (q + d - Rp)$ where d is the vector of

[1]With a slight abuse of language $x \in \mathbb{R}^K$ will also be referred to as a portfolio. No confusion should arise.

dividend payments and q is the vector of future prices of the assets at which he sells in the second period of his life. In order to obtain well defined asset demand, it is assumed that there exists a (sufficiently low) uniform bound $\kappa \in -(\mathbb{R}^K_+)$ on short sales for all assets and that consumers cannot obtain credit. Let

$$B(p, e^a) := \left\{ x \in \mathbb{R}^K \,|\, p \cdot x \leq e^a \,,\, \kappa \leq x \right\} \tag{2.1}$$

denote the budget set of consumer a.

Since the primary purpose of this analysis is to study the influence of subjective expectation for future prices on current prices, it is assumed that the sequence of events and the market mechanism are such that every young consumer knows the buying price as well as the dividend payment to be received when he trades the assets. Thus, when deciding on his portfolio, he treats both arguments parametrically, so that all remaining uncertainty for the return of the portfolio rests with the future price q at which he sells his assets[2]. Let $\nu^a \in \text{Prob}(\mathbb{R}^K_+)$ denote the subjective probability measure held by the young consumer a regarding the future sales price[3] of the asset. Then his asset demand is defined as a portfolio which maximizes expected utility of his future wealth on his budget set, i.e.

$$\tilde{\varphi}^a(p, d, \nu^a) = \arg \max_{x \in B(p, e^a)} \int_{\mathbb{R}^K_+} u^a(Re^a + x \cdot (q + d - Rp))\nu^a(dq). \tag{2.2}$$

Assumption 2.1 *Preferences and expectations of each consumer $a \in A$ are such that the following hold:*

(i) The utility function $u^a : \mathbb{R} \longrightarrow \mathbb{R}$ is twice continuously differentiable, strictly monotonically increasing, strictly concave, and bounded.

(ii) Each distribution ν^a is a Borel measure on $\mathbb{R}^K_+$.

Clearly, $\tilde{\varphi}^a(p, d, \nu^a)$ is well defined for all positive prices and positive endowment e^a if assumption 2.1 holds. Notice that prices p, dividends d, and the safe rate R enter in a linear way as $d - Rp$ in the utility function, a term which measures the discounted premium of the dividend of each asset over the purchase price if the future sales price is zero. Define this vector as the dividend premium $\pi := d - Rp \in \mathbb{R}^K$. Standard arguments imply that any optimal portfolio in the interior of the budget set is a function of the dividend premium alone. Thus, if portfolios of consumers are interior, it suffices to consider the reduced asset demand function

$$\varphi^a(\pi, \nu^a) := \arg \max_x \int_{\mathbb{R}^K_+} u^a(Re^a + x \cdot (q + \pi))\nu^a(dq). \tag{2.3}$$

[2] for a more detailed discussion of the time structure see Böhm, Deutscher & Wenzelburger (2000)

[3] The analysis of consumer behavior as well as the market analysis which follows would be the same, if expectation were considered *cum dividends* with no essential structural implications for the model.

Lemma 2.1 *If assumption 2.1 holds, individual asset demand* $\tilde{\varphi}^a$ *is a continuous function of* (p, d, R)*, while the reduced demand function* φ^a *is a continuous function of the dividend premium* π*, for every Borel measure* ν^a *.*

Define aggregate asset demand[4] by young consumers as $\varphi(\pi,\nu) := \sum_{a\in A} \varphi^a(\pi,\nu^a)$ where $\nu := (\nu^a)_{a\in A}$. Let $\bar{x} \in \mathbb{R}^K_+$ denote aggregate supply of assets offered by old consumers in the economy. Since young consumers take the dividend premium parametrically the asset market is in equilibrium if and only if for some π

$$\bar{x} = \sum_{a\in A} \varphi^a(\pi,\nu^a) = \varphi(\pi,\nu),$$

which implies a *deterministic* equilibrium premium π. As a consequence, one obtains the following lemma describing equilibrium asset prices in each period.

Lemma 2.2 *Let assumption 2.1 on preferences be satisfied and assume that aggregate demand is globally invertible with respect to the dividend premium. Then, there exists a continuous mapping*

$$S : \mathcal{D} \times \left(Prob(\mathbb{R}^K)\right)^{|A|} \longrightarrow \mathbb{R}^K \qquad (d, (\nu^a)_{a\in A}) \longmapsto S(d, (\nu^a)_{a\in A})$$

determining endogenous asset prices at each time t *by*

$$p = S(d,\nu) := \frac{1}{R}\left[d - \pi(\nu)\right] \tag{2.4}$$

where $\pi(\nu) := \varphi^{-1}(\bar{x},\nu)$*.*

Therefore, if aggregate asset demand is globally invertible there exists a unique equilibrium dividend premium which clears asset markets in every period implying that the equilibrium dividend premium in every period is a deterministic function of individual characteristics (preferences and beliefs) of young consumers. As a consequence, individual portfolios are non random and functions of these characteristics as well. However, the equilibrium price of assets will be a random variable determined by (2.4) if and only if dividends are random. Note also, that prices would be non random, if expectations were formed with respect to *cum dividend* prices.

To complete the description of the model, assume that the dividends $d \in \mathbb{R}^K$ will be subjected to exogenous noise which is modeled in the following way. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and $\vartheta : \Omega \to \Omega$ a measurable invertible mapping with measurable inverse. The map ϑ is measure preserving with respect to $\mathbb{P}$ and $\mathbb{P}$ is ergodic with respect to ϑ such that the collection $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta^t\})$ becomes an *ergodic* dynamical system,

[4] In order to investigate the structural features of the asset price process, all of the remaining analysis assumes that consumer behavior is represented by interior portfolio choices, ignoring and avoiding the associated boundary problems. This is a well justified approach for mean–variance preferences. The issue of possibly negative asset prices is treated explicitly at the appropriate locations.

(cf.Arnold (1998) for details). In addition let $D : \Omega \to \mathbb{R}^K$ denote a measurable map such that the exogenous perturbation is given by

$$D \circ \vartheta^t : \Omega \longrightarrow \mathbb{R}^K, \qquad t \in \mathbb{N} \tag{2.5}$$

which defines a so called real noise process on $(\Omega, \mathcal{F}, \mathbb{P})$.

Assumption 2.2
The dividend process $(D_t)_{t\in\mathbb{N}}$ defined by $D_t := D \circ \vartheta^t$ is a stationary Markov process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $D_t : \Omega \to \mathcal{D}$ where $\mathcal{D} := [d_{min}, d_{max}] \subset \mathbb{R}^K_+$, with $0 \ll d_{min} \ll d_{max}$

Given the price law (2.4), assumption 2.2, and fixed beliefs $\nu = (\nu^a)_{a\in A}$, then prices $\{p_t\}_{t\in\mathbb{N}}$ evolve over time according to a stochastic process $S(D_t(\cdot), \bar{x}, \nu) : \Omega \to \mathbb{R}^K$ defined by

$$p_t = S(D_t(\cdot), \nu) = \frac{1}{R}\left[D_t(\cdot) - \pi(\nu)\right]. \tag{2.6}$$

The mapping (2.4) resp. (2.6) is an economic law in the sense of Böhm & Wenzelburger (1997b, 1999) with an expectations feedback. Such mappings incorporate the interaction of the given expectations and of the realization of the random dividend process at any point in time.

The mapping (2.4) has several distinctive structural features. First, the mapping S is of the cobweb type, i. e. S does not contain the price itself for which expectations are formed as an argument, which is the common feature of all cobweb models. As is well known for such mappings, this implies that the dynamics of the system are exclusively driven by the expectations formation process. Second, the dating of the expectations term $\nu = (\nu^a)_{a\in A}$ relative to the dividend process and the determination of the price in each period shows that (2.4) has an expectational lead, i. e. expectations are formed with respect to the realization of prices one period ahead of the mapping S. As a consequence, the predictors (the functions) describing the expectation formation process possess an extra delay of one period: predictions made in t for period $t+1$ are based on information up to the previous period $t-1$ (cf. Böhm & Wenzelburger (1997a)).

Third, the algebraic form of the economic law is worth noting. It has a systematic (deterministic) component determined by the equilibrium dividend premium, aggregate supply of shares $\bar{x}$, and by the subjective beliefs $\nu = (\nu^a)_{a\in A}$ of young consumers for future prices. These in turn determine deterministic quantities of assets traded by each consumer. Therefore, the randomness of the asset price is generated by an additive noise term driven by the dividend process. Thus, the price process inherits all of the properties of the dividend process with two immediate consequences. For every $t \in \mathbb{N}$, the conditional mean of asset prices is given by

$$\mathbb{E}_{t-1} p_t = \mathbb{E}_{t-1}\left[S(D_t(\cdot), \bar{x}, \nu)\right] = \frac{1}{R}\left[\mathbb{E}_{t-1} d_t - \pi(\nu)\right], \tag{2.7}$$

and its conditional covariance

$$\mathbb{V}_{t-1} p_t = \mathbb{V}_{t-1}\left[S(D_t(\cdot), \bar{x}, (\nu^a)_{a\in A})\right] = \frac{1}{R}\mathbb{V}_{t-1} d_t. \tag{2.8}$$

Notice that the conditional mean depends strongly on subjective expectations for the future price development whereas the conditional covariance is independent of individual preferences and beliefs. These features of the asset price process are quite general and independent of the characteristics of consumers. They are essentially consequences of the market clearing mechanism and of the additive nature of the definition of the return on assets. They show that the asymmetry of the price uncertainty among the different generations has a structural impact on the nature of the return process.

It is clear that the price process will be a direct mirror image of the dividend process if expectations do not change across generations. If, however, agents change/update their expectations, this will induce an additional interaction between the expectations formation process and the dividend process. Thus, in general, actual prices and dividends may follow quite different paths. As a consequence the description of the development of endogenous asset prices is complete only if the expectation formation process as well as the dividend process are specified. Then, as will be shown below, the interaction of the dynamic forces of expectations formation with the random dividend process induces a random dynamical system in the sense of Arnold (1998).

2.1 Returns and Risk premia

Before discussing the dynamics of the economic system, it is useful to analyze some consequences of the price process (2.6) on the rates of return and on risk premia. Given an arbitrary portfolio $(y, x) \in \mathbb{R}^{K+1}$, its total return would be $Ry + x \cdot (q + d)$ where the vector $(R, q + d) \in \mathbb{R}^{K+1}$ describes the rates of return for each of the $K + 1$ assets. Taking account of the budget constraint $y + p \cdot x = e$ and making the time structure explicit, the return (measured in monetary units) on a portfolio $(e - p_t \cdot x_t, x_t)$ purchased in period t is defined as

$$w_t = Re + x_t \cdot (p_{t+1} + d_t - Rp_t) = Re + x_t \cdot (p_{t+1} + \pi_t). \tag{2.9}$$

Here, the vector $(p_{t+1} + \pi_t) \in \mathbb{R}^K$ is the vector of premia over the safe rate R earned by each risky asset and $x_t \cdot (p_{t+1} + \pi_t)$ is the total premium earned by the asset portfolio.

Consider the return on consumer a's portfolio if he belongs to an arbitrary generation $t - 1$. Let $W^a_{t-1}(\cdot)$ denote his actual final wealth at the beginning of period t. As before write $\nu := (\nu^a)_{a\in A}$ and denote by ν^a_t resp. $\nu_t := (\nu^a_t)_{a\in A}$ the beliefs of young consumers in period t. Then (2.9) implies

$$W^a_{t-1}(\cdot) \quad = \quad Re^a + \varphi^a(\pi(\nu_{t-1}), \nu^a_{t-1}) \cdot \left[\frac{1}{R}\Big(D_t(\cdot) - \pi(\nu_t)\Big) + \pi(\nu_{t-1})\right]. \tag{2.10}$$

The risk premium earned by consumer a of generation $t-1$ (measured as the total rate on investment minus the safe rate R) becomes

$$(2.11) \qquad \begin{aligned} \mathcal{R}^a_{t-1}(\cdot) &= \frac{1}{e^a} W^a_{t-1}(\cdot) - R \\ &= \frac{1}{e^a}\varphi^a(\pi(\nu_{t-1}), \nu^a_{t-1}) \cdot \left[\frac{1}{R}\Big(D_t(\cdot) - \pi(\nu_t)\Big) + \pi(\nu_{t-1})\right]. \end{aligned}$$

Equations (2.10) and (2.11) describe the stochastic and intertemporal factors which determine the random development of the return process. It is transparent that, due to the overlapping generations structure, each generation influences strongly the buying price of its shares through its expectations, whereas the selling price is determined by the expectations of the following generation. Thus, wealth, rates of return, and risk premia of any agent in any generation will depend on a pair of expectations from two consecutive generations and on the stochastic nature of the dividend process. It is obvious therefore, that the evolution of individual beliefs over time will play an important role in the determination of actual asset prices and returns.

Finally, consider the development of aggregate wealth

$$(2.12) \qquad \sum_{a \in A} W^a_{t-1}(\cdot) = R \sum_{a \in A} e^a + \bar{x} \cdot \left[\frac{1}{R}\Big(D_t(\cdot) - \pi(\nu_t)\Big) + \pi(\nu_{t-1})\right]$$

and its return resp. risk premium

$$(2.13) \qquad \mathcal{R}_{t-1}(\cdot) = \left(\sum e^a\right)^{-1} \bar{x} \cdot \left[\frac{1}{R}\Big(D_t(\cdot) - \pi(\nu_t)\Big) + \pi(\nu_{t-1})\right].$$

Before studying the dynamics of expectations formation and its impact on the price process, it is useful to consider the special case when beliefs are stationary, i. e. when expectations of consumer types are the same across generations, i. e. for any $a \in A$, $\nu^a_t = \nu^a_{t-1}$ holds for all t. As a consequence dividend premia and individual portfolios are constant through time. For the wealth process one obtains

$$(2.14) \qquad W^a_{t-1}(\cdot) = Re^a + \frac{1}{R}\varphi^a(\pi(\nu), \nu^a) \cdot [D_t(\cdot) + (R-1)\pi(\nu)].$$

Similarly, the risk premium for consumers of type a becomes

$$(2.15) \qquad \mathcal{R}^a_{t-1}(\cdot) = \frac{1}{Re^a}\varphi^a(\pi(\nu), \nu^a) \cdot [D_t(\cdot) + (R-1)\pi(\nu)].$$

The development of aggregate wealth under stationary beliefs is described by

$$(2.16) \qquad \sum_{a \in A} W^a_{t-1}(\cdot) = R \sum_{a \in A} e^a + \frac{1}{R}\bar{x} \cdot [D_t(\cdot) + (R-1)\pi(\nu)].$$

Thus, the risk premium process of the market portfolio under stationary beliefs is

$$\mathcal{R}_{t-1}(\cdot) = \left(\frac{1}{R\sum e^a}\right)\bar{x}\cdot[D_t(\cdot)+(R-1)\pi(\nu)] \tag{2.17}$$

The derivation of the preceding equations shows that, under stationary beliefs, the stochastic processes of returns and premia are affine functions, linear in the dividend with a systematic deterministic (constant) term depending on individual and market characteristics. As a consequence, their conditional mean values depend on individual preferences and beliefs whereas their conditional covariances do not. Thus, under stationary beliefs, the volatility of all processes (prices, returns, and premia) will reflect essentially the features of the underlying production process from which the real dividends are generated.

2.2 Mean–Variance Preferences

The situation when agents make portfolio choices on the basis of mean–variance preferences constitutes a much studied class of models which have been used widely and successfully in financial theory. They form the basis of the classical capital asset pricing model (CAPM) the results of which serve as a fundamental guideline to the understanding in evaluating the trade off between returns and risk in asset markets[5].

The primary importance of mean–variance preferences within the classical asset pricing theory stems from the fact that they supply a convenient structure to analyze asset demand behavior explicitly. The case with quadratic utility and normally distributed returns yields the well known standard CAPM pricing formula of Sharpe–Lintner–Mossin[6].

In other more general situations, as is known with quadratic utility, mean–variance preferences induce globally invertible demand functions which are often solvable algebraically. In the general equilibrium context this may yield explicit functional forms of the equilibrium price map, which are needed if an explicit description of the asset price process is the goal.

Consider now the general case of mean–variance preferences taken as a general primitive concept to represent preferences under risk. Assume that consumers form their future price expectations ν^a using a fixed two parameter family of measures with mean $\mu^a \in \mathbb{R}^K$ and covariance matrix $v^a \in \mathbb{R}^K \times \mathbb{R}^K$. Thus we can identify ν^a with the pair $(\mu^a, v^a) \in \mathbb{R}^K \times (\mathbb{R}^K \times \mathbb{R}^K)$.

Assumption 2.3 *For every $a \in A$, beliefs $(\mu^a, v^a) \in \mathbb{R}^K \times (\mathbb{R}^K \times \mathbb{R}^K)$ satisfy:*

(i) $0 \ll \mu_{min} \leqq \mu^a \in \mathbb{R}^K_+$,

(ii) v^a *is symmetric, non singular, and positive definite.*

[5] The relationship/restrictions the mean–variance model imposes are well understood if consistency with expected utility maximization is required, see for example Brennan (1989), Eichberger & Harper (1994), and Dumas & Allaz (1996)).

[6] Sharpe (1964), Lintner (1965), and Mossin (1966)

Property (i) imposes a strictly positive lower bound on expected prices. Let $x \in \mathbb{R}^K$ be arbitrary and $(e^a - p \cdot x, x) \in \mathbb{R}^{K+1}$ denote the associated portfolio. Then, for arbitrary beliefs, $(\mu^a, v^a) \in \mathbb{R}^K \times (\mathbb{R}^K \times \mathbb{R}^K)$

$$M^a(\mu^a + \pi, x) := Re^a + x \cdot (\mu^a + \pi) \tag{2.18}$$

defines the subjective expected return of the portfolio and

$$V^a(x) := x' v^a x \tag{2.19}$$

its variance. Then, consumer a's asset demand function is defined by

$$\varphi^a(\mu^a + \pi) := \arg\max_x U^a(M^a(\mu^a + \pi, x), V^a(x)). \tag{2.20}$$

Since the characteristics (v^a, Re^a) are kept constant throughout the remainder of the analysis they are suppressed as arguments of the demand function. It is well known and straight forward to show that the demand for risky assets φ^a is a well defined function. The following proposition states a sufficient condition under which individual as well as aggregate asset demand is globally invertible in the mean premium $(\mu^a + \pi)$.

Proposition 2.1 *Let the concave mean–variance utility U^a be twice continuously differentiable and additively separable. If the characteristics of consumers satisfy Assumption 2.3, then, for every $v^a \in \mathbb{R}^K \times \mathbb{R}^K$, the asset demand function φ^a is globally invertible.*

Proof: (standard) □

Lemma 2.3 *Let the concave mean–variance utility U^a be twice continuously differentiable, linear in the mean and strictly concave in the variance. If, in addition, the characteristics of consumers satisfy Assumption 2.3, then aggregate demand $\sum_a \varphi^a(\mu^a + \pi)$ is globally invertible in π for any given $(\mu^a)_{a \in A}$.*

Proof: (see Appendix) □

It follows now from the results of the previous section, that under the conditions of **Lemma 2.3**, the model has a well defined price law given by a mapping

$$S : \mathcal{D} \times \left(\mathbb{R}^K\right)^{|A|} \to \mathbb{R}^K$$

defined by

$$S(d, \mu) = \frac{1}{R}\left[d - \pi(\mu)\right] \tag{2.21}$$

with $\mu := (\mu^a)_{a \in A}$.

The situation where consumers have the linear utility function

$$U^a(M,V) := M - \frac{\alpha^a}{2} V \tag{2.22}$$

where for each $a \in A$, $\alpha^a > 0$ measures risk aversion and $1/\alpha^a$ measures risk tolerance, provides an explicit algebraically solvable example for the price mapping. In this case individual asset demand is given by

$$\begin{aligned} \varphi^a(\mu^a + \pi) &= \frac{1}{\alpha^a}(v^a)^{-1}(\mu^a + \pi) && (2.23)\\ &= \tau^a(\mu^a + \pi) && (2.24) \end{aligned}$$

where $\tau^a := \frac{1}{\alpha^a}(v^a)^{-1} \in \mathbb{R}^K \times \mathbb{R}^K$ is the risk adjusted inverse of the subjective covariance matrix. Invoking asset market clearing

$$\sum_{a \in A} \varphi^a(\mu^a + \pi) = \bar{x},$$

one obtains for the equilibrium dividend premium

$$\pi(\mu) := \left(\sum_a \tau^a \right)^{-1} \left(\bar{x} - \sum_a \tau^a \mu^a \right). \tag{2.25}$$

This yields the asset price law as

$$\begin{aligned} p &= S(d, \mu) && (2.26)\\ &= \frac{1}{R}\left[d + \left(\sum_a \tau^a \right)^{-1} \left(\sum_a \tau^a \mu^a - \bar{x} \right) \right]. && (2.27) \end{aligned}$$

Notice that the price law is an affine function in expected means (determined by the dividend premium, aggregate supply of shares, preferences, and by arbitrary but given subjective beliefs $(\tau^a, \mu^a)_{a \in A}$ of young consumers) with additive noise. If dividends are zero and/or expectations were formed *cum* dividends, prices would not be random. Observe the similarity of the price map with the standard Sharpe–Lintner–Mossin equation (see for example Stapleton & Subrahmanyam 1978). However, individual portfolios are proportional to the market portfolio $\bar{x}$ if and only if all consumers hold the same mean expectations μ *and* the same covariances.

It is also apparent that, without further restrictions, asset prices may become negative if dividends are too low. Therefore, some additional restrictions are needed to obtain non–negative prices.

Lemma 2.4 *Let the conditions of lemma 2.3 be satisfied and assume that*

$$d_{min} \gg \left(\sum_a \tau^a\right)^{-1} \bar{x}.$$

Then, asset prices

$$p = \frac{1}{R}\left[d + \left(\sum_a \tau^a\right)^{-1} \left(\sum_a \tau^a \mu^a - \bar{x}\right)\right]$$

are positive for all non–negative mean expectations $(\mu^a)_{a\in A}$.

2.3 Expectations formation and the dynamics of random asset prices

The dynamical features of the economy are only specified completely, if the way in which predictions are made by the different generations are described, i. e. how consumer a of generation t determines the measure ν_t^a. It is obvious, in view of the forward recursive structure of dynamical systems, that agents at time t have observed previous states p_τ and exogenous perturbations $D_\tau = D(\vartheta^\tau \omega)$ only up to time $\tau \leq t-1$. In addition, the information set I_t at t may also include the forecasts made by previous generations ν_τ, $\tau \leq t-1$.

Usually, two distinct scenarios are considered when the explicit formation of expectations is described in economic models. The first one adopts a *stationary* framework. This defines forecasting mechanisms or expectations functions (often called *predictors*) as time invariant forward recursive mappings using the information I_t. Such rules are typically (best) modeled by Markov kernels. These include for example Bayesian updating, many econometric updating schemes (like the Kalman filter or OLS estimation), and a large number of other adaptive rules. It is known, that this class also contains the predictors generating rational expectations. They differ primarily in the form and in the extent to which they use past information. They share the common stationarity property of being time invariant mappings updating the predictions at each time in a recursive fashion. The key implication of using such predictors in conjunction with an economic law is that the resulting dynamical system becomes autonomous.

The second scenario uses non stationary rules as in many models of (non–autonomous/non–adaptive) learning or in evolutionary models. Their dominant feature is that the updating mechanism of subjective beliefs becomes a function of time. Since the general theory of (non–autonomous) learning in non linear random models is still in its infancy, this paper will use exclusively the Markovian/stationary approach. Here, the structure proposed in Böhm & Wenzelburger (1997a, 1997b) will be followed directly.

Assume that each consumer a of a generation determines his subjective probability distribution of future prices using a Markov kernel

$$(2.28)\quad \Psi^a : \mathcal{D} \times \mathbb{R}^K \times \text{Prob}(\mathbb{R}^K) \times \mathcal{B}(\mathbb{R}^K) \longrightarrow [0,1] \qquad (d,p,\nu,B) \mapsto \Psi^a(d,p,\nu)(B)$$

with $\mathcal{B}(\mathbb{R}^K)$ denoting the σ-algebra of Borel sets. Thus, $\Psi^a(d,p,\nu)(B)$ is consumer a's subjective probability that future prices are in B[7]. Thus, the predictor Ψ^a described in (2.28) can be viewed as a function

$$\Psi^a : \mathcal{D} \times \mathbb{R}^K \times \mathrm{Prob}(\mathbb{R}^K) \longrightarrow \mathrm{Prob}(\mathbb{R}^K)$$

predicting probability distributions, in other words, $\nu_t^a := \Psi^a(d_{t-1}, p_{t-1}, \nu_{t-1}^a) \in \mathrm{Prob}(\mathbb{R}^K)$ is the subjective probability distribution used by consumer a of generation t when determining his optimal portfolio.

Let $\Psi := ((\Psi^a)_{a\in A})$ denote the list of the Markov kernels for all $a \in A$. Inserting them into the price law (2.4) defines a mapping

$$S_\Psi := \begin{cases} \mathcal{D} \times \mathcal{D} \times \mathbb{R}^K \times (\mathrm{Prob}(\mathbb{R}^K))^{|A|} & \longrightarrow \quad \mathbb{R}^K \\ (d, d_{-1}, p_{-1}, \nu_{-1}) & \mapsto \quad S_\Psi(d, d_{-1}, p_{-1}, \nu_{-1}) \end{cases}$$

determining endogenous prices in any period

$$p = S(d, \Psi(d_{-1}, p_{-1}, \nu_{-1}))$$

as a function of current dividends as well as past dividends and past prices while assuming that agents use the kernels Ψ to make their forecasts. As a consequence, the pair of mappings

$$(S_\Psi, \Psi) : \Omega \times \mathbb{R}^K \times \mathrm{Prob}(\mathbb{R}^K)^{|A|} \longrightarrow \mathbb{R}^K \times \mathrm{Prob}(\mathbb{R}^K)^{|A|}$$

defined by

$$p_t = S_\Psi(D_t(\cdot), D_{t-1}(\cdot), p_{t-1}, \nu_{t-1}) \tag{2.29}$$

$$\nu_t = \Psi(D_{t-1}(\cdot), p_{t-1}, \nu_{t-1})) \tag{2.30}$$

defines the time-one map of a discrete time random dynamical system in the sense of Arnold (1998), which governs the evolution of prices and beliefs.

It is obvious from the structure of these equations that the form of the predictors $(\Psi^a)_{a\in A}$ will have a dominant influence on the actual price development. Thus, under stationary updating, the evolution of prices will not be a simple image of the dividend process. However, this random dynamical system has two specific features. First, note that only the price law is subject to the current random perturbations, while the predictor is a deterministic difference equation corresponding to a normal statistical updating procedure. Second, the vector of past prices enters into the time one map only through the predictor. Thus, if past prices are not taken into account by consumers when making their predictions, then the random dynamical system will consist of the prediction process alone, while prices evolve according to the stochastic process induced by equation (2.4).

[7] It is assumed for simplicity that consumers in each generation use the beliefs of their own predecessors only when making their forecasts. The more general case where they know also all other beliefs is a straightforward extension which can be integrated easily.

One of the primary objectives of this paper is to analyze the long run behavior of the random dynamical system given by (2.29) and (2.30). In deterministic dynamical systems the concepts used to describe the long run behavior are stable fixed points and attractors. These concepts have been extended to the random case. Here the concept of a stable random fixed point will be used to characterize stable random evolutions of asset prices, cf. Arnold (1998), Schmalfuß (1996, 1998).

Definition 2.1 *Consider a random dynamical system given by the continuous mapping* $F : X \times \mathbb{R}^m \longrightarrow X$ *with real noise process* $u_t = u \circ \vartheta^t$, $u : \Omega \longrightarrow \mathbb{R}^m$ *measurable, over the ergodic dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta^t))$. *A* ***random fixed point*** *of* F *is a random variable* $x_* : \Omega \longrightarrow X$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that almost surely*

$$x_*(\vartheta\omega) = F(x_*(\omega), u(\omega)). \tag{2.31}$$

Some implications of the definition can be observed directly. If F is independent of the perturbation ω, then the Definition 2.1 coincides with the one of a deterministic fixed point.

Definition 2.1 implies that $x_*(\vartheta^{t+1}\omega) = F(x_*(\vartheta^t\omega), u(\vartheta^t\omega))$ for all times t. Therefore, the orbit $\{x_*(\vartheta^t\omega)\}_{t\in\mathbb{N}}$, $\omega \in \Omega$ generated by x_* solves the random difference equation

$$x_{t+1} = F(x_t, u_t(\omega)).$$

It follows from stationarity and ergodicity of ϑ that the process$\{x_*(\vartheta^t)\}_{t\in\mathbb{N}}$ is stationary and ergodic. If, in addition, $\mathbb{E}||x_*|| < \infty$, then

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T} 1_B(x_*(\vartheta^t\omega)) = x_*\mathbb{P}(B) := \mathbb{P}\left\{\omega \in \Omega | x_*(\omega) \in B\right\}$$

for every $B \in \mathcal{B}(X)$. In other words, the empirical law of an orbit is well defined and is equal to the distribution $x_*\mathbb{P}$ of x_*. Finally, if the perturbation corresponds to an i. i. d. process the orbit will be an ergodic Markov equilibrium in the usual sense (cf. Duffie, Geanakoplos, Mas-Colell & McLennan 1994).

The concept of stability to be used in the context of a random dynamical system is as follows.

Definition 2.2 *A random fixed point* x_* *is called* ***attracting*** *on some set* $\mathcal{U} \subset \Omega \times X$ *if*

$$\lim_{t\to\infty} ||x_t(\omega) - x_*(\vartheta^t\omega)|| = 0 \quad \textit{for all } (\omega, x_0(\omega)) \in \mathcal{U}.$$

Thus, a random fixed point is attracting if nearby orbits converge to the orbit of the random fixed point. This is clearly a rather strong property for random difference equations. However, as it turns out, this stability property can be verified for many of the economic systems under investigation here.

3 The dynamics with unbiased predictions – rational expectations

One of the intriguing questions dealt with in economic models with an expectations feedback is to understand whether perfect forecasts can be made in such environments at all times. Within the context of models with exogenous disturbances the concept of rational expectations equilibrium is the most widely used one to describe long run equilibrium along time paths. It is well known for many economic models that the two requirements of ongoing equilibrium *and* of rational expectations at all times cannot be fulfilled.

It is clear that the evaluation of the performance of predictors (Markov kernels) can be carried out on several levels. Böhm & Wenzelburger (1997b) provide a general abstract method using the notion of a pseudo metric on the space of probability measures. In most applications, however, it is rare to have full knowledge of the measure, and more often it is useful and practicable to compare moments of (conditional) distributions. In fact, the notion of rational expectations compares only first moments of actual conditional and of subjective distributions. Since this is also the major objective here, the analysis will be confined to the specific situation of the comparison of conditional means only.

With a slight abuse of notation, let $\Psi^a : \mathcal{D} \times \mathbb{R}^K_+ \times \mathbb{R}^K_+ \to \mathrm{Prob}(\mathbb{R}^K)$ denote a kernel, whose arguments are past dividends and prices as well as the subjective expected mean value of the previous generation. In order to characterize conditional subjective expectations consider the *mean value predictor* associated with Ψ^a defined by the function

$$\psi^a : \mathcal{D} \times \mathbb{R}^K_+ \times \mathbb{R}^K_+ \longrightarrow \mathbb{R}^K_+, \qquad (d,p,q) \mapsto \int_{\mathbb{R}^K} x \Psi^a(d,p,q,dx), \tag{3.1}$$

making $q^a_t := \psi^a(d_{t-1}, p_{t-1}, q^a_{t-1})$ the predicted *mean value* for the realization p_{t+1} given the information $(d_{t-1}, p_{t-1}, q^a_{t-1})$. The mean value predictor therefore is a point predictor rather than a predictor for distributions. As before, let $\psi := (\psi^a)_{a\in A}$ denote the list of mean value predictors used by agents $a \in A$. Then, the pair of mappings

$$(S_\Psi, \psi) : \mathcal{D} \times \mathbb{R}^K \times \mathcal{D} \times (\mathbb{R}^K)^{|A|} \longrightarrow \mathbb{R}^K \times (\mathbb{R}^K)^{|A|} \tag{3.2}$$

determines prices and predicted means in every period as functions of current and past dividends, past actual and predicted prices. Therefore,

$$(S_\Psi, \psi) : \Omega \times \mathbb{R}^K \times \mathcal{D} \times (\mathbb{R}^K)^{|A|} \longrightarrow \mathbb{R}^K \times (\mathbb{R}^K)^{|A|} \tag{3.3}$$

yields the random dynamical system with the two component maps

$$p_t = S_\Psi(D_t(\cdot), D_{t-1}(\cdot), p_{t-1}, q_{t-1}) \tag{3.4}$$

$$q_t = \psi(D_{t-1}(\cdot), p_{t-1}, q_{t-1})). \tag{3.5}$$

It is obvious that the consumers $a \in A$ can be unbiased simultaneously only if they all make the *same mean* prediction. Using the notions proposed in Böhm & Wenzelburger (1997b) and taking account of the expectational lead (see Wenzelburger (1999)) yields the following definition.

Definition 3.1 *A list of predictors* $\Psi_* := ((\Psi_*^a)_{a\in A})$ *with associated mean value predictors* $\psi_*^a : \mathcal{D} \times \mathbb{R}_+^K \times \mathbb{R}_+^K \longrightarrow \mathbb{R}_+^K$, *is called* ***unbiased*** *if, for all* $a \in A$ *and all* t*:*

$$\mathbb{E}_{t-1} S_{\Psi_*}(D_t(\cdot), d_{t-1}, p_{t-1}, q_{t-1}) \;=\; q_{t-1}^a = \psi_*^a(d_{t-2}, p_{t-2}, q_{t-2}). \tag{3.6}$$

Thus, predictors are unbiased if the true conditional expectation of the price process coincides with the point forecast for the mean value of the price made by all agents for all points in time. Hence, all orbits with an unbiased predictor have rational expectations in the usual sense.

For a study of the case with unbiased predictions it is useful to impose further restrictions on agents' expectations. Assume, for example, that all agents choose from a fixed (not necessarily the same) family of probability measures parameterized in their means, when making their predictions. In this case, asset demand will depend on the subjective mean parametrically. Unbiased predictions for all agents then requires that all choose the same mean prediction $\mu \in \mathbb{R}_+^K$, so that aggregate demand for assets is a function of the simple form $\varphi(\pi, \mu)$. Then, the equilibrium dividend premium can be written as $\pi(\mu) = \varphi^{-1}(\bar{x}, \mu)$, so that the economic law (2.4) $S : \mathcal{D} \times \mathbb{R}_+^K \times \mathbb{R}_+^K \to \mathbb{R}^K$ is given by

$$S(d, \mu) = \frac{1}{R}\left[d - \pi(\mu)\right]. \tag{3.7}$$

3.1 Unbiased predictions with mean–variance preferences

Consider the case with mean–variance preferences as described by (2.21) assuming that consumers make mean–variance predictions with the *same* expected mean $\mu = \mu^a$ for all $a \in A$. Then, assuming that aggregate asset demand is globally invertible, one finds that the equilibrium premium is linear in μ, i. e.

$$\pi(\mu) := \varphi^{-1}(\bar{x}, \mu) = \phi(\bar{x}) - \mu, \tag{3.8}$$

so that the price law is of the special form

$$S(d, \mu) = \frac{1}{R}\left[d + \mu - \phi(\bar{x}\right]. \tag{3.9}$$

Using the techniques developed in Böhm & Wenzelburger (1997b) one obtains the following result.

Theorem 3.1 *Let consumers have mean–variance preferences and assume that consumers know the conditional mean of the dividend process (or let dividends follow a martingale process with* $\mathbb{E}_{t-1} D_t = d_{t-1}$*). If aggregate asset demand is globally invertible, there exists a unique unbiased predictor* ψ_**. Moreover,* ψ_* *is an affine map with additive noise independent of actual prices.*

Proof: Assume that members of successive generations make the same mean price forecasts, i. e. $\mu^a_{-1} = \mu_{-1}$, $\mu^a = \mu$ for all $a \in A$, and consider the conditional mean error by the old generation in any arbitrary period

$$\begin{aligned} \mathbb{E}_{t-1}\left[S(D_t(\cdot),\mu)\right] - \mu_{-1} &=: \frac{1}{R}\left[\mathbb{E}_{t-1}D_t - \pi(\mu)\right] - \mu_{-1} && (3.10)\\ &= \frac{1}{R}\left[\mathbb{E}_{t-1}D_t + \mu - \phi(\bar{x})\right] - \mu_{-1} \\ &=: e^{\mathbb{E}}_S(\mathbb{E}_{t-1}D_t, \mu, \mu_{-1}). && (3.11) \end{aligned}$$

The mapping $e^{\mathbb{E}}_S : \mathbb{R}^3 \to \mathbb{R}$ (3.11) defines the mean forecast error as a function for arbitrary values $(\mathbb{E}_{t-1}D_t, \mu, \mu_{-1})$ of conditional dividend means and pairs of mean expectations by two successive generations. Therefore, the forecast μ by any young generation makes the forecast μ_{-1} by the preceding generation unbiased if and only if the mean forecast error of the latter is zero. In other words, $e^{\mathbb{E}}_S(\mathbb{E}_{t-1}D_t, \mu, \mu_{-1}) = 0$ if and only if

$$\mu - \phi(\bar{x}) = R\mu_{-1} - \mathbb{E}_{t-1}D_t.$$

Solving for μ, one obtains the unbiased predictor

$$\psi_*(\mathbb{E}_{t-1}D_t, \mu_{-1}) := \phi(\bar{x}) + R\mu_{-1} - \mathbb{E}_{t-1}D_t. \tag{3.12}$$

Thus, knowing the conditional mean (or setting $\mathbb{E}_{t-1}D_t = d_t$ if dividends follow a martingale) one obtains the result.

□

Thus, the unbiased predictor associated with the economic law (2.21) is an affine map independent of prices. Combining (3.12) with (2.21) yields the random dynamical system (S_{ψ_*}, ψ_*) with unbiased prediction given by

$$\begin{aligned} p_t &= \frac{1}{R}\left[D_t(\cdot) + R\mu_{t-1} - \mathbb{E}_{t-1}D_t\right], && (3.13)\\ \mu_t &= \phi(\bar{x}) + R\mu_{t-1} - \mathbb{E}_{t-1}D_t. && (3.14) \end{aligned}$$

Since the unbiased predictor is independent of past prices, the random dynamical system reduces to equation (3.14), generating prices via the stochastic process (3.13) which one may also write as

$$p_t = \frac{1}{R}\left[D_t(\cdot) + \mu_t - \varphi^{-1}(\bar{x}, v)\right]. \tag{3.15}$$

Its conditional mean is equal to μ_{-1} while its conditional covariance is given by

$$\mathbb{V}_{t-1}S_{\psi_*}(D_t(\cdot), \psi_*(d_{t-1}, \mu_{t-1})) = \frac{1}{R}\mathbb{V}_{t-1}D_t(\cdot) \tag{3.16}$$

confirming the property derived in (2.8). Notice in particular that it is independent of subjective expectations, and thus of the predictor. It varies inversely with the safe rate R, i. e. higher R implies a lower conditional covariance. Notice, however, that the price process will exhibit serial correlation through the expectations feedback *and* through the dividend process itself. For example, even if the dividend process is i. i. d. , price expectations follow a Markov process. As a consequence, prices would be serially correlated.

It is apparent from Theorem 3.1 that the dynamic properties of the price process under unbiased prediction originate exclusively from the expectations process (3.14), which is an affine random dynamical system with additive noise. For such systems, existence and stability of a unique globally attracting random fixed point follow from well established results (see Arnold 1998) which imply the following theorem.

Theorem 3.2 *The random dynamical system given by equation* (3.14) *has a unique random fixed point* μ_* *if and only if* $R \neq 1$. *Furthermore,* μ_* *is globally attracting if and only if* $0 < R < 1$.

The instability of the fixed point for a rate of return of the safe asset greater than one is an immediate consequence of the structure of the model, namely the cobweb nature and the expectational lead. This phenomenon is well known in the comparable class of deterministic models under perfect foresight with monotonic economic laws (see for example Chiarella 1988).

To obtain globally defined positive prices as well as positive price expectations in the stable case, some further restrictions on the structure of the economy (aggregate demand) and on the dividend process have to be imposed.

Theorem 3.3 *Let* $0 < R < 1$ *and assume* $d_{max} \gg d_{min} \gg 0$.

1. *There exists a non empty forward invariant interval* $[\underline{\mu}, \overline{\mu}] \subset \mathbb{R}^K$, *for the mapping* (3.14) *defined by*

$$\underline{\mu} := \frac{1}{1-R}\left[\phi(\bar{x}) - d_{max}\right], \qquad \overline{\mu} := \frac{1}{1-R}\left[\phi(\bar{x}) - d_{min}\right]. \tag{3.17}$$

2. *Price expectations are positive along all orbits if*

$$\mu_0 \geq \phi(\bar{x}) - d_{max} \gg 0. \tag{3.18}$$

3. *If, in addition,*

$$\phi(\bar{x}) \gg \max\left\{d_{max}, \frac{1}{R}\left[d_{max} - (1-R)d_{min}\right]\right\} \tag{3.19}$$

holds, then asset prices generated by (3.13) *are strictly positive if*

$$\mu_0 \gg \frac{d_{max} - d_{min}}{R}. \tag{3.20}$$

Proof: By straightforward calculations. □

Due to the linearity of the expectations process, one can calculate directly the basic properties of the so called empirical law of asset prices and of price expectations of the random dynamical system (3.13) and (3.14). Since all orbits converge to the random fixed point μ_* the long run behavior is completely characterized by its statistical properties. Stationarity and ergodicity of the mapping θ and the definition of a random fixed point (2.1) imply that

$$\mathbb{E}\mu_*(\omega) = \mathbb{E}\mu_*(\theta\omega) \tag{3.21}$$

$$= \mathbb{E}\left\{\phi(\bar{x}) + R\mu_*(\omega) - \mathbb{E}_{t-1}D(\omega)\right\}, \tag{3.22}$$

which yields the mean of the fixed point as

$$\mathbb{E}\mu_*(\omega) = \frac{1}{1-R}\left[\phi(\bar{x}) - \mathbb{E}D(\omega)\right]. \tag{3.23}$$

Therefore, ergodicity implies

$$\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T}\mu_t(\omega) = \mathbb{E}\mu_*(\omega) = \frac{1}{1-R}\left[\phi(\bar{x}) - \mathbb{E}D(\omega)\right]. \tag{3.24}$$

In other words, for all initial conditions μ_0 satisfying (3.18), the mean of all sample paths of (3.14) converges to the long run mean (3.23).

To complete the description of the long run features of the economy, notice that the behavior all other variables of the economy (prices, premia, 1portfolios, returns, etc.) is also determined by uniquely defined sample paths each converging to a stationary random variable. Let p_* denote the stationary random variable for the price process induced by the unbiased fixed point μ_* which is defined as

$$p_*(\omega) := \frac{1}{R}\left[D(\omega) + \mu_*(\omega) - \phi(\bar{x})\right]. \tag{3.25}$$

One finds that p_* and μ_* have the same mean, i. e.

$$\mathbb{E}p_*(\omega) = \mathbb{E}\mu_*(\omega) = \frac{1}{1-R}\left[\phi(\bar{x}) - \mathbb{E}D(\omega)\right],$$

as it should be. By the same token, let

$$\pi_*(\omega) := D_*(\omega) - Rp_*(\omega) = \phi(\bar{x}) - \mu_*(\omega)$$

denote the stationary dividend premium. This implies that

$$\pi_*(\omega) + \mu_*(\omega) = \phi(\bar{x}),$$

the vector of net expected returns, is non random. As a consequence, the stationary state of the economy will have constant individual portfolios of all young consumers. Notice, however, that stationary individual wealth, rates of return, and equity premia will still be agent specific stationary random variables, although all consumers have rational expectations.

The situation where consumers have linear mean–variance preferences (see 2.23) provides an explicitly solvable example of the price law and the unbiased predictor. With consumer characteristics given by $(\tau^a, \mu^a)_{a\in A}$, one obtains the economic law given in equation (2.26).

$$p \;=\; \frac{1}{R}\left[d + \left(\sum_a \tau^a\right)^{-1}\left((\sum_a \tau^a\mu^a) - \bar{x}\right)\right].$$

With $\mu^a = \mu$ for all $a \in A$, this simplifies to

$$p \;=\; \frac{1}{R}\left[d + \mu - \left(\sum_a \tau^a\right)^{-1}\bar{x}\right], \tag{3.26}$$

which implies the following corollary to Theorem 3.1.

Corollary 3.1 *Let the conditions of Theorem 3.1 be satisfied and assume that consumers have CARA preferences with given subjective risk tolerance. There exists a unique unbiased predictor ψ_* which is an affine map independent of actual prices.*

Therefore, the random dynamical system (S_{ψ_*}, ψ_*) under unbiased prediction

$$p_t \;=\; \frac{1}{R}\left[D_t(\cdot) + R\mu_{t-1} - \mathbb{E}_{t-1}D_t(\cdot)\right] \tag{3.27}$$

$$\mu_t \;=\; \left(\sum_a \tau^a\right)^{-1}\bar{x} + R\mu_{t-1} - \mathbb{E}_{t-1}D_t(\cdot). \tag{3.28}$$

reduces to equation (3.28), generating prices via the stochastic process (3.27). The results of Theorem 3.2 and of Theorem 3.3 translate directly into the case with CARA utilities implying the following corollary.

Corollary 3.2
Let $0 < R < 1$ and let the conditions of Corollary 3.1 be given. If

$$\left(\sum_a \tau^a\right)^{-1}\bar{x} \gg \max\left\{d_{max}, \frac{1}{R}(d_{max} - d_{min}) + d_{min}\right\} \tag{3.29}$$

holds, then random asset prices will be positive for all initial conditions μ_0 satisfying

$$\mu_0 \gg \underline{\mu} := \frac{d_{max} - d_{min}}{R}. \tag{3.30}$$

3.2 Unbiased predictions with AR(1) dividends

This section presents results of numerical simulations carried out for an economy with one risky asset and heterogeneous consumers with linear mean–variance preferences, to illustrate the results of Theorem 3.3. All simulations were carried out using MACRODYN , a software package (cf. Böhm, Lohmann & Middelberg 1998) designed for the analysis of discrete time dynamical systems (see also Böhm & Schenk-Hoppé 1998).

Consider the associated general price law from equation 2.26

$$p \;=\; \frac{1}{R}\left[d + \left(\sum_a \tau^a\right)^{-1} \left(\sum_a \tau^a\,\mu^a - \bar{x}\right)\right].$$

With $\mu^a = \mu$ for all $a \in A$, this simplifies to

(3.31)
$$p \;=\; \frac{1}{R}\left[d + \mu - \left(\sum_a \tau^a\right)^{-1} \bar{x}\right].$$

Thus, if agents make the same mean forecast, the vector $\phi(\bar{x}) = \left(\sum_a \tau^a\right)^{-1} \bar{x}$ captures all of the remaining exogenous parameters (including the heterogeneity of the consumption sector, i. e. the number of consumers, their risk tolerance, subjective covariance, as well as the total number of assets). For the single asset case here, the value $\phi(\bar{x}) = .55$ will be used for all simulations along with $R = 1.01$.

Consider a situation where dividends follow an AR(1) process modeled as

(3.32)
$$D_{t+1} = .8\,D_t + \zeta_t, \quad \text{where} \quad \zeta_t \sim \text{uniform i.i.d. over } [.01, .13]$$

g

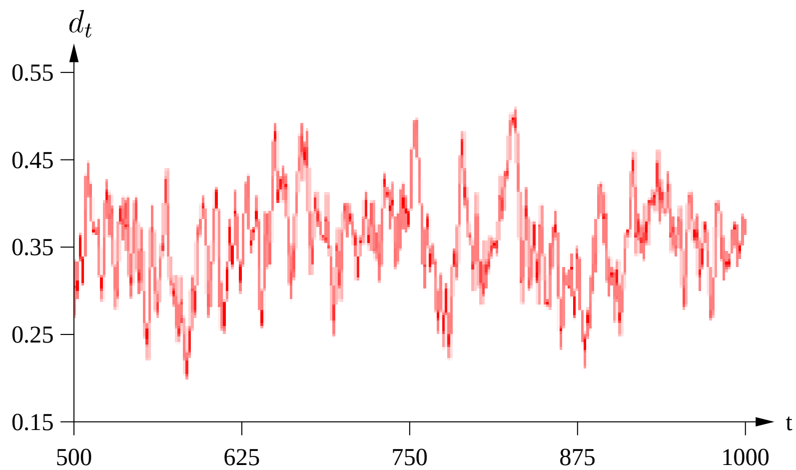

Figure 3.1: AR(1) dividends: $D_{t+1} = .8\,D_t + \zeta_t, \quad \zeta_t \sim$ uniform i.i.d. over $[.01, .13]$

The dividend process (3.32) is itself a random dynamical system with a unique globally attracting random fixed point D_* with mean $\mathbb{E}D_* = .35$. Figure 3.1 shows a typical time window for the dividend process, while Figures 3.2 – 3.4 display additional features.

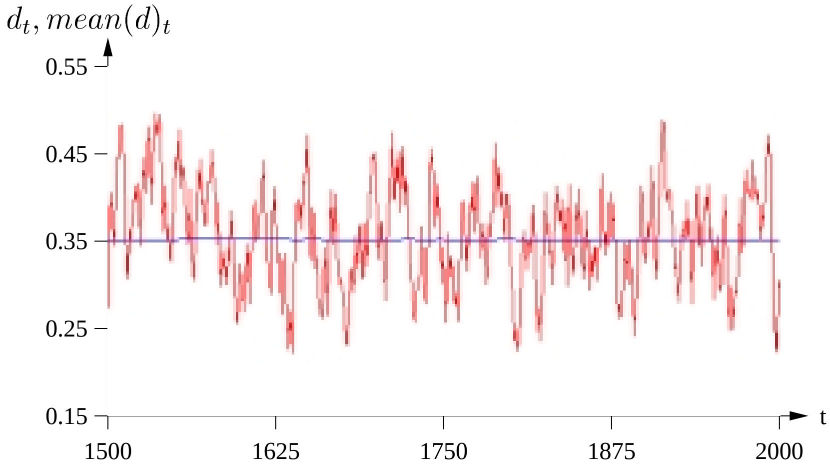

Figure 3.2: AR(1) Dividends and recursive mean

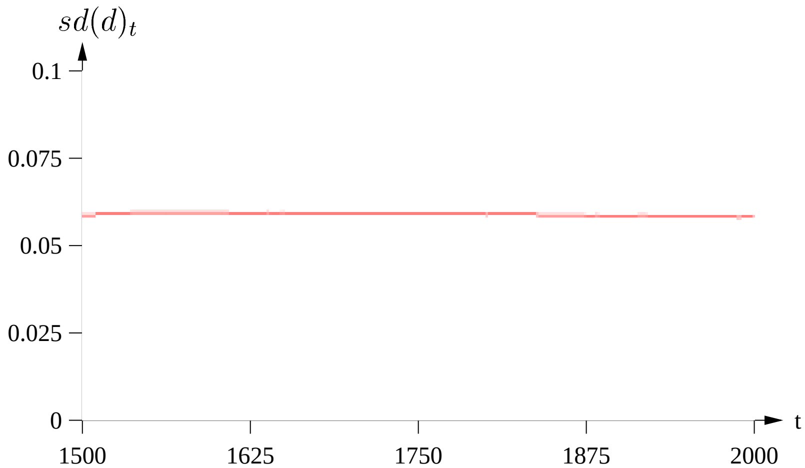

Figure 3.3: Recursive standard deviation of AR(1) dividends

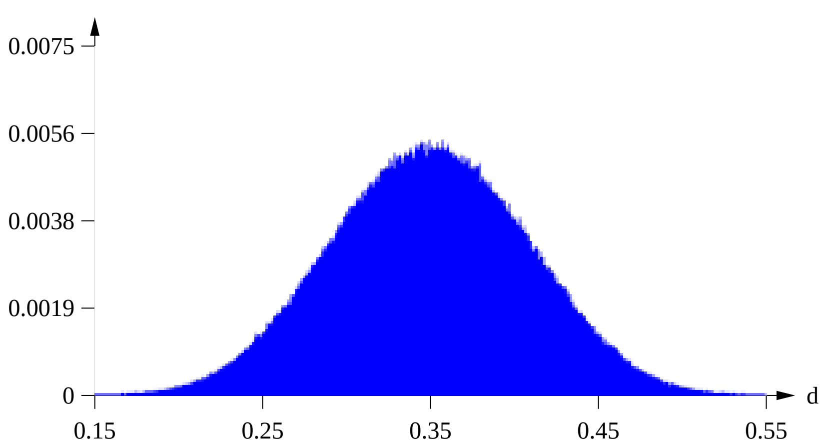

Figure 3.4: Dividends: mean=0.350047; variance=0.00331875; sd =0.0576086; $T = 10^6$

The quantity

$$mean(d)_t := \frac{1}{t+1} \sum_{\tau=0}^{t} d_\tau$$

.

is the recursive mean dividend along the orbit and

$$sd(d)_t := \frac{1}{t+1} \sum_{\tau=0}^{t} \left(\sqrt{\frac{1}{\tau+1} \sum_{i=0}^{\tau} \left(d_i - mean(d)_i\right)^2} \right)$$

is the recursive standard deviation. The next three diagrams show the behavior of asset prices. Figure 3.5 portrays the stability of the price process indicating that convergence of prices is obtained (numerically) within less than 500 periods for arbitrary initial prices. Figures 3.6 and 3.7 show quite distinct phases of the orbit of the random fixed point.

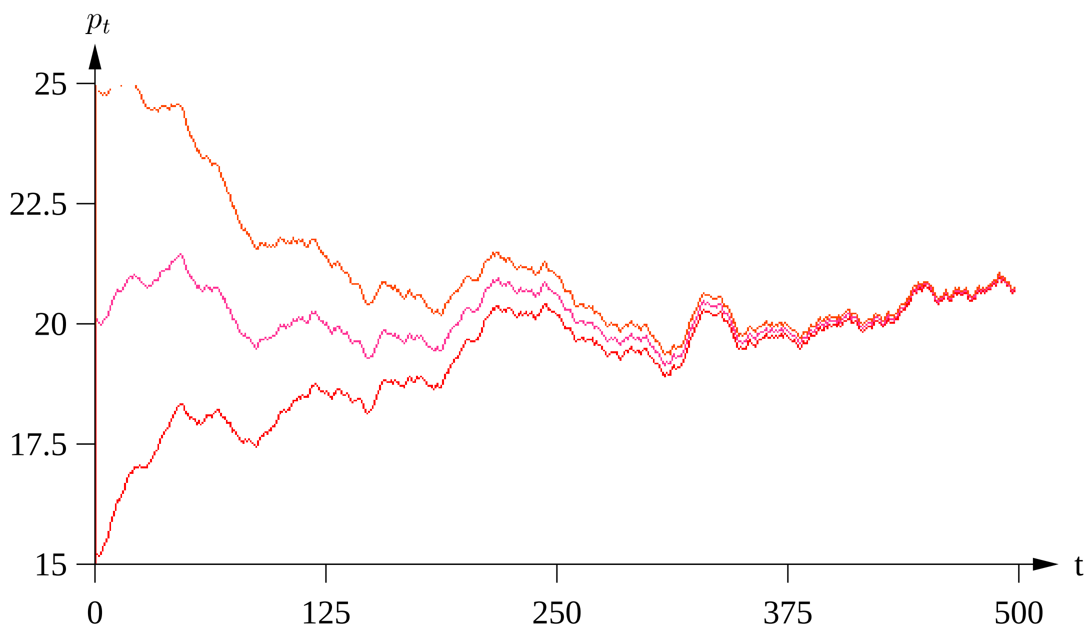

Figure 3.5: Convergence of prices under unbiased prediction: AR(1) dividends

Figures 3.8 – 3.10 show some of the long run statistical properties of the fixed point. The cyclical movement of the recursive deviation in Figure 3.9 is a clear indication of volatility clustering of prices, in spite of the fact that the long run density of prices, calculated on the basis of 10^6 periods, is symmetric.

The following three diagrams (Figures 3.11 – 3.13) show the performance of the unbiased predictor. They depict the sample path, recursive mean and standard deviation, and the density of the forecasting errors $e_t := p_t - \mu_{t-1}$ which is approximately uniform, an indication of the fact that errors are uncorrelated. Notice, while the unbiased predictor induces a zero conditional forecast error at any point in time, the diagrams show that the long run average error is strictly positive.

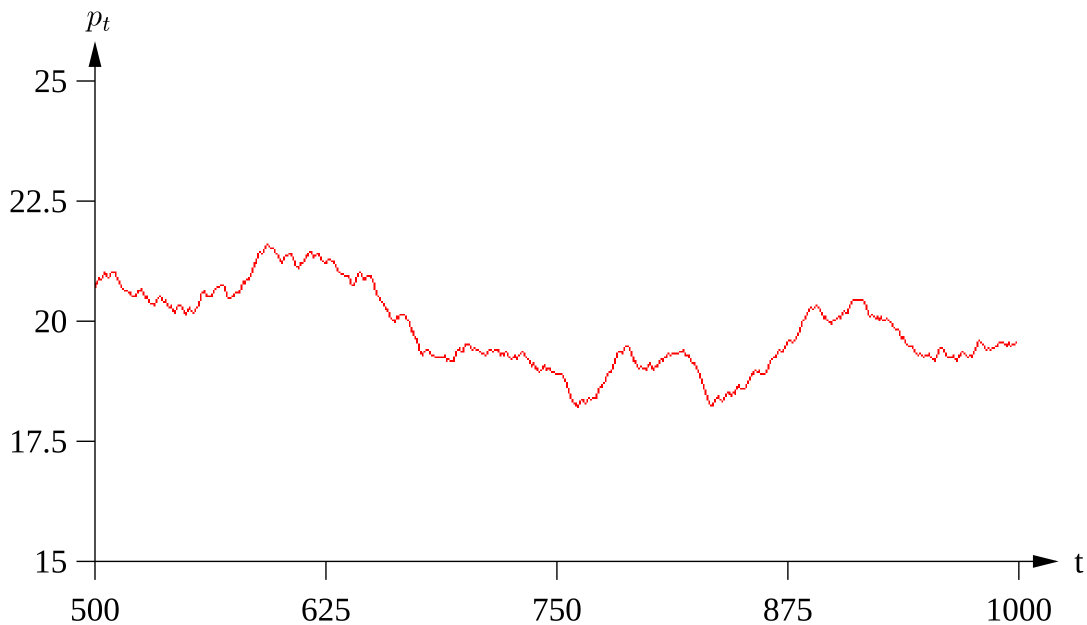

Figure 3.6: Prices with AR(1) dividends

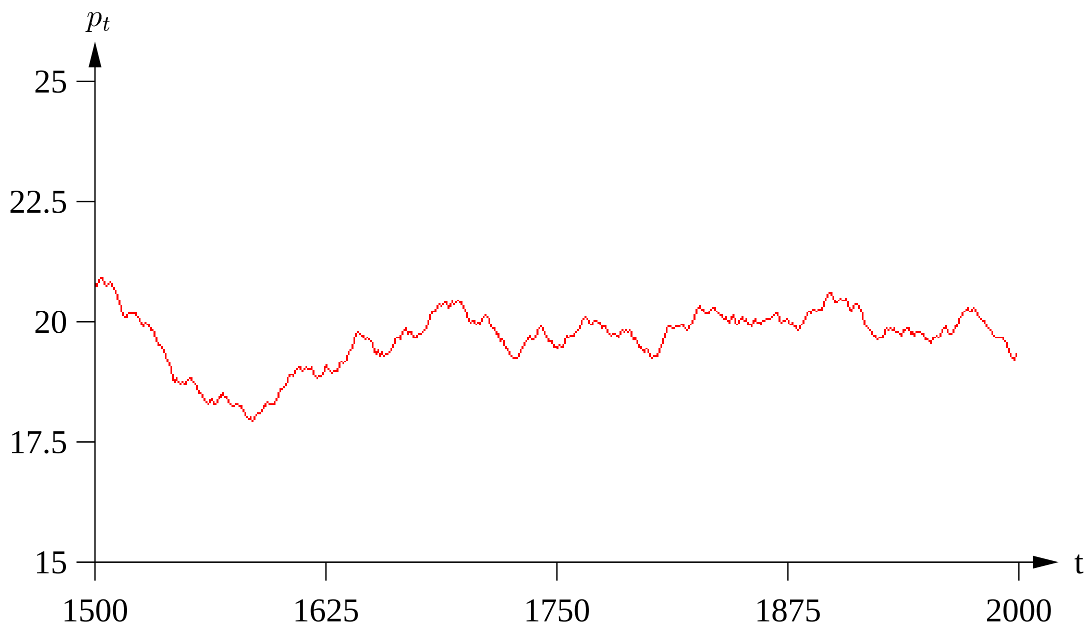

Figure 3.7: prices with AR(1) dividends

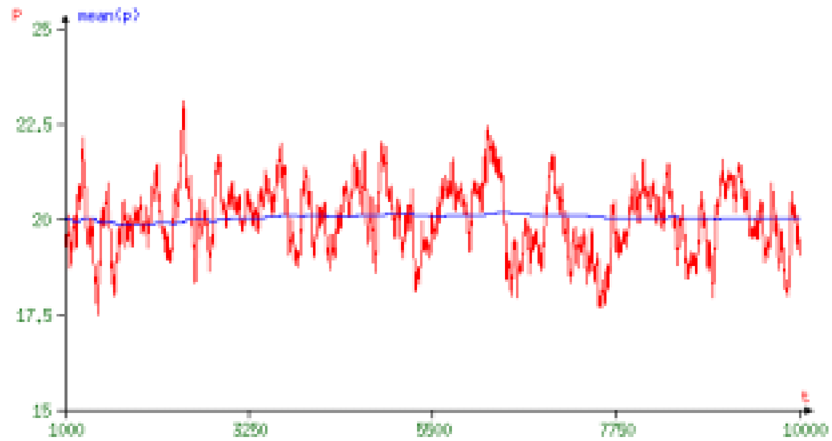

Figure 3.8: Prices and recursive mean with AR(1) dividends

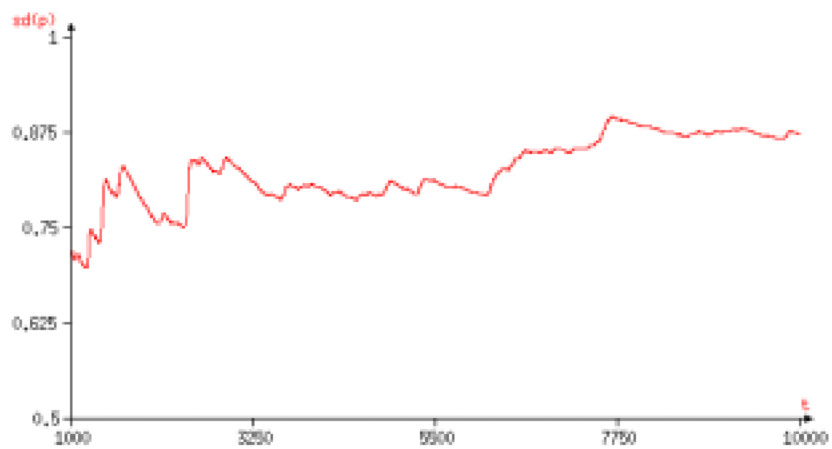

Figure 3.9: Recursive standard deviation of prices with AR(1) dividends

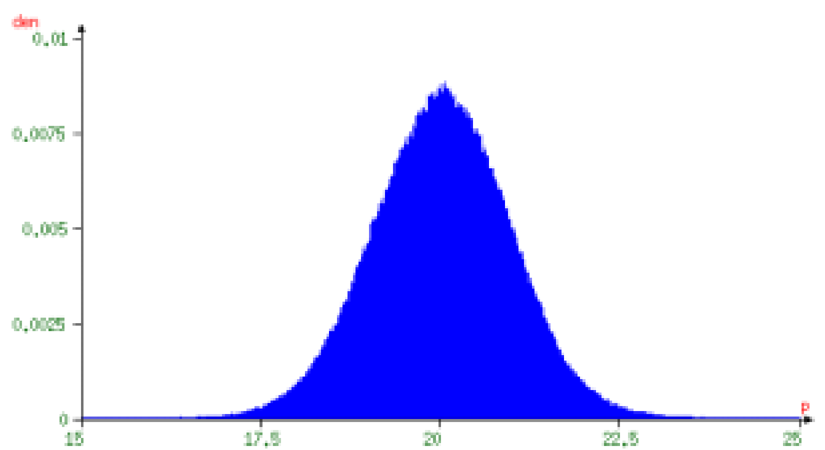

Figure 3.10: mean=20.0064; variance=0.919327; sd=0.958815; $T = 10^6$

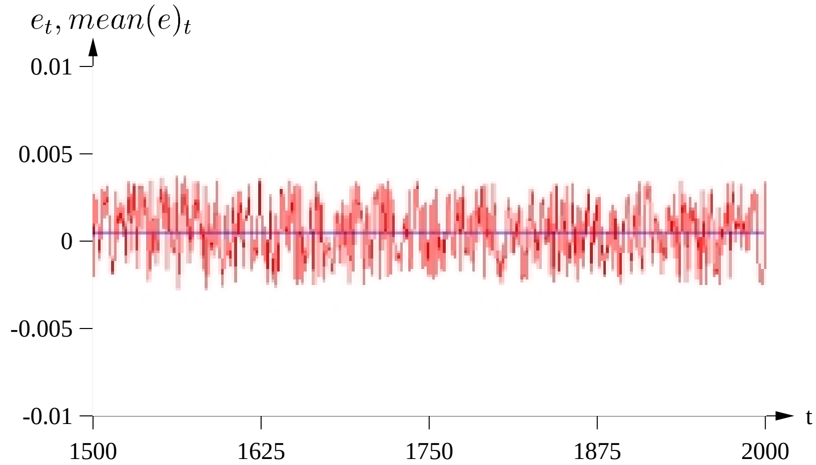

Figure 3.11: Forecast error and recursive mean with AR(1) dividends

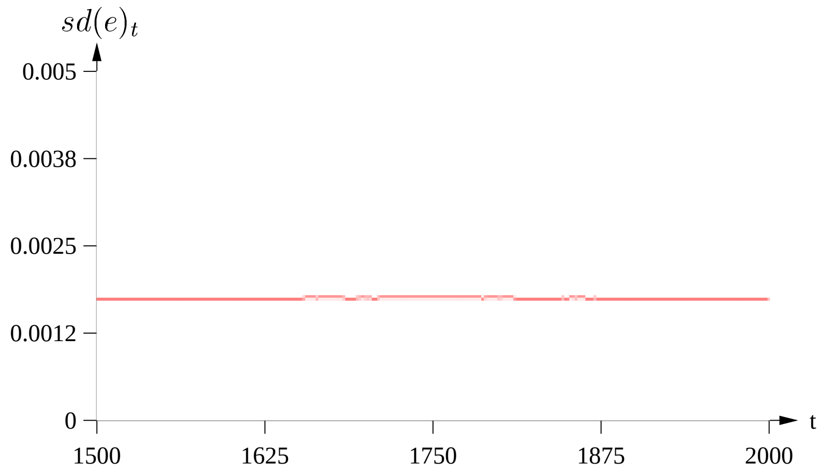

Figure 3.12: Recursive standard deviation of error with AR(1) dividends

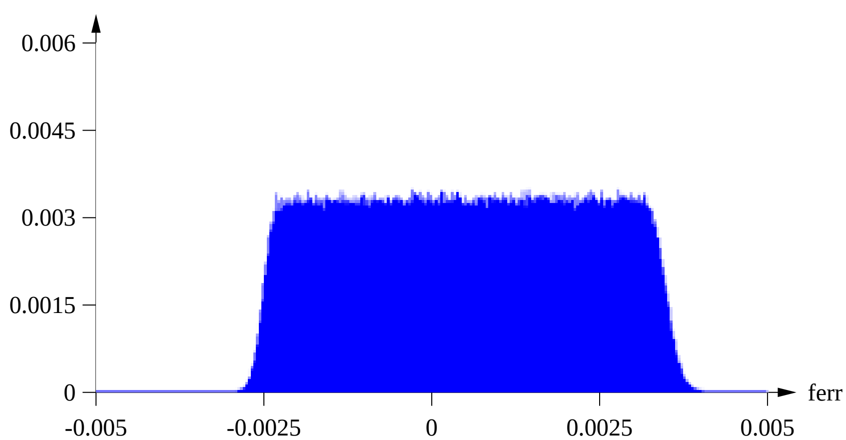

Figure 3.13: mean=0.000503457; variance=3.07008e-06; sd =0.00175216; $T = 10^6$

The remaining diagrams are designed to provide a characterization of the volatility of prices (Figures 3.14 – 3.16) and of the predictions (3.17 – 3.19). They show the sample paths of first differences of prices $(\Delta p)_t := p_t - p_{t-1}$ and of predictions $(\Delta \mu)_t := \mu_t - \mu_{t-1}$. Notice that prices fluctuate systematically more than predictions. This is due to the fact that the volatility of asset prices under unbiased predictions is made up of the volatility of the predictions and of the dividends themselves. This confirms the intuition that under unbiased predictions (i. e. when expected conditional forecast errors are zero) predictions of prices fluctuate less and deviate less from the long run mean than actual prices.

Finally, the diagrams Figures 3.20 through 3.22 show different time profiles of the return of the asset and its long run characteristics (mean and standard deviation).

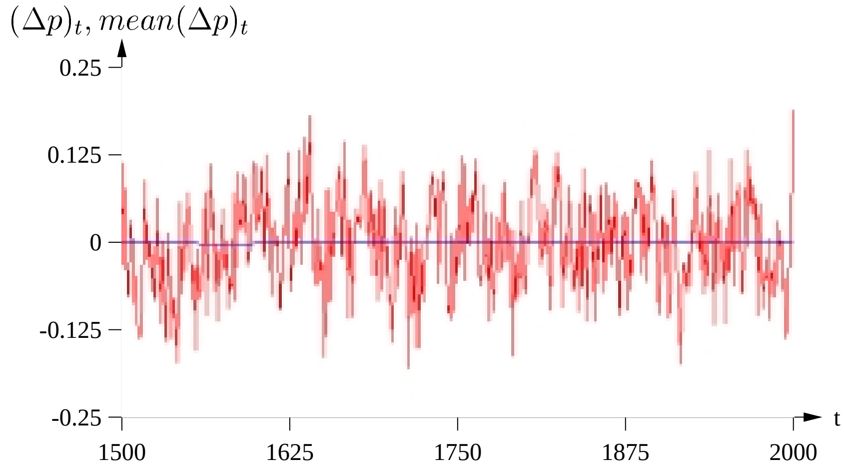

Figure 3.14: Volatility of prices and recursive mean with AR(1) dividends

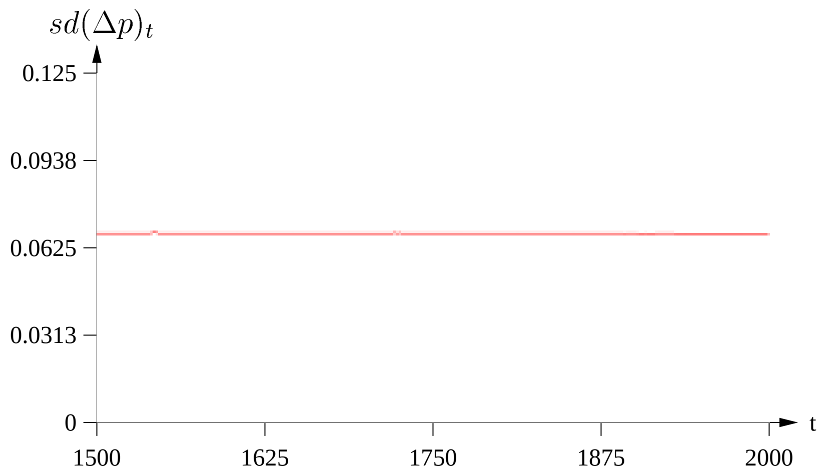

Figure 3.15: Recursive standard deviation of Δp with AR(1) dividends

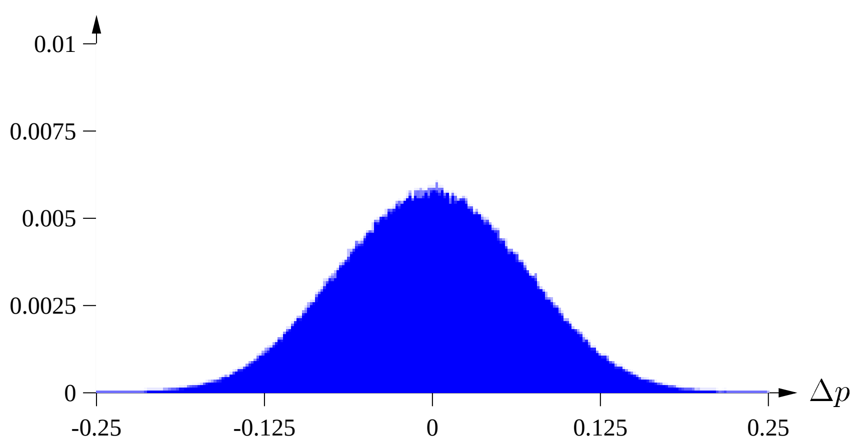

Figure 3.16: Δp: mean=-2.65849e-07; variance=0.00449621; sd =0.0670537; $T = 10^6$

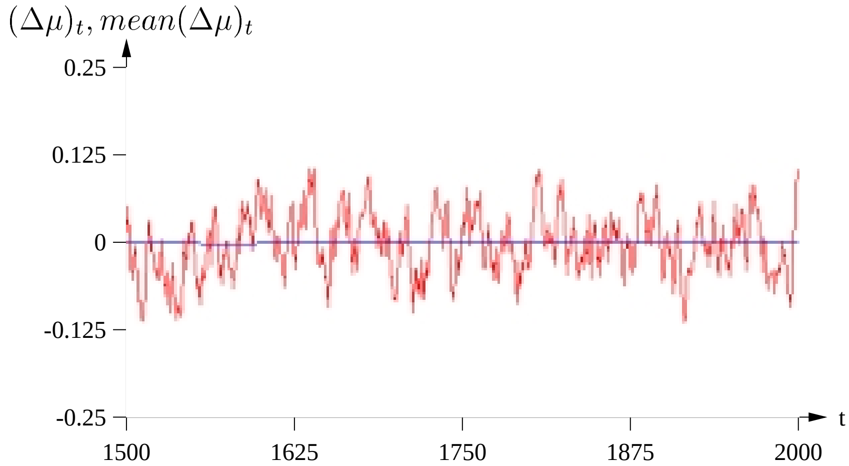

Figure 3.17: Volatility of predictions and recursive mean

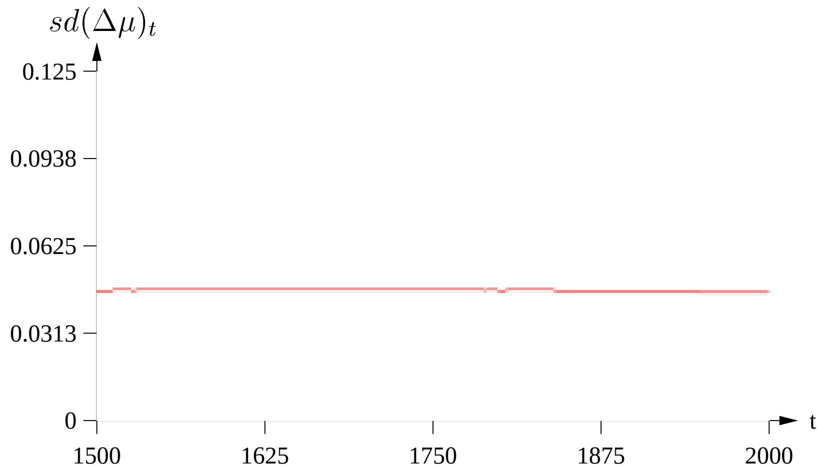

Figure 3.18: Recursive standard deviation of $\Delta\mu$

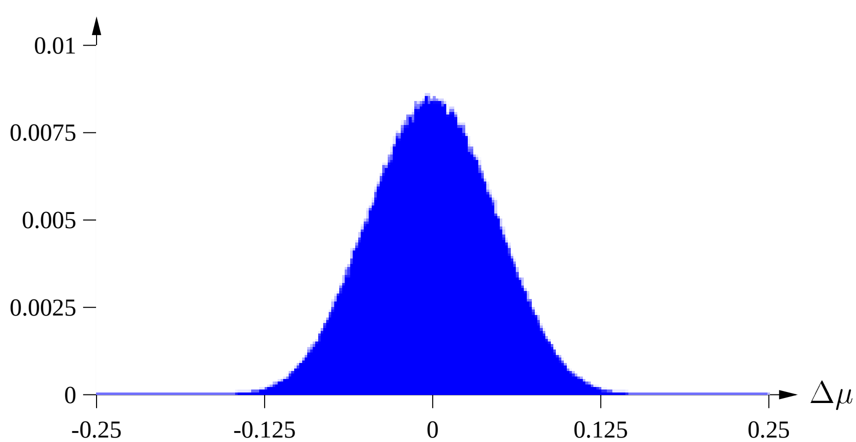

Figure 3.19: $\Delta\mu$:mean=-3.3544e-07; variance=0.00205272; sd =0.0453069; $T = 10^6$

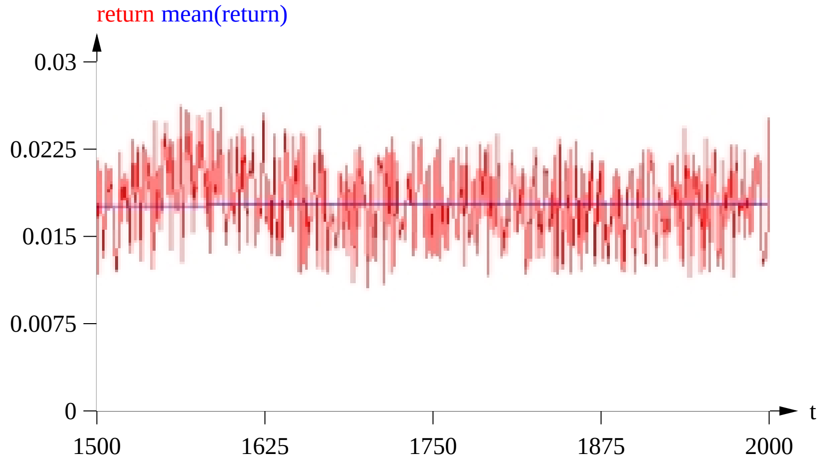

Figure 3.20: Returns and recursive mean with AR(1) dividends

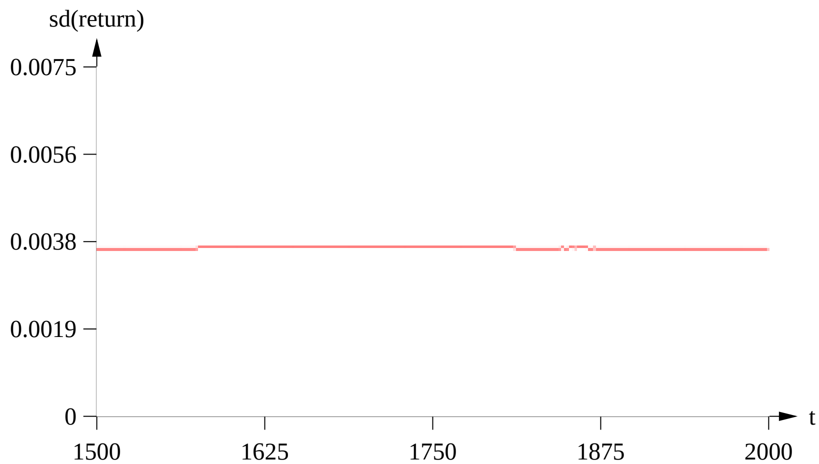

Figure 3.21: Recursive standard deviation of returns with AR(1) dividends

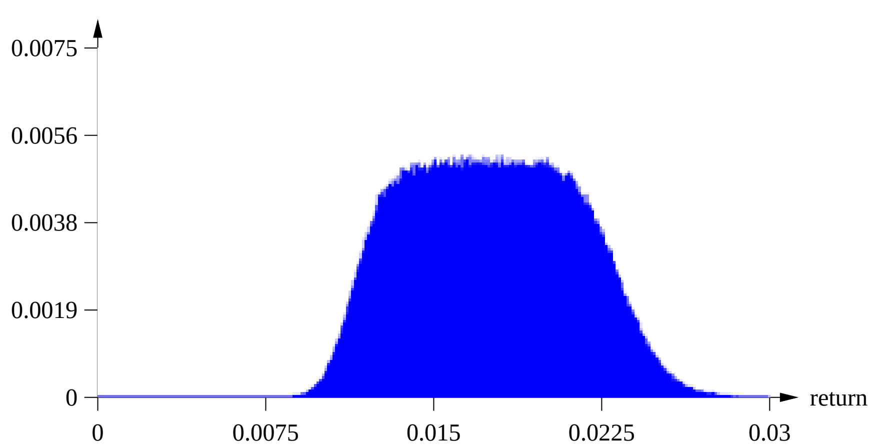

Figure 3.22: mean=0.017549; variance=1.38042e-05; sd =0.00371539; $T = 10^6$

4 The dynamics with adaptive expectations formation and learning

This last section is designed to explore the dynamical properties of the asset price process for some commonly used predictors, portraying some typical stochastic features which one might expect, but also demonstrating the importance of the expectations feedback when some standard and commonly used predictors are employed. The exercise is also partly of an expository nature designed to exhibit the power of numerical and analytical possibilities once an explicit sequential model of asset price formation is obtained. The three cases analyzed numerically are those of (1) naive error correction, (2) unweighted averaging, and (2) so called OLS learning.

4.1 Naive expectations formation and averaging

Recall the price law with mean–variance preferences of the CARA type from equation (2.26) when all types make the same mean prediction μ,

$$p = \frac{1}{R}\left[d + \mu - \left(\sum_a \tau^a\right)^{-1} \bar{x}\right], \tag{4.1}$$

and consider first a predictor employing a simple error correction principle

$$\mu_t = p_{t-1} + \alpha(\mu_{t-2} - p_{t-1}) \tag{4.2}$$

with $0 \le \alpha \le 1$. Notice that this formulation includes the two special cases of constant predictions ($\alpha = 1$, i. e. $\mu_t = \mu_{t-2} = \mu_0$) and of naive prediction ($\alpha = 0$, i. e. $\mu_t = p_{t-1}$). Combining this with the price law (4.1) yields the random dynamical system (an affine stochastic delay system of order 2)

$$p_t = \frac{1}{R}\left[D_t(\cdot) + p_{t-1} + \alpha(\mu_{t-2} - p_{t-1}) - \left(\sum_a \tau^a\right)^{-1} \bar{x}\right] \tag{4.3}$$

$$\mu_t = p_{t-1} + \alpha(\mu_{t-2} - p_{t-1}) \tag{4.4}$$

with additive noise. Hence, the results on the existence of a unique globally attracting random fixed point apply if the deterministic part of the mapping is a contraction. For the situation with one asset only, it is straightforward to show that the deterministic mapping of the system 4.3 and 4.3 has three distinct real roots, $-1 < \lambda_1 < \lambda_2 = 0 < \lambda_3$. One finds that $\lambda_3 < 1$ if and only if

$$\sqrt{(1-\alpha)^2 + 4\alpha R^2} < 2R - (1 - \alpha).$$

This requires $R > 1$ if $0 < \alpha < 1$. In order to keep the numerical results with adaptive expectations formation comparable with the results under unbiased predictions, all other parameters of the model are kept at the same levels as in the unbiased case. The following simulations take $\alpha = .1$ and $R = 1.1$ while maintaining the value $\phi(\bar{x}) = \left(\sum_a \tau^a\right)^{-1} \bar{x} = .55$ and the parameters of the AR(1) dividend process.

Figures 4.1 – 4.3 display the stable sample paths of prices, their recursive means and standard deviations and the histogram, followed by the same set of diagrams for the return and the forecasting error. Notice that prices fluctuate substantially around a negative mean with a slowly but not monotonically converging recursive mean and non monotonic recursive standard deviation. This indicates that the orbit of the fixed point displays distinct phases of volatility clustering. The negativity of prices is caused by the shift parameter $\phi(\bar{x})$ and by $R > 1$. There exists a random fixed point with $0 < R < 1$ which is unstable. Notice further that individual portfolios will not be constant along the fixed point.

One also observes that the returns are permanently negative. Prediction errors fluctuate substantially with a long run negative mean, indicating that the error adjustment rule underpredicts more often than not. This seems to be one of the causes why returns are negative, in spite of the fact that the safe rate is positive. Thus, the prediction performance and the portfolio performance of the error correction mechanism are disastrous taking into account that *ceteris paribus* each generation could have obtained a positive return by investing in the safe asset alone.

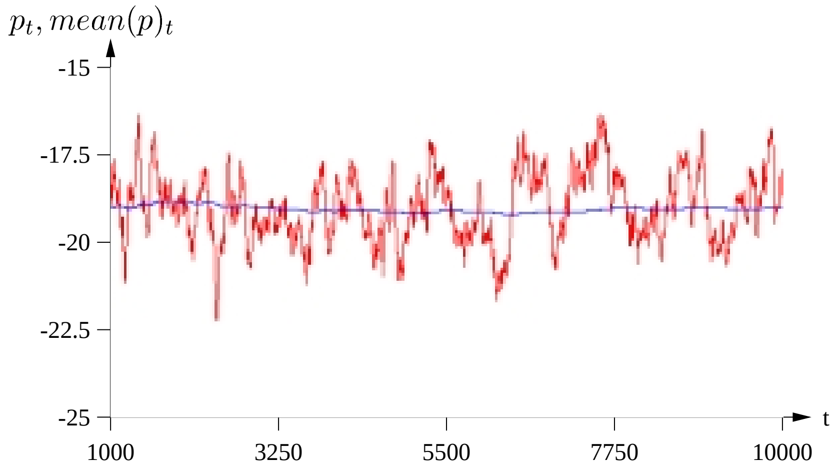

Figure 4.1: naive prices 1000–10000

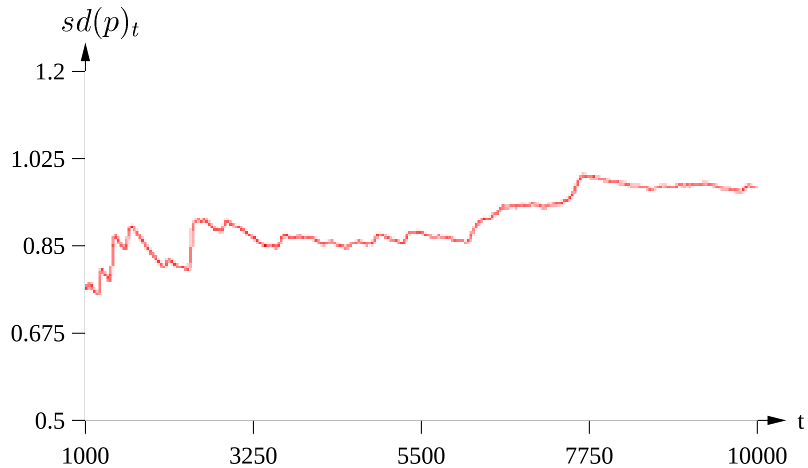

Figure 4.2: sd naive prices 1000–10000

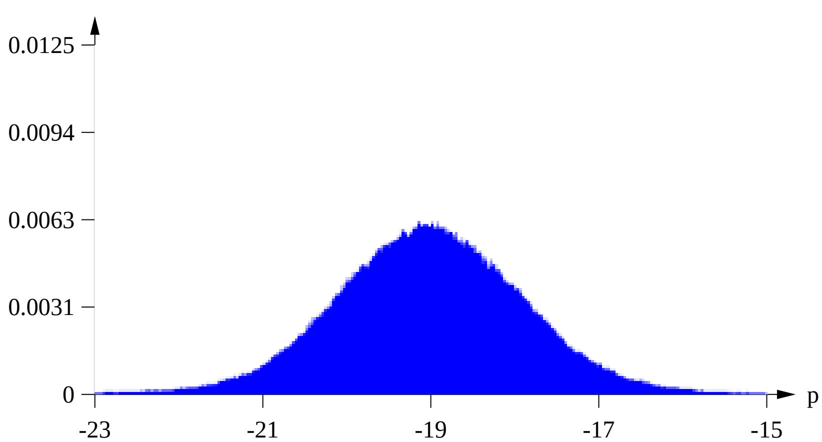

Figure 4.3: naive prices: mean= -18.9954; variance=1.17046; sd=1.08188; $T = 10^6$

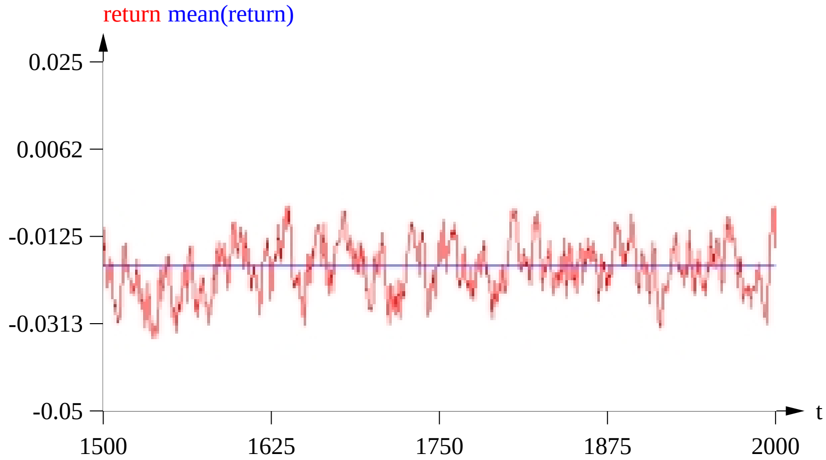

Figure 4.4: naive returns 1500–2000

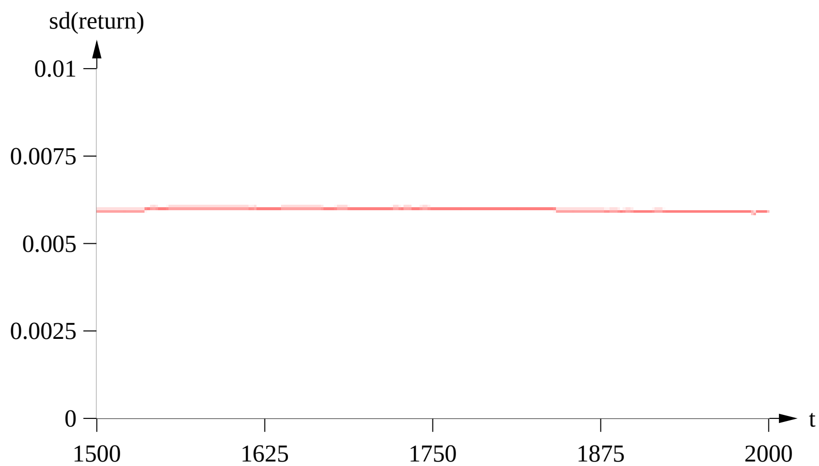

Figure 4.5: sd naive returns 1500–2000

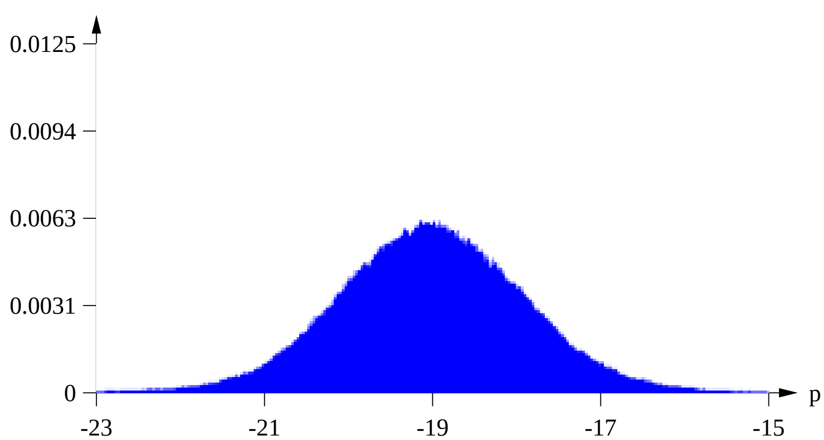

Figure 4.6: naive returns: mean=-0.0185303; var=3.42112e-05; sd=0.00584904; $T = 10^6$

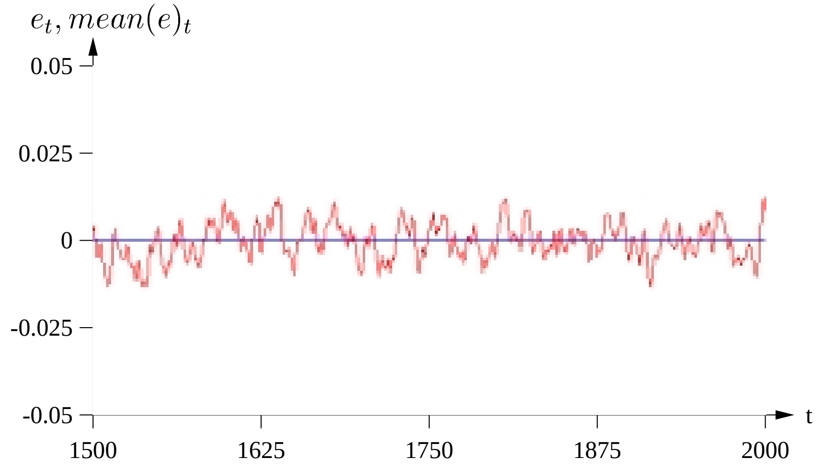

Figure 4.7: naive errors 1500–2000

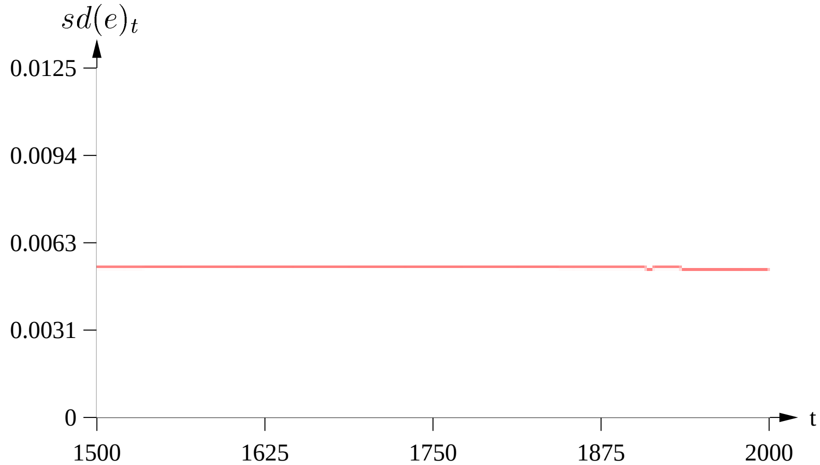

Figure 4.8: sd naive errors 1500–2000

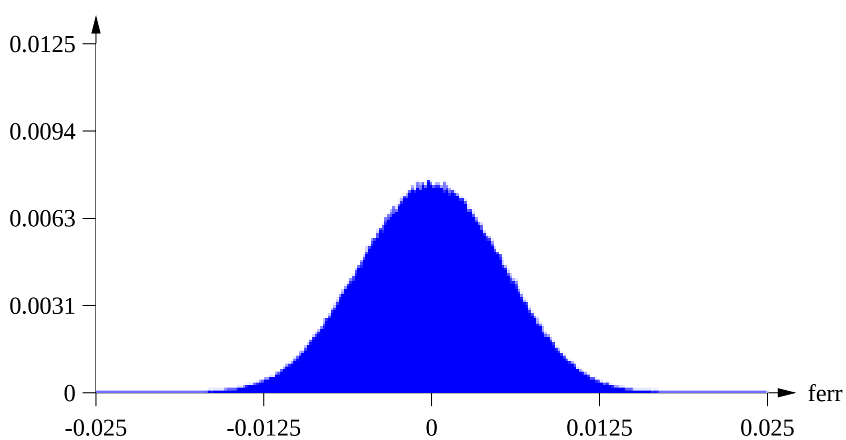

Figure 4.9: naive errors: mean=-1.47409e-05; var=2.66943e-05; sd=0.00516665; $T = 10^6$

The second forecasting scheme assumes that all agents choose the average of past prices with a memory of length $L \geq 1$ and some trend parameter $\beta \geq 1$ to be the expected future price. Hence, the predicted mean price μ is defined as

$$\mu_t = \frac{\beta}{L}\sum_{i=1}^{L} p_{t-i}. \tag{4.5}$$

Substituting (4.5) into (4.1) yields the random dynamical system for prices (an affine stochastic delay equation of order L) with additive noise

$$p_t = \frac{1}{R}\left[D_t(\cdot) + \frac{\beta}{L}\sum_{i=1}^{L} p_{t-i} - \left(\sum_a \tau^a\right)^{-1} \bar{x}\right]. \tag{4.6}$$

It is well known that stability of the random fixed fixed point requires

$$-1 < \frac{\beta}{R} < L, \tag{4.7}$$

Two numerical experiments with a short lag $L = 2$ and a long one $L = 20$ are carried out with $\beta = 1$ and $R = 1.01$ while keeping the other parameters the same including the AR(1) dividend process. As in the naive case, the unique random fixed point for $0 < R < 1$ is unstable.

General economic folklore developed from deterministic systems tends to support the view that higher memory in predictions tends to reduce fluctuations if mappings are linear. However, it is not known whether this is a monotonic relation in general[8].and whether this dampening occurs in stochastic systems with a decisive expectations feedback. The effects of different memory in the model here are not clear and are not easily established. The numerical results go against that intuition for the price orbit: while for $L = 2$, prices fluctuate mildly with no volatility clustering (Figure 4.10) the orbit with the longer lag shows clear clustering (cf. Figures 4.19 and 4.20). The diagrams show no clustering for the returns which are negative for the entire orbit (cif. Figures 4.13 – 4.15 and 4.22 – 4.24). The means for the two lags are essentially the same with a slightly smaller variance for the longer memory. Thus, on average both lags perform equally poorly relative to the unbiased and to the safe rate. The prediction errors are systematically smaller with smaller variance for the longer lag, while both have negative means.

[8]For many nonlinear deterministic models of the cobweb type, simple (or more complex) averaging rules may generate stable as well as complex behavior, but where longer memory does not always stabilize (cf. for example Chiarella & Khomin (1996), Balasko & Royer (1996), Stiefenhofer (1998, 1999)).

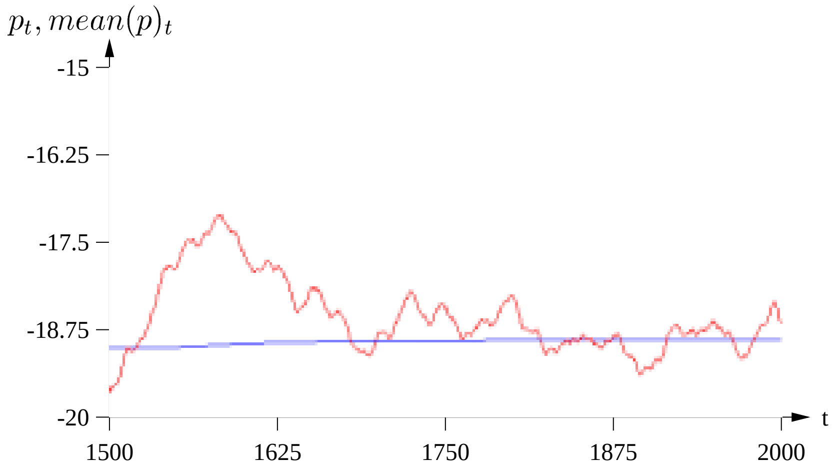

Figure 4.10: MA L2 prices 1500–2000

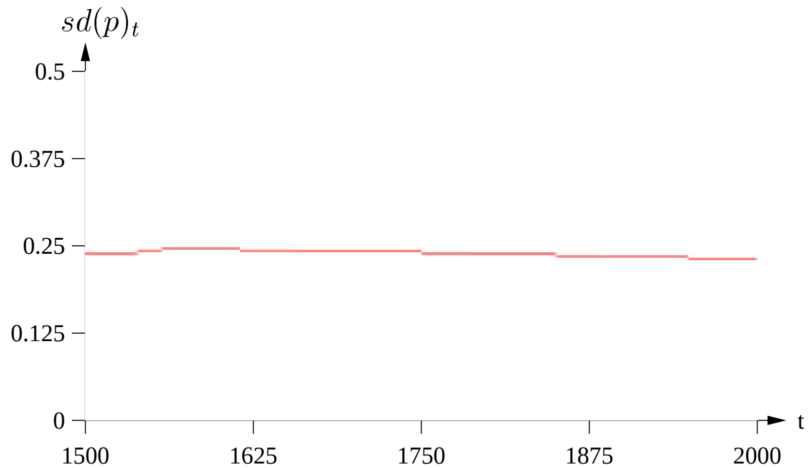

Figure 4.11: MA L2 prices 1500–1000

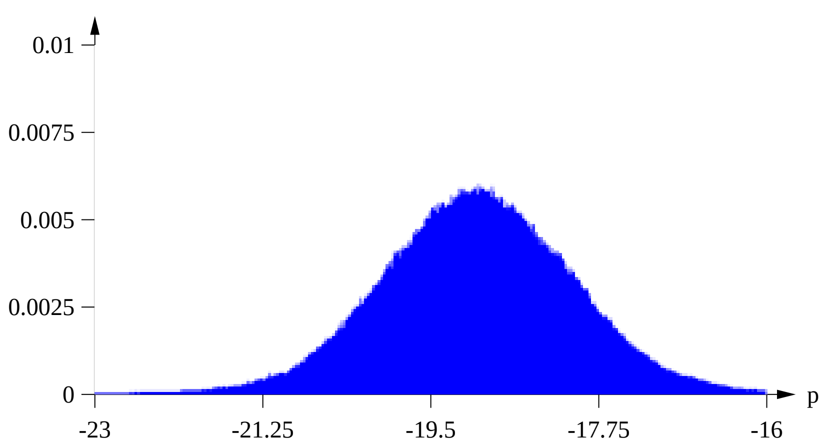

Figure 4.12: MA L2 prices: mean=-18.9956; variance=0.959744; sd=0.979665; $T = 10^6$

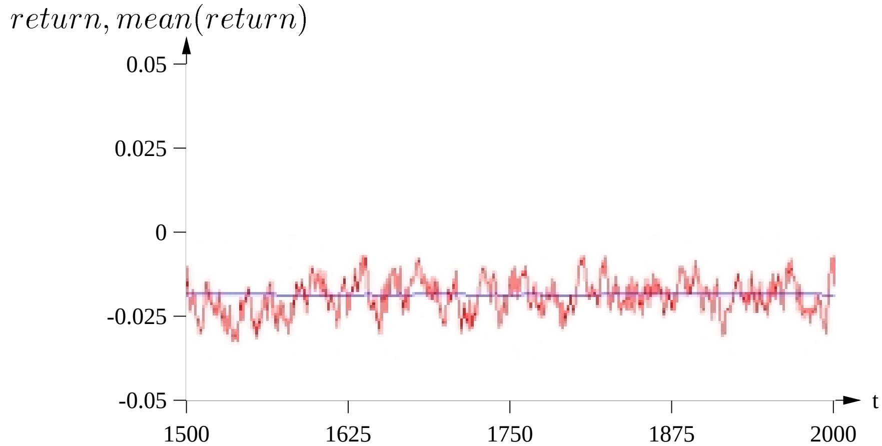

Figure 4.13: MA L2 returns 1500–2000

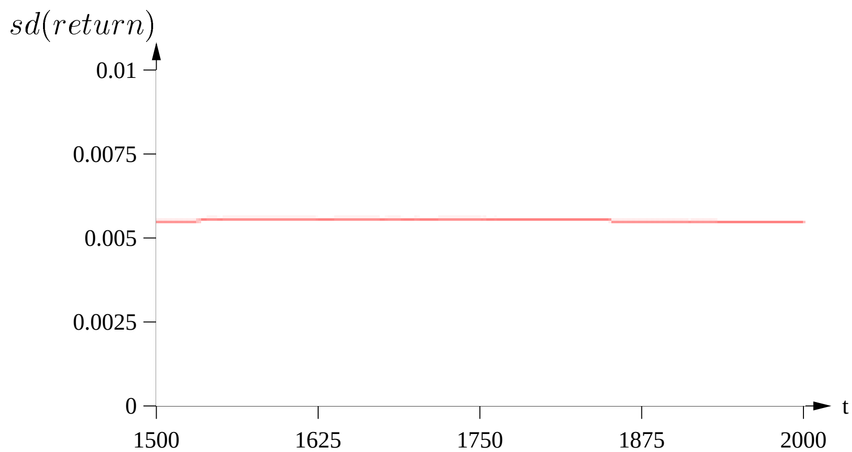

Figure 4.14: MA L2 returns 1500–2000

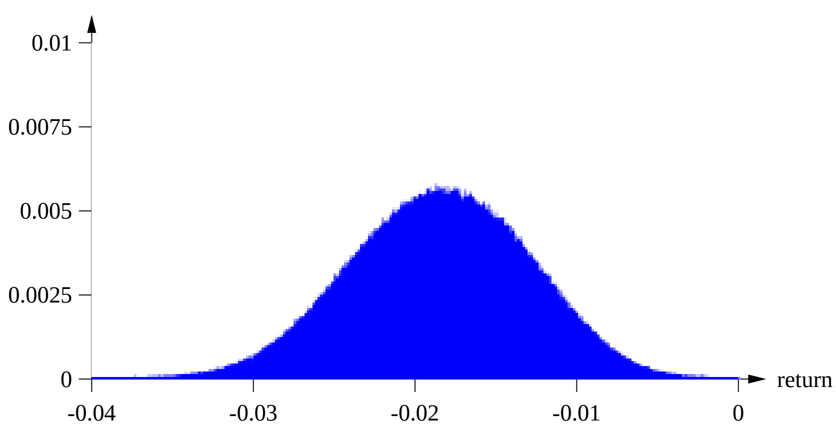

Figure 4.15: MA L2 returns: mean=-0.0185119; var=2.93938e-05; sd=5.42161e-03; $T = 10^6$

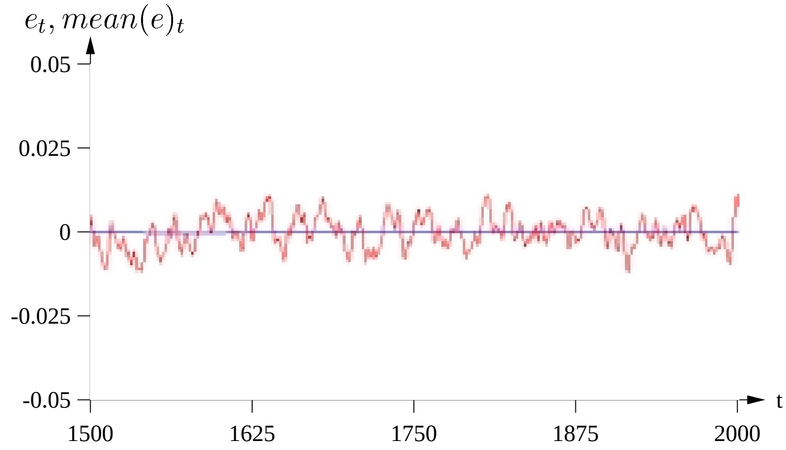

Figure 4.16: MA L2 error 1500–2000

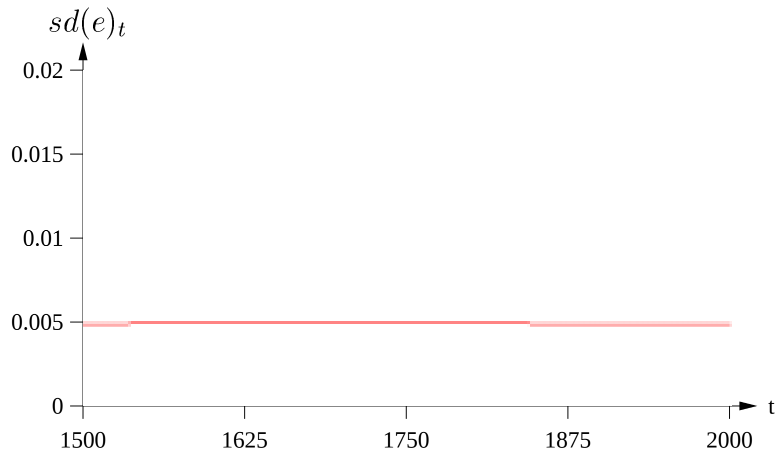

Figure 4.17: MA L2 error 1500–2000

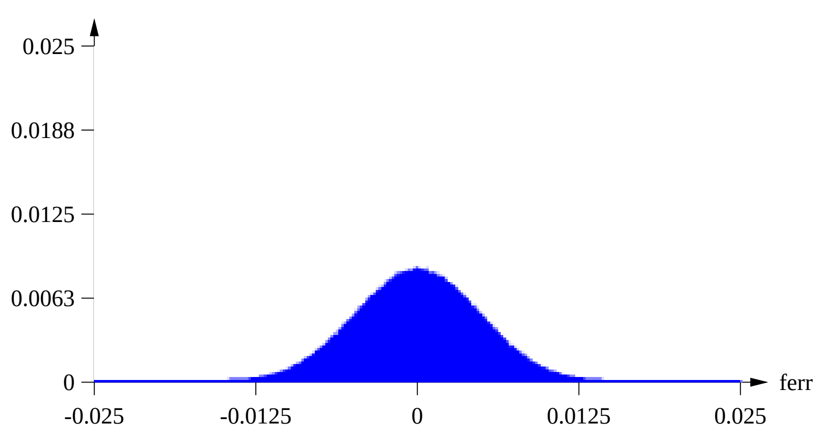

Figure 4.18: MA L2 error: mean=-0.0185118; variance=2.93936e-05; sd=0.00542158$T = 10^6$

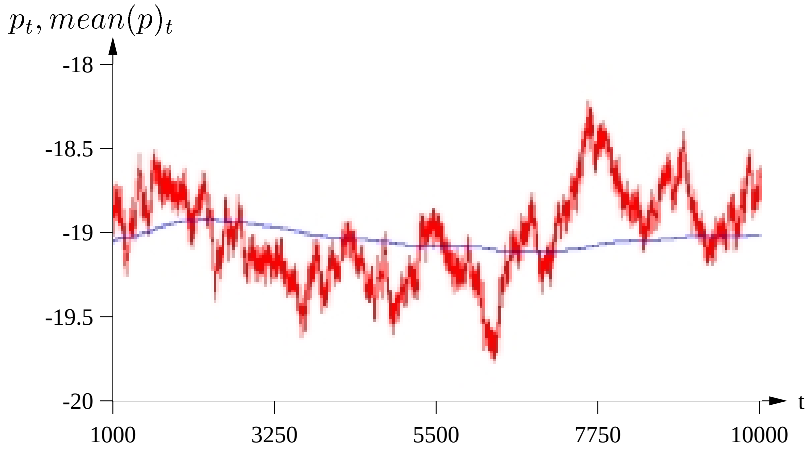

Figure 4.19: MA L20 prices 1000–10000

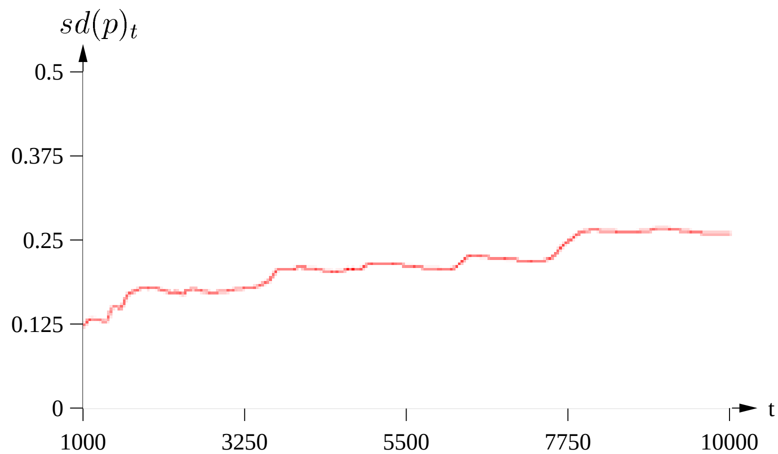

Figure 4.20: sd MA L20 prices 1000–10000

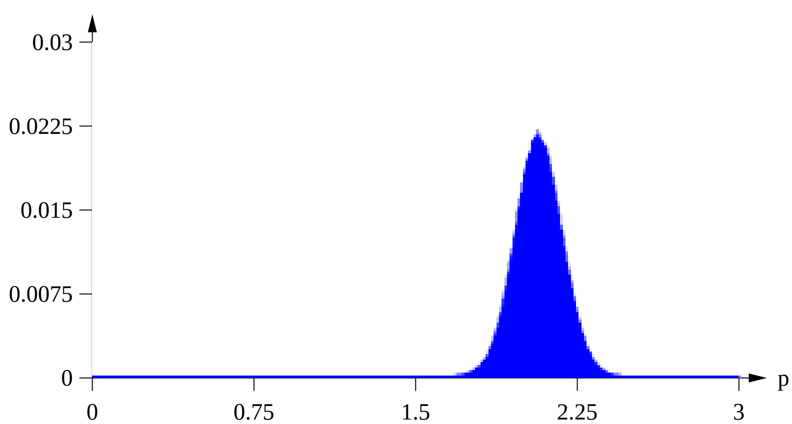

Figure 4.21: MA L20 prices: mean=2.07182; var=0.0121042 sd=0.110019; $T = 10^6$

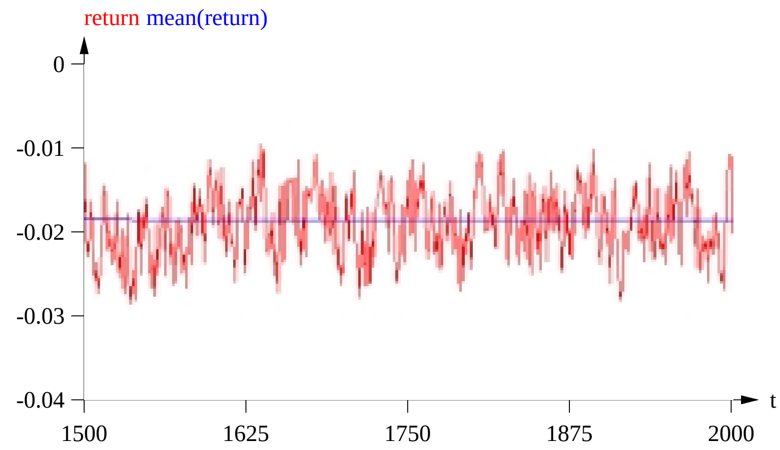

Figure 4.22: MA L20 returns 1500–2000

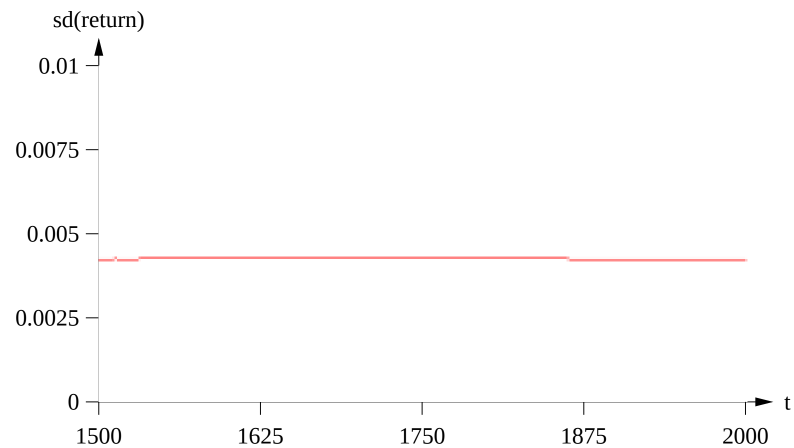

Figure 4.23: MA L20 returns 1500–2000

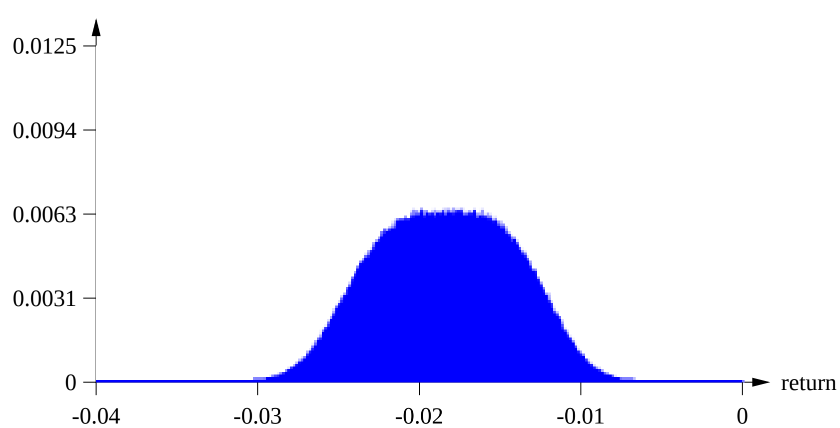

Figure 4.24: MA L20 returns: mean=-0.0184478; variance=1.78104e-05; sd=0.00422024 $T = 10^6$

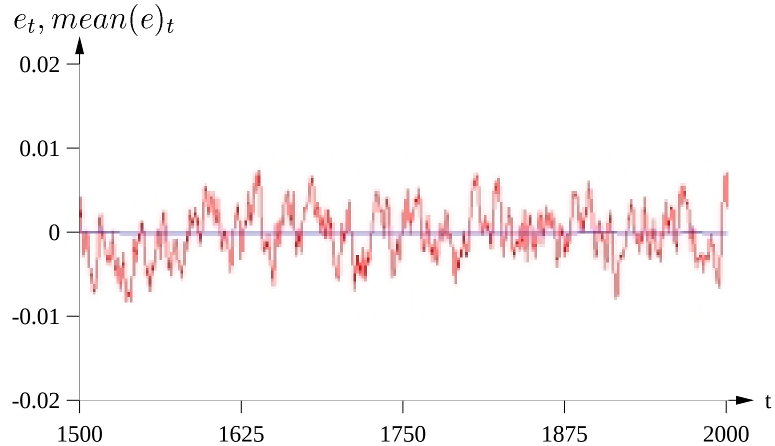

Figure 4.25: MA L20 errors 1500–2000

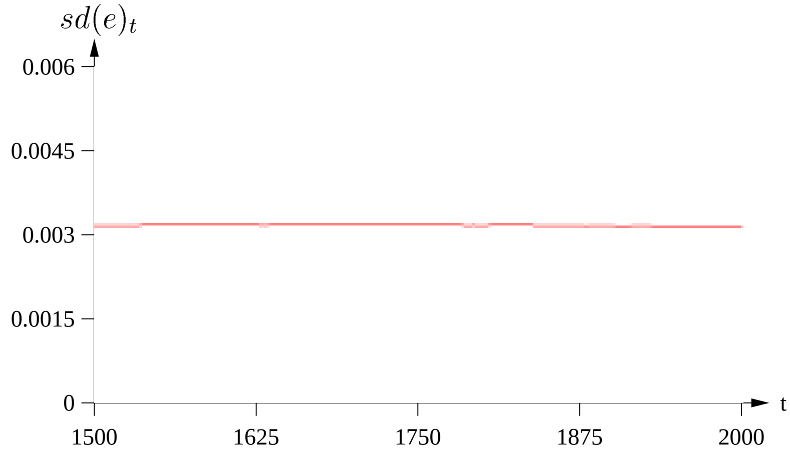

Figure 4.26: sd MA L20 errors 1500–2000

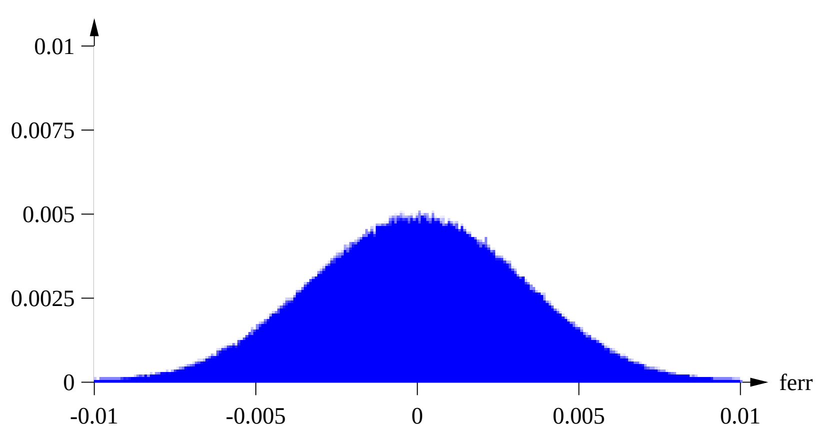

Figure 4.27: MA L20 errors: mean=-7.97631e-06; var=9.96226e-06; sd=0.00315631 $T = 10^6$

4.2 Recursive Least Squares Predictions

One of the most widely used expectations formation principles is associated with ordinary least squares estimates (OLS), often referred to as OLS learning dynamics (see for example the recent survey Evans & Honkapohja (1999) and references therein). The underlying assumption of this procedure is that the observed time series is generated by a deterministic linear model with an additive noise process and specific statistical properties. Then, forecasts are made using the updated OLS estimate of the unknown linear mapping (matrix). Under relatively weak additional assumptions predictions using the OLS estimate are unbiased and have acceptable and reliable statistical properties if the true model is linear and without an expectations feedback.

To formulate the associated OLS estimation procedure for the model here, let

$$x_{n+1} = \Theta x_n + w_n(\cdot) \tag{4.8}$$

describe the perceived linear model of the asset price process of the forecasting agent, where $\Theta \in \mathbb{R}^K \times \mathbb{R}^K$ is a $K \times K$ matrix and $w_n(\cdot)$ an i. i. d. process of perturbations. Let $x_j \in \mathbb{R}^K$, $j = 1, \ldots, n$ denote the list of observed prices, then the n-th OLS estimate $\hat{\Theta}_n$ for Θ is defined as

$$\hat{\Theta}_n := \arg\min_{\Theta} \sum_{j=0}^{n-1} ||x_{j+1} - \Theta x_j|| \tag{4.9}$$

which yields the well known solution

$$\hat{\Theta}_n = \left(\sum_{j=0}^{n-1} x_j x_j^T \right)^+ \sum_{j=0}^{n-1} x_j x_{j+1}^T, \tag{4.10}$$

where the matrix $X_n := \left(\sum_{j=0}^{n-1} x_j x_j^T \right)^+$ is the so called Moore–Penrose generalized inverse. Following Chen & Guo (1991) one obtains the recursive least squares procedure RLS for the OLS estimate as

$$\begin{aligned}
\hat{\Theta}_{n+1} &= \hat{\Theta}_n + \lambda_n X_n \left(x_n (x_{n+1} - \hat{\Theta}_n x_n)^T \right) && (4.11)\\
\lambda_n &= (1 + x_n^T X_n x_n)^{-1} && (4.12)\\
X_{n+1} &= X_n + \lambda_n X_n (x_n x_n^T) X_n && (4.13)
\end{aligned}$$

with $X_0 = \lambda Id$, $0 < \lambda < (1/e)$, and $\hat{\Theta}_0$ arbitrary. Then, at any stage n the one step ahead forecast

$$x_{n+1}^e = \hat{\Theta}_n x_n \tag{4.14}$$

is unbiased since $\hat{\Theta}_n$ is a best linear unbiased estimator if the system is linear. As a consequence, the two step ahead unbiased forecast would be $\hat{\Theta}_n^2 x_n$.

Applying the RLS procedure to the asset market model here, (which has an expectational lead!), the associated predictor for the mean price uses equations (4.11) - (4.13) to predict μ_{t+1} in period t on the basis of observed prices p_{t-1} and estimated $\hat{\Theta}_{t-1}$, i. e.

$$\mu_{t+1} = \hat{\Theta}^2_{t-1} p_{t-1} \tag{4.15}$$

Assume again, as in the examples before, that all agents use the same RLS forecast (which may be one given by some official forecasting institute!). Then, combining the RLS predictor with the price law (3.9) with identical mean predictions, one obtains the time–one map of a random dynamical system

$$F : \Omega \times \ \mathbb{R}^K \times (\mathbb{R}^K \times \mathbb{R}^K)^2 \longrightarrow \mathbb{R}^K \times (\mathbb{R}^K \times \mathbb{R}^K)^2$$

defined by

$$\begin{aligned}
(4.16) \qquad p_t &= \frac{1}{R}\left[D_t(\cdot) - \pi(\hat{\Theta}^2_{t-1} p_{t-1}, v)\right] \\
&= \frac{1}{R}\left[D_t(\cdot) + \hat{\Theta}^2_{t-1} p_{t-1} - \varphi^{-1}(\bar{x}, v)\right] \\
(4.17) \qquad \hat{\Theta}_t &= \hat{\Theta}_{t-1} + \lambda_{t-1} X_{t-1}\left(p_{t-1}(p_t - \hat{\Theta}_{t-1} p_{t-1})^T\right) \\
(4.18) \qquad X_t &= X_{t-1} + \lambda_{t-1} X_{t-1} (p_{t-1} p^T_{t-1}) X_{t-1}
\end{aligned}$$

where

$$\lambda_{t-1} = \frac{1}{(1 + p^T_{t-1} X_{t-1} p_{t-1})}. \tag{4.19}$$

These equations show quite clearly why, in general, the RLS forecasts cannot generate the statistical properties expected to prevail for the actual sample path. At first sight, equation (4.16) looks like a first order linear difference equation in prices with an additive noise term. However, the estimated coefficient $\hat{\Theta}^2_{t-1}$ is a highly non–linear function of prices which feeds back into the price law (3.9). Thus, this non–linear expectations feedback induces a so called ARMAX process, violating the underlying structural assumption of an ARMA process needed for the success of the OLS estimation procedure. Therefore, the direct application of the OLS learning approach to the asset price process in the ARMAX setting here in general cannot be expected to lead to successful predictions.

Many authors have used the OLS forecasting rule in non–linear deterministic cobweb environments (Marcet & Sargent (1989, 1989a), Bullard (1994), Schönhofer (1999, 1999a), and others). They have found numerous scenarios where the OLS procedure fails to predict efficiently. This is hardly surprising. Structurally one would expect that the OLS forecasting technique fails in situations with a decisive (non–linear) feedback since it is designed for systems without it. Moreover, the above contributions indicate that OLS learning often destabilizes the steady state and may generate almost any kind of deterministic dynamic behavior. Therefore, it would not be too surprising, if one finds in the stochastic environment here, that sample paths do not confirm the statistical properties typically claimed for RLS in an ARMA setting and/or that the system becomes unstable.

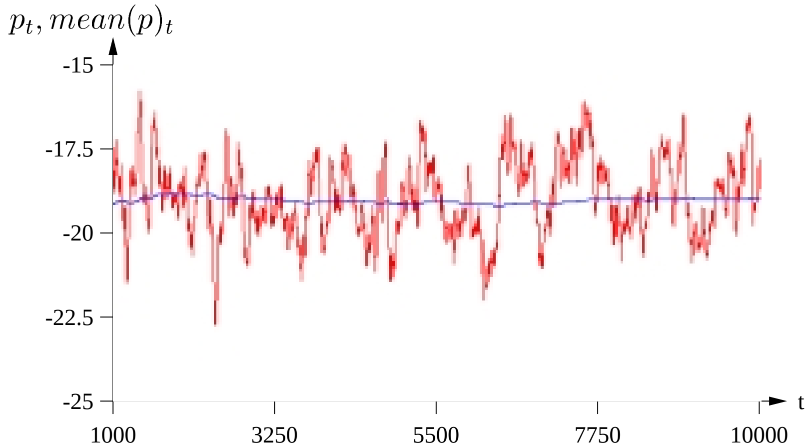

Figure 4.28: RLS prices 1000–10000

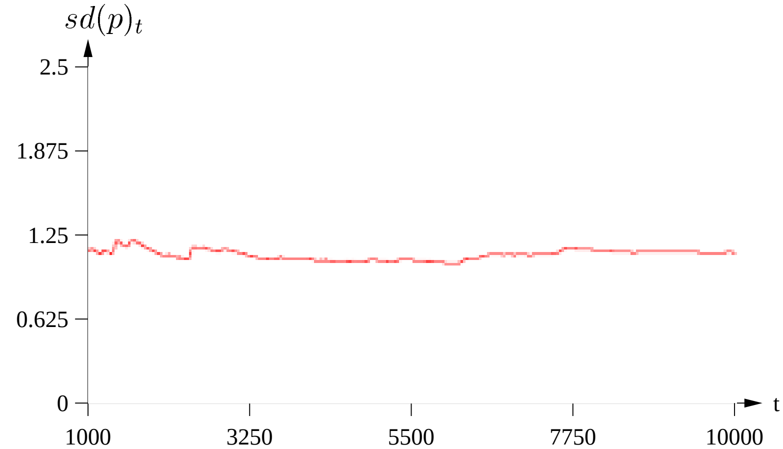

Figure 4.29: sd RLS prices 1000–10000

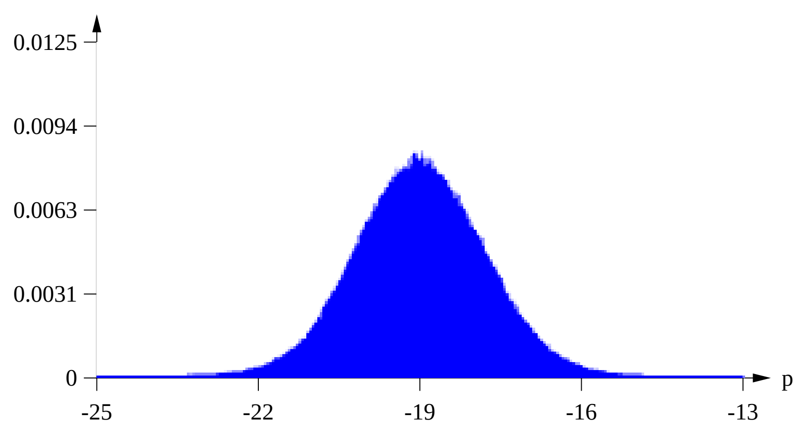

Figure 4.30: RLS prices: mean=-18.9866; variance=1.42164; sd=1.19233; $T = 10^6$

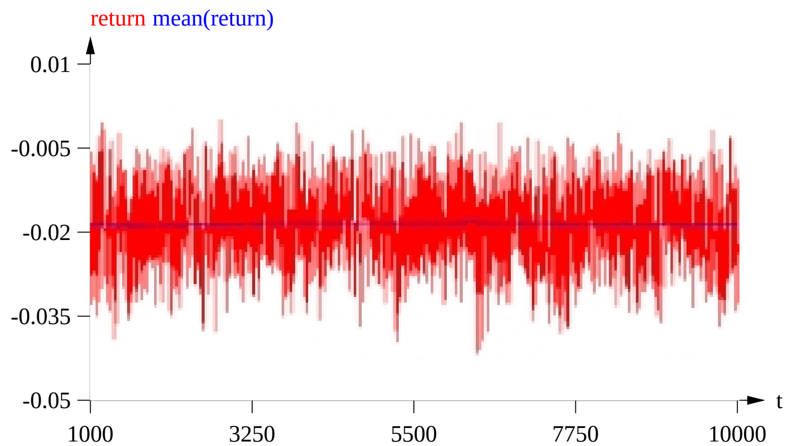

Figure 4.31: RLS returns 1000–10000

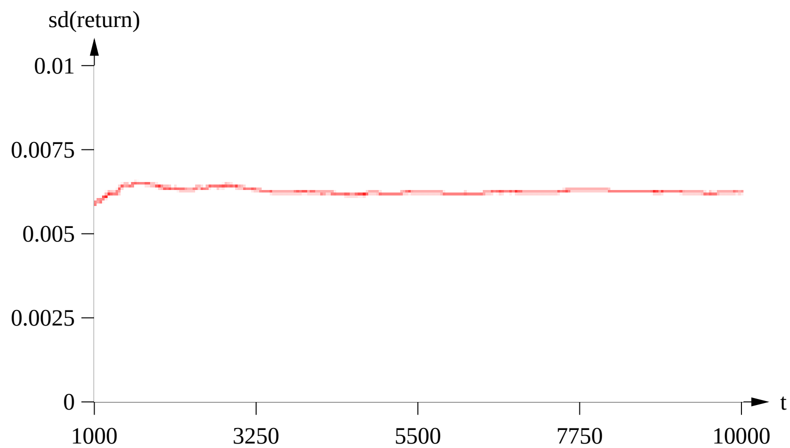

Figure 4.32: sd RLS returns 1000–10000

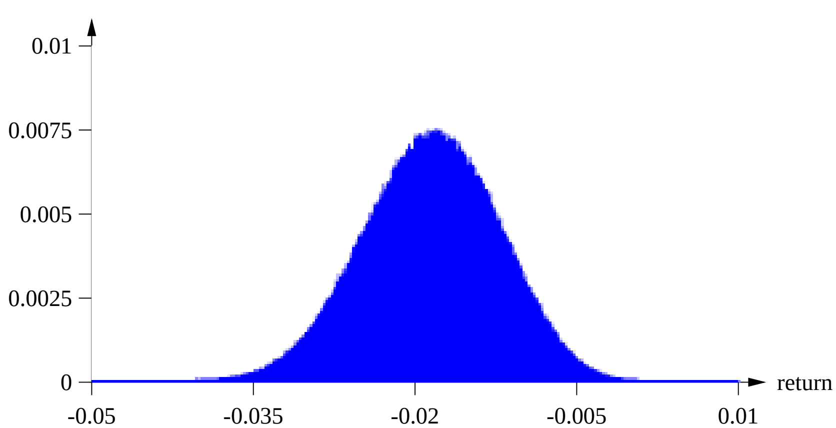

Figure 4.33: RLS returns: mean=-0.018559; variance=3.91099e-05; sd=0.00625379; $T = 10^6$

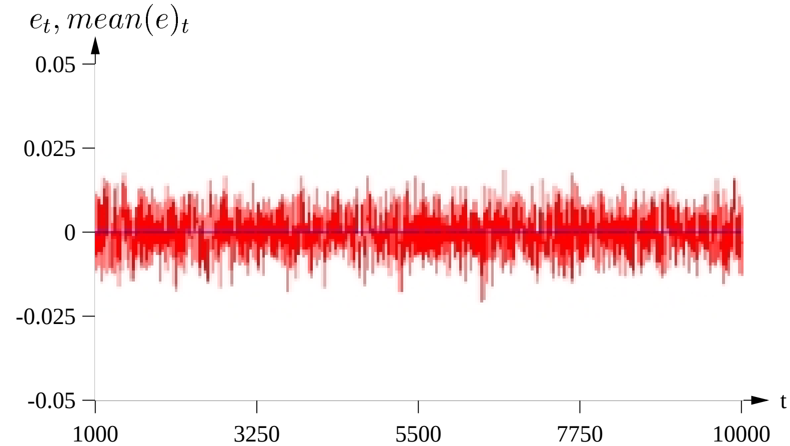

Figure 4.34: RLS errors 1000–10000

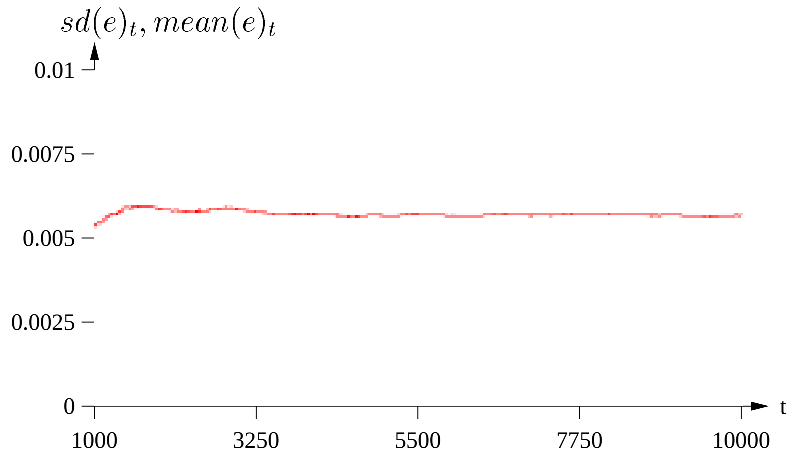

Figure 4.35: sd RLS errors 1000–10000

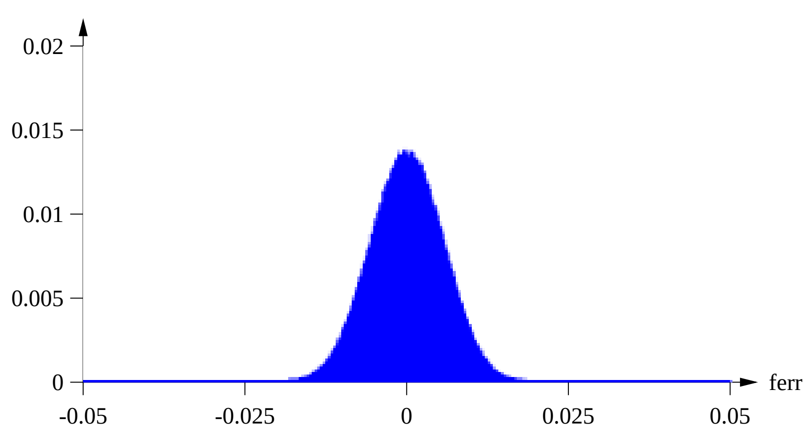

Figure 4.36: RLS errors: mean=-1.12051e-05; variance=3.16049e-05; sd= 0.00562182; $T = 10^6$

In spite of this criticism, it is a worth while exercise to investigate the effects of the RLS predictors for the asset pricing model and to compare its performance with the unbiased predictor as well as with the other adaptive schemes. Since the system (4.16) – (4.18) is non linear, the stability results from affine random dynamical systems are no longer applicable. A general proof of existence and stability of a random fixed point is out of reach at this point. However, the numerical analysis for $R > 1$ yields stable behavior for the one asset model analyzed so far. In order to keep the results comparable the values $R = 1.01$ and $\phi(\bar{x}) = .55$ were used along with the same AR(1) dividend process of equation (3.32).

Figures 4.28 – 4.36 show the sets of diagrams characterizing prices, returns, and forecast errors. All three sample paths show some small volatility clustering (non constant recursive standard deviations), prices more so than returns and errors. Mean prices and returns are negative as with all other adaptive predictors. The prediction performance has a low mean error but a high variance while the portfolio performance has the highest long run standard deviation among the adaptive predictors.

Tables (4.39 and 4.37) and Figure 4.38 summarize the results of the numerical analysis and provide a comparison of the performance of all predictors. They show very clearly that the portfolio performance of the adaptive procedures is poor relative to the unbiased predictor. In all periods consumers receive a negative return on their portfolio when *ceteris paribus* a safe asset with a positive return is available. The long memory moving average predictor ($L = 20$) seems to be the worst performer with the highest (absolute) mean-to-sd ratio while the RLS predictor has the lowest ratio, but with the highest long run variance of the portfolio. Considering the forecasting errors, RLS has a low mean error, but it performs poorly with the highest variance and the smallest mean-to-sd ratio among the adaptive predictors (see Figure 4.38.

Portfolio Performance: Returns

	mean	variance	sd	mean/sd
unbiased AR1	+0.017549	$1.38042{\cdot}10^{-5}$	0.00371539	+4.723326488
naive AR1	-0.0185303	$3.42112{\cdot}10^{-5}$	0.00584904	-3.168092542
MA L 2	-0.0185119	$2.93938{\cdot}10^{-5}$	0.00542161	-3.414465445
MA L 20	-0.0184478	$1.78104{\cdot}10^{-5}$	0.00422024	-4.371267985
RLS	-0.018559	$3.91099{\cdot}10^{-5}$	0.00625379	-2.967640423

Figure 4.37: Statistics of returns

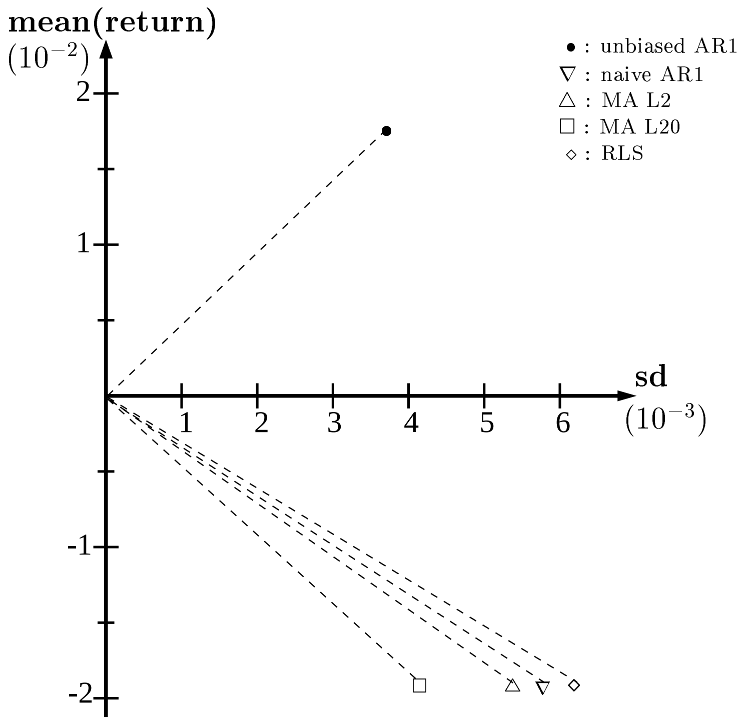

Figure 4.38: Mean–Standard Deviation of returns

Prediction Performance: Forecasting errors

	mean	variance	sd	mean/sd
unbiased AR1	$+5.03457 \cdot 10^{-3}$	$3.07008 \cdot 10^{-6}$	0.00175216	+0.28733506
naive AR1	$-1.47409 \cdot 10^{-5}$	$2.66943 \cdot 10^{-5}$	0.00516665	$-2.85308662 \cdot 10^{-3}$
MA L2	$-1.85118 \cdot 10^{-2}$	$2.93936 \cdot 10^{-5}$	0.00542158	-3.414465894
MA L20	$-7.97631 \cdot 10^{-6}$	$9.96226 \cdot 10^{-6}$	0.00315631	$-2.5270996 \cdot 10^{-3}$
RLS	$-1.12051 \cdot 10^{-5}$	$3.16049 \cdot 10^{-5}$	0.00562182	$-1.9931445 \cdot 10^{-3}$

Figure 4.39: Statistics of errors

5 Conclusions

The extension of the model developed in Böhm, Deutscher & Wenzelburger (2000) to the case of an arbitrary finite number of assets has led to the analysis of a fully explicit and sequential model whose temporary structure coincides with that of the traditional capital asset pricing model. This paper presented a full dynamic analysis of the endogenous formation of random asset prices providing existence and stability results for adaptive as well as unbiased expectations formation. These results show quite strikingly that the long run random behavior of asset prices results from the dynamic interaction of agents' preferences and behavior towards risk, their expectation formation procedures, the market mechanism, and of exogenous random forces. All four factors are essential determinants of the final process. More precisely, the random movement of prices and returns derives from the joint interaction of the exogenous forces *and* an induced randomness from the expectation formation procedure used by agents. This implies, for example, that prices and returns reflect the stochastic nature of the exogenous dividend process fully *only* if predictions of agents about prices are constant over time. If, however, agents adjust their predictions based on new information every period, the stochastic features of prices and returns above that of the dividends is essentially generated by the agents themselves. As a consequence, the notion of a *fundamental law* of asset prices should be viewed always in connection with a particular forecasting principle employed by agents in the market.

It is one of the remarkable results of the above analysis that sequential models of this type generate a unique stable long run (stationary) price process supported by an unbiased prediction mechanism for large classes of models under general structural economic assumptions. Along any generated orbit, one has full rational expectations, thus the sample paths are generalizations of Markov equilibria, a concept used extensively in economics when exogenous noise processes are assumed to be i. i. d. The results on uniqueness and stability were obtained using some new mathematical results from the theory of random dynamical systems (cf. Arnold 1998), which proves their adequacy and usefulness for economic applications. Finally, it is evident from the results obtained in the numerical simulations that the efficiency and predictive power of additional forecasting rules can be analyzed within such models, in order to examine their role and usefulness for econometric forecasting.

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Mathematical Appendix

Proof of Lemma 2.3

It suffices to show that

$$\big(\varphi(\pi) - \varphi(\pi')\big)^T (\pi - \pi') > 0$$

holds for all $\pi \neq \pi'$. Let

$$U^a(M, V) = M - \frac{\alpha}{2}\, w(V)$$

be given with $w : \mathbb{R} \to \mathbb{R}$ strictly increasing, strictly convex, and C^2. For any symmetric positive definite matrix v, the mapping $W : \mathbb{R}^K \to \mathbb{R}$ defined by

$$W(x) := \frac{\alpha}{2}\, w(x^T\, v\, x)$$

1. is strictly convex,
2. has a gradient $DW(x) = \alpha\, \big[w\prime(x^T\, v\, x)\big]\; v\, x$,
3. which satisfies $(x - y)^T (DW(x) - DW(y)) > 0$ for all $x, y \in \mathbb{R}^K$ with $x \neq y$.

Individual asset demand φ^a is globally invertible since U^a is quasi linear in M and w is strictly convex. Therefore, $\pi \neq \pi'$ implies: $\varphi^a(\mu^a + \pi) = x \neq y = \varphi^a(\mu^a + \pi')$. Then, using the first order conditions,

$$\begin{aligned} \mu^a + \pi &= \alpha\, \big[w\prime(x^T\, v^a\, x)\big]\; v^a\, x = DW(x) \\ \mu^a + \pi' &= \alpha\, \big[w\prime(y^T\, v^a\, y)\big]\; v^a\, y = DW(y) \end{aligned}$$

one has

$$\begin{aligned} \big(\varphi^a(\mu^a + \pi) - \varphi^a(\mu^a + \pi')\big)^T (\pi - \pi') &= \\ (x - y)^T \big(\alpha\, [w'(x\, v^a\, x)]\; v^a\, x) - \alpha\, [w'(y\, v^a\, y)]\; v^a\, y\big) &= \\ (x - y)^T \big(DW(x) - DW(y)\big) &> 0 \end{aligned}$$

Therefore, $\varphi(\pi) := \sum_{a \in A} \varphi^a(\mu^a + \pi)$ satisfies

$$\big(\varphi(\pi) - \varphi(\pi')\big)^T (\pi - \pi') > 0$$

holds for all $\pi \neq \pi'$.

□