Option Pricing via Stochastic Volatility Models: an Empirical Comparison

Gianna Figà-Talamanca[*] Maria Letizia Guerra[†]

Abstract

Many empirical analysis suggest that market prices dynamics are not well captured by Black and Scholes model. A valid generalization is attained by allowing volatility to change randomly and different approaches have been proposed in literature since the pioneering model by Hull and White [11].

The aim of this paper is to compare different volatility specifications focusing on their capability in pricing options.

1 Volatility Models

A wide number of volatility models is described in [1] and [16]. In Figure 1 some of the most important models are summed up. They are collected into three classes.

One specific model in each class is analyzed: the complete model recently proposed by Hobson and Rogers in [10], the bivariate model by Heston [9] and the Garch model as in [3].

1.1 Endogenous Source of Risk

In the endogenous or univariate case the volatility is allowed to depend also on the underlying security price, i.e. $\sigma(t; S(t))$. Among the univariate volatility models, the model recently suggested by Hobson and Rogers in [10] seems

[*]Istituto Economico, Università della Tuscia
[†]Istituto di Scienze Economiche, Università di Urbino

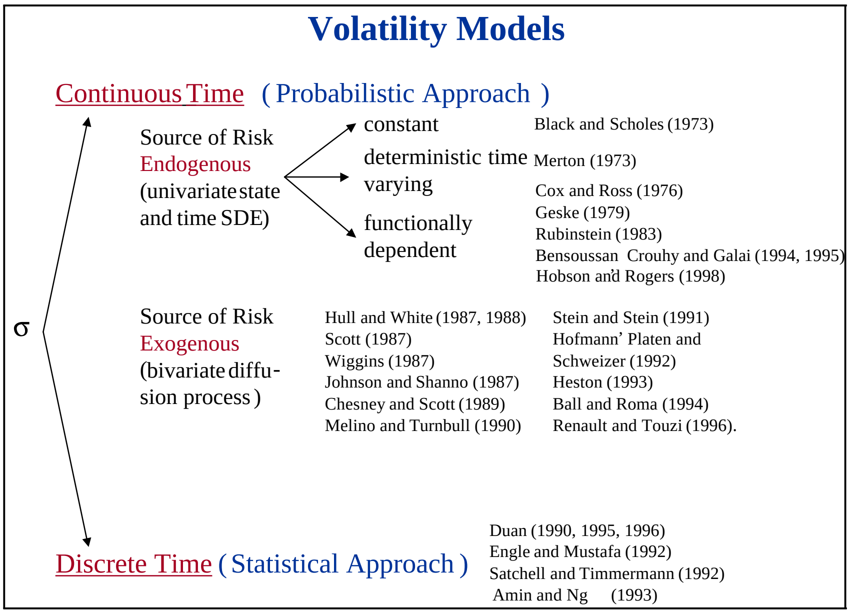

Figure 1: Survey of volatility models

particularly interesting. In such a model P_t is the price process and Z_t is the discounted log-price process given by:

$$Z_t = \log(P_t \exp(-rt)) \tag{1}$$

Under some technical condition the first order offset function is defined as:

$$X_t^{(1)} = \int_0^{\infty} \lambda \exp(-\lambda u)(Z_t - Z_{t-u})^1 \, du \tag{2}$$

where the constant λ is a parameter of the model describing the rate at which past information is discounted.

Volatility depends only on the first order offset $X_t^{(1)}$ and the dynamics for $\sigma(x)$ are given by:

$$\sigma(x) = \eta\sqrt{1 + \varepsilon^2 x^2} \tag{3}$$

This choice is justified in terms of some useful properties of $\sigma(x)$: it is even, bounded and the model itself captures the fact that volatility changes are correlated with price changes.

The relation of Z_t with the offset function is:

$$dZ_t = dX_t - \nu X_t dt \tag{4}$$

and X_t itself is supposed to be driven by a general SDE:

$$dX_t = -\frac{\eta^2}{2}\left(1 + \varepsilon^2 X_t^2\right) dt + \eta\sqrt{1 + \varepsilon^2 X_t^2}\, dB_t \tag{5}$$

The call option price can be written as usual as

$$call(P_t; X_t; T - t) = e^{-r(T-t)} E\left[(P_T - K; 0)^+\right] \tag{6}$$

1.2 Exogenous Source of Risk

When the source of risk is exogenous, volatility is assumed to evolve as a stochastic process. The standard framework assumes a bivariate diffusion process in which the processes of the underlying asset S and the volatility σ have to be jointly specified.

In the model suggested by Heston in [9], the stock price and the volatility are assumed to be driven by the following bivariate SDE:

$$dS_t = \phi S_t dt + \sigma_t S_t dB_t^1 \tag{7}$$

$$dV_t = k(\theta - V_t)dt + c\sqrt{V_t}\, dB_t^2 \tag{8}$$

where $V_t = \sigma_t^2$ and $k; \theta; c$ are positive constants. The positivity of the variance process V_t is guaranteed by suitable constraints on parameters.

The derivation of option prices depends upon the parameters values which thus have to be estimated.

The estimation of $k; \theta$ and c is particularly difficult since there is no asset that is clearly instantaneously perfectly correlated with the variance process σ_t^2; i.e. the volatility is neither directly observable or deducible from market data.

1.3 Garch Option Pricing

Duan's discrete time model for option pricing (see [3] and [4]) is briefly described.

Let S_t be the asset price at time t for $t = 1, 2, \ldots, T$.

The log-returns of the asset are supposed to evolve according to a linear model with Garch innovations, that is:

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda\sqrt{V_t} - \frac{V_t}{2} + \varepsilon_t \tag{9}$$

where ε_t, conditionally on the information until time $t-1$, is a centered normal random variable with variance V_t; r is the constant risk-free rate in the market and λ can be interpreted as a "risk premium".

The conditional variance is described by the following garch relationship:

$$V_t = \gamma + \alpha\varepsilon^2_{t-1} + \beta V_{t-1} \tag{10}$$

where parameters are non negative and $\alpha + \beta < 1$ guarantees the stationarity of the process.

Duan's model can be seen as an extension of a discrete time Black and Scholes model, when imposing $\alpha = \beta = 0$ and $\gamma = \sigma^2$.

In this setting Duan generalizes the traditional risk neutral valuation principle defining the "locally risk neutral valuation relationship" and the corresponding sufficient conditions on the agent's preferences. By this local relationship, Duan obtains an option pricing formula which is locally, but not globally, independent on the agent's preferences.

Under a local risk neutral probability measure Q the log-return of the asset follows the following discrete time process:

$$\log\left(\frac{S_t}{S_{t-1}}\right) = r - \frac{V_t}{2} + \xi_t \tag{11}$$

where ξ_t, conditionally on the information until time $t-1$, is a centered normal random variable with variance V_t.

The conditional variance is here described by the following modified garch relationship:

$$V_t = \gamma + \alpha(\xi_{t-1} - \lambda\sqrt{V_{t-1}})^2 + \beta V_{t-1} \tag{12}$$

where the parameters are the same as in (10). This process correspond to a non-linear asymmetric garch specification (NGARCH) introduced in [6].

The price process, under the modified measure Q; is thus given by:

$$S_T = S_t \exp\left\{ (T - t)r - \frac{1}{2}\sum_{s=t+1} V_s + \sum_{s=t+1} \xi_s \right\}$$

and $e_t^{-rt}S$ is a $Q -$ martingale.

By the local risk neutral principle, the price of an European call option with expiration date T and strike price K is given by:

$$C_t = e^{-r(T-t)} E_t[\max(X_T - K; 0)] \qquad (13)$$

Siamo sicuri che in questa call ci deve essere Xt???? Nel caso endogeno Xt è l'offset!!

It does not exist a closed expression for the price in (13) and it can be obtained by simulation methods. In this paper the alternative approach of giving an analytical approximation to the pricing formula is used, as suggested in [5].

2 Estimation's Methodologies

Parameter estimation is carried out according to a suitable methodology for each one of the analyzed stochastic volatility models.

The usual Garch maximum likelihood procedure is applied for the estimation of Duan's option pricing model while two recent techniques, which deserve a brief description, are used for estimating Hobson-Rogers and Heston models.

For the Hobson-Rogers' case the diffusion coefficient is first estimated following the procedure proposed in [2], and the corresponding parameter estimates are then derived. More precisely, given a general diffusion process S_t; defined as a solution to the SDE:

$$dS_t = \mu(t; S_t)\, dt + \sigma(t; S_t)\, dB_t \qquad (14)$$

after many computations Chesney, Elliott, Madan and Yang (briefly called Cemy) obtain the estimation of $\sigma(t; s_t)^2$ in the general form:

$$\hat{\sigma}_t^2 = \frac{2f'(y_t)}{f''(y_t)\,y_t}\left(\frac{f(y_{t+\Delta t}) - f(y_t)}{f'(y_t)\,y_t} - \frac{y_{t+\Delta t} - y_t}{y_t}\right)\frac{1}{\Delta t} \tag{15}$$

where $y_t = \exp(s_t)$:

Cemy consider then the particular function:

$$f(y) = y^{1+\gamma}; \tag{16}$$

so that the equation (15) becomes:

$$\hat{\sigma}_{t;\gamma}^2 = \frac{2}{\gamma}\left[\frac{y_{t+\Delta t}^{1+\gamma} - y_t^{1+\gamma}}{(1+\gamma)\,y_t^{1+\gamma}} - \frac{y_{t+\Delta t} - y_t}{y_t}\right]\frac{1}{\Delta t}. \tag{17}$$

Pastorello in [15] corrects the value of the parameter γ that minimizes the conditional variance of $\hat{\sigma}_{t;\gamma}^2$, it is:

$$\gamma = -\frac{19}{11} - \frac{12\mu(t;s_t)}{11\sigma(t;s_t)^2} \tag{18}$$

In Hobson-Rogers model the Cemy equation (17) becomes:

$$\left(\eta\sqrt{1+\lambda X_t^2}\right)^2 = \frac{2}{\gamma}\left[\frac{y_{t+\Delta t}^{1+\gamma} - y_t^{1+\gamma}}{(1+\gamma)\,y_t^{1+\gamma}} - \frac{y_{t+\Delta t} - y_t}{y_t}\right]\frac{1}{\Delta t} \tag{19}$$

The solution $(\hat{\eta};\hat{\lambda})$ represents the estimated values of the two parameters and it

$$\min_{\eta;\lambda}\left(\left(\eta\sqrt{1+\lambda X_t^2}\right)^2 - \frac{2}{\gamma}\left[\frac{y_{t+\Delta t}^{1+\gamma} - y_t^{1+\gamma}}{(1+\gamma)\,y_t^{1+\gamma}} - \frac{y_{t+\Delta t} - y_t}{y_t}\right]\frac{1}{\Delta t}\right)^2 \tag{20}$$

A pair $(\hat{\eta};\hat{\lambda})$ is computed for every $t \in [t_0;T]$.

The estimation procedure for the bivariate Heston model is based on some limit results in [7].

By introducing the "log-prices" $Y_t = \log S_t$, Heston model can be specified by means of the following bivariate SDE:

$$
\begin{aligned}
dY_t &= (\mu - \frac{V_t}{2})dt + \sqrt{V_t}dB_t; \\
dV_t &= k(\theta - V_t)dt + c\sqrt{V_t}dW_t.
\end{aligned} \tag{21}
$$

Under some technical conditions, the estimation procedure suggested in the quoted paper provides consistent estimates of the variance parameters.

Suppose that the available dataset for the log-prices is given by $Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}$, equally spaced by Δ.

Define the variables:

$$
X_i = X_i^n = \frac{1}{\sqrt{\Delta}}(Y_{t_i} - Y_{t_{i-1}}) \tag{22}
$$

where $t_i = t_i^n = i\Delta;\ i = 1, 2, n.$

Conditionally on $F = \sigma(V_s;\ s \geq 0)$ the distribution function of the "rescaled returns" $(X_1, X_2, \ldots, X_n)$ is known to be n-dimensional Gaussian with zero mean and covariance matrix:

$$
\Sigma = \begin{bmatrix}
\overline{V_1} & 0 & \cdots & 0 \\
0 & \overline{V_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \overline{V_n}
\end{bmatrix} \tag{23}
$$

where

$$
\overline{V_i} := \frac{1}{\Delta}\int_{\Delta(i-1)}^{\Delta i} V_s ds; \quad i = 1, 2, \ldots n. \tag{24}
$$

Vector $(X_1, X_2, \ldots, X_n)$ is thus a variance Gaussian mixture with the stationary distribution of vector $(\overline{V_1}, \overline{V_2}, \ldots \overline{V_n})$. Using the same notation as in [7], we denote this mixture distribution, depending on θ, with Q_θ^n.

Setting

$$
\begin{aligned}
Q_\theta(f) &:= \int_{R^d} f(u_1, u_2, \ldots, u_d) Q_\theta(du_1 du_2 \cdots du_d) \\
\hat{P}_n(f) &:= \frac{1}{n}\sum_{i=0}^{n-d} f(X_{i+1}^n, X_{i+2}^n, \ldots, X_{i+d}^n)
\end{aligned}
$$

with f in a specific class of functions on R^d, the following main results are proved in [7]:

A)

$$\hat{P}_n(f) \xrightarrow{a.s.} Q_\theta(f); \tag{25}$$

as n goes to infinity;

B)

$$\sqrt{n}\left(\hat{P}_n(f) - Q_\theta(f)\right) \xrightarrow{d} N\left(0; V_\theta(f; f)\right); \tag{26}$$

as n goes to infinity.

For estimation purposes function f is essentially chosen as some mixed moment, that is "power functions", of the variables X_i, which belong to the class for which the stated results hold.

In particular A) implies that:

$$\frac{1}{n}\sum_{1}^{n} X_i^2 \xrightarrow{a.s.} E(\bar{V}_1) \tag{27}$$

$$\frac{1}{n}\sum_{1}^{n} X_i^4 \xrightarrow{a.s.} 3E(\bar{V}_1^2)$$

$$\frac{1}{n}\sum_{1}^{n-1} X_i^2 X_{i+1}^2 \xrightarrow{a.s.} E(\bar{V}_1 \bar{V}_2)$$

Since the "stationary" moments of variables $\bar{V}_i$ can be expressed depending explicitly on the value of the variance process parameters, their estimates are obtained through the solution of a non-linear system of equation based on the convergence results stated above. The limit results guarantee that the parameters estimates are consistent and have an asymptotic Gaussian distribution.

3 Numerical Results: a comparison

The market data[1] taken into account are call options prices on the FTSE100 Index and on the exchange rate SF/USD, with different strike prices and

[1] Market data has been provided by Datastream

di®erent maturities. To obtain parameters estimation, ¯ve years of FTSE100 Index and SF/USD data have been considered.

The price of a European call option is thus computed according to the di®erent stochastic volatility model assumed for the dynamics of the underlying.

In the models by Hobson-Rogers and Duan option prices are obtained trough Monte Carlo simulation of 1000 di®erent price trajectories and Milstein scheme is adopted for the discretization of the continuous model.

On the contrary, Heston provides a closed formula for the computation of option prices, which depends only upon the estimated parameters.

Some results are summed up in the next table, which reports Market Prices for a call option on the FTSE100 Index, with 1-month time to maturity, for di®erent values of the strike price[2]. The corresponding Model Prices, also collected in the table, are derived according to four di®erent model assumptions: Black and Scholes, Hobson and Rogers, Heston and Duan[3].

K	Market	BS	HOB-ROG	HESTON	DUAN
5825	360,5	365,835	313,0511	343,0252	361,145
5925	277	292,94	224,4709	264,8033	285,083
5975	238,5	259,71	182,5812	229,384	250,447
6025	202	228,778	137,4089	196,709	218,348
6075	168,5	200,196	117,8864	166,9251	188,441
6125	138	174,002	98,356	140,1161	160,926
6175	111	150,193	67,985	116,2987	136,306
6225	87	128,732	55,443	95,4228	114,396
6275	67	109,553	33,848	77,3763	94,976
6325	50	92,562	22,464	61,9939	78,708
6375	36,5	77,637	14,427	49,0681	64,681
6425	23	64,644	10,183	/	52,791

Figure 2: FTSE100 Call: one month to maturity

[2]Call Option prices on March, twenty-sixth 1999 , matutity April 1999. The strike prices reported in the table are those for wich call options have been e®ectively traded

[3]The risk-free rate r is ¯xed at 0% and the parameter ° in (2)) is ¯xed equal to 5

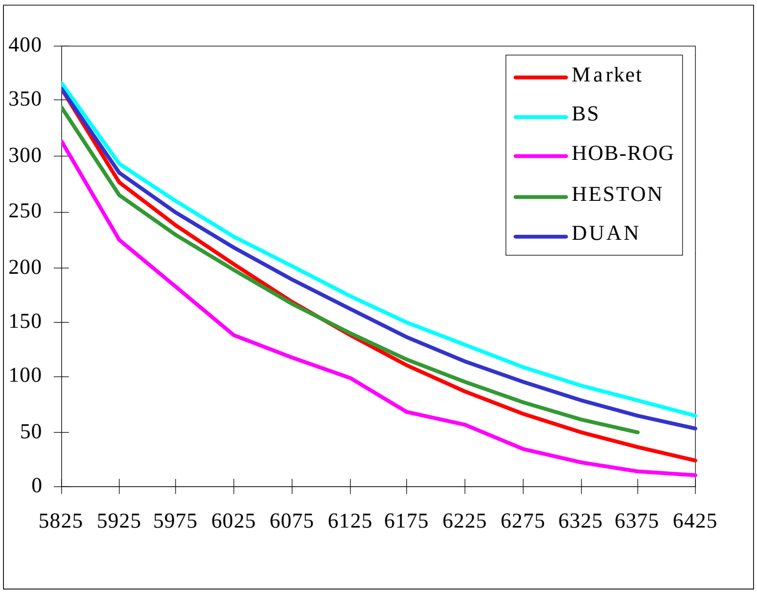

Figure 3: FTSE100 Call: comparing model and market prices

Di®erent results are summed up in the next table, which reports Market Prices for a call option on the SF/USD change, with 3-months time to maturity, for di®erent values of the strike price

K	Market	BS	HOB-ROG	HESTON	DUAN
66	2,71	2,919	1,543	1,828	1,492
67	2,67	3,689	1,279	2,519	2,134
68	2,0	4,572	0,845	3,361	2,959
69	1,66	5,563	1,731	4,351	3,969
70	2,81	6,657	2,954	5,481	5,151
71	2,83	7,843	3,701	6,737	6,69
72	0,52	9,111	4,602	8,101	7,939

Figure 4: Call Option SF/USD Maturity: June 1999

4 Conclusions

The analysis given in this paper suggests some important considerations. First of all the necessity of further investigations in volatility modelling due to the fact that real data exhibit special features like: almost no correlation, heavy tail and high threshold exceedances in clusters.

These observations becomes particularly true when looking back at the "hot" period from August to October 1998: no one of models suggested could forecast the occurrence of those events. More in practice, the empirical comparison has to be done with very di®erent real data to understand if there exists some relationship between a special kind of model and certain data type.

References

[1] Ball C. A., "A Review of Stochastic Volatility Models with Application to Option Pricing", Financial Markets, Institutions&Instruments, 2, 5, 1993.

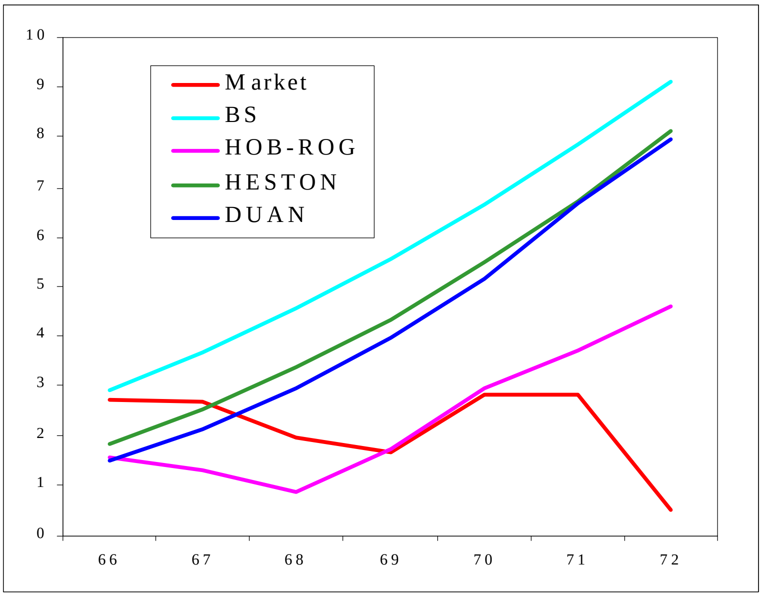

Figure 5: SF/USD Call: comparing model and market prices

[2] Chesney M. Elliott R. J. Madan D. Yang H., Di®usion coe±cient estimation and Asset Pricing when Risk Premia and sensitivities are time varying, Mathematical Finance, 3, 2, 1993.

[3] Duan J.-C., "The GARCH Option Pricing Model", Matematical Finance, 5, 1 13-32, 1995.

[4] Duan J.-C., "A Uni¯ed Theory of Option Pricing under Stochastic Volatility - from GARCH to Di®usion", working paper, Hong Kong University, 1996.

[5] Duan J.-C., Gauthier, G, Simonato, J., "An Analytical Approximation for the GARCH Option Pricing Model", 1998.

[6] Engle, R., Ng, V., "Measuring and Testing of the Impact of News on Volatility", Journal of ¯nance 48, 1749-1778, 1993;

[7] Genon-Catalot, V., Jeantheau, T., Laredo, C.,, "Parameter Estimation of Discretely Observed Stochastic Volatility Models", paper

presented in The Interplay between Finance, Insurance and statistics, Cortona, june 1998.

[8] Guerra M. L., "Numerical Simulations for Volatility Estimation: the case of an Arch Process in Continuous Time", in Achim Sydow (Ed.), Commputational Mathematics, 443-448, Wissenschaft & Technick Verlag, Berlino, 1997.

[9] Heston S.L., "A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options", The Review of Financial Studies, 6, 2, 327-343, 1993.

[10] Hobson D. Rogers L.C.G., "Complete Models with Stochastic Volatility", Mathematical ¯nance, 8, 1, 1998.

[11] Hull J. White A., "The pricing of Options on Assets with Stochastic Volatility", Journal of Finance, 42, 2, 281-300, 1987.

[12] Hull J. White A., "An Analysis of the Biais in Option Pricing Caused by a Stochastic Volatility", Advances in Futures and Options Research, 3, 29-61, 1988.

[13] Kloeden P. E. Platen E. Schurz H., Numerical Solution of SDE Through Computer Experiments, Spriger Verlag, 1994.

[14] Nelson D.B., "ARCH Models as Di®usion Approximations", Journal of Econometrics, 45, 7-38, 1990.

[15] Pastorello S., Di®usion coe±cient estimation and Asset Pricing when Risk Premia and sensitivities are time varying: a comment, Mathematical Finance, 6, 1, 1996.

[16] Taylor S.J., "Modeling Stochastic Volatility: a Review and Comparative Study", Mathematical Finance, 4, 2, 183-204, 1994.