ELSEVIER SCIENCE LTD., [DTD 4.1.0]
JOURNAL CHAOS ARTICLE No. 1738
PAGES 1-21 DISPATCH 25 January 2001

CHAOS 1738 | A
PROD. TYPE: FROM DISK

PERGAMON

Chaos, Solitons and Fractals 000 (2001) 000–000

CHAOS
SOLITONS & FRACTALS

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Nonlinear duopoly games with positive cost externalities due to spillover effects

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Accepted 3 January 2001

Abstract

A Cournot duopoly game is proposed where the interdependence between the quantity-setting firms is not only related to the selling price, determined by the total production through a given demand function, but also on cost-reduction effects related to the presence of the competitor. Such cost reductions are introduced to model the effects of know-how spillovers, caused by the ability of a firm to take advantage, for free, of the results of competitors' Research and Development (R&D) results, due to the difficulties to protect intellectual properties or to avoid the movements of skilled workers among competing firms. These effects may be particularly important in the modeling of high-tech markets, where costs are mainly related to R&D and workers' training. The results of this paper concern the existence and uniqueness of the Cournot–Nash equilibrium, located at the intersection of non-monotonic reaction curves, and its stability under two different kinds of bounded rationality adjustment mechanisms. The effects of spillovers on the existence of the Nash equilibrium are discussed, as well as their influence on the kind of attractors arising when the Nash equilibrium is unstable. Methods for the global analysis of two-dimensional discrete dynamical systems are used to study the structure of the basins of attraction.

1. Introduction

An *oligopoly* is a market structure where a few producers, each of appreciable size, manufacture the same commodity, or homogeneous commodities (i.e. perfect substitutable goods). The fewness of firms gives rise to *interdependencies*, that is, each producer must take into account the actions of the competitors in choosing its own action.

In the industrial organization literature, one of the most widely used mathematical representations of oligopoly competition is the Cournot model, first introduced by the French Mathematician Cournot about 160 years ago, which describes a market where N quantity-setting firms, producing homogeneous goods, update their production strategies in order to maximize their profits. In the original work of Cournot, as well as in many of the subsequent papers, the above-mentioned interdependence only depends on the fact that the retail price p is determined by the total supply on the market, $Q = q_1 + q_2 + \cdots + q_N$, according to a given inverse demand function, $p = f(Q)$. But also other sources of interdependencies can be considered, for example originated by positive cost externalities, i.e. a reduction of production cost due to the presence of competitors. This may appear rather paradoxical, but several reasons can be given to support such a cost-reduction effect, due to technological and intellectual spillovers between companies, related to exchanges of information on technological innovations, skilled labor, results of Research and Development

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(R&D) investments. Indeed, as stressed by many authors, see e.g. [4,5,36,19], information may spill from one firm to another, due to the difficulties of protecting intellectual properties. Moreover, when a firm operates in a district where other firms producing the same goods already exist, it is more easy to find skilled workers (thus giving a reduction of costs for workers training) and the presence of suitable structures for transportation and other services may contribute to lower the costs for goods delivery. In particular, we consider the fact that a firm producing goods for high-tech markets must invest a lot in R&D issues, and when it increases its production it often dedicates more resources to R&D. But information may spill from one firm to another, due to the difficulties of protecting intellectual properties, so it often happens that such results on technological innovations become common industrial knowledge. This fact can be seen as a positive cost externality, which can be used to model the trivial statement that successful inventions of rivals can be imitated by a firm at lesser cost than if they are reinvented by itself (see e.g. [19]).

All these facts introduce cost externalities which change the standard way of modeling Cournot oligopoly competition. In this paper we consider the simplest oligopoly market, where just two producers are present, called *duopoly*, and in order to model the presence of spillover effects we assume that the cost function of firm i has the form:

$$c_i(q_1, q_2) = \frac{k_i q_i + s_i(q_i)}{1 + \gamma_{ij} q_j}, \quad i, j = 1, 2, \;\; j \neq i, \tag{1}$$

where the parameter $\gamma_{ij} \geqslant 0$ characterizes the positive cost externality in the cost of producer i related to the presence of producer j, $k_i \geqslant 0$ represent the unitary cost of firm i without taking into account the presence of the spillovers and the function $s_i(q_i)$ incorporates the extra costs (if any) which firm i pay to avoid spillovers. In fact, as stressed by some authors (see e.g. [3,34]) a firm may adopt some actions to avoid that its R&D results, as well as its skilled workers, can spill over the competitors, and we assume that these costs can be represented as an additive cost proportional to its own firm's production. Indeed, if R&D efforts of a firm are proportional to its own production also the costs to protect them will be proportional to the production, and the same can be said for the salary increases necessary to avoid the movement of skilled workers towards competing firms. For sake of simplicity we assume a linear function $s_i(q_i)$, so that these costs can be included in a unique linear function at the numerator. Moreover, in order to focus our attention to the role of positive externalities due to spillover effects, we shall assume a linear demand function, expressed by $p = a - b(q_1 + q_2)$, where a and b are positive parameters. So, the profit of firm i becomes

$$\pi_i(q_1, q_2) = q_i[a - b(q_1 + q_2)] - \frac{c_i q_i}{1 + \gamma_{ij} q_j}, \quad i, j = 1, 2, \;\; i \neq j. \tag{2}$$

Our goal is to investigate the effect of increasing values of the spillover parameters γ_{ij} on the existence and stability of the Nash equilibria [1] of the duopoly game. The assumption of a linear demand function, together with the fact that also the cost functions become linear if spillover effects are neglected, i.e. $\gamma_{12} = \gamma_{21} = 0$, allow us to obtain linear reaction functions, [2] which is the simplest case proposed by all the standard textbooks. In other words, the only cause for the non-linearity and (as we shall see in Section 2) of non-monotonicity of the reaction curves is due to the presence of cost externalities due to spillover effects.

Other papers where cost externalities are considered in duopoly games are [28,36]. In both cases multiplicity of Nash equilibria is obtained (see also [13]). Indeed, in our simple Cournot duopoly game, even if the introduction of spillover effects, in the form of cost externalities, has the effect of changing the reaction curves from lines to strictly concave curves, which are unimodal for sufficiently high values of spillover parameters, it is easy to see that at most one Nash equilibrium exists. This is proved in Section 2, where we also show that the conditions to ensure the existence of a Nash equilibrium are weaker than in the linear case.

[1] A Nash equilibrium is a profile of strategies such that each firm's strategy is an optimal response to the other firms' strategies. In the Nash equilibrium none of the firms has an incentive to deviate, since each firm's strategy is that firm's best response to the other firms' predicted strategies.

[2] A *reaction function* describes the profit-maximizing production of a firm given the production decision of the other firms. A Nash equilibrium can be defined as an intersection point of the reaction functions of the oligopolists.

In Sections 3 and 4, in order to investigate the effects of the spillovers on the stability of the Nash equilibrium, we analyze the Nash Equilibrium from an evolutionary point of view, i.e. we consider how the equilibrium arises as the outcome of a dynamic adjustment process occurring when less that fully rational players play the game repeatedly (see e.g. [22], or [7, Chapter 9).

Several kinds of boundedly rational adjustment processes may be considered, all sharing the same Nash equilibrium but with different methods to update productions when the system is out of it. Of course, these boundedly rational games are based on some kinds of profit increasing mechanism adopted by the firms, but no fully rational optimization solutions are obtained. This means that the players generally do not reach a Nash equilibrium immediately, but play the game repeatedly in order to approach it. In Section 3 a kind of boundedly rational adjustment mechanism is proposed, known in the literature as gradient dynamics (or myopic adjustment, see [9,21,40,41]). In this section we give results on the stability of the Nash equilibrium and the local bifurcations through which it becomes unstable, and we also investigate on the kind of attractors arising and the structure of their basins of attraction. In Section 4 we propose another classical dynamic adjustment, which was originally proposed by Cournot himself, known as best reply with naive expectations, and we show that in this case the Nash equilibrium is always stable. We end the paper with a discussion in Section 5.

2. The reaction curves and the Nash equilibrium

As explained in Section 1, a Cournot duopoly game is based on the assumption that each player decides, given the competitor's action, its own production in order to maximize the expected profit

$$\max_{q_i} \pi_i(q_1, q_2), \quad i = 1, 2, \tag{3}$$

where π_i is the profit that firm i expects by selling a production of q_i units of good and assuming that the competitor decides to produce a quantity q_j.

From the first-order conditions $\partial\pi_i/\partial q_i = 0$, we can easily get the solution of (3) with profit functions given by (2), expressed by the reaction functions

$$q_i = r_i(q_j) = \frac{1}{2b}\left(a - bq_j - \frac{c_i}{1 + \gamma_{ij} q_j}\right). \tag{4}$$

A simple check of the second derivatives testifies that these solutions indeed represent local profit maxima, provided that the quantities are non-negative. Accordingly, in the following we shall call reaction curves R_1 and R_2 the portions, inside the positive orthant, of the functions $q_1 = r_1(q_2)$ and $q_2 = r_2(q_1)$, respectively. Hence, for a given expected production of the competitor, R_i represents the "Best Reply" of the quantity-setting firm i according to the optimization problem (3).

Every intersection between the two reaction curves, being an optimal choice for both firms, is characterized by the fact that no firm has an incentive to unilaterally deviate from its chosen strategy given the choice of its rival. As mentioned above, such a point is called a Cournot–Nash equilibrium in the economics literature. A Nash equilibrium might then serve as a prediction of what outcome will be observed in an oligopoly market with fully rational players.

For $\gamma_{12} = \gamma_{21} = 0$ the reaction functions become linear, so the well-known case of linear reaction curves is obtained, for which the unique Nash equilibrium

$$E_* = \left(\frac{a + c_2 - 2c_1}{3b}, \frac{a + c_1 - 2c_2}{3b}\right) \tag{5}$$

exists, provided that the constant unitary costs are not too high, namely

$$2c_1 - c_2 < a \quad \text{and} \quad 2c_2 - c_1 < a. \tag{6}$$

A necessary condition for (6) to be satisfied is that $c_i < a$, $i = 1, 2$, which, in the case of a linear cost function, corresponds to the trivial statement that each unitary production cost must be less than the maximum selling price. In the following we shall assume that this condition is always satisfied in order to rule out uninteresting situations.

In the presence of spillovers, i.e. with $\gamma_{ij} > 0$, the reaction curves are concave branches of hyperbolae. Let us consider, for example, R_2. It is strictly concave, intersects the q_2 axis in $r_2(0) = (a - c_2)/2b$ (as in the linear case), the q_1 axis in $q_1 = q_1^0$, with

$$q_1^0 = \frac{a\gamma_{21} - b + \sqrt{(a\gamma_{21} - b)^2 + 4b\gamma_{21}(a - c_2)}}{2b\gamma_{21}} \tag{7}$$

and has a maximum for $q_1 = \widehat{q}_1 = (\sqrt{\gamma_{21}c_2/b} - 1)/\gamma_{21}$ provided that $\gamma_{21} > b/c_2$. Of course, the same description also holds for the reaction curve R_1 just swapping the indexes 1 and 2 (see Fig. 1).

So, in this simple Cournot game, the introduction of spillover effects in the form of cost externalities has the effect of changing the reaction curves R_i, $i = 1, 2$, from straight lines to strictly concave curves, which are unimodal for sufficiently high values of spillover parameters γ_{ij}. We now show that in the presence of spillovers the conditions on the parameters c_i (now representing the maximum unitary costs) in order to ensure the existence of a Nash equilibrium E_* are weaker than in the linear case and, like in the linear case, when E_* exists it is unique. This result is not obvious at a first sight, because in the presence of non-monotonic reaction curves multiple Nash equilibria may exist, see e.g. [13,28,39]. In [36] the fact that cost externalities may give multiplicity of Nash equilibria is extensively discussed. However, in our case, even if an analytical computation of the positive solutions of the equations $q_1 = r_1(q_2)$ and $q_2 = r_2(q_1)$ is not easy, the following proposition can be proved.

Proposition 1. *Let $c_i < a$, $i = 1, 2$. A unique Nash equilibrium $E_* = (q_1^*, q_2^*)$ exists with $0 < q_1^* < q_1^0$ and $q_2^* = r_1(q_1^*)$, where q_1^0 is given in (7) if and only if*

$$\gamma_{21} > \frac{2b(2c_2 - c_1 - a)}{a^2 - c_1^2}, \quad \gamma_{12} > \frac{2b(2c_1 - c_2 - a)}{a^2 - c_2^2}. \tag{8}$$

The proof is in Appendix A.

Notice that if

$$\gamma_{21} > \frac{2b(2c_2 - c_1 - a)}{a^2 - c_1^2} \quad \text{and} \quad \gamma_{12} = \frac{2b(2c_1 - c_2 - a)}{a^2 - c_2^2}$$

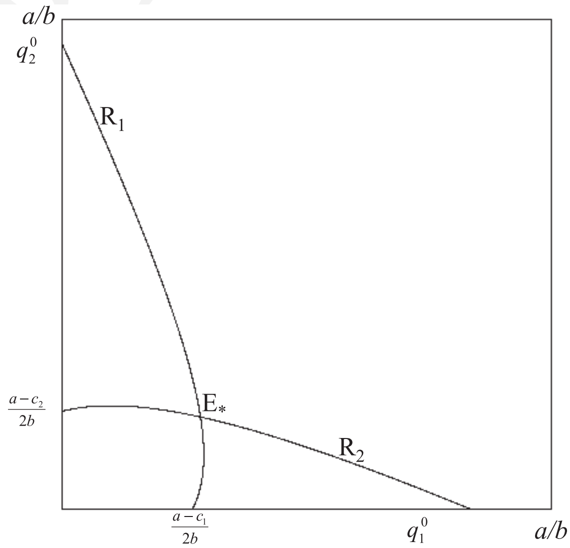

Fig. 1. Graph of the reaction curves R_1 and R_2 of equation $q_1 = r_1(q_2)$ and $q_2 = r_2(q_1)$, respectively.

then the Nash equilibrium collapses to a monopoly situation given by $E_* = (0, (a - c_2)/2b)$, where only firm 2 produces, whereas if

$$\gamma_{21} = \frac{2b(2c_2 - c_1 - a)}{a^2 - c_1^2} \quad \text{and} \quad \gamma_{12} > \frac{2b(2c_1 - c_2 - a)}{a^2 - c_2^2}$$

then the Nash equilibrium collapses into a monopoly situation given by $E_* = ((a - c_1)/2b, 0)$, where only firm 1 produces.

The presence of spillover effects enlarges, in the space of parameters, the region of existence of the Nash equilibrium. In fact, positive parameters γ_{ij} ensure the positivity of E_* even when conditions (6) are not satisfied. Moreover, a simple analysis of the reaction curves reveals that if γ_{ij} is increased then (ceteris paribus) q_i^* also increases, that is, as expected, a greater ability to take advantage of competitor's results allows one to improve production (and profits).

The arguments given above only concern the existence of the Nash equilibrium, but nothing is said about its stability. In order to investigate the effects of the spillovers on the stability of the Nash equilibrium we must consider how the equilibrium arises as the outcome of a dynamic adjustment process occurring when less than fully rational players play the game repeatedly (see e.g. [22], or [7], Chapter 9). This "evolutionary" concept of the stability of a Nash equilibrium was already stated by Nash himself: we can attain such a optimal equilibrium solution not as a result of a fully rational choice (i.e. with the help of Adam Smith "invisible hand") but as the asymptotic (i.e. long-run) outcome of a repeated game played by boundedly rational players (see e.g. [31]).

Indeed, a fully rational game is based on the following assumptions:

(i) each firm, in taking its optimal production decision, knows beforehand its rival's production decision;

(ii) each firm has a complete knowledge of the profit function.

Under these conditions of full information, the system moves straight (in one shot) to a Nash equilibrium, if it exists, independently of the initial status of the market, so that no dynamic adjustment process is needed. However, it seems unlikely that firms would immediately coordinate on such an equilibrium. Indeed, as stressed by many authors, firms are not so rational and often use simpler (and less expensive) "rules of thumb" in their decision-making processes (see e.g. [6]). Nevertheless, even a not fully rational game may gradually move to a Nash equilibrium if it is played many and many times, so that the long-run outcome is the same as if the player were fully rational. This idea has been largely confirmed after the advent of experimental economics, where human agents typically find their way to a Nash equilibrium by using trial and error methods. So, it is interesting to ask if, even relaxing assumptions (i) and/or (ii) competitors would learn to play according to a Nash equilibrium profile over time. This naturally leads to an analysis of the stability properties of the Nash equilibria and to the consideration of various dynamic adjustment processes.

Of course, several kinds of boundedly rational adjustment processes may be considered, by weakening assumptions (i) and (ii).

3. Bounded rationality adjustment based on marginal profits

In this section we propose a repeated Cournot duopoly game where two boundedly rational players update their production strategies at discrete time periods by an adjustment mechanism based on a local estimate of the marginal profit $\partial \pi_i / \partial q_i$: At each time period t a firm decides to increase (decrease) its production for period $t + 1$ if it perceives positive (negative) marginal profit on the basis of information held at time t, according to the following dynamic adjustment mechanism (see e.g. [9]):

$$q_i(t+1) = q_i(t) + \alpha_i(q_i(t)) \frac{\partial \pi_i}{\partial q_i}(q_1(t), q_2(t)); \quad i = 1, 2, \tag{9}$$

where $\alpha_i(q_i)$ is a positive function which gives the extent of production variation of ith firm following a given profit signal. With this kind of adjustment dynamics both the assumptions (i) and (ii) are relaxed: in fact, in order to follow this local adjustment mechanism the two producers are not requested to have a complete knowledge of the demand and cost functions, since they only need to infer how the market will respond to small production changes by an estimate of the marginal profit, which may be obtained by brief experiments of small (or local) production variations performed at the beginning of period t (see e.g. [41]). Of course, this local estimate of expected marginal profits is much easier to obtain than a global knowledge of the demand function (involving values of q_i that may be very different from the current ones).

We also notice that a Nash equilibrium, defined by the first order conditions $\partial\pi_i/\partial q_i = 0$ is a stationary point of the dynamical system defined by (9), but the converse is not necessarily true, as we shall see below.

This adjustment mechanism, which is sometimes called *myopic* (see [20,21]) has been recently proposed by many authors, see e.g. [6,14,15,23,40,41], mainly with continuous time and constant α_i. However, following [9,12], we believe that a discrete time decision process is more realistic since in real economic systems production decisions cannot be revised at every time instant. Moreover, we assume linear functions $\alpha_i(q_i) = v_i q_i$, $i = 1, 2$, since this assumption captures the fact that *relative* production variations are proportional to marginal profits, i.e.

$$\frac{q_i(t+1) - q_i(t)}{q_i(t)} = v_i\left(\frac{\partial\pi_i}{\partial q_i}\right),$$

where v_i is a positive *speed of adjustment*, which represents firm's i speed of reaction to profit signals per unitary production. With these assumptions, together with the profit functions given in (2), we obtain a discrete dynamical system of the form $(q_1(t+1), q_2(t+1)) = T(q_1(t), q_2(t))$, with the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{aligned} q_1' &= q_1 + v_1 q_1\left[a - 2bq_1 - bq_2 - \frac{c_1}{1+\gamma_{12}q_2}\right] \\ q_2' &= q_2 + v_2 q_2\left[a - 2bq_2 - bq_1 - \frac{c_2}{1+\gamma_{21}q_1}\right], \end{aligned} \tag{10}$$

where $'$ denotes the unit-time advancement operator, that is, if the right-hand side variables represent the productions at time period t then the left-hand side represents the productions at time $(t+1)$. Of course, only non-negative trajectories obtained by the iteration of (10) are interesting from the point of view of economic applications.

Besides the Equilibrium point E_*, located at the intersections of the reaction curves (4), the map (10) has three boundary equilibria located along the coordinate axes

$$E_0 = (0,0), \quad E_1 = \left(\frac{a-c_1}{2b}, 0\right), \quad E_2 = \left(0, \frac{a-c_2}{2b}\right). \tag{11}$$

The fixed points E_1 and E_2 can be denoted as *monopoly equilibria* provided that $c_i < a$, $i = 1, 2$.

It is worth to note that the coordinate axes $q_i = 0$, $i = 1, 2$, are invariant submanifold, i.e. if $q_i = 0$ then $q_i' = 0$. This means that starting from an initial condition on a coordinate axis (*monopoly case*) the dynamics are trapped into the same axis for each t, thus giving *monopoly dynamics*, governed by the restriction of the map T to that axis. Such a restriction is given by the following one-dimensional map, obtained from (10) with $q_i = 0$

$$q_j = (1 + v_j(a - c_j))q_j - 2bv_j q_j^2 \quad j \neq i. \tag{12}$$

This map is conjugate to the standard logistic map $x' = \mu x(1-x)$ through the linear transformation

$$q_j = \frac{1 + v_j(a - c_j)}{2bv_j}x \tag{13}$$

from which we obtain the relation $\mu = 1 + v_j(a - c_j)$.

If $\gamma_{12} = \gamma_{21} = 0$ the dynamic game (10) reduces to the one studied in [9], which may be considered as a benchmark case in this context. As shown in [9], unbounded trajectories are obtained if the initial condition is taken sufficiently far from the Nash equilibrium, [3] hence E_* cannot be globally stable. Moreover, even if bounded trajectories are obtained, they may fail to converge to the Nash equilibrium since they may continue to move around it, on some more complex (periodic or chaotic) attractor or converge to the boundary equilibria.

3.1. Local stability analysis for boundary (monopoly) equilibria and the Nash equilibrium

In this section we perform the standard study of the local stability of the fixed points of the map (10), based on the localization, on the complex plane, of the eigenvalues of the Jacobian matrix

$$DT(q_1, q_2) = \begin{bmatrix} 1 + av_1 - 4v_1bq_1 - v_1bq_2 - \frac{v_1c_1}{1+\gamma_{12}q_2} & v_1q_1\left(\frac{c_1\gamma_{12}}{(1+\gamma_{12}q_2)^2} - b\right) \\ v_2q_2\left(\frac{c_2\gamma_{21}}{(1+\gamma_{21}q_1)^2} - b\right) & 1 + av_2 - 4v_2bq_2 - v_2bq_1 - \frac{v_2c_2}{1+\gamma_{21}q_1} \end{bmatrix}. \tag{14}$$

The main results are summarized in the following proposition:

Proposition 2. *Let $c_i < a$, $i = 1, 2$. Then*

(i) *the fixed point $E_0 = (0, 0)$ is a repelling node*;

(ii) *the monopoly equilibrium E_1 is stable if*

$$v_1(a - c_1) < 2 \quad \text{and} \quad \gamma_{21} < \frac{2b(2c_2 - c_1 - a)}{a^2 - c_1^2}, \tag{15}$$

where the first inequality corresponds to the condition for attractivity along the invariant axis $q_2 = 0$ and the second inequality is the condition for attractivity along a direction transverse to the invariant axis. At $v_1(a - c_1) = 2$ a flip bifurcation occurs which creates a cycle along the q_1 axis, at

$$\gamma_{21} = \frac{2b(2c_2 - c_1 - a)}{a^2 - c_1^2}$$

a transcritical bifurcation occurs at which E_1 and E_* merge and exchange the stability along the transverse direction.

(iii) *the monopoly equilibrium E_2 is stable if*

$$v_2(a - c_2) < 2 \quad \text{and} \quad \gamma_{12} < \frac{2b(2c_1 - c_2 - a)}{a^2 - c_2^2}, \tag{16}$$

where the first inequality corresponds to the condition for attractivity along the invariant axis $q_1 = 0$ and the second inequality is the condition for attractivity along a direction transverse to the invariant axis. At $v_2(a - c_2) = 2$ a flip bifurcation occurs which creates a cycle along the q_2 axis, at

$$\gamma_{12} = \frac{2b(2c_1 - c_2 - a)}{a^2 - c_2^2}$$

a transcritical bifurcation occurs at which E_2 and E_* merge and exchange the stability along the transverse direction.

(iv) *the fixed point E_* is a stable node if conditions* (8) *are satisfied and v_1, v_2 are sufficiently small*; *it becomes a saddle point*, *through a supercritical flip bifurcation*, *as v_1 or v_2 are increased*, *or for increasing values of γ_{12} or γ_{21} provided that v_1 and v_2 are not too small. For*

[3] From an economic point of view, diverging trajectories do not represent interesting evolutions, as they can be interpreted as an irreversible departure from optimality.

$$\gamma_{12} = \frac{2b(2c_1 - c_2 - a)}{a^2 - c_2^2}, \quad \left(\gamma_{21} = \frac{2b(2c_2 - c_1 - a)}{a^2 - c_1^2}\right)$$

a transcritical bifurcation occurs at which $E_* = E_2$ $(E_* = E_1)$.

The proof, based on the standard analysis of the eigenvalues, is given in Appendix A.

From the Proposition 2 we can deduce that whenever the Nash equilibrium exists, i.e. E_* is inside the positive orthant, the monopoly equilibria are transversely unstable (saddle points or repelling nodes according to the first inequalities in (15) and (16) are satisfied or not). This implies that when the Nash equilibrium exists then the duopoly does not collapse into a monopoly, i.e. coexistence of firms is preserved. Of course, coexistence in the long run does not necessarily mean that the game will converge to E_*, since E_* may be a saddle point and some non-stationary dynamics of the game may be observed around it.

This proposition also confirms the role of spillovers to help the coexistence in the market of the two firms involved in the duopoly competition, in the sense that greater spillover effects not only contribute to ensure the existence of the positive Nash equilibrium, where both the firms produce and share the market, but also contribute to make the monopoly equilibria more repelling in the direction transverse to the coordinate axes. This means that if, at a certain time period, the production of a firm is close to zero, say $q_i \simeq 0$, then a sufficiently high γ_{ij} helps this firm to increase its production for the next period. By using a term from ecology, we may say that spillovers help the *persistence* of producers in the market.

3.2. Effects of the spillovers on global dynamics

Up to now, we only considered questions related to the existence and local stability of the equilibria. However, other issues are important in the study of long-run dynamic behavior of the duopoly game considered. Firstly, the question of what happens when the positive equilibrium exists, but it is not stable. Do the trajectories starting near the unstable Nash equilibrium remain close to it, thus giving some kind of bounded and positive dynamics (characterized, for example, by periodic or chaotic oscillations) or do they irreversibly depart from optimality?

Secondly, the question of the extension, in the strategy space $\mathbb{R}^2_+$, of the set of initial conditions which generate *economically feasible trajectories* (i.e. bounded and positive trajectories, which may or not converge to the Nash equilibrium). Does every initial condition generate an economically feasible time evolution of the game (10) or only a subset of points located around the Nash equilibrium?

Both these questions require a global analysis of the dynamical system represented by the iteration of the map (10). As we shall see, this study can be performed through a continuous dialogue between analytic, geometric and numerical methods. This is typical of the study of the global properties of non-linear dynamical systems of dimension greater than one, as clearly emphasized, among others, in [16,32,38].

If $\gamma_{12} = \gamma_{21} = 0$ the dynamic game (10) reduces to the one studied in [9], which will constitute our benchmark (no-spillovers) case in this section. So, for both the questions outlined above, the effect of the spillovers will be evaluated in comparison with the results given in [9], where it is shown that when the Nash equilibrium is unstable, feasible attractors may still exist around it, on which the productions of the two firms exhibit periodic or chaotic time paths. In the following we shall see that similar dynamic situations are still present with spillover effects, but changes in the spillover parameters may have remarkable effects on the creation and the structure of the complex attractors, and consequently on the qualitative properties of the time evolutions of the duopoly game. Concerning the second question, in [9] it is proved that diverging and negative trajectories (hence economically unfeasible) are obtained if the initial condition is taken sufficiently far from the Nash equilibrium. Moreover, the boundary which separates the set of initial strategies giving feasible trajectories from the complementary set is studied, and it is proved that in the simplest case such boundary is a quadrilateral whose sides are given by portions of the invariant coordinate axes and their rank-one preimages. However, global bifurcations are detected which lead to more complex structures of the boundary as the speeds of adjustment v_i are varied. As we shall see below, similar results hold with positive spillover parameters, and we shall evaluate the main differences caused by the increase of these parameters.

3.2.1. Effects of spillovers on the properties of complex attractors

As stated in Proposition 2, with the dynamic adjustment considered in this section, based on gradient dynamics, the introduction of spillover effects has a destabilizing role, in the sense that starting from situations in which the game has a stable Nash equilibrium with $\gamma_{12} = \gamma_{21} = 0$, it may lose stability and become a saddle point for increasing values of one (or both) γ_{ij}, and more complex attractors appear around E_*. Moreover, when the duopoly dynamics fail to converge to E_*, an increase in the spillover parameters may have some particular effects on the kind of oscillatory dynamics of the duopoly game.

We now show some of the numerical experiments which support the statements given above. In Fig. 2(a), obtained with parameters $a = 10$, $b = 0.5$, $v_1 = v_2 = 0.23$, $c_1 = c_2 = 2$, we consider the benchmark case of no spillover effects, i.e. $\gamma_{12} = \gamma_{21} = 0$. With this set of parameters, the Nash equilibrium

$$E_* = \left(\frac{a + c_2 - 2c_1}{3b}, \frac{a + c_1 - 2c_2}{3b} \right) = (5.\overline{3}, 5.\overline{3})$$

is stable, because the stability condition given in [9] for the no-spillovers case, i.e. $3b^2 q_1^* q_2^* v_1 v_2 - 4bq_1^* v_1 - 4bq_2^* v_2 + 4 < 0$, holds true. Its basin is represented by the white region, whereas the grey region represents the basin of infinity, i.e. the set of initial conditions which generates unbounded trajectories. As proved in [9], the basin of E_* is the interior of the quadrilateral $OO_{-1}^{(1)} O_{-1}^{(3)} O_{-1}^{(2)}$, where $O = (0,0)$ and the other three vertexes are its rank-1 preimages, i.e. the points such that $T(O_{-1}^{(i)}) = O$, $i = 1, 2, 3$, given by

$$O_{-1}^{(1)} = \left(\frac{1 + v_1(a - c_1)}{2bv_1}, 0 \right), \quad O_{-1}^{(2)} = \left(0, \frac{1 + v2(a - c_2)}{2bv_2} \right) \tag{17}$$

and

$$O_{-1}^{(3)} = \left(\frac{v_1 v_2 (a + c_2 - 2c_1) + 2v_2 - v_1}{3bv_1 v_2}, \quad \frac{v_1 v_2 (a + c_1 - 2c_2) + 2v_1 - v_2}{3bv_1 v_2} \right). \tag{18}$$

We now increase γ_{21} with $\gamma_{12} = 0$, in order to model an asymmetric situation where only one firm (firm 2 in this case) is able to take advantage from the rival's developments by exploiting its know-how. We obtain that the Nash equilibrium flip bifurcates as γ_{21} is increased, and in the situation shown in Fig. 2(b), obtained with $\gamma_{21} = 1$ and all the other parameters with the same values as in Fig. 2(a), the Nash equilibrium $E_* = (4.25, 7.49)$ is a saddle point. The generic feasible trajectory is attracted by a stable cycle of period two,

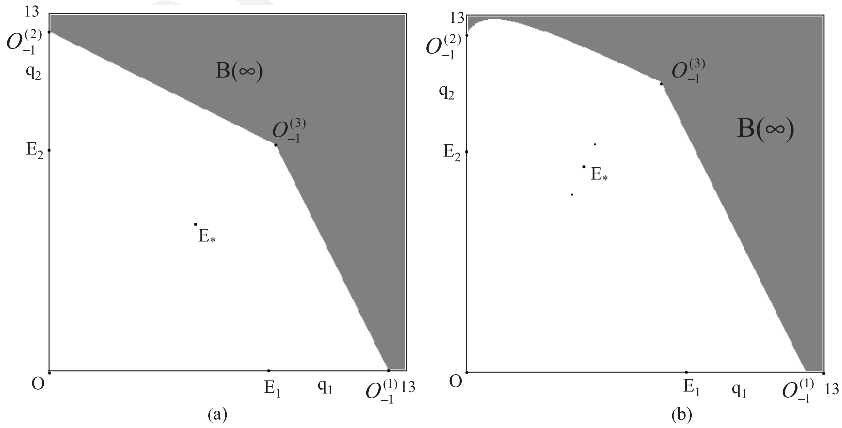

Fig. 2. Numerical representation of the attractors and the basins of attraction for the duopoly game with myopic adjustment. (a) For the benchmark (no spillover) case with parameters $a = 10$, $b = 0.5$, $v_1 = v_2 = 0.23$, $c_1 = c_2 = 2$, $\gamma_{12} = \gamma_{21} = 0$, the Nash equilibrium E_* is stable (a stable node). (b) With the same parameters a, b, v_i, c_i as in (a) and asymmetric spillover parameters $\gamma_{12} = 0$ and $\gamma_{21} = 1$ the Nash equilibrium is unstable (a saddle point) and the generic trajectory starting from the white region converges to a stable cycle of period 2. The two figures are obtained by taking a grid of initial conditions and generating, for each of them, a numerically computed trajectory of the duopoly map. If the trajectory is diverging then a grey dot is painted in the point corresponding to the initial condition, otherwise a white dot is painted.

given by $\mathscr{C}_2 = ((3.81, 6.46), (4.64, 8.27))$. It can be noticed that the increase of γ_{21} induces an increase of q_2^*, as remarked in Section 2, and the same is true "on the average" during the long-run periodic motion on the cycle $\mathscr{C}_2$. Another effect which can be noticed in Fig. 2b concerns a change of the shape of the boundary which separates the basin $\mathscr{B}$ of the feasible trajectories from the basin of infinity, denoted by $\mathscr{B}(\infty)$. Indeed, $\mathscr{B}$ becomes larger in the direction of q_2, so that the duopoly system seems to be less vulnerable with respect to perturbations of q_2. We shall analyze this question in the next subsection where we shall give the analytic expression of the boundary.

Here we are interested in the numerical exploration of the effects of the spillover parameters on the qualitative properties of the complex attractors which exist around the unstable Nash equilibrium. In Fig. 3a we consider a different set of parameters, with slightly higher values of the speeds of adjustment, given by $v_1 = v_2 = 0.32$ and, again, an asymmetric situation for the spillover parameters, given by $\gamma_{12} = 0$ and $\gamma_{21} = 0.25$. So, the two firms only differ in the asymmetric behavior with respect to spillovers. Due to the higher values of v_i, in this case the Nash equilibrium becomes unstable for smaller values of γ_{21} with respect to the case analyzed in Fig. 2. Indeed, in the situation shown in Fig. 3a chaotic dynamics occur, but the shape of the chaotic area implies a certain degree of correlation, in the sense that high (low) productions of firm 1 are associated with high (low) productions of firm 2 in the same period. An increase in the asymmetric spillover leads to a progressive loss of correlation, as it can be seen in Fig. 3b, obtained with $\gamma_{21} = 1$. In fact, in this case the large chaotic area suggests a very low correlation between the two production choices, in the sense that low production of a firm may be associated with low or high production of its competitor. So, even if in both the cases shown in Fig. 3 chaotic time series are obtained for the production choices of the two competitors, an higher asymmetry in spillover parameters introduces a loss of predictability because any correlation between the two production strategies is lost.

It can also be noticed that in Fig. 3b the boundary of the chaotic area is rather close to the basin boundary. Indeed a further increase of γ_{21} will lead to a contact between the chaotic area and the basin boundary which will cause the disappearance of the chaotic area (see [25,26]) and after the contact the generic trajectory will be divergent. Such a global bifurcation is called *final bifurcation* in [1,32] or *boundary crisis* in [24]. This confirms the destabilizing effects of too high spillover parameters.

We now consider, again, the set of parameters a, b, v_i, and c_i as in Fig. 3, and a more symmetric situation with respect to spillover parameters, namely $\gamma_{12} = 0.2$ and $\gamma_{21} = 0.25$. In this case the chaotic area becomes larger, as shown in Fig. 4a, but the density of the iterated points inside the chaotic area is mainly concentrated along the diagonal $q_1 = q_2$, i.e. a generic trajectory inside the chaotic area visits much more often the region around the diagonal with respect to the portions of the chaotic area which are far from the diagonal. This property reveals the occurrence of so-called *on-off intermittency* dynamics (see [35]) which typically arise in symmetric and quasi-symmetric dynamic duopoly games, see [10,12,29]. In order to explain better the kind of dynamics occurring in such a situation we show, in Fig. 4b, the versus time plot of

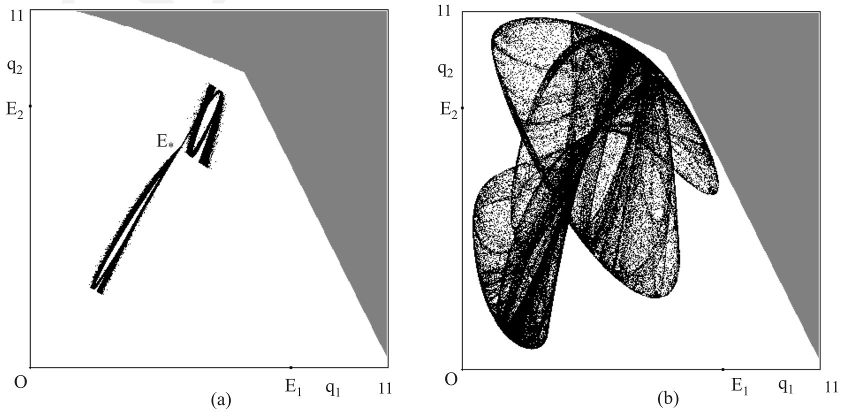

Fig. 3. (a) For $a = 10$, $b = 0.5$, $v_1 = v_2 = 0.32$, $c_1 = c_2 = 2$, $\gamma_{12} = 0$, $\gamma_{21} = 0.25$, the Nash equilibrium E_* is unstable and an attracting chaotic area exists around it. (b) With the same parameters a, b, v_i, c_i as in (a) and $\gamma_{12} = 0$, $\gamma_{21} = 1$ a larger chaotic area exists around the unstable Nash equilibrium

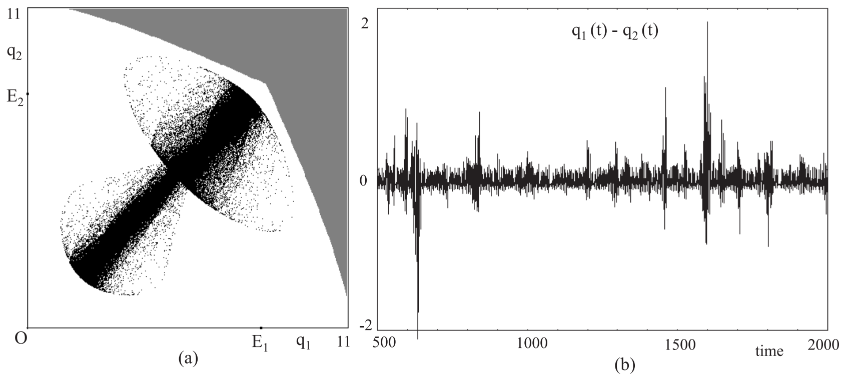

Fig. 4. (a) With the same parameters a, b, v_i, c_i as in Fig. 3 and $\gamma_{12} = 0.2$, $\gamma_{21} = 0.25$ the portion of the chaotic area close to the diagonal is more frequently visited by the iterated point. (b) With the same set of parameters as in (a) the versus time representation of the difference $q_1(t) - q_2(t)$ is represented along a generic chaotic trajectory.

the difference $q_1(t) - q_2(t)$ along a typical trajectory inside the chaotic area of Fig. 4a. It can be seen that the two productions are almost *synchronized*, i.e. $q_1(t) \simeq q_2(t)$, for several time periods, but sudden bursts occur sometimes at which some periods are characterized by very different production choices, which may be called periods of *asynchronous* production. The time periods at which such asynchronous bursts occur are randomly distributed along the time axis, so it is very difficult to make forecastings about their occurrence. However, their maximum amplitude can be determined through the study of critical curves of the non-invertible map (10) as described in [11] and [10] (see Appendix B for a definition of critical curve). Indeed, as shown in [11], the critical curves can be used to obtain the boundary of the chaotic area, so they define a sort of bounded vessel inside which the asymptotic dynamics are trapped, thus giving an upper bound for the on-off intermittency phenomena (see also [29]).

3.2.2. Effects of spillovers on the boundary of the set of feasible trajectories

In the following we denote by $\mathscr{B}$ the set of points which generate feasible trajectories, i.e. trajectories which are constituted by sequences of positive and bounded values of the state variables q_1 and q_2. A feasible trajectory may converge to the Nash equilibrium E_*, to another more complex attractor [4] inside $\mathscr{B}$ or to a one-dimensional invariant set embedded inside a coordinate axis. The last occurrence means that one of the two competitors exits the market, i.e. a monopoly situation is reached. However, we already know that when the Nash equilibrium exists, i.e. conditions (8) are satisfied, the coordinate axes are transversely unstable, so they behave as repelling sets with respect to trajectories approaching them from the interior of the non-negative orthant, and consequently evolutions of the duopoly game toward monopoly situations are excluded. Trajectories starting outside the set $\mathscr{B}$ represent exploding (or collapsing) evolutions of the economic system, because trajectories which start out of $\mathscr{B}$ always involve negative values and diverge. [5] In other words, the iterated map (10) has an attractor at infinite distance, and we denote the complementary of the set $\mathscr{B}$ as $\mathscr{B}(\infty)$. This can be interpreted by saying that the adjustment mechanism expressed by the dynamical system (10) is not suitable to model the time evolution of a duopoly system with initial productions outside the set $\mathscr{B}$.

An exact determination of the boundary $\partial\mathscr{B}$ which separates $\mathscr{B}$ from $\mathscr{B}(\infty)$, and the study of the qualitative changes of its structure as some parameters are let to vary, are important in the understanding of the dynamic behavior of the duopoly game proposed. This is the main goal of this subsection. The same problem has been studied in [9] in the case $\gamma_{12} = \gamma_{21} = 0$, where it is shown that $\partial\mathscr{B}$ is included in the set formed by the union of the coordinate axes and all their preimages, i.e. the set of all the points which are

[4] Several attractors may coexist inside $\mathscr{B}$, each with its own basin of attraction, although this has not been observed in our numerical explorations.

[5] This has been proved in [9] for the benchmark case with no spillovers, but similar arguments also apply to the model with spillovers.

mapped into the coordinate axes after a finite number of iterations of the map T. Indeed, the same result also applies to our case, as we explain below.

Let us first consider the dynamics of T restricted to the invariant axis $q_2 = 0$. From the one-dimensional restriction defined in (12), we can deduce that bounded trajectories along that invariant axis are obtained for $v_1(a - c_1) \leqslant 3$ (corresponding to $\mu \leqslant 4$ in (13)), provided that the initial conditions are taken inside the segment $\omega_1 = OO_{-1}^{(1)}$, where $O_{-1}^{(1)}$ is the rank-1 preimage of the origin O computed according to the restriction (12), i.e.

$$O_{-1}^{(1)} = \left(\frac{v_1(a - c_1)}{2bv_1}, 0 \right) \tag{19}$$

and divergent trajectories along the invariant q_1 axis are obtained starting from an initial condition out of the segment ω_1. Analogously, when $v_2(a - c_2) \leqslant 3$, bounded trajectories along the invariant q_2 axis are obtained provided that the initial conditions are taken inside the segment $\omega_2 = OO_{-1}^{(2)}$, where

$$O_{-1}^{(2)} = \left(0, \frac{v_2(a - c_2)}{2bv_2} \right) \tag{20}$$

and, also in this case, divergent trajectories along the q_2 axis are obtained starting from an initial condition out of the segment ω_2.

Consider now the region bounded by the segments ω_1 and ω_2 and their rank-1 preimages, say ω_1^{-1} and ω_2^{-1}, respectively. Such preimages can be analytically computed as follows. Let $X = (x, 0)$ be a point of ω_1. Its preimages are the real solutions (q_1, q_2) of the algebraic system obtained from (10) with $(q_1', q_2') = (x, 0)$:

$$\begin{aligned} q_1 \left[1 + v_1 \left(a - 2bq_1 - bq_2 - \frac{c_1}{1 + \gamma_{12} q_2} \right) \right] &= x, \\ q_2 \left[1 + v_2 \left(a - 2bq_2 - bq_1 - \frac{c_2}{1 + \gamma_{21} q_1} \right) \right] &= 0. \end{aligned} \tag{21}$$

From the second equation it is easy to see that the preimages of the points of ω_1 are either located on the same invariant axis $q_2 = 0$ or on the curve of equation

$$q_2 = r_2(q_1) + \frac{1}{2bv_2}, \tag{22}$$

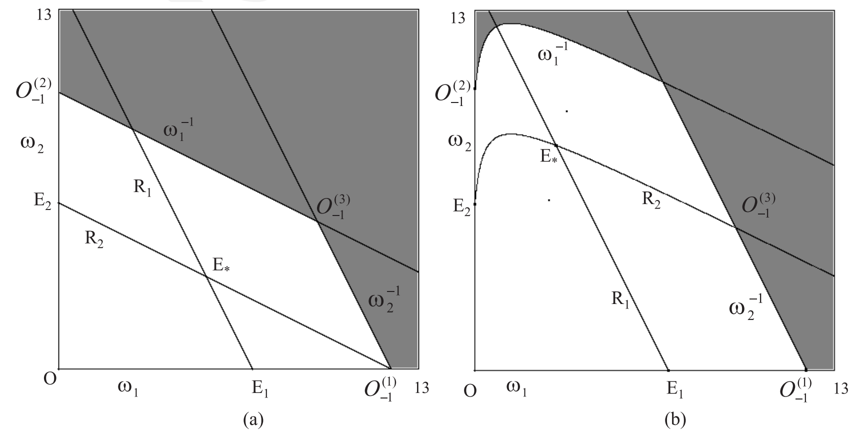

Fig. 5. (a) The reaction curves R_i, $i = 1, 2$, and the lines ω_i^{-1}, $i = 1, 2$, of Eqs. (22) and (23), respectively, are represented for the benchmark (no spillover) case with parameters $a = 10$, $b = 0.5$, $v_1 = v_2 = 0.25$, $c_1 = 3$, $c_2 = 4$, $\gamma_{12} = \gamma_{21} = 0$. (b) The reaction curves R_i, $i = 1, 2$, and the lines ω_i^{-1}, $i = 1, 2$, are represented for the same parameters a, b, v_i, c_i as in (a) and asymmetric spillover parameters $\gamma_{12} = 0$ and $\gamma_{21} = 3$.

where r_2 is the reaction function defined in (4). Analogously, the preimages of a point $Y = (0, y)$ of ω_2 belong to the same invariant axis $q_1 = 0$ or to the curve of equation

$$q_1 = r_1(q_2) + \frac{1}{2bv_1}, \tag{23}$$

where r_1 is the reaction function defined in (4). It is straightforward to see that curve (22) intersects the q_2 axis in the point $O_{-1}^{(2)}$ and curve (23) intersects the q_1 axis in the point $O_{-1}^{(1)}$. Moreover, the two curves intersect at a point $O_{-1}^{(3)}$, which is another rank-1 preimage of $O = (0, 0)$. These four rank-1 preimages of the origin are the vertexes of a "quadrilateral" $OO_{-1}^{(1)}O_{-1}^{(3)}O_{-1}^{(2)}$, whose sides are ω_1, ω_2 and their rank-1 preimages located on the curves of Eqs. (22) and (23), respectively, denoted by ω_1^{-1} and ω_2^{-1} in the Figs. 5 and 6. It is evident that the sides $O_{-1}^{(2)}O_{-1}^{(3)}$ and $O_{-1}^{(3)}O_{-1}^{(1)}$, given by ω_1^{-1} and ω_2^{-1} of Eqs. (22) and (23), respectively, are parallel translations of the reaction curves R_2 and R_1, shifted of $1/2bv_i$, $i = 2, 1$, respectively. All the points outside this quadrilateral cannot generate feasible trajectories. In fact, the points located on the right of ω_2^{-1} are mapped into points with negative q_1 after one iteration, as can be easily deduced from the first component of (10), and the points located above ω_1^{-1} are mapped into points with negative q_2 after one iteration, as can be deduced from the second component of (10).

For $\gamma_{12} = \gamma_{21} = 0$ the curves ω_1^{-1} and ω_2^{-1} reduce to straight lines, as already proved in [9]. This situation is shown in Fig. 5(a), obtained with $v_1 = 0.2$, $v_2 = 0.25$, $c_1 = 3$, $c_2 = 4$ and $\gamma_{12} = \gamma_{21} = 0$. With this set of parameters the Nash equilibrium E_* is stable, and the set $\mathscr{B}$ coincides with the basin of E_*. As it can be seen in Fig. 5(a), where the numerically computed basin of E_* is represented by the white region and the basin of infinity by the grey one, the boundary $\partial\mathscr{B}$ is formed by ω_1, ω_2 and their rank-1 preimages ω_1^{-1} and ω_2^{-1} of Eqs. (22) and (23), respectively, which are parallel to the reaction curves R_2 and R_1 (shown in Fig. 5). In Fig. 5(b) one of the spillover parameters is positive, namely $\gamma_{21} = 3$, and the other parameters are the same as in Fig. 5(a). It can be noticed that in this case the upper boundary, belonging to the curve ω_1^{-1}, is concave.

As proved in [9], the boundary of $\partial\mathscr{B}$ is given, in general, by the union of all the preimages, of any rank, of the segments ω_1 and ω_2

$$\partial\mathscr{B}(\infty) = \left(\bigcup_{n=0}^{\infty} T^{-n}(\omega_1)\right) \cup \left(\bigcup_{n=0}^{\infty} T^{-n}(\omega_2)\right), \tag{24}$$

where $T^{-n}(\omega_i)$ represents the set of all the points which are mapped into a point of ω_i after n iterations of the map T ($T^0(\omega_i)$ represents ω_i). However, the simple shape of $\partial\mathscr{B}$ shown in Fig. 5 is due to the fact that only preimages of rank-1 of ω_i exist. In fact, ω_1^{-1} and ω_2^{-1} are entirely included inside a region of the plane whose points have no preimages. The situation is different when the values of the parameters are such that some portions of these curves belong to regions whose points have preimages, which constitute preimages

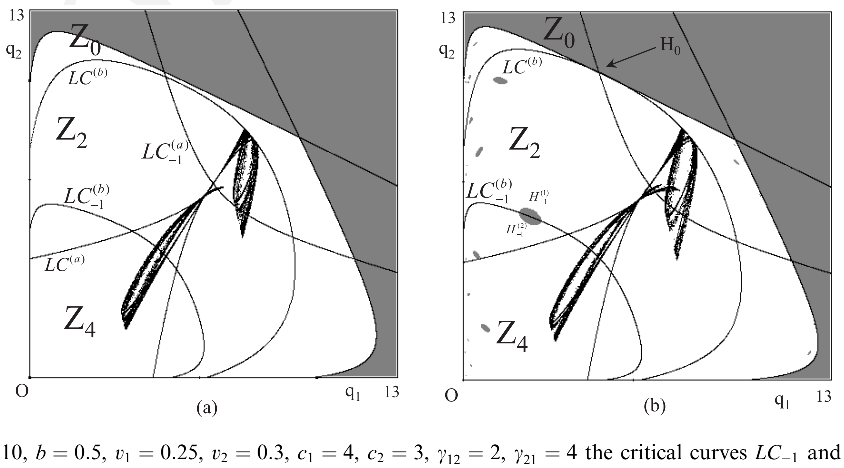

Fig. 6. (a) For $a = 10$, $b = 0.5$, $v_1 = 0.25$, $v_2 = 0.3$, $c_1 = 4$, $c_2 = 3$, $\gamma_{12} = 2$, $\gamma_{21} = 4$ the critical curves LC_{-1} and LC are represented, together with the boundaries which separate $\mathscr{B}$ from $\mathscr{B}(\infty)$. (b) With the same parameters a, b, v_i, c_i as in (a) and $\gamma_{12} = 3$, $\gamma_{21} = 7$ a portion of $\mathscr{B}(\infty)$ belongs to the region Z_2, and consequently "lakes" of $\mathscr{B}(\infty)$ are nested inside $\mathscr{B}$.

of rank higher than one of the segments ω_i. In this case the set $\mathscr{B}$ has a more complex topological structure, due to the fact that the map T is non-invertible (see Appendix B). The transitions between qualitatively different structures of the boundary $\partial\mathscr{B}$, as some parameters are varied, occur through so-called *contact bifurcations* (see e.g. [32]) which can be described in terms of contacts between $\partial\mathscr{B}$ and arcs of *critical curves*, as described below.

The map T defined in (10) is a non-invertible map (see Appendix B for definitions). In fact, given a point $(q'_1, q'_2) \in \mathbb{R}^2$ its preimages are computed by solving, with respect to q_1 and q_2, the following sixth degree algebraic system obtained from (10)

$$\begin{aligned} q_1\left[1 + v_1\left(a - 2bq_1 - bq_2 - \tfrac{c_1}{1+\gamma_{12}q_2}\right)\right] &= q'_1, \\ q_2\left[1 + v_2\left(a - 2bq_2 - bq_1 - \tfrac{c_2}{1+\gamma_{21}q_1}\right)\right] &= q'_2, \end{aligned} \tag{25}$$

which may have up to six real solutions. For example, as shown above, the origin $O = (0, 0)$ can have four rank-1 preimages, given by O itself and $O^{(i)}_{-1}$, $i = 1, 2, 3$.

For a given set of parameters, the critical curves of the map (10) can be easily obtained numerically following the procedure outlined in Appendix B. In fact, being the map (10) continuously differentiable, the set LC_{-1} can be obtained numerically as the locus of points (q_1, q_2) for which the Jacobian determinant $\det DT$ vanishes, where DT is given in (14). Then the critical curves LC, which separate regions Z_k whose points have different numbers of preimages, are obtained by computing the images of the points of LC_{-1}, i.e. $LC = T(LC_{-1})$. For example, for the set of parameters used to obtain Fig. 6(a), i.e. $v_1 = 0.25$, $v_2 = 0.3$, $c_1 = 4$, $c_2 = 3$ and $\gamma_{12} = 2$, $\gamma_{21} = 4$, the numerically computed set of points at which the Jacobian vanishes is formed by the union of two branches, denoted by $LC^{(a)}_{-1}$ and $LC^{(b)}_{-1}$ in Fig. 6(a). Also $LC = T(LC_{-1})$ is formed by two branches, denoted in Fig. 6(a) by $LC^{(a)} = T(LC^{(a)}_{-1})$ and $LC^{(b)} = T(LC^{(b)}_{-1})$. By definition (see Appendix B) each branch of the critical curve LC separates the phase plane of T into regions whose points have the same number of distinct rank-1 preimages: in our case $LC^{(b)}$ separates the region Z_0, whose points have no preimages, from the region Z_2, whose points have two distinct rank-1 preimages, and $LC^{(a)}$ separates the region Z_2 from Z_4, whose points have four distinct preimages.

The curve $LC^{(b)}_{-1}$ intersects the q_i axis at the point of maximum of restriction (12), given by $M^i_{-1} = 1 + v_i(a - c_i)/4bv_i$, and the curve $LC^{(b)}$ intersects the q_i axis at the corresponding maximum value $M^i = [1 + v_i(a - c_i)]^2/8bv_i$ of restriction (12).

As it can be seen in Fig. 6(a), the simple structure of the set $\mathscr{B}$, which is a simply connected set with the boundary $\partial\mathscr{B}$ having the "quadrilateral shape" described above, is due to the fact that the preimages ω_i^{-1}, $i = 1, 2$, of the invariant axes, are entirely included inside the region Z_0, so that no preimages of higher rank exist. The situation would be different if some portions of these lines were inside the regions Z_2 or Z_4. Indeed, the fact that a portion of $LC^{(b)}$ is close to $\partial\mathscr{B}$ suggests that a contact bifurcation may occur if some parameter is varied. In fact, if a portion of $\mathscr{B}(\infty)$ enters Z_2 after a contact of $\partial\mathscr{B}$ with $LC^{(b)}$, then new preimages of that portion will appear near $LC^{(b)}_{-1}$ and such preimages must belong to $\mathscr{B}(\infty)$. This is the situation illustrated by Fig. 6(b), obtained after an increase of the spillover parameters, i.e. $\gamma_{12} = 3$ and $\gamma_{21} = 7$. In fact, after a contact between $\partial\mathscr{B}$ and $LC^{(b)}$, a portion of $\mathscr{B}(\infty)$, say H_0 (bounded by a portion of ω_1^{-1} and LC) which was in region Z_0 before the bifurcation, enters inside Z_2. The points belonging to H_0 have two distinct preimages, located at opposite sides with respect to the line LC_{-1}, with the exception of the points of the curve $LC^{(b)}$ inside $B(\infty)$ whose preimages, according to the definition of LC, merge on $LC^{(b)}_{-1}$. Since H_0 is part of $\mathscr{B}(\infty)$ also its preimages belong to $\mathscr{B}(\infty)$. In other words, the rank-1 preimages of H_0 are formed by two areas joining along LC_{-1} and constitutes a *hole* of $\mathscr{B}(\infty)$ nested inside $\mathscr{B}$ (this hole is also called "lake" in [33]). This is the largest hole appearing in Fig. 6(b), and is called the *main hole*. It lies inside region Z_2, hence it has 2 preimages, which are smaller holes bounded by preimages of rank 3 of ω_1. Even these are both inside Z_2, so each of them has two further preimages inside Z_2, and so on. Now the boundary $\partial\mathscr{B}$ is formed by the union of an external part, given by the coordinate axes and their rank-1 preimages (22) and (23), and the boundaries of the holes, which are sets of preimages of higher rank of ω_1. So, the global bifurcation just described transforms a *simply connected* basin into a *multiply connected* one, with a countable infinity of holes, called *arborescent sequence of holes*, inside it (see [32,33] for a rigorous treatment of this type of global bifurcation, or [1] for a simpler and charming exposition).

To sum up, our numerical results show that the structure of the basins may become more complex as the spillover parameters γ_{ij} are increased. Moreover, the size of the holes of $\mathscr{B}(\infty)$ increases as one or both γ_{ij} become larger and larger. This leads to a higher probability of obtaining unfeasible trajectories, and this indicates that we are moving to unrealistic values of the spillover parameters.

We end this section by stressing that both in Figs. 6(a) and (b) segments of LC bound the upper portion of the chaotic area. Indeed, by drawing images of LC of higher rank, i.e. $LC_k = T^k(LC)$, an exact delimitation of the chaotic attractor can be obtained (see e.g. [1,32] or [38]), but we do not exploit such a property in this paper.

4. Best reply dynamics with naive expectations

We now consider a different kind of boundedly rational dynamic adjustment, based on the assumption that the two firms have a global knowledge of the profit function, so that they are able to compute their best reply to the expected production choice of the competitor, given by

$$q_i(t+1) = r_i\left(q_j^{(e)}(t+1)\right) \; i,j = 1,2 \quad i \neq j, \tag{26}$$

where $q_j^{(e)}$ represents the expectation of producer i about the next period production of producer j. However, we assume that the two firms are not so rational to be able to know in advance the competitor's choices, and like in the original Cournot paper [17], we assume that each firm adopts a very simple (or *naive*) expectation, by guessing that the production of the other firm will remain the same as in current period, i.e. $q_i^{(e)}(t+1) = q_i(t)$. This assumption, together with (26), leads to the following dynamical system $(q_1(t+1), q_2(t+1)) = T(q_1(t), q_2(t))$ where the map T is now given by

$$T : \begin{cases} lq_1' = r_1(q_2), \\ q_2' = r_2(q_1). \end{cases} \tag{27}$$

This dynamical system describes the so-called *best reply dynamics with naive expectations*. With this kind of dynamic adjustment, only the assumption (i), given in Section 2, is relaxed, whereas (ii) is now assumed to hold.

The equation $q_i(t+1) = q_i(t)$, which defines the steady states, is only satisfied at the intersections between the two reaction curves, hence the positive fixed points of the map (27) are Nash equilibria and vice versa. The stability properties of the Nash equilibrium, as well as the global dynamics of the dynamical system (27), are very simple. Indeed, due to the simple structure of the Jacobian matrix of (27), given by

$$DT(q_1, q_2) = \begin{bmatrix} 0 & r_1'(q_2) \\ r_2'(q_1) & 0 \end{bmatrix}$$

it is rather easy to prove, after some algebraic manipulations, that the eigenvalues,

$$z_{1,2} = \pm\sqrt{r_1'(q_2^*)r_2'(q_1^*)} = \pm\frac{1}{2}\sqrt{\frac{\left(b(1+q_2^*\gamma_{12})^2 - c_1\gamma_{12}\right)\left(b(1+q_1^*\gamma_{21})^2 - c_2\gamma_{21}\right)}{b^2(1+q_2^*\gamma_{12})^2(1+q_1^*\gamma_{21})^2}}$$

have modulus less than 1 whenever a positive Nash equilibrium $E_* = (q_1^*, q_2^*)$ exists.

Moreover, in this case also the delimitation of the feasible set $\mathscr{B}$ is quite straightforward: a feasible trajectory is generated if and only if the initial condition is taken in the rectangle

$$\mathscr{B} = \left[0, q_1^0\right] \times \left[0, q_2^0\right], \tag{28}$$

where q_1^0 and q_2^0 are the intersections of the reaction curves with the axes q_1 and q_2, respectively given by (7) and the expression obtained from it just swapping the indexes 1 and 2. In fact, from (27) follows that for $0 < q_2 < q_2^0$ we have $q_1' > 0$ and for $0 < q_1 < q_1^0$ we have $q_2' > 0$, and all the successive iterations give positive values being $\max q_1' = \max r_1(q_2) = r_1(\widehat{q}_2) < q_1^0$ and, symmetrically, $\max q_2' = \max r_2(q_1) =$

$r_2(\widehat{q}_1) < q_2^0$. Instead, for $q_1 > q_1^0$ we have $q_2' < 0$, i.e. a non-feasible trajectory, and, symmetrically, for $q_2 > q_1^0$ we have $q_1' < 0$.

However, we may assume that whenever $q_i(t) < 0$ a zero production decision occurs, i.e. we put $q_i(t) = 0$. With this assumption the best-reply dynamics gives $q_j(t+1) = r_j(0) = (a - c_j)/2b$ and $q_i(t+2) = r_i(q_j(t+1) = r_i((a - c_j)/2b) > 0$. In other words, with this kind of adjustment the coordinate axes are not trapping, so that the duopoly market is maintained even if some periods of no-production choices occur.

On the basis of the reasoning given above, and supported by numerical explorations, we can say that the generic trajectory starting from a positive initial condition converges to the unique Nash equilibrium E_* provided that it exists, i.e. (8) hold.

This behavior is very similar to the one observed in the benchmark game without spillover effects, which in this case is given by the classical linear Cournot game with naive expectations, that is, the standard example considered in any elementary textbooks. However, the result was not obvious. In fact, when spillovers are considered, and the parameters γ_{ij} are sufficiently large so that non-monotonic reaction functions are got (as explained in Section 2) things could be not so trivial. In fact, best reply dynamics with unimodal reaction functions may exhibit very complex behaviors, as was clearly proved in [39]. Indeed, as shown in [8], the dynamics of discrete dynamical systems of form (27) with non-monotonic functions r_i may be extremely rich, being characterized by the coexistence of many periodic and chaotic attractors with very intermingled basins of attraction. Particular economic situations where unimodal reaction functions are obtained as a consequence of non-linearities in demand or cost functions have been described by Dana and Montrucchio (see [18,28,37]). So, our assumption on cost externalities may be seen as a straightforward way to obtain unimodal reaction functions starting from a simple economic situation, but our conclusions show that no complexity is introduced if best reply dynamics with naive expectations is considered.

5. Conclusions

Starting from a standard Cournot duopoly game, we introduced positive cost externalities in order to investigate, in a simple and well-known framework, the effects of spillovers in a high-tech market. These effects are mainly related to the fact that firms which invest in R&D are not able to exclude that the benefits obtained by their own research spill over to competitors, due, for example, to employees which change firms or informal communication occurring during the innovation processes. From the analysis of the reaction curves of the game, our results indicate that the presence of spillovers generally helps coexistence at (or around) a Nash equilibrium, in the sense that they may contribute to avoid that a duopoly collapses into a monopoly.

If we extend these results to an oligopoly situation, where more than two firms producing homogeneous goods are present in the same district, the fact that spillover effects help to avoid the elimination of firms may be stated by saying that the existence of spillover effects may help the formation of clusters of firms producing a given good in the same district. Indeed, it is well known that flow of informations and facilities among firms of the same region is one of the main reasons for the creation of clusters of competing firms which operate in the same district, and *regional planners* interested in building up a certain industry cluster can help this process by providing infrastructure and incentives for the emergence of spillovers (see e.g. [4,5]).

In the spirit of the evolutionary games, we have also considered the problem of stability of the Nash equilibrium of the duopoly game under two different adjustment processes with boundedly rational players, both extensively supported by the current literature: one based on local (or myopic) profit maximization, obtained by following the direction of increasing marginal profits, and one based on the best reply dynamics with naive expectations (à la Cournot). With the first kind of dynamic adjustment we have shown that the introduction of spillover effects has a destabilizing role, in the sense that starting from situations in which the game has a stable Nash equilibrium, it fails to converge for increasing values of spillover parameters, and more complex attractors are obtained. Moreover, also the structure of the basins may become more complex as the spillover effects increase.

Instead, with the second type of adjustment, the Nash equilibrium remains stable even in the presence of spillovers. The fact that the stability under bounded rationality depends on the kind of adjustment considered is well known, and our results confirm this. Of course, the type of adjustment process which is suitable to describe a given duopoly depends on the market one is considering. In the recent literature many authors stress that firm behaviors in real markets (and in experimental economics) are often characterized by myopic strategies, like the one modeled by gradient dynamics (see e.g. [6,21,40] to cite a few).

This paper also collocates in the stream of non-linear duopoly games with non-monotonic reaction curves, which starting from the paper by Rand [39] gave rise to a flourishing literature (see e.g. [18,28,37] to cite a few).

6. Uncited reference

[2]

Acknowledgements

We thank the participants to the Workshop MDEF-2000 in Urbino for their interesting comments and suggestions, in particular Luca Colombo and Laura Gardini. This work has been performed under the auspices of CNR, Italy, and under the activity of the national research project "non-linear Dynamics and Stochastic Models in Economics and Finance", MURST, Italy.

Appendix A. Proofs

Proof of Proposition 1. In order to prove that there can be at most one Nash equilibrium, let us assume that a Nash equilibrium $E_* = (q_1^*, q_2^*)$ exists, and let us consider the line l through E_* with slope -1, i.e. the line of equation $q_1 + q_2 = q_1^* + q_2^*$, which can be written as

$$q_2 = l(q_1) = -q_1 + \frac{1}{2}q_1^* + \frac{a}{2b} - \frac{c_2}{2b(1+\gamma_{21}q_1^*)}$$

being $q_2^* = r_1(q_1^*)$. We now prove that the points of R_2 with $0 < q_1 < q_1^*$ lie below the line l and the points of R_2 with $q_1 > q_1^*$ lie above that line. In fact, $r_2(q_1) < l(q_1)$ if and only if

$$\frac{1}{2}q_1 + \frac{c_2}{2b(1+\gamma_{21}q_1)} > q_1 - \frac{1}{2}q_1^* + \frac{c_2}{2b(1+\gamma_{21}q_1^*)},$$

which is true for $0 < q_1 < q_1^*$ being, in this case,

$$\frac{c_2}{2b(1+\gamma_{21}q_1)} > \frac{c_2}{2b(1+\gamma_{21}q_1^*)} \quad \text{and} \quad \frac{1}{2}q_1 = q_1 - \frac{1}{2}q_1 > q_1 - \frac{1}{2}q_1^*$$

and the reverse inequalities hold if $q_1 > q_1^*$. Symmetrically, every point of R_1 is below the line l for $0 < q_2 < q_2^*$ and above l for $q_2 > q_2^*$. Therefore, there cannot be another point of intersection between R_1 and R_2.

In order to prove that at least one equilibrium exists if and only if conditions (8) hold, we recall that a Nash equilibrium $E_* = (q_1^*, q_2^*)$ exists if and only if q_1^* is a positive solution of $F_1(q_1) = q_1$, where $F_1(q_1) = r_1(r_2(q_1))$, and q_2^* is a positive solution of $F_2(q_2) = q_2$, where $F_2(q_2) = r_2(r_1(q_2))$. Let us first consider the equation $F_1(q_1) - q_1 = 0$. We have $F_1(0) \geqslant 0$ iff

$$\gamma_{12} \geqslant \frac{2b(2c_1 - c_2 - a)}{a^2 - c_2^2} \quad \text{and} \quad F_1(q_1^0) - q_1^0 \leqslant 0$$

iff $\gamma_{21} \geqslant 2b(2c_2 - c_1 - a)/a^2 - c_1^2$. So, if (8) hold then a solution $q_1^* \in (0, q_1^0)$ exist and vice versa (due to the already proved uniqueness). Symmetrically, $F_2(0) \geqslant 0$ iff

$$\gamma_{21} \geqslant \frac{2b(2c_2 - c_1 - a)}{a^2 - c_1^2} \quad \text{and} \quad F_2(q_2^0) - q_2^0 \leqslant 0$$

iff $\gamma_{12} \geqslant 2b(2c_1 - c_2 - a)/a^2 - c_2^2$. This proves the statement on existence. □

Proof of Proposition 2. We first consider the boundary equilibria E_i , $i = 0, 1, 2$. At $E_0 = (0, 0)$ the Jacobian matrix (14) becomes diagonal

$$DT(0,0) = \begin{bmatrix} 1 + v_1(a - c_1) & 0 \\ 0 & 1 + v_2(a - c_2) \end{bmatrix}$$

whose eigenvalues, given by the diagonal entries, are greater than 1 if $c_1 < a$ and $c_2 < a$. Thus, under the given assumptions, E_0 is a repelling node with eigendirections along the coordinate axes.

At $E_1 = (\frac{a-c_1}{2b}, 0)$ the Jacobian matrix is given by the triangular matrix

$$DT(E_1^*) = \begin{bmatrix} 1 - v_1(a - c_1) & v_1 \frac{a-c_1}{2b}(c_1\gamma_{12} - b) \\ 0 & 1 + v_2\left(\frac{(a+c_1)}{2} - \frac{c_2}{1+\gamma_{21}\left(\frac{a-c_1}{2b}\right)}\right) \end{bmatrix}$$

whose eigenvalues, given by the diagonal entries, are $z_1 = 1 - v_1(a - c_1)$, with eigenvector $\mathbf{r}_1^{(1)} = (1, 0)$ along the q_1 axis, and

$$z_2 = 1 + v_2\left(\frac{(a + c_1)}{2} - \frac{c_2}{1 + \gamma_{21}\left(\frac{a-c_1}{2b}\right)}\right)$$

with eigenvector

$$\mathbf{r}_1^{(2)} = \left(1, \frac{2b(\lambda_2 - \lambda_1)}{v_1(a - c_1)(c_1\gamma_{12} - b)}\right).$$

So, the condition for the stability along the invariant axis $q_2 = 0$ is $v_1(a - c_1) < 2$, and when the reverse inequality holds the well-known bifurcation scenario of a logistic map occurs, as can be easily deduced from the topological conjugacy given in (13). The stability condition $z_2 < 1$ can be written as the second of the (15). Moreover, it is straightforward to see that for $\gamma_{21} = 2b(2c_2 - c_1 - a)/a^2 - c_1^2$ we have $z_2 = 1$ and $q_1^{(0)} = (a - c_1)/2b$, so that $E_* = E_1$. Hence, this corresponds to a typical transcritical bifurcation (see e.g. [27] or [30]).

For E_2 symmetric considerations hold just swapping the indexes 1 and 2.

To study the local stability of the fixed point $E_* = (q_1^*, q_2^*)$, we consider the Jacobian matrix (14) which, by using the fact that $E_* \in R_1 \cap R_2$, i.e.

$$2bq_i^* = a - bq_j^* - \frac{c_i}{1 + \gamma_{ij}q_j^*} \; i, j = 1, 2, \quad i \neq j,$$

becomes

$$DT(E_*) = \begin{bmatrix} 1 - 2v_1bq_1^* & v_1q_1^*\left(\frac{c_1\gamma_{12}}{\left(1+\gamma_{12}q_2^*\right)^2} - b\right) \\ v_2q_2^*\left(\frac{c_2\gamma_{21}}{\left(1+\gamma_{21}q_1^*\right)^2} - b\right) & 1 - 2v_2bq_2^* \end{bmatrix} \tag{A.1}$$

Let Tr^* and Det^* be, respectively, the trace and the determinant of the matrix A.1. Then the characteristic equation becomes

$$P(z) = z^2 - \mathrm{Tr}^* \cdot z + \mathrm{Det}^* = 0$$

and a set of sufficient conditions for the stability of E_*, i.e. for the eigenvalues to be inside the unit circle of the complex plane, is given by

$$P(1) = 1 - \mathrm{Tr}^* + \mathrm{Det}^* > 0; \; P(-1) = 1 + \mathrm{Tr}^* + \mathrm{Det}^* > 0; \; 1 - \mathrm{Det}^* > 0 \tag{A.2}$$

From the analysis of the boundary equilibria we already know that when one of conditions (8) becomes an equality then we have $P(1) = 0$, i.e. $z = 1$ is an eigenvalue, and these equalities correspond to the occurrence of transcritical bifurcations related to the merging of E_* with E_1 and E_2, respectively. After some algebraic manipulations [6] it is possible to show that when conditions (8) hold the eigenvalues are real, i.e. $\mathrm{Tr}^* - 4\mathrm{Det}^* > 0$, $P(1) > 0$ whereas $P(-1)$ may change its sign. In particular, $P(-1) > 0$ is satisfied for sufficiently small values of v_1 or v_2, since $P(1) > 0$ as at least one of the v_i tends to zero, whereas it changes sign if one of them is increased with the other one fixed at a positive value. Moreover, for fixed positive values of both v_1 and v_2, $P(-1)$ becomes negative as one of the parameters γ_{ij} are increased. These sign changes of $P(-1)$ give rise to flip (or period doubling) bifurcations (see e.g. [27] or [30]).

Appendix B. Non-invertible maps and critical curves

In this appendix, we give some basic definitions and a minimal vocabulary concerning non-invertible maps of the plane and the method of critical curves. [7]

Let us consider a two-dimensional map $T : (x, y) \rightarrow (x', y')$ written in the form

$$(x', y') = T(x, y) = (f(x, y), g(x, y)) \tag{A.3}$$

where $(x, y) \in \mathbb{R}^2$ and f, g are assumed to be real valued continuous functions. The point $(x', y') \in \mathbb{R}^2$ is called rank-1 image of the point (x, y) under T, and (x, y) is called rank-1 preimage of the point (x', y'). The point $(x_t, y_t) = T^t(x, y)$, $t \in \mathbb{N}$, is called image of rank-t of the point (x, y), where T^0 is identified with the identity map and $T^t(\cdot) = T(T^{t-1}(\cdot))$. A point (x, y) such that $T^t(x, y) = (x_t, y_t)$ is called rank-t preimage of (x_t, y_t).

The map T is said to be non-invertible (or "many-to-one") if distinct points $(x_a, y_a) \neq (x_b, y_b)$ exist which have the same image, $T(x_a, y_a) = T(x_b, y_b) = (x, y)$. This can be equivalently stated by saying that points (x, y) exist which have several rank-1 preimages, i.e. the inverse relation $T^{-1}(x, y)$ is multivalued.

As the point (x, y) varies in the plane, the number of its rank-1 preimages can change, and according to the number of distinct rank-1 preimages associated with each point of $\mathbb{R}^2$, the plane can be subdivided into regions, denoted by Z_k, whose points have k distinct preimages. Generally pairs of real preimages appear or disappear as the point (x', y') crosses the boundary separating regions characterized by a different number of rank-1 preimages. Accordingly, such boundaries are generally characterized by the presence of two coincident (merging) preimages. This leads us to the definition of *critical curves*, one of the distinguishing features of non-invertible maps. The critical curve of rank-1, denoted by LC (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages. These preimages are located in a set called critical curve of rank-0, denoted by LC_{-1}. The curve LC is the two-dimensional generalization of the notion of critical value (local minimum or maximum value) of a one-dimensional map, and LC_{-1} is the generalization of the notion of critical point (local extremum point). From the definition given above it is clear that the relation $LC = T(LC_{-1})$ holds, and the points of LC_{-1} in which the map is continuously differentiable are necessarily points where the Jacobian determinant vanishes:

$$LC_{-1} \subseteq \left\{(x, y) \in \mathbb{R}^2 \middle| \det DT = 0\right\} \tag{A.4}$$

In fact, as LC_{-1} is defined as the locus of coincident rank-1 preimages of the points of LC, in any neighborhood of a point of LC_{-1} there are at least two distinct points mapped by T in the same point near LC. This means that the map T is not locally invertible in the points of LC_{-1} and, if the map T is continuously differentiable, it follows that $\det DT$ necessarily vanishes along LC_{-1}.

[6] The algebraic calculations, performed by the package Mathematica, are available from the authors.

[7] For a deeper treatment see [32]; see also [38] for several applications of the method of critical curves to non-invertible maps arising in dynamic economic modeling.

Portions of LC separate regions Z_k of the phase space characterized by a different number of $rank-1$ preimages, for example Z_k and Z_{k+2} (this is the standard occurrence). This property is at the basis of the contact bifurcations which give rise to complex topological structures of the basins, like those formed by non-connected sets or multiply connected sets. In fact, if a parameter variation causes a crossing between a basin boundary and a critical set which separates different regions Z_k so that a portion of a basin enters a region where an higher number of inverses is defined, then new components of the basin may suddenly appear at the contact.

Geometrically, the action of a non-invertible map T can be expressed by saying that it ‘‘folds and pleats’’ the plane, so that two or more distinct points are mapped into the same point, or, equivalently, that several inverses are defined which ‘‘unfold’’ the plane.

So, the backward iteration of a non-invertible map *repeatedly unfolds* the phase plane, and this implies that a basin may be non-connected, i.e. formed by several disjoint portions.

Instead, the fact that the forward iteration of a non-invertible map *repeatedly folds* the phase plane along the critical curves and their images, gives the property that segments of the critical curves LC, together with a suitable number of their images $LC_k = T^k(LC)$, may be used to bound a trapping regions, called absorbing areas in [32], which act like a bounded vessels inside which the asymptotic dynamics of the bounded trajectories are ultimately confined. In particular, this property of the critical curves allows one to obtain the boundaries of the chaotic areas, and practical procedures are given in the literature in order to obtain the boundary of a chaotic area by segments of critical curves (see e.g. [1,11,32,38].

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