Fractional Diffusion in Finance: Basic Theory*

Rudolf Gorenflo[(1)], **Francesco Mainardi**[(2)]

Marco Raberto[(3)] **and Enrico Scalas**[(4)]

[(1)] Erstes Mathematisches Institut, Freie Universität Berlin,
Arnimallee 3, D-14195 Berlin, Germany; E-mail: gorenflo@math.fu-berlin.de

[(2)] Dipartimento di Fisica, Università di Bologna,
Via Irnerio 46, I-40126 Bologna, Italy; E-mail: mainardi@bo.infn.it

[(3)] Dipartimento di Ingegneria Biofisica ed Elettronica, Università di Genova,
via dell'Opera Pia 11a, I–16145 Genova, Italy; E-mail: raberto@dibe.unige.it

[(4)] Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale,
via Cavour 84, I–15100 Alessandria, Italy; E-mail: scalas@unipmn.it

Abstract

In this paper we present a rather general phenomenological theory of tick-by-tick dynamics in financial markets, based on the continuous time random walk (CTRW) model. The theory can take into account the possibility of the non-Markovian character of financial time series by means of a generalized master equation with a time fractional derivative. We present predictions on the behaviour of the waiting-time probability density whose decay interpolates from a stretched exponential at small times to a power-law for long times. A proper transition to the so-called diffusion or hydrodynamic limit is also discussed by using scaling arguments. It turns out that the probability density function obeys a generalized diffusion equation of fractional order both in space and in time. Finally, a general representation of the fundamental solution of the fractional diffusion equation is given, which leads to a general scaling property for the the probability density function, henceforth to a statistical self-similarity for the limiting process.

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1. Introduction

The importance of random walks in finance has been known since the seminal work of Bachelier [1] which was completed at the end of the XIXth century, nearly a hundred years ago. The ideas of Bachelier were further carried out and improved by many scholars see *e.g.* Mandelbrot [37], Cootner [11], Samuelson [48], Balck and Scholes [4], Merton [39], Mantegna and Stanley [38], Bouchaud and Potters [5].

In a series of recent papers, see Scalas et al. [49], Mainardi et al. [36], Raberto et al. [45], Gorenflo et al. [26], the authors have argued that the continuous time random walk (CTRW) model, formerly introduced in Statistical Mechanics by Montroll and Weiss [42], can provide a phenomenological description of tick-by-tick dynamics in financial markets and they have discussed some applications concerning high frequency exchanges of bond futures. Here, we review our theoretical arguments along with financial applications.

The paper is divided as follows. Section 2 is devoted to revisit the theoretical framework of the CTRW model. We provide the most appropriate form for the general master equation, which is expected to govern the evolution of the probability density for non-local and non-Markovian processes. In Section 3 the conditions for the derivation of the time-fractional master equation are given to characterize non-Markovian processes with long memory. In this respect, we outline the central role played by the Mittag-Leffler function which exhibits an algebraic tail consistent with such processes. Section 4 is devoted to explain how the CTRW model can be used in describing the financial time series of the log-prices of an asset, for which the time interval between two consecutive transactions varies stochastically. In particular we test the theoretical predictions on the waiting-time distribution against empirical market data. The empirical analysis concerns high-frequency prices time series of German and Italian bond futures. In Section 5, we briefly discuss the transition to the *diffusion* (or *hydrodynamic*) *limit* in the time-fractional master equation. This leads to the *space-time fractional diffusion equation*. Section 6 is devoted to the scaling properties of the fundamental solution (Green function) of the space-time fractional diffusion, which are derived from its Fourier-Laplace representation. This is equivalent to provide the general *scaling form* for the probability density function of finding the log-price of an asset at a given time. Finally, the main conclusions are drawn in Section 7.

Appendices A and B introduce the correct notions of fractional derivatives in time and space, respectively, in a simple way, which enter our fractional diffusion equations.

2. The CTRW model in statistical physics

We recall that the CTRW model leads to the general problem of computing the probability density function (*pdf*) $p(x,t)$ $(x \in \mathbf{R},\ t \in \mathbf{R}^+)$ of finding, at position x at time t, a particle (the walker) which performs instantaneous random jumps $\xi_i = x(t_i) - x(t_{i-1})$ at random instants t_i with $i = 1, 2, \dots$. We denote by $\tau_i = t_i - t_{1-1}$ the (so-called) *waiting times*. As usual, it is assumed that the particle is located at $x_0 = 0$ for $t_0 = 0$, which means $p(x,0) = \delta(x)$. We denote by $\varphi(\xi,\tau)$ the *joint probability density* for jumps and waiting times.

The CTRW is generally defined through the requirement that the ξ_i and τ_i are independent identically distributed (i.i.d.) random variables with *pdf*'s independent of each other, so that we have the factorization $\varphi(\xi,\tau) = w(\xi)\,\psi(\tau)$, which implies $w(\xi) = \int_0^\infty \varphi(\xi,\tau)\,d\tau$, $\psi(\tau) = \int_{-\infty}^{+\infty} \varphi(\xi,\tau)\,d\xi$. The marginal probability densities w and ψ are called *jump pdf* and *waiting-time pdf*, respectively. Of course, all the probability densities are assumed to be non negative and subjected to the normalization conditions.

We now provide further details on the densities $w(\xi)$, $\varphi(\tau)$ in order to derive their relation with the *pdf* $p(x,t)$.

The *jump pdf* $w(\xi)$ represents the *pdf* for transition of the walker from a point x to a point $x + \xi$, so it is also called the *transition pdf*. The *waiting-time pdf* represents the *pdf* that a step is taken at the instant $t_{i-1} + \tau$ after the previous one that happened at the instant t_{i-1}, so it is also called the *pausing-time pdf*. Therefore, the probability that $\tau \le t_i - t_{i-1} < \tau + d\tau$ is equal to $\psi(\tau)\,d\tau$. The probability that a given waiting interval is greater or equal to τ will be denoted by $\Psi(\tau)$, which is defined in terms of $\psi(\tau)$ by

$$\Psi(\tau) = \int_\tau^\infty \psi(t')\,dt' = 1 - \int_0^\tau \psi(t')\,dt'\,, \quad \psi(\tau) = -\frac{d}{d\tau}\Psi(\tau)\,. \tag{2.1}$$

We note that $\int_0^\tau \psi(t')\,dt'$ represents the probability that at least one jump is taken at some instant in the interval $[0,\tau)$, hence $\Psi(\tau)$ is the probability that the walker is sitting in x at least during the time interval of duration τ after a jump. Recalling that $t_0 = 0$, we also note that $\Psi(t)$ represents the so called *survival probability*, namely the probability of finding the walker at the initial position $x_0 = 0$ until time instant t.

Now, only based upon the previous probabilistic arguments, we can derive the evolution equation for the *pdf* $p(x,t)$, that we shall call the *master equation* of the CTRW. In fact, we are led to write

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \psi(t-t')\,\left[\int_{-\infty}^{+\infty} w(x-x')\,p(x',t')\,dx'\right]\,dt'\,, \tag{2.2}$$

where we recognize the role of the *survival probability* $\Psi(t)$ and of the *pdf*'s $\psi(t)$, $w(x)$. The first term in the RHS of Eq. (2.2) expresses the persistence (whose strength decreases with increasing time) of the initial position $x = 0$. The second term (a spatio-temporal convolution) gives the contribution to $p(x,t)$ from the walker sitting in point $x' \in \mathbf{R}$ at instant $t' < t$ jumping to point x just at instant t, after stopping (or waiting) time $t - t'$. Furthermore, as a check for the correctness of Eq. (2.2) we can easily verify that $p(x,t) \geq 0$ for all $t \geq 0$ and $x \in \mathbf{R}$, and $\int_{-\infty}^{+\infty} p(x,t)\, dx = 1$ for all $t \geq 0$.

Originally the *master equation* was derived by Montroll and Weiss in 1965, see [42], recurring to the tools of the Fourier-Laplace transforms. These authors showed that the Fourier-Laplace transform of $p(x,t)$ satisfies a characteristic equation, now called the *Montroll-Weiss equation*, which reads

$$\widehat{\widetilde{p}}(\kappa, s) = \widetilde{\Psi}(s)\, \frac{1}{1 - \widehat{w}(\kappa)\, \widetilde{\psi}(s)}\,, \quad \text{with} \quad \widetilde{\Psi}(s) = \frac{1 - \tilde{\psi}(s)}{s}\,. \tag{2.3}$$

Here, we have adopted the following standard notation for the generic Fourier and Laplace transforms:

$$\mathcal{F}\,\{f(x); \kappa\} = \widehat{f}(\kappa) = \int_{-\infty}^{+\infty} \mathrm{e}^{\,i\kappa x}\, f(x)\, dx\,, \quad \mathcal{L}\,\{g(t); s\} = \widetilde{g}(s) = \int_0^{\infty} \mathrm{e}^{\,-st}\, g(t)\, dt\,,$$

where $f(x)$ $(x \in \mathbf{R})$ and $g(t)$ $(t \in \mathbf{R}^+)$ are sufficiently well-behaved functions of their arguments. It is straightforward to verify the equivalence between the Eqs. (2.2) and (2.3) by recalling the well-known properties of the Fourier and Laplace transforms with respect to the space and time convolution.

Hereafter, we present an alternative form to Eq. (2.2), formerly proposed by Mainardi et al. [36], which involves the first time derivative of $p(x,t)$ (along with an additional auxiliary function), so that the resulting equation can be interpreted as an *evolution* equation of *Fokker-Planck-Kolmogorov* type. To this purpose we re-write Eq. (2.3) as

$$\widetilde{\Phi}(s)\, \left[s\, \widehat{\widetilde{p}}(\kappa, s) - 1\right] \;=\; \left[\widehat{w}(\kappa) - 1\right]\, \widehat{\widetilde{p}}(\kappa, s)\,, \tag{2.4}$$

where

$$\widetilde{\Phi}(s) = \frac{1 - \widetilde{\psi}(s)}{s\, \widetilde{\psi}(s)} = \frac{\widetilde{\Psi}(s)}{\widetilde{\psi}(s)} = \frac{\widetilde{\Psi}(s)}{1 - s\widetilde{\Psi}(s)}\,. \tag{2.5}$$

Then our master equation reads

$$\int_0^t \Phi(t - t')\, \frac{\partial}{\partial t'} p(x, t')\, dt' \;=\; -p(x,t) + \int_{-\infty}^{+\infty} w(x - x')\, p(x', t)\, dx'\,, \tag{2.6}$$

where the "auxiliary" function $\Phi(t)$, being defined through its Laplace transform in Eq. (2.5), is such that $\Psi(t) = \int_0^t \Phi(t - t')\, \psi(t')\, dt'$. We remind the reader that Eq. (2.6), combined with the initial condition $p(x, 0) = \delta(x)$, is equivalent to

Eq. (2.4), and then its solution represents the Green function or the fundamental solution of the Cauchy problem for Eq. (2.6).

From Eq. (2.6) we recognize the role of $\Phi(t)$ as a "memory function" . As a consequence, the CTRW turns out to be in general a non-Markovian process. However, the process is "memoryless", namely "Markovian" if (and only if) the above memory function degenerates into a delta function (multiplied by a certain positive constant) so that $\Psi(t)$ and $\psi(t)$ may only differ by a multiplying positive constant. By appropriate choice of the unit of time we assume $\widetilde{\Phi}(s) = 1$, so $\Phi(t) = \delta(t)$, $t \geq 0$. In this case we derive

$$\widetilde{\psi}(s) = \widetilde{\Psi}(s) = \frac{1}{1+s}, \quad \text{so} \quad \psi(t) = \Psi(t) = \mathrm{e}^{-t}, \; t \geq 0. \tag{2.7}$$

Then Eq. (2.6) reduces to

$$\frac{\partial}{\partial t} p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\, p(x',t)\, dx', \quad p(x,0) = \delta(x). \tag{2.8}$$

This is, up to a change of the unit of time (which means multiplication of the RHS by a positive constant), the most general *master equation* for a *Markovian* CTRW; it is called the *Kolmogorov-Feller equation* in [46].

We note that the form (2.6), by exhibiting a weighted first-time derivative, is original as far as we know; it allows us to characterize in a natural way a peculiar class of non-Markovian processes, as shown in the next Section. Furthermore, Eq. (2.6) represents for us the suitable starting point to derive from it the generalized diffusion equations of fractional order in time and/or in space that will be treated later.

In closing this Section we note that several authors have treated the CTRW model and/or the passage to the fractional diffusion equation. The reader, for example, may refer to the following list of selected papers [43, 55, 41, 29, 54, 28, 30, 27, 51, 2, 52, 3, 40], and to the references therein quoted.

3. The time-fractional master equation for "long-memory" processes

Let us now consider "long-memory" processes, namely non-Markovian processes characterized by a memory function $\Phi(t)$ exhibiting a power-law time decay. To this purpose a natural choice is

$$\Phi(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad t \geq 0, \quad 0 < \beta < 1. \tag{3.1}$$

Thus, $\Phi(t)$ is a weakly singular function that, in the limiting case $\beta = 1$, reduces to $\Phi(t) = \delta(t)$, according to the formal representation of the Dirac generalized function, $\delta(t) = t^{-1}/\Gamma(0)$, $t \geq 0$ (see *e.g.* [15]).

As a consequence of the choice (3.1), we see that (in this peculiar non-Markovian situation) our *master equation* (2.6) contains a time fractional derivative. In fact, by inserting into Eq. (2.4) the Laplace transform of $\Phi(t)$, $\widetilde{\Phi}(s) = 1/s^{1-\beta}$, we get

$$s^\beta \,\widehat{\widetilde{p}}(\kappa, s) - s^{\beta-1} = [\widehat{w}(\kappa) - 1]\, \widehat{\widetilde{p}}(\kappa, s)\,, \quad 0 < \beta < 1\,, \tag{3.2}$$

so that the resulting Eq. (2.6) can be written as

$$\frac{\partial^\beta}{\partial t^\beta} p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\, p(x',t)\, dx'\,, \quad p(x,0) = \delta(x)\,, \tag{3.3}$$

where $\partial^\beta/\partial t^\beta$ is the pseudo-differential operator explicitly defined in the Appendix A, that we call the *Caputo* fractional derivative of order β. Thus Eq. (3.3) can be considered as the time-fractional generalization of Eq. (2.8) and consequently can be called the *time-fractional Kolmogorov-Feller equation*. We note that this derivation differs from that presented in [49] and references therein, in that here we have pointed out the role of the long-memory processes rather than that of scaling behaviour in the hydrodynamic limit. Furthermore, here the *Caputo* fractional derivative appears in a natural way without use of the *Riemann-Liouville* fractional derivative.

Our choice for $\Phi(t)$ implies peculiar forms for the functions $\Psi(t)$ and $\psi(t)$ that generalize the exponential behaviour (2.7) of the Markovian case. In fact, working in the Laplace domain we get from (2.5) and (3.1)

$$\widetilde{\Psi}(s) = \frac{s^{\beta-1}}{1+s^\beta}\,, \quad \widetilde{\psi}(s) = \frac{1}{1+s^\beta}\,, \quad 0 < \beta < 1\,, \tag{3.4}$$

from which by inversion we obtain for $t \ge 0$

$$\Psi(t) = E_\beta(-t^\beta)\,, \quad \psi(t) = -\frac{d}{dt} E_\beta(-t^\beta)\,, \quad 0 < \beta < 1\,, \tag{3.5}$$

where E_β denotes an entire transcendental function, known as the Mittag-Leffler function of order β, defined in the complex plane by the power series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta\, n+1)}\,, \quad \beta > 0\,, \quad z \in \mathbf{C}\,. \tag{3.6}$$

For detailed information on the Mittag-Leffler-type functions and their Laplace transforms the reader may consult *e.g.* [12, 20, 21, 34]. We note that for $0 < \beta < 1$ and $1 < \beta < 2$ the function $\Psi(t)$ appears in certain relaxation and oscillation processes, then called *fractional relaxation* and *fractional oscillation* processes, respectively (see *e.g.* [20, 21, 32, 33] and references therein).

Hereafter, we find it convenient to summarize the features of the functions $\Psi(t)$ and $\psi(t)$ most relevant for our purposes. We begin to quote their series expansions and asymptotic representations:

$$\Psi(t) \begin{cases} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta\, n+1)}\,, & t \ge 0 \\ \sim \displaystyle\frac{\sin(\beta\pi)}{\pi}\,\frac{\Gamma(\beta)}{t^{\beta}}\,, & t \to \infty\,, \end{cases} \tag{3.7}$$

and

$$\psi(t) \begin{cases} = \displaystyle\frac{1}{t^{1-\beta}} \sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta\, n+\beta)}\,, & t \ge 0 \\ \sim \displaystyle\frac{\sin(\beta\pi)}{\pi}\,\frac{\Gamma(\beta+1)}{t^{\beta+1}}\,, & t \to \infty\,. \end{cases} \tag{3.8}$$

The expression for $\psi(t)$ can be shown to be equivalent to that one obtained by Hilfer and Anton [27] in terms of the generalized Mittag-Leffler function in two parameters. We consider it conceptually more economical to remain (as long as possible) in the kingdom of Mittag-Leffler functions with one parameter only.

In the limit for $\beta \to 1$ we recover the exponential functions of the Markovian case. We note that for $0 < \beta < 1$ both functions $\psi(t)$, $\Psi(t)$, even if losing their exponential decay by exhibiting power-law tails for large times, keep the "completely monotonic" character. Complete monotonicity of the functions $\psi(t)$, $\Psi(t)$, $t > 0$, means:

$$(-1)^n \frac{d^n}{dt^n}\,\Psi(t) \ge 0\,, \quad (-1)^n \frac{d^n}{dt^n}\,\psi(t) \ge 0\,, \quad n = 0, 1, 2, \ldots \tag{3.9}$$

or equivalently, their representability as (real) Laplace transforms of non-negative functions. In fact, it can be shown for $0 < \beta < 1$:

$$\Psi(t) = \frac{\sin(\beta\pi)}{\pi} \int_0^{\infty} \frac{r^{\beta-1}\, \mathrm{e}^{-rt}}{r^{2\beta} + 2\, r^{\beta}\, \cos(\beta\pi) + 1}\, dr\,, \quad t \ge 0\,, \tag{3.10}$$

and

$$\psi(t) = \frac{\sin(\beta\pi)}{\pi} \int_0^{\infty} \frac{r^{\beta}\, \mathrm{e}^{-rt}}{r^{2\beta} + 2\, r^{\beta}\, \cos(\beta\pi) + 1}\, dr\,, \quad t \ge 0\,. \tag{3.11}$$

A special case is $\beta = \frac{1}{2}$ for which it is known that

$$E_{1/2}(-\sqrt{t}) = \mathrm{e}^{\,t}\, \mathrm{erfc}(\sqrt{t}) = \mathrm{e}^{\,t}\, \frac{2}{\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} \mathrm{e}^{-u^2}\, du\,, \quad t \ge 0\,, \tag{3.12}$$

where erfc denotes the *complementary error* function.

It may be instructive to note that for sufficiently small times $\Psi(t)$ exhibits a behaviour similar to that of a stretched exponential; in fact we have

$$E_{\beta}(-t^{\beta}) \simeq 1 - \frac{t^{\beta}}{\Gamma(\beta+1)} \simeq \exp\{-t^{\beta}/\Gamma(1+\beta)\}\,, \quad 0 \le t \ll 1\,. \tag{3.13}$$

4. The CTRW model in statistical finance

The price dynamics in financial markets can be mapped onto a random walk whose properties are studied in continuous, rather than discrete, time, see *e.g.* [39].

As a matter of fact, there are various ways in which to embed a random walk in continuous time. Here, we shall base our approach on the CTRW discussed in Sect. 2, in which time intervals between successive steps are i.i.d. random variables.

Let $S(t)$ denote the price of an asset or the value of an index at time t. In finance, returns rather than prices are considered. For this reason, in the following we shall take into account the variable $x(t) = \log S(t)$, that is the logarithm of the price. Indeed, for a small price variation $\Delta S = S(t_i) - S(t_{i-1})$, the return $r = \Delta S/S(t_{i-1})$ and the logarithmic return $r_{log} = \log [S(t_i)/S(t_{i-1})]$ virtually coincide. The statistical physicist will recognize in x the position of a random walker jumping in one dimension. Thus, in the following, we shall use the language and the notations of Sect. 2.

In financial markets, prices are fixed when demand and offer meet and a transaction occurs. In this case, we say that a trade takes place. As a consequence, not only prices but also waiting times between two consecutive transactions can be modelled as random variables. In agreement with the assumptions of Sect. 2, we consider the returns $\xi_i = x(t_i) - x(t_{i-1})$ as i.i.d random variables with *pdf* $w(\xi)$ and the waiting times $\tau_i = t_i - t_{i-1}$ as i.i.d. random variables with *pdf* $\psi(\tau)$. In real processes of financial markets this independence hypothesis may not strictly hold for their duration or not be verified at all. Therefore, it may be considered with caution.

In the following we limit ourselves to investigate the consistency of the long-memory process analyzed in Sect. 3 with respect to the empirical data concerning exchanges of certain financial derivatives.

We have considered the waiting time distributions of certain futures traded at LIFFE in 1997 and estimated the corresponding empirical survival probabilities. LIFFE stands for *London International Financial Futures (and Options) Exchange*. It is a London-based derivative market; for further information, see `http://www.liffe.com`. Futures are derivative contracts in which a party agrees to sell and the other party to buy a fixed amount of an underlying asset at a given price and at a future delivery date.

As underlying assets we have chosen German and Italian Government bonds, called BUND and BTP respectively, for both of which the delivery dates are June and September 1997*.

* BUND and BTP (*Buoni del Tesoro Poliennali*) are respectively the German and Italian word for BOND (middle and long term Government bonds with fixed interest rate).

Hereafter we summarize the results obtained by Mainardi et al. [36] and Raberto et al. [45].

Usually, for a future with a certain maturity, transactions begin some months before the delivery date. At the beginning, there are few trades a day, but closer to the delivery there may be more than 1 000 transactions a day. For each maturity, the total number of transactions is greater than 160 000.

In Figs. 4.1-4.4 we plot $\Psi(\tau)$ for the four cases (June and September delivery dates for BUND and BTP).

The circles refer to market data and represent the probability of a waiting time greater than the abscissa τ. We have determined about 500 values of $\Psi(\tau)$ for τ in the interval between $1\,s$ and $50\,000\ s$, neglecting the intervals of market closure.

The solid line is a two-parameter fit obtained by using the Mittag-Leffler type function

$$\Psi(\tau) = E_\beta\left[-(\gamma\tau)^\beta\right]\,, \tag{4.1}$$

where β is the index of the Mittag-Leffler function and γ is a time-scale factor, depending on the time unit. The dash-dotted line is the stretched exponential function $\exp\{-(\gamma\tau)^\beta)/\Gamma(1+\beta)\}$, see the RHS of Eq. (3.13), whereas the dashed line is the power law function $(\gamma\tau)^{-\beta}/\Gamma(1-\beta)$, see the RHS of the second Eq. in (3.7), noting that $\Gamma(\beta)\,\sin(\beta\pi)/\pi = 1/\Gamma(1-\beta)$.

The Mittag-Leffler function well interpolates between these two limiting behaviours: the stretched exponential for small time intervals, and the power law for large ones.

As regards the BUND futures we can summarize as follows. For the June delivery date we get an index $\beta = 0.96$ and a scale factor $\gamma = \frac{1}{12}$, whereas, for the September delivery date, we have $\beta = 0.95$ and $\gamma = \frac{1}{12}$. The fit in Fig. 4.1 has a reduced chi square $\widetilde{\chi}^2 \simeq 0.26$, whereas the reduced chi square of the fit in Fig. 4.2 is $\widetilde{\chi}^2 \simeq 0.25$.

As regards the BTP futures we can summarize as follows. For the both the June and September delivery dates we get the same index $\beta = 0.96$ and the same scale factor $\gamma = \frac{1}{13}$. The fits in Fig. 4.3 and 4.4 have a reduced chi square $\widetilde{\chi}^2 \simeq 0.2$.

We may note the common behaviour of the survival probabilities found from the trading of the above assets. This might be corroborated or not in other cases. To the possible objection that, in all four cases here treated, β does not differ significantly from 1 and so the process still could be Markovian, we answer that then we would have $\Psi(\tau) = \exp(-\gamma\tau)$ and the graph of $\Psi(\tau)$ would look completely different for sufficiently long times.

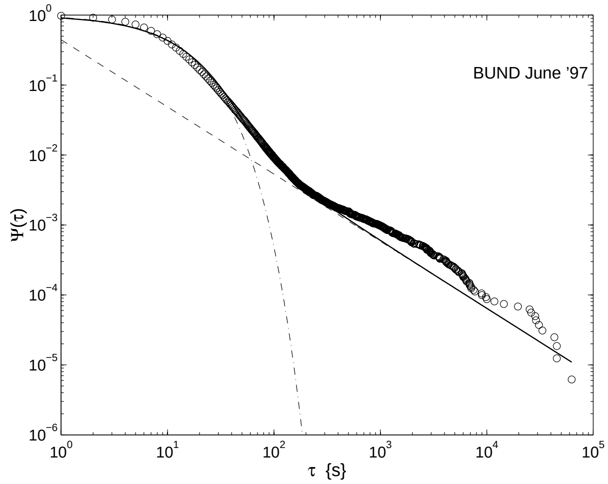

Fig. 4.1

Survival probability for BUND futures with delivery date: June 1997.
The Mittag-Leffler function (solid line) of index $\beta = 0.96$ and scale factor $\gamma = 1/12$
is compared to the stretched exponential (dash-dotted line) and the power law (dashed line).

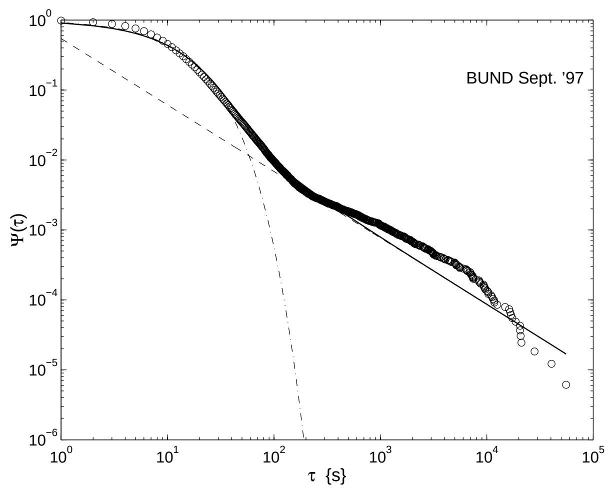

Fig. 4.2

Survival probability for BUND futures with delivery date: September 1997.
The Mittag-Leffler function (solid line) of index $\beta = 0.95$ and scale factor $\gamma = 1/12$
is compared to the stretched exponential (dash-dotted line) and the power law (dashed line).

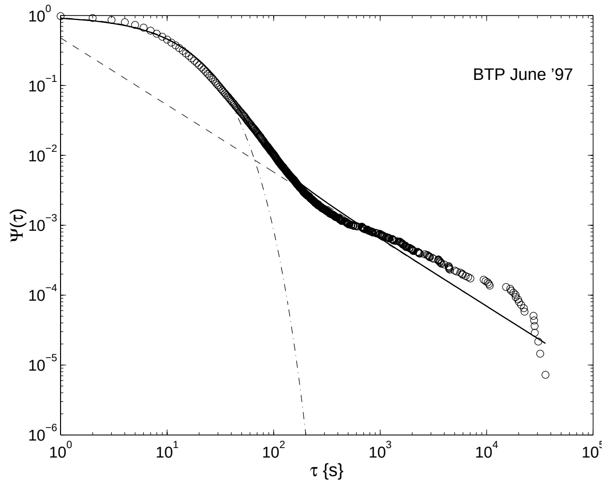

Fig. 4.3

Survival probability for BTP futures with delivery date: June 1997.
The Mittag-Leffler function (solid line) of index $\beta = 0.96$ and scale factor $\gamma = 1/13$
is compared to the stretched exponential (dash-dotted line) and the power law (dashed line).

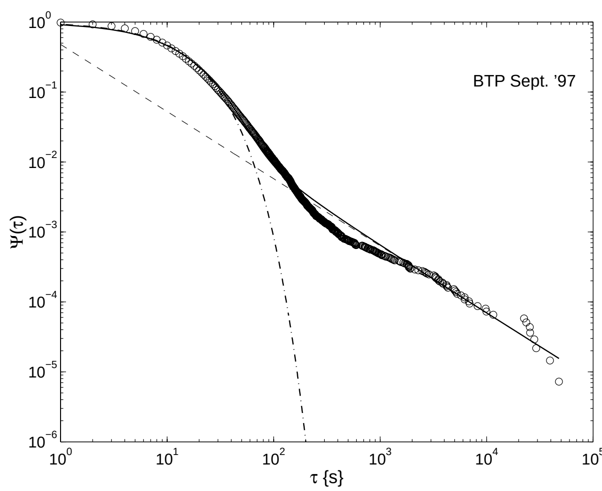

Fig. 4.4

Survival probability for BTP futures with delivery date: September 1997.
The Mittag-Leffler function (solid line) of index $\beta = 0.96$ and scale factor $\gamma = 1/13$
is compared to the stretched exponential (dash-dotted line) and the power law (dashed line).

5. The transition to the space-time fractional diffusion

Let us now consider the so called *diffusion* or *hydrodynamic limit* for the transition from the time-fractional master equation (3.3) to a space-time fractional diffusion equation. Our aim is to prove this transition in a correct and transparent way by using proper scaling arguments. In other words, we would like to show which scaling assumptions ensure the validity of the fractional diffusion limit. Indeed, the correct transition to this limit needs a special care and involves the introduction of a vanishing length scale h (a limit of infinitely fine discretization in the CTRW).

In our opinion, and to our knowledge, the approaches to the diffusion limit, appearing in the literature, even if usually good enough for practical purposes, may present flaws from the mathematical point of view. Usually (see *e.g.* Scalas et al. [49] and references therein) the transition to the diffusion limit is obtained by approximating the Fourier transform of the jump *pdf* and the Laplace transform of the waiting-time *pdf* as follows,

$$\widehat{w}(\kappa) \sim 1 - |\kappa|^{\alpha}\,, \quad \kappa \to 0\,, \quad 0 < \alpha \le 2\,; \tag{5.1}$$

$$\widetilde{\psi}(s) \sim 1 - s^{\beta}\,, \quad s \to 0\,, \quad 0 < \beta \le 1\,. \tag{5.2}$$

In order to properly carry out the transition, we follow the arguments stated by Gorenflo et al. [26] (see also Scalas et al. [50]). We start from Eqs. (2.8) and (3.3) and pass through their Fourier-Laplace counterpart. The process described by these equations originates via a sequence of jumps, each jump being a sample of the real random variable Y. The particle is at position $Y_1 + Y_2 + \ldots Y_n$ during the time interval $t_n \le t < t_{n+1}$ $(n = 1, 2, \ldots)$, at position 0 in the interval $t_0 \le t < t_1$ (in agreement with the empty sum convention). The Y_j $(j = 1, 2, \ldots)$ are i.i.d. random variables all having, like Y, the *pdf* $w(x)$, and, consequently, the characteristic function $\widehat{w}(\kappa)$. To be precise, we require that the *pdf* $w(x)$ is such that, if $\alpha = 2$:

$$\sigma^2 = \int_{-\infty}^{+\infty} x^2\, w(x)\, dx < \infty\,, \tag{5.3}$$

whereas, if $0 < \alpha < 2$:

$$w(x) = (b + \epsilon(|x|))\, |x|^{-(\alpha+1)}\,, \quad b > 0\,, \quad \epsilon(|x|) \to 0 \text{ as } |x| \to \infty\,, \tag{5.4}$$

where $\epsilon(|x|)$ is assumed to be bounded and $\mathcal{O}\left(|x|^{-\eta}\right)$ with some $\eta > 0$ as $|x| \to \infty$.

We now consider a sequence of random processes with *pdf*'s $p_h(x,t)$ resulting from jumps of size $h\,Y_k$ instead of Y_k, and from an acceleration of the process by a factor (*the scaling factor*) $\mu^{-1/\beta}\, h^{-\alpha/\beta}$, where μ must fulfill some conditions to be stated later. The *pdf* of the jump size is $w_h(x) = w(x/h)/h$, so that its characteristic function is $\widehat{w_h}(\kappa) = \widehat{w}(\kappa h)$. For $0 < \alpha \le 2$ and $0 < \beta \le 1$ we replace Eq. (3.3) (including (2.8) in the special case $\beta = 1$) by the sequence of equations

$$\mu h^{\alpha}\,\frac{\partial^{\beta}}{\partial t^{\beta}}p_h(x,t) = -p_h(x,t) + \int_{-\infty}^{+\infty} w_h(x-x')\,p_h(x',t)\,dx'\,. \tag{5.5}$$

Applying the Fourier-Laplace transform and recalling the Laplace transform of the Caputo time-fractional derivative, see (A.1), we get

$$\mu h^{\alpha}\,\left\{s^{\beta}\,\widehat{\widetilde{p}}_h(\kappa,s) - s^{\beta-1}\right\} = \left[\widehat{w}_h(\kappa) - 1\right]\,\widehat{\widetilde{p}}_h(\kappa,s)\,. \tag{5.6}$$

Let us now be guided by the *classical central limit theorem* or by the *Gnedenko limit theorem*, see Gnedenko & Kolmogorov [16] and Feller [14], both expressed in terms of the characteristic functions. We recall that the Gnedenko limit theorem provides a suitable generalization of the classical central limit theorem for space *pdf*′s with infinite variance, decaying according to condition (5.4).

The transition to the *diffusion limit* is based on the following *Lemma* due to Gorenflo (for the proof see [25]):
With the scaling parameter

$$\mu = \frac{\sigma^2}{2}\,, \quad \text{if} \quad \alpha = 2\,, \tag{5.7}$$

$$\mu = \frac{b\,\pi}{\Gamma(\alpha+1)\,\sin(\alpha\pi/2)}\,, \quad \text{if} \quad 0 < \alpha < 2\,, \tag{5.8}$$

we have the relation

$$\lim_{h\to 0}\frac{\widehat{w}(\kappa h) - 1}{\mu\, h^{\alpha}} = -|\kappa|^{\alpha}\,, \quad 0 < \alpha \le 2\,, \quad \kappa \in \mathbf{R}\,. \tag{5.9}$$

Now, it is possible to set

$$\rho_h(\kappa) = \frac{\widehat{w}(\kappa h) - 1}{\mu h^{\alpha}}\,, \tag{5.10}$$

and the sequence of equations (5.6) reads:

$$s^{\beta}\,\widehat{\widetilde{p_h}}(\kappa,s) - s^{\beta-1} = \rho_h(\kappa)\,\widehat{\widetilde{p_h}}(\kappa,s)\,. \tag{5.11}$$

Then, passing to the limit $h \to 0$, thanks to (5.8), we get:

$$s^{\beta}\,\widehat{\widetilde{p_0}}(\kappa,s) - s^{\beta-1} = -|\kappa|^{\alpha}\,\widehat{\widetilde{p_0}}(\kappa,s)\,, \quad 0 < \alpha \le 2\,, \quad 0 < \beta \le 1\,. \tag{5.12}$$

By inversion and using the Fourier transform of the Riesz space-fractional derivative defined in Eq. (B.1), we finally obtain the equation:

$$\frac{\partial^{\beta}}{\partial t^{\beta}}p_0(x,t) = \frac{\partial^{\alpha}}{\partial |x|^{\alpha}}p_0(x,t)\,, \quad p_0(x,0) = \delta(x)\,, \tag{5.13}$$

which is the required space-time fractional diffusion equation. In view of the initial condition as a delta function, the corresponding solution $p_0(x,t)$ is thus the fundamental solution or Green function of the space-time fractional diffusion equation. In the limiting cases $\beta = 1$ and $\alpha = 2$, Eq. (5.13) reduces to the standard diffusion equation.

In summary, we have shown a formally correct transition to the diffusion limit starting from the general master equation of the CTRW, namely Eq. (2.2) or Eq. (2.6). Indeed, by invoking the continuity theorem of probability theory, see *e.g.* Lukacs [31], we can convince ourselves that the random variable whose density is $p_h(x,t)$ converges in distribution ("weakly" or "in law") to the random variable whose density is $p_0(x,t)$.

Solving Eq. (5.11) for $\widehat{\widetilde{p_h}}(\kappa,s)$, Eq. (5.12) for $\widehat{\widetilde{p_0}}(\kappa,s)$, gives

$$\widehat{\widetilde{p_h}}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta - \rho_h(\kappa)}\,, \quad \widehat{\widetilde{p_0}}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha}\,. \tag{5.14}$$

Inverting the Laplace transforms in Eqs (5.14) implies

$$\widehat{p_h}(\kappa,t) = E_\beta\left(\rho_h(\kappa)t^\beta\right)\,, \quad \widehat{p_0}(\kappa,t) = E_\beta\left(-|\kappa|^\alpha t^\beta\right)\,, \tag{5.15}$$

where E_β denotes the Mittag-Leffler function of index β, see Eqs. (3.4)-(3.6). By (5.9) we obtain $\rho_h(\kappa) \to -|\kappa|^\alpha$ as $h \to 0$, hence

$$p_h(x,t) \to p_0(x,t)\,, \quad \text{for} \quad t>0\,, \quad h \to 0\,. \tag{5.16}$$

6. The scaling properties of the fractional diffusion

In this Section we are going to complement our analysis by presenting the fundamental solution of the fractional diffusion equation (5.13) with particular attention to its scaling properties. We agree to rename this solution as $p_{\alpha,\beta}(x,t)$ to point out its dependence on the parameters α $(0<\alpha\le 2)$ and β $(0<\beta\le 1)$. We shall also consider the relevant particular cases $\{\alpha=2,\,\beta=1\}$ (*standard diffusion*), $\{0<\alpha<2,\,\beta=1\}$ (*space-fractional diffusion*), $\{\alpha=2,\,0<\beta<1\}$ (*time-fractional diffusion*).

From (5.14) and (5.15) we recall the Fourier-Laplace transform and the Fourier transform of the fundamental solution $p_{\alpha,\beta}(x,t)$, which read respectively

$$\widehat{\widetilde{p}}_{\alpha,\beta}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha}\,, \tag{6.1}$$

$$\widehat{p}_{\alpha,\beta}(\kappa,t) = E_\beta\left(-|\kappa|^\alpha\, t^\beta\right)\,, \quad 0<\alpha\le 2\,, \quad 0<\beta\le 1\,. \tag{6.2}$$

By using the known scaling rules for the Fourier ($\mathcal{F}$) and Laplace ($\mathcal{L}$) transforms

$$\mathcal{F}\left[f(ax)\right] = a^{-1}\,\widehat{f}(\kappa/a)\,, \quad a>0\,, \qquad \mathcal{L}\left[f(bt)\right] = b^{-1}\,\widetilde{f}(s/b)\,, \quad b>0\,,$$

we can infer directly from (6.1) (thus without inverting the two transforms) the following *similarity property* of the fundamental solution,

$$p_{\alpha,\beta}(ax\,,\,bt) = b^{-\gamma} p_{\alpha,\beta}(ax/b^\gamma\,,\,t)\,, \quad \gamma = \beta/\alpha\,. \tag{6.3}$$

Consequently, we can write

$$p_{\alpha,\beta}(x,t) = t^{-\gamma}\, K_{\alpha,\beta}(x/t^{\gamma})\,, \quad \gamma = \beta/\alpha\,, \tag{6.4}$$

where

$$K_{\alpha,\beta}(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-iqu}\, E_\beta(-|q|^\alpha)\, dq\,. \tag{6.5}$$

We note that for $0 < \alpha \le 1$ and $0 < \beta < 1$ the integral in (6.5) is intended as a Cauchy Principal Value. In fact the function $E_\beta(-|q|^\alpha)$, being completely monotonic in $\mathbf{R}^+$, turns out to be positive and decreasing to zero like $|q|^{-\alpha}/\Gamma(1-\beta)$. For $u = 0$ we obtain, after some calculations on the Mittag-Leffler function, see *e.g.* [35, 17],

$$K_{\alpha,\beta}(0) = \begin{cases} \infty & \text{if } 0<\alpha\le 1 \text{ and } 0<\beta<1\,, \\ \dfrac{1}{\pi}\,\dfrac{\Gamma(1+1/\alpha)\,\Gamma(1-1/\alpha)}{\Gamma(1-\beta/\alpha)} & \text{if } 1<\alpha\le 2 \text{ and } 0<\beta<1\,, \\ \dfrac{1}{\pi}\,\Gamma(1+1/\alpha) & \text{if } 0<\alpha\le 2 \text{ and } \beta=1\,. \end{cases} \tag{6.6}$$

<u>For $\alpha = 2$ and $\beta = 1$</u> we recover the known Green function of the *standard diffusion equation*: in fact Eq. (6.2) reduces to

$$\widehat{p}_{2,1}(\kappa,t) = \exp(-\kappa^2\, t)\,, \tag{6.7}$$

then we obtain

$$p_{2,1}(x,t) = t^{-1/2}\,\frac{1}{2\sqrt{\pi}}\exp(-x^2/(4t)) = t^{-1/2}\, G(x/t^{1/2})\,, \tag{6.8}$$

where $G(x)$ denotes the Gaussian *pdf*

$$G(x) = \frac{1}{2\sqrt{\pi}}\exp(-x^2/4)\,. \tag{6.9}$$

We note that

$$K_{2,1}(0) = G(0) = \frac{1}{2\sqrt{\pi}}\,. \tag{6.10}$$

<u>For $0 < \alpha < 2$ and $\beta = 1$</u> (*space-fractional diffusion equation*) Eq. (6.2) reduces to

$$\widehat{p}_{\alpha,1}(\kappa,t) = \mathrm{e}^{-|\kappa|^\alpha\, t}\,, \tag{6.11}$$

so we obtain

$$p_{\alpha,1}(x,t) = t^{-1/\alpha}\, L_\alpha\left(x/t^{1/\alpha}\right)\,, \quad 0<\alpha<2\,, \tag{6.12}$$

where $L_\alpha(x)$ is the (non Gaussian) Lévy stable *pdf* of index α. For this class of probability densities we refer *e.g.* to [16], [14].

The Green function (6.12) has been discussed in several papers, for example in [23, 24], and references therein. We note that

$$K_{\alpha,1}(0) = L_\alpha(0) = \frac{1}{\pi}\,\Gamma(1+1/\alpha)\,. \qquad (6.13)$$

It is worthwhile to remember the *algebraic* decay of the non-Gaussian stable distributions; in fact the asymptotic behaviour of $L_\alpha(x)$ is given by

$$L_\alpha(x) \sim \frac{\sin(\alpha\pi/2)}{\pi}\,\frac{\Gamma(\alpha+1)}{|x|^{\alpha+1}}\,, \quad |x|\to\infty\,, \quad 0<\alpha<2\,. \qquad (6.14)$$

We note that for $\alpha = 2$ the asymptotics (6.14) breaks down (the coefficient of the power becoming zero so that an exponentially decreasing remainder comes into play).

For $\alpha = 2$ and $0 < \beta < 1$ (*time-fractional diffusion equation*) Eq. (6.2) reduces to

$$\widehat{p}_{2,\beta}(\kappa,t) = E_\beta\left(-\kappa^2\, t^\beta\right)\,, \qquad (6.15)$$

so, we get

$$p_{2,\beta}(x,t) = \frac{1}{2} t^{-\beta/2}\, M_{\beta/2}\left(|x|/t^{\beta/2}\right)\,, \qquad (6.16)$$

where $M_{\beta/2}$ denotes a function of Wright-type of index $\beta/2$, introduced by Mainardi, see *e.g.* [32, 33]. For a given $\nu \in (0,1)$ the function $M_\nu(z)$ is defined in the whole complex plane by the power series

$$M_\nu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma[-\nu n + (1-\nu)]}\,, \quad 0<\nu<1\,, \quad z\in\mathbf{C}\,. \qquad (6.17)$$

For a detailed discussion on the Wright-type functions and related Laplace transforms we refer to the recent papers by Gorenflo, Luchko and Mainardi, see [18, 19], and references therein quoted. We note that

$$K_{2,\beta}(0) = \frac{1}{2}\, M_{\beta/2}(0) = \frac{1}{2\,\Gamma(1-\beta/2)}\,. \qquad (6.18)$$

This Wright-type function has an *exponential* decay; in fact its asymptotic behaviour is

$$M_{\beta/2}(|x|) \sim A(\beta)\, x^{\,d(\beta)}\, \exp\left[-B(\beta)\, x^{\,c(\beta)}\right]\,, \quad |x|\to\infty\,, \qquad (6.19)$$

where A and B are certain positive constants depending on β and

$$c(\beta) = 2/(2-\beta)\,, \quad d(\beta) = (\beta-1)/(2-\beta)\,. \qquad (6.20)$$

We note that $d(\beta) = c(\beta)/2 - 1$; in particular, as β varies from 0 to 1, $c(\beta)$ increases from 1 to 2 whereas $d(\beta)$ increases from $-1/2$ to 0. When $\beta = 1$ (standard diffusion) formula (6.19) is no longer asymptotic but provides the exact expression (6.9) in terms of the Gaussian.

Furthermore, as a consequence of Eq. (6.19), all the spatial moments of $p_{2,\beta}(x,t)$ of order $\delta > -1$ are finite. In particular, considering the moments of even (integer) order, we have

$$\int_{-\infty}^{+\infty} x^{2n}\, p_{2,\beta}(x,t)\, dx = \frac{\Gamma(2n+1)}{\Gamma(\beta\, n+1)}\, t^{\beta\, n}\,, \quad n = 0, 1, 2, \ldots \tag{6.21}$$

In particular, we interpret $p_{2,\beta}(x,t)$ as a spatial *pdf* evolving in time, which exhibits a finite variance σ^2 (the moment of order 2) proportional to the β-th power of time, t^β (*anomalous slow diffusion*, being $0 < \beta < 1$).

We recall that Mainardi [32, 33] has shown that the Green function (6.16) for the *time-fractional diffusion equation* can be obtained also from the Laplace transform

$$\widetilde{p}_{2,\beta}(x,s) = \frac{1}{2}\, s^{\beta/2-1}\, \mathrm{e}^{-x s^{\beta/2}}\,. \tag{6.22}$$

We note that such solution is valid also for $1 < \beta < 2$ when the time-fractional equation, describing an intermediate process between diffusion and wave propagation, can be referred to as the *time-fractional diffusion-wave equation.* Also in this case the Green function can be interpreted as a spatial *pdf* evolving in time, but now we have *anomalous fast diffusion* as the exponent β (related to time in the variance) exceeds 1.

We now present a composition rule which allows us to express the general Green function of the space-time fractional diffusion equation as an integral involving the two Green functions corresponding to space-fractional and time-fractional diffusion equations. To this purpose we note that the Fourier Laplace transform of the Green function (6.1) can be re-written in integral form as suggested by Saichev & Zaslavski [46] of the *space-time-fractional diffusion equation*, as

$$\widetilde{\widehat{p}}_{\alpha,\beta}(\kappa,s) = \frac{s^{\beta-1}}{s^\beta + |\kappa|^\alpha} = s^{\beta-1} \int_0^\infty \mathrm{e}^{-r(s^\beta + |\kappa|^\alpha)}\, dr\,. \tag{6.23}$$

Now, our result (6.22) coupled with (6.11) allows us to interpret Eq. (6.23) with a better insight than in [46]. In fact, in view of (6.11), (6.22) we can express (6.23) as

$$\widetilde{\widehat{p}}_{\alpha,\beta}(\kappa,s) = 2 \int_0^\infty \widehat{p}_{\alpha,1}(\kappa,r)\, \widetilde{p}_{2,2\beta}(r,s)\, dr\,, \tag{6.24}$$

so, by inversion,

$$p_{\alpha,\beta}(x,t) = 2 \int_0^\infty p_{\alpha,1}(x,r)\, p_{2,2\beta}(r,t)\, dr\,. \tag{6.25}$$

Note the presence of $p_{2,2\beta}$ instead of $p_{2,\beta}$. Hence Eq. (6.25) is a sort of a formula of separation of variables stating that the Green function for the space-time-fractional diffusion equation of order $\{\alpha,\beta\}$ can be expressed in terms of the Green function for the space-fractional diffusion equation of order α and the Green function for the time-fractional diffusion-wave equation of order 2β .

Because of Eqs. (6.12) and (6.16), we can write Eq. (6.25) as

$$p_{\alpha,\beta}(x,t) = t^{-\beta} \int_0^\infty r^{-1/\alpha}\, L_\alpha\left(x/r^{1/\alpha}\right)\, M_\beta\left(r/t^\beta\right)\, dr\,. \tag{6.26}$$

Of course, the formulas (6.12) and (6.16) corresponding to the particular cases $\{\alpha, 1\}$ and $\{2, \beta\}$ are recovered from (6.26) as follows:

$$p_{\alpha,1}(x,t) = t^{-1} \int_0^\infty r^{-1/\alpha}\, L_\alpha\left(x/r^{1/\alpha}\right)\, \delta\left(r/t - 1\right)\, dr = t^{-1/\alpha}\, L_\alpha\left(x/t^{1/\alpha}\right), \tag{6.27}$$

and

$$p_{2,\beta}(x,t) = \frac{t^{-\beta}}{2\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-x^2/4r}\, M_\beta\left(r/t^\beta\right)\, dr = \frac{t^{-\beta/2}}{2}\, M_{\beta/2}\left(x/t^{\beta/2}\right). \tag{6.28}$$

Eq. (6.27) is a direct consequence of the property that, as $\nu \to 1^-$, we have $M_\nu(r) \to \delta(r-1)\,,\; r > 0$. Eq. (6.28) is a sort of (integral) duplication formula with respect to the index for the function $M_\nu(r)\,,\; r > 0$.

In a recent paper by Mainardi, Luchko and Pagnini [35], that was inspired by a previous one by Gorenflo, Iskenderov and Luchko [17], the authors have given a general representation of the fundamental solution of the space-time fractional diffusion equation (including asymmetry effects in the space-fractional derivative by a skewness parameter θ) in terms of Mellin-Barnes integrals. Indeed they have derived explicit formulae (based on convergent series matched with asymptotic expansions) which allow them to plot the fundamental solution for different values of the relevant parameters α, θ, β in any space domain.

7. Conclusions

In this paper we have reviewed our phenomenological theory of tick-by-tick dynamics in financial markets, based on the continuous time random walk (CTRW) model. The theory can take into account the possibility of the non-Markovian character of financial time series by means of a generalized master equation with a time fractional derivative. We have presented predictions on the behaviour of the waiting-time probability density by introducing a special function of Mittag-Leffler type whose decay interpolates from a stretched exponential at small times to a power-law for long times. This function has been successfully applied in the empirical analysis of high-frequency prices time series of German and Italian bond futures.

Furthermore, we have proposed a scaling method to derive, in a correct way, the transition to the diffusion limit from the CTRW master equation governing a stochastic process. It turns out that the probability density function obeys a generalized diffusion equation, of fractional order both in space and in time, with self-similarity properties.

Appendix A: The Caputo time-fractional derivative

For readers' convenience, here we present an introduction to the *Caputo* fractional derivative starting from its representation in the Laplace domain and contrasting it to *Riemann-Liouville* fractional derivative. In so doing we avoid the subtleties lying in the inversion of fractional integrals.

If $f(t)$ is a (sufficiently well-behaved) function with Laplace transform $\mathcal{L}\{f(t);s\} = \widetilde{f}(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,dt$, we have

$$\mathcal{L}\left\{\frac{d^\beta}{dt^\beta} f(t); s\right\} = s^\beta\, \widetilde{f}(s) - s^{\beta-1}\, f(0^+)\,, \quad 0 < \beta < 1\,, \tag{A.1}$$

if we define

$$\frac{d^\beta}{dt^\beta}\, f(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{df(\tau)}{d\tau}\, \frac{d\tau}{(t-\tau)^\beta}\,. \tag{A.2}$$

We can also write, in each of the following two ways,

$$\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)}\, \frac{d}{dt} \left\{ \int_0^t [f(\tau) - f(0^+)]\, \frac{d\tau}{(t-\tau)^\beta} \right\}, \tag{A.3}$$

$$\frac{d^\beta}{dt^\beta} f(t) = \frac{1}{\Gamma(1-\beta)}\, \frac{d}{dt} \left\{ \int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\, d\tau \right\} - \frac{t^{-\beta}}{\Gamma(1-\beta)}\, f(0^+)\,. \tag{A.4}$$

The Eqs. (A.1-4) can be extended to any non integer $\beta > 1$, see *e.g.* the survey by Gorenflo & Mainardi [21]. We refer to the fractional derivative defined by (A.2) as the *Caputo* fractional derivative, since it was formerly applied by Caputo in the late sixties for modelling dissipation effects in *Linear Viscoelasticity*, see *e.g.* [6, 7, 10]. Several applications have been treated by Caputo himself up to nowadays, see *e.g.* [8, 9] and references therein.

The reader should observe that the Caputo's definition differs from the usual one named after Riemann and Liouville, which is given by the first term in the RHS of (A.4), see *e.g.* the treatise on Fractional Calculus by Samko, Kilbas & Marichev [47]. The *Caputo* fractional derivative is of course more restrictive than the *Riemann-Liouville* fractional derivative in that the first-order derivative is required to exist and be absolutely Laplace transformable.

The *Caputo* fractional derivative, practically ignored in the mathematical treatises, represents a sort of regularization in the time origin for the *Riemann-Liouville* fractional derivative. Recently, it has been extensively investigated by Gorenflo & Mainardi [21] and by Podlubny [44] in view of its major utility in treating problems of physical interest, which require standard initial conditions. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed at $t = 0$ by the field variable and its derivatives of integer order, despite the fact that the governing evolution equation may be a generic integro-differential equation and therefore, in particular, a fractional differential equation.

Appendix B: The Riesz space-fractional derivative

If $f(x)$ is a (sufficiently well-behaved) function with Fourier transform

$$\mathcal{F}\{f(x);\kappa\} = \hat{f}(\kappa) = \int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, f(x)\, dx\,, \quad \kappa \in \mathbf{R}\,,$$

we have

$$\mathcal{F}\left\{\frac{d^\alpha}{d|x|^\alpha} f(x);\kappa\right\} = -|\kappa|^\alpha\, \hat{f}(\kappa)\,, \quad 0<\alpha<2\,, \qquad (B.1)$$

if we define

$$\frac{d^\alpha}{d|x|^\alpha} f(x) = \Gamma(1+\alpha)\,\frac{\sin(\alpha\pi/2)}{\pi}\int_0^\infty \frac{f(x+\xi)-2f(x)+f(x-\xi)}{\xi^{1+\alpha}}\, d\xi\,. \qquad (B.2)$$

In other words $\frac{d^\alpha}{d|x|^\alpha}$ is the pseudo-differential operator with symbol $-|\kappa|^\alpha$. Let us recall that a generic pseudo-differential operator A, acting with respect to the variable $x \in \mathbf{R}$, is defined through its Fourier representation, namely $\int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x}\, A\,[f(x)]\, dx = \hat{A}(\kappa)\, \hat{f}(\kappa)$, where $\hat{A}(\kappa)$ is referred to as symbol of A, given as $\hat{A}(\kappa) = \left(A\,\mathrm{e}^{-i\kappa x}\right)\,\mathrm{e}^{+i\kappa x}$.

The fractional derivative defined by (B.2) can be referred to as the *Riesz fractional derivative* since it is obtained from the inversion of the fractional integral originally introduced by Marcel Riesz in the late 1940's, known as the *Riesz potential*, see *e.g.* [47]. It is based on a suitable regularization of a hyper-singular integral, according to a method formerly introduced by Marchaud in 1927. The representation (B.2) can be found in [47] as formula (12.1') and is more explicit and convenient than other ones available in the literature, see *e.g.* [46], [53], in that it is valid in the whole range $0<\alpha<2$. We have used it in [25], where we have shown that it holds also in the singular case $\alpha = 1$.

For $\alpha = 1$ the Riesz derivative is related to the Hilbert transform, as pointed out by Feller in 1952 in his pioneering paper [13], resulting in the formula

$$\frac{d}{d|x|} f(x) = -\frac{1}{\pi}\,\frac{d}{dx}\int_{-\infty}^{+\infty} \frac{f(\xi)}{x-\xi}\, d\xi\,. \qquad (B.3)$$

For $\alpha = 2$ the Riesz derivatives reduces to the standard derivative of order 2 since $-|\kappa|^2 = -\kappa^2$ is known to be its symbol. We note, by writing $-|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$, that the Riesz derivative of order α can be interpreted as the negative of the $\alpha/2$ power of the (positive definite) operator $-D^2 = -\frac{d^2}{dx^2}$, namely

$$\frac{d^\alpha}{d|x|^\alpha} = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2}\,. \qquad (B.4)$$

The notation here adopted is due to Zaslavsky, see *e.g.* [46]. A different notation which allows asymmetric effects is motivated by Feller's paper [13] and is due to Gorenflo & Mainardi, see *e.g.* [23], [24].

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