Quadratic Dynamic Programming*

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Abstract

The dynamic programming problem with one-period concave quadratic returns is analysed. As well-known, in this context the fundamental Blackwell's contraction theorem cannot be applied since the return functions are unbounded from below. In this paper, we establish results on the existence of the fixed points of the Bellman operator and properties on the convergence of the iterative processes generated by this operator. The main tool for obtaining these results is a contraction theorem given by the Thompson metric.

Keywords: dynamic programming, quadratic function, contraction theorem, Thompson metric, Bellman operator.

*Presented at the Workshop MDEF2000, Urbino, September 28-29-30, 2000. Research partially supported by M.U.R.S.T., National Group on "Non-linear Dynamics in Economics and Social Sciences", Italy.

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1 Introduction

This paper studies the somewhat classical problem of quadratic dynamic programming over an unbounded horizon which can be formulated as:

$$\begin{array}{l} v(x_0) = \sup \sum_{t=0}^{+\infty} q(x_t, x_{t+1})\beta^t \\ \text{s.t. } x_t \in \Re^n, \, x_0 \text{ given in } \Re^n \end{array} \tag{1}$$

where $v(x)$ is the optimal value function, $\beta \in (0,1)$ is the intertemporal discount factor and the one-period return is a concave quadratic function $q(x,y) = (x,y)'Q(x,y)$. The $2n$-order matrix Q can be partitioned in the form

$$Q = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}$$

with A, B and C n-order matrices and, in addition, A and C symmetric. Besides this, it will also analyse the complete quadratic problem obtained by adding a linear function to the one-period return of the previous case, *i.e.* $q(x,y) = (x,y)'Q(x,y) + (a,b)'(x,y) + k$ with $a, b \in \Re^n$.

Though these classes of optimization problems have a long story and are often adopted for approximating more complex nonlinear programming models, we want to underline that their treatment is far from being entirely completed. In spite of the fact that the issue is evidently of an algebraic nature. As a matter of fact, the $2n$-order matrix Q encloses entirely all the properties of the purely quadratic programming.

One of the main reasons of the analytical difficulties met within this context is that the quadratic returns $(x,y)'Q(x,y)$ are necessarily unbounded from below. We cannot therefore invoke the fundamental Blackwell's contraction theorem which requires the return function to be bounded. In fact, our main results will be obtained by employing a different contraction theorem related to the Thompson metric and recently established by Montrucchio [4] to study the differentiability property of policy functions arising in nonlinear dynamic programming.

Before going into the specific results presented here, let us recall a few basic ingredients related to the technique of dynamic programming and that are valid for returns $q(x,y)$ non necessarily quadratic.

It is very well known that the value function $v(x)$ of sequential problem (1) satisfies the Bellman functional equation ([7], [2], [5])

$$v(x) = \sup_{y \in \Re^n} \left[q(x, y) + \beta v(y) \right]. \tag{2}$$

Put differently, if we define the Bellman operator

$$(Tf)(x) = \sup_{y \in \Re^n} \left[q(x, y) + \beta f(y) \right] \tag{3}$$

associated with (2), the value function $v(x)$ turns out to be a fixed point of this operator. That is to say, $Tv = v$ must hold. Another important aspect concerns the computational procedure $v_{n+1} = Tv_n$ that is hoped to converge to the value function v for a convenient initial function v_0. Actually, Blackwell's result shows that the operator T is a contraction, whenever $q(x, y)$ is bounded, and therefore the iterative process $v_{n+1} = Tv_n$ converges uniformly to v for any initial bounded function v_0 (see Stokey et Alii [7]).

Let us turn to our class of problems. To fix ideas, we shall mention here the purely quadratic case $q(x, y) = (x, y)'Q(x, y)$. We list below questions which arise naturally in this context and that will be answered in the present paper. It is worth mentioning that we left the matrix Q to be singular. The nonsingular case, *i.e.,* Q negative definite, is slightly simpler and already sufficiently investigated in the literature.

1. Is the value function $v(x)$ a quadratic function? Of course, the answer is yes, but an elementary proof of this fact is not known to us, unless in a general setting.

2. It is straightforward that the operator T maps the cone of concave quadratic functions into itself. It is also easily verified by means of examples that not always T has a unique fixed point given by the value function. Is it possible to characterize algebraically the matrices Q for which the fixed point is unique? More generally, how many fixed points can T exhibit? Which algebraic properties of Q are involved?

3. The "uniqueness versus multiplicity of fixed points for T" is closely related to the convergence of iterative process $v_{n+1} = Tv_n$ to the value v. Which subclass of Q leads to the global convergence to v? Otherwise, which does initial quadratic functions v_0 assure the convergence to v?

These and other related issues will be investigated in this paper. The plan of the paper is the following. In Section 2 we shall present the results on the purely quadratic case. The main result is the Theorem 1 which provides statements on the existence of fixed points of Bellman operator and the convergence of $v_{n+1} = Tv_n$. An important consequence is Theorem 2 which establishes a sufficiently general subclass of problems having an unique globally convergent fixed point. Section 3 is dedicated to studying the case $q(x, y) = (x, y)'Q(x, y) + (a, b)'(x, y) + k$. The proofs of theorem are gathered in Section 4.

2 Results

2.1 Notation

Vector $x \in \Re^n$ is viewed as column vector and x' denotes its transposed. The symbol $|x|$ denotes the Euclidean norm, *i.e.*, $|x|^2 = x'x$. The set $\mathcal{S} = \left\{x \in \Re^n : |x|^2 = 1\right\}$ means the unit sphere of $\Re^n$.

$\mathcal{B}(Y)$ is the vector space of bounded functions $f : Y \to \Re$, where Y is an arbitrary set. The vector space $\mathcal{B}(Y)$ is endowed with the uniform convergence norm $\|f\| = \sup_{y \in Y} |f(y)|$. In $\mathcal{B}(Y)$ there is the point-wise partial order: $f \leq g$ if and only if $f(y) \leq g(y)$ for all $y \in Y$. $\mathcal{K}_{-}(Y)$ is the cone of non-positive functions of $\mathcal{B}(Y)$. We define the interval $[f, g] = \{h \in \mathcal{B}(Y) : f \leq h \leq g\}$.

The cone of n-order symmetric negative semidefinite matrices will be denoted by $Sym_{-}(n)$. A matrix $H \in Sym_{-}(n)$ can be identified with the quadratic form $f(x) = x'Hx$, $x \in \Re^n$. Moreover, since a one-to-one correspondence exists between f and its restriction over the unit sphere $\mathcal{S}$ of $\Re^n$, the set $Sym_{-}(n)$ may be viewed as a subset of $\mathcal{K}_{-}(\mathcal{S})$. The partial order induced on $Sym_{-}(n)$ by $\mathcal{B}(\mathcal{S})$ is the natural order among symmetric matrices, *i.e.*, $A \geq B$ if and only if $A - B$ is positive semidefinite. If $H \in Sym_{-}(n)$, the norm of uniform convergence is $\|H\| = \sup_{|x|=1} |x'Hx|$, *i.e.*, the spectral norm of the matrix. In fact, $\|H\| = \rho(H) = \max_i |\lambda_i|$, where λ_i are its eigenvalues.

A function $f : \Omega \to \Re$, where Ω is a convex set of $\Re^n$, is said α-concave if $f(x) + \frac{1}{2}\alpha |x|^2$ is concave over Ω. Following Montrucchio [3], a function $f(x, y)$ is called (α, γ)-concave whenever $f(x, y) + \frac{1}{2}\alpha |x|^2 + \frac{1}{2}\gamma |y|^2$ is concave.

If $A : X \to Y$ is a linear operator, $\mathcal{R}(A) \subset Y$ is the range of A. The sets $\mathcal{N}(A) = \{x : Ax = 0\}$ and $\mathcal{N}(A)^{\perp}$ are the null-space of A and its orthogonal complement, respectively. A^{+} denotes the pseudoinverse matrix of A. For the main properties of A^{+} see LUENBERGER [1]; in particular we shall often use of property: $A^{+}AA^{+} = A^{+}$.

In view of providing an algebraic characterization of quadratic form, for a matrix $Q \in Sym_{-}(2n)$ we define by induction the following sequence of vector subspaces of $\Re^n$:

$$\begin{array}{l} N_0 = pr_1\mathcal{N}(Q) = \{x \in \Re^n; \exists y \in \Re^n \text{ s. t. } (x, y) \in \mathcal{N}(Q)\} \\ N_{r+1} = \{x \in N_r; \exists y \in N_r \text{ s. t. } (x, y) \in \mathcal{N}(Q)\}, \qquad \forall r \geq 0 \end{array}$$

Since $N_0 \supset N_1 \supset N_2 \supset ...$, an index i will exist such that $N_i = N_{i+1} = N_{i+2} = ...$. Such a vector subspace will be denoted by N_Q. That to say: $N_Q = \cap_i N_i$.

Obviously this set N_Q satisfies the property: if $x \in N_Q$ then there exists a vector $y \in N_Q$ such that $(x, y) \in \mathcal{N}(Q)$. Moreover, N_Q is the *maximal* set satisfying this property, *i.e.*,: for every set M for which:

$$x \in M \Longrightarrow \exists y \in M \text{ such that } (x, y) \in \mathcal{N}(Q),$$

we have $M \subset N_Q$. This fact is easily seen by observing that $M \subset N_r$ for each r, so that $M \subset \cap_r N_r$.

In the following, for a vector subspace M of $\Re^n$, $Sym_{-}(n; M) \subset Sym_{-}(n)$ will denote the convex set of all symmetric negative semidefinite matrices H such that $\mathcal{N}(H) = M$.

2.2 Main results: purely quadratic case

The first theorem is directed at solving the purely quadratic dynamic programming (1) illustrated in the introduction. Here $q(x, y) = (x, y)' Q (x, y)$ and $Q \in Sym_{-}(2n)$. Some more words are needed to understand the statements formulated below. The Bellman operator T defined in (3) is acting over the function space $f(x)$, $x \in \Re^n$. We shall restrict it to the quadratic functions $f(x) = x'Hx$, with $H \in Sym_{-}(n)$. More importantly, given the one-to-one correspondence between quadratic forms and matrices, we can think of T as a mapping $T : Sym_{-}(n) \to Sym_{-}(n)$. For ease of notation we shall use the same symbol T for both operators. By resorting to pseudoinverses, it is not difficult to write down this new operator. In fact, we have $TH = A - B\,(C + \beta H)^{+}\,B'$ (see the proof of Lemma 7 in section 4.2).

Theorem 1

i) *Any purely quadratic problem (1) admits optimal solutions for all initial condition* x_0;

ii) *its value function* $v(x)$ *is quadratic:* $v(x) = x'H^*x$ *where:*

iii) *the negative semidefinite matrix* H^* *is the only one fixed point of the Bellman operator* T *belonging to* $Sym_{-}(n; N_Q)$;

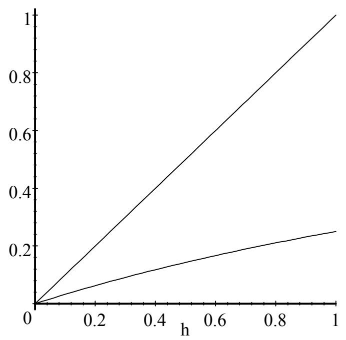
iv) *if* $\hat{H} \in Sym_{-}(n)$ *is any other fixed point of* T*, then* $\hat{H} \in Sym_{-}(n; M)$*, where* M *is a vector subspace of* $\Re^n$ *satisfies the following property:* $x \in M \Longrightarrow \exists y \in M$ *such that* $(x, y) \in \mathcal{N}(Q)$*. Moreover the maximality property* $\hat{H} \leq H^*$ *holds;*

v) *given any initial matrix* H_0 *such that* $\mathcal{N}(H_0) \supset N_Q$ *and* $H_1 \leq H_0$*, the iterative system* $H_{m+1} = TH_m$ *converges to* H^*.

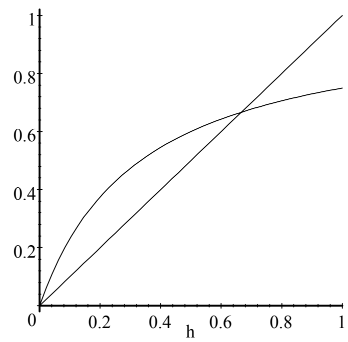
It is worth underlining that the previous theorem does not establish the convergence of $H_{m+1} = TH_m$ to the fixed point H^* for every initial condition. Indeed, the iterative process may converge to another fixed point $\hat{H}$ for many initial $H_0 \in Sym_{-}(n)$. The following simple example illustrates this aspect. Let $q(x, y) = -\frac{1}{2}(x - cy)^2$, with $c > 0$. The value function turns out to be $v(x) = 0$ for all discount factors. $\mathcal{N}(Q) = \{(x, y)\,; x = cy\}$. Therefore $N_Q = \Re$. Of course, there is also the invariant vector space $M = \{0\}$. The iterative system $H_{m+1} = TH_m$ is

$$-\frac{1}{2}h_{m+1}x^2 = \max_y \left[-\frac{1}{2}(x - cy)^2 - \frac{1}{2}\beta h_m y^2\right]$$

where we have set $x'H_m x = -\frac{1}{2}h_m x^2$ and $h_m \geq 0$. After some calculations, we obtain $h_{m+1} = \beta h_m \left(c^2 + \beta h_m\right)^{-1}$. If $\beta \leq c^2$, the iterative system has the only one fixed point $h^* = 0$ which is globally attractive. For $\beta > c^2$, another fixed point $\hat{h} = 1 - c^2\beta^{-1}$ arises. This fixed point attracts all the initial conditions different from zero (see figure below).



$\beta = .25 < c^2 = .75$



$\beta = .75 > c^2 = .25$

In the previous theorem the existence of the only one fixed point and its attractive property is not guaranteed. It is of interest to single out special subclasses of problems having stronger properties. The next theorem provides one of these subclasses.

Theorem 2

If the matrix Q has the following property: $\exists\gamma \in (0,1)$ such that the matrix

$$\begin{bmatrix} \gamma A & B \\ B' & C \end{bmatrix}$$

is still negative semidefinite, then T has one and only one fixed point H^*. The sequence of iterates $H_{m+1} = TH_m$ converges uniformly to H^*, from any starting $H_0 \in Sym_{-}(n)$.

It is worth remarking that this class of matrices includes the case in which the quadratic form $(x,y)'\,Q\,(x,y)$ is $(\alpha,0)$-concave and, in turn, the non-singular matrices. In fact, if $(x,y)'\,Q\,(x,y)$ is $(\alpha,0)$-concave, we have $A - BC^{+}B' \le -\alpha I$ (see Lemma 5). Thus $\gamma A + (1-\gamma)\,A - BC^{+}B' \le -\alpha I$, *i.e.*, $\gamma A - BC^{+}B' \le -\alpha I - (1-\gamma)\,A$. Since $\alpha > 0$, for γ sufficiently close to 1 the matrix $-\alpha I - (1-\gamma)\,A$ is negative definite and consequently $\gamma A - BC^{+}B' \le 0$ that implies that Q belongs to the class of Theorem 2 (see Lemma 5).

3 Complete quadratic return

This section is devoted to studying the complete quadratic case:

$$q(x, y) = (x, y)'Q(x, y) + (a, b)'(x, y) + k. \tag{4}$$

Here, the operator (3) acts on the space of functions $f(x) = x'Hx + h'x + \gamma$. In addition to T, we shall introduce the operator T_Q:

$$(T_Q f)(x) = \sup_{y \in \Re^n} [(x, y)'Q(x, y) + \beta f(y)] \tag{5}$$

where it is acting over the purely quadratic function f.

The next theorem gives the relationship between the pair T and T_Q. In order to obtain this result we need an additional assumption given below:

A) the Euler equation of problem (1) admits at least one stationary solution, that is to say, some $\bar{x}$ exists such that

$$D_2 q(\bar{x}, \bar{x}) + \beta D_1 q(\bar{x}, \bar{x}) = 0 \tag{6}$$

Proposition 3 *If assumption (A) holds, then for any quadratic fixed point $x'\hat{H}x$ of T_Q, there exists at least one quadratic function $x'\hat{H}x + h'x + \gamma$ which is a fixed point of T. More precisely the quadratic function is*

$$(x - \bar{x})'\hat{H}(x - \bar{x}) + u'(x - \bar{x}) + \frac{1}{1 - \beta} q(\bar{x}, \bar{x})$$

where $\bar{x}$ satisfies (6) and $u = 2A\bar{x} + 2B\bar{x} + a$.

4 Proofs

This section is dedicated to the proofs of the statements claimed in Sections 2 and 3. We need several lemmas and properties both of algebraic nature and on dynamic optimization. It will be essential to provide as well a contraction theorem which will be used in the proofs.

4.1 Algebraic properties

Lemma 4 *A is a matrix belonging to $Sym_{-}(n)$, c is a vector of $\Re^n - \{\mathbf{0}\}$ and $f(x) = \frac{1}{2}x'Ax - c'x$. If $L = \sup_{x\in\Re^n} f(x) < +\infty$ then $L = \max_{x\in\Re^n} f(x)$.*

Proof. Since $f(x)$ is concave, a stationary point is a maximum. The first order condition is $Ax = c$ or equivalently $c\in\mathcal{R}(A)$.

Let us suppose for absurd that the maximum is not achieved. This implies $c\notin\mathcal{R}(A)$. Since it holds $\mathcal{R}(A) = \mathcal{N}(A)^{\perp}$, $c\notin\mathcal{N}(A)^{\perp}$ .Therefore, there will exist an element $x_0 \in \mathcal{N}(A)$ such that $c'x_0 \neq 0$. If we now consider the vectors $x = \lambda x_0$, we have $f(x) = \frac{1}{2}x'Ax - c'x = -\lambda c'x_0$. But this function is not bounded when $\lambda \in R$. This contradicts the assumption made and thus the lemma is proved. ■

The next proposition provides a characterization of the quadratic form $(x,y)'\,Q\,(x,y)$ by means of the elements of the partitioned matrix.

Lemma 5 *The quadratic function $(x,y)'\,Q\,(x,y)$ is concave iff matrix C is negative semidefinite, $A - BC^{+}B'$ is negative semidefinite and $\mathcal{R}(C) \supset \mathcal{R}(B')$. It is $(\alpha,0)$-concave for some real number $\alpha > 0$ iff in addition $A - BC^{+}B'$ is negative definite.*

Proof. We only prove the second part of the lemma. The quadratic function will be $(\alpha,0)$-concave if and only if the following inequality holds

$$k'Ak + h'Ch + 2k'Bh \leq -\alpha\,|k|^2 \tag{7}$$

$\forall\,(k,h) \in \Re^n$.

From (7) with $k = 0$, it follows $h'Ch \leq 0$ and therefore C is negative semidefinite.

Let us calculate $\max_h k'Ak + h'Ch + 2k'Bh$. First of all, $\mathcal{R}(C) \supset \mathcal{R}(B')$ is a necessary condition in order that the superior is finite. The maximum value can easily be written down through the pseudoinverse matrix C^{+}. In fact we get:

$$\max_h k'Ak + h'Ch + 2k'Bh = k'\left[A - BC^{+}B'\right]k.$$

Thus the (7) becomes $k'\left[A - BC^{+}B'\right]k \leq -\alpha\,|k|^2$ and this last inequality amount to saying that $A - BC^{+}B'$ is negative definite. It should be noted that the first part of lemma is obtained by setting $\alpha = 0$. ■

4.2 Dynamic Programming

We collect here the main properties related to infinite horizon problem (1) and which are extensively exploited throughout the paper. For more details we refer to STOKEY ET ALII [7]. We underline that all these properties hold under very general assumptions made on the return $q(x,y)$. One must only postulate that the objective series is well defined and the value function is finitely valued. However, in our case $q(x,y) = (x,y)' Q(x,y)$ this happens to be true. In fact, $q(x_t, x_{t+1}) \leq 0$ and thus the series $\sum_{t=0}^{+\infty} q(x_t, x_{t+1})\beta^t$ is well defined and non-positive. Hence $v(x) \leq 0$. Furthermore, by evaluating the objective functional along the sequence $(x, 0, 0, 0...)$ we can infer that $v(x) \geq x'Ax$ regardless to the value of the discount factor. Consequently, the value function $v(x)$ is finitely-valued independently of the discount factor.

We must recall two properties of the operator (3). First, T is monotonic, i.e., $Tf \leq Tg$, if $f \leq g$. Second, T is convex. That to say:

$$T[\alpha f + (1-\alpha) g] \leq \alpha Tf + (1-\alpha) Tg$$

for $\alpha \in [0,1]$.

In fact, we have

$$T[\alpha f + (1-\alpha) g](x) = \sup_{y \in \Re^n} [q(x,y) + \alpha\beta f(y) + (1-\alpha)\beta g(y)] =$$

$$= \sup_{y \in \Re^n} [\alpha q(x,y) + \alpha\beta f(y) + (1-\alpha) q(x,y) + (1-\alpha)\beta g(y)] \leq$$

$$\leq \alpha \sup_{y \in \Re^n} [q(x,y) + \beta f(y)] + (1-\alpha) \sup_{y \in \Re^n} [q(x,y) + \beta g(y)] =$$

$$= \alpha (Tf)(x) + (1-\alpha)(Tg)(x).$$

An other useful property will be frequently invoked is the following [5]:

Proposition 6 *Let some function $\hat{v}(x)$ satisfy the following properties:*

1. $T\hat{v} \leq \hat{v}$;
2. $\liminf_{t\to\infty} \beta^t \hat{v}(x_t) \leq 0$ *for each feasible sequence;*
3. $\sum_{t=0}^{\infty} q(x_t, x_{t+1})\beta^t \leq \hat{v}(x_0)$, $\forall x_0 \in \Re^n$ *and for each feasible sequence*;
4. *the function* $w(x) = \lim_{m\to\infty}(T^m \hat{v})(x)$ *is a fixed point of* T*;*

then w is the value function of (1) .

Whenever $q(x,y)$ in (1) is quadratic, T sends quadratic functions to quadratic functions. Below we formulate a sharper statement.

Lemma 7 *The Bellman operator (3) sends the cone $Sym_-(n)$ into the interval $[A, A - BC^+B']$.*

Proof. Let $f(x) = x'Hx$, with $H \in Sym_-(n)$. The supremum of (3) is attained in view of Lemma 4. We have $(Tf)(x) = x'Ax - x'B(C+\beta H)^+B'x$. Moreover $(Tf)(x) \geq x'Ax$ because $B(C+\beta H)^+B'$ is negative semidefinite. Since T is monotonic, $(Tf)(x) \leq (T0)(x) = x'(A - BC^+B')x$ and hence T sends $Sym_-(n)$ into interval $[A, A - BC^+B']$. ■

4.3 Thompson metric and contraction property

We present a contraction theorem which uses Thompson's distance. We must first add some more notation.

Two functions $f, g \in \mathcal{K}_-(Y) - \{0\}$ are said *comparable* if two real numbers α, $\beta > 0$ exist such that $\alpha f \leq g \leq \beta f$.

If f and g are comparable, we define

$$M(f \mid g) = \inf\{\alpha > 0; g \geq \alpha f\}.$$

The Thompson's distance is given by

$$d(f,g) = \max\{\ln M(g \mid f),\ \ln M(f \mid g)\}.$$

We refer to Thompson [8] for further details. The following theorem [4] is a contraction theorem valid for a certain class of operators acting over $\mathcal{K}_-(Y)$.

Theorem 8 *Let $[f_0, f_1]$ be an interval of $\mathcal{K}_-(Y) - \{0\}$, with f_0 and f_1 two comparable functions. If T is an operator mapping $[f_0, f_1]$ into itself, having the following two properties:*

1. *T is monotonic,* i.e., *$f \leq g$ implies $Tf \leq Tg$, for all f, $g \in [f_0, f_1]$,*
2. *$T(\alpha f + (1-\alpha) f_1) \leq \alpha Tf + (1-\alpha) Tf_1$ for $\alpha \in [0,1]$ and $f \in [f_0, f_1]$,*

then T turns out to be a contraction for Thompson's distance. More precisely, $d(Tf, Tg) \le \gamma d(f, g)$ holds for all f and g in $[f_0, f_1]$, where $\gamma = 1 - \mu^{-1} < 1$ and $\mu = M(f_0 \mid f_1)$.

The two assumptions postulated in the previous theorem hold for the Bellman operator T. We saw that T is monotonic. Moreover, the convexity property of T implies assumption 2. It should be noted that all what will be needed in the present paper is inclosed in the next corollary that can be easily derived from Theorem 8 (see [4]).

Corollary 9 *Under the same conditions of Theorem 8, T has an unique fixed point f^*. For any initial condition $f \in [f_0, f_1]$, the sequence $T^m f$ of iterates converges uniformly to f^*,* i.e., $\|T^m f - f^*\| \to 0$.

We need to add an important remark. Theorem 8 is valid for functions $f_0, f_1 \in \mathcal{K}_-(Y) - \{0\}$. Indeed, in our applications f_0, f_1 are quadratic functions. In other words, $f_0, f_1 \in Sym_-(n)$. It should be noted that every thing has unchanged and of course the fixed point f^* is a quadratic function as well. All this follows from the fact that the subset $[f_0, f_1] \cap Sym_-(n)$ is a complete subspace of $[f_0, f_1]$ with respect of Thompson's distance.

4.4 Proofs of theorems

Lemma 10 *Given the iterative process $H_{m+1} = TH_m$, with $H_0 \in Sym_-(n)$, if $\mathcal{N}(H_0) \supset N_Q$ and $H_1 \le H_0$, then after a finite number i of steps, $H_m \in Sym_-(n; N_Q)$, $\forall m \ge i$. If H_i is the first matrix belonging to $Sym_-(n; N_Q)$, then it follows that $TH \in Sym_-(n; N_Q)$ for each $H \le H_i$.*

Proof. Since $H_1 \le H_0$, it follows that $H_{m+1} \le H_m$, $\forall m$, from the monotonicity of T. From this last inequality we get $\mathcal{N}(H_{m+1}) \subset \mathcal{N}(H_m)$, $\forall m$. Hence an index i exists such that $\mathcal{N}(H_i) = \mathcal{N}(H_{i+1}) = ... = M$. Moreover, the vector space $\mathcal{N}(H_{m+1})$ can be defined as

$$\mathcal{N}(H_{m+1}) = \{x \in \Re^n; \exists y \in \mathcal{N}(H_m) \text{ s.t. } (x, y) \in \mathcal{N}(Q)\}.$$

In fact, $x'H_{m+1}x = \sup_{y \in \Re^n} [q(x, y) + \beta y'H_m y]$. Now, if for one x we have $H_{m+1}x = 0$, Lemma 4 implies that some y exists for which $q(x, y) = 0$ and $y'H_m y = 0$.

Hence the vector space M satisfies the following property: $x \in M \Longrightarrow \exists y \in M$ such that $(x,y) \in \mathcal{N}(Q)$. Therefore, $M \subset N_Q$.

Being $\mathcal{N}(H_0) \supset N_Q$, let us prove that $\mathcal{N}(H_m) \supset N_Q$, $\forall m \geq 0$, by induction. Let $\mathcal{N}(H_m) \supset N_Q$. If $x \in N_Q$, then some $y \in N_Q \subset \mathcal{N}(H_m)$ exists such that $q(x,y) = 0$. But $y' H_m y = 0$, hence it follows $x' H_{m+1} x = 0$, *i.e.* $x \in \mathcal{N}(H_{m+1})$. Thus $\mathcal{N}(H_{m+1}) \supset N_Q$ and the induction argument is concluded.

From the previous property, we infer that $M \supset N_Q$ and thus $M = N_Q$. So that, after a finite number i of steps, $H_m \in Sym_{-}(n; N_Q)$, $\forall m \geq i$.

Let us now prove the second part of this lemma. If $H \leq H_i$ then $TH \leq H_{i+1} \leq H_i$. Therefore $\mathcal{N}(TH) \subset N_Q$. Since $H \in Sym_{-}(n; N_Q)$, then $\mathcal{N}(TH) \supset N_Q$ and this shows that $TH \in Sym_{-}(n; N_Q)$. ■

Lemma 11 *If M is a vector subspace, any two elements H and K of $Sym_{-}(n; M)$ are comparable.*

Proof. The proof will be an immediate consequence of the following property. Let H be a negative semidefinite matrix with $\mathcal{N}(H) = M$ and $(\mu_1, \mu_2, ..., \mu_k, 0)$ be its eigenvalues with $\mu_1 \leq \mu_2 \leq ... \leq \mu_k < 0$ and $k \leq n$. If P denotes the orthogonal projection on the subspace $M^{\perp}$ then, obviously, $\mathcal{N}(P) = M$. By the diagonalization method, it is possible to prove that for any x the following inequalities hold:

$$\mu_1 \|Px\|^2 \leq x' H x \leq \mu_k \|Px\|^2 .$$

From this we infer that any $H \in Sym_{-}(n; M)$ is comparable to $-P'P$, since it holds that $\mu \|Px\|^2 = -|\mu| \, x' P' P x$. ■

Proof of Theorem 1: Let us take $H_0 = 0$. By virtue of Lemma 10, we have $H_i = T^i 0 \in Sym_{-}(n; N_Q)$ for some index i. Let $\sigma = \rho(A)$ be the spectral radius of A. Consider the matrix $-\sigma P'P \in Sym_{-}(n; N_Q)$, where P is the orthogonal projection on $N_Q^{\perp}$ (see proof of Lemma 11).

Let us show that the Bellman operator T sends the interval $[-\sigma P'P, \, T^i 0]$ into itself. In fact, if $H \in [-\sigma P'P, \, T^i 0]$, then $\mathcal{N}(H) = N_Q$. Now, since $H \leq T^i 0$, $TH \leq T^{i+1} 0 \leq T^i 0$. Moreover we know that $TH \geq A \geq -\sigma I$ and this last property means that all the eigenvalues of TH are greater than $-\sigma$. By Lemma 11 it follows that $TH \geq -\sigma P'P$ and this proves the invariance of the interval $[-\sigma P'P, \, T^i 0]$.

By Theorem 8 we can conclude that T is a contraction and consequently a unique fixed point H^* exists within $[-\sigma P'P,\, T^i 0]$.

Considering the sequence $T^i 0$, this converges uniformly to the fixed point H^* respect to the Thompson metric. Therefore, in view of the Proposition 6, $x'H^*x$ turns out to be the value function. This proves (ii).

It is also straightforward to show that $H^{\hat{}} \in Sym_{-}(n;M)$ for some M. Let us prove that $H^{\hat{}} \leq H^*$. For any given x_0, if $H^{\hat{}}$ satisfies the Bellman equation then some x_1 exists such that $x_0'H^{\hat{}}x_0 = q(x_0,x_1) + \beta x_1' H^{\hat{}} x_1$. By iterating this procedure, a sequence of vectors is obtained for which $x_0'H^{\hat{}}x_0 = \sum_{t=0}^{N-1} q\left(x_t,x_{t+1}\right)\beta^t + \beta^N x_N' H^{\hat{}} x_N \leq \sum_{t=0}^{N-1} q\left(x_t,x_{t+1}\right)\beta^t$. Consequently $x_0'H^{\hat{}}x_0 \leq \sum_{t=0}^{N-1} q\left(x_t,x_{t+1}\right)\beta^t \leq x_0'H^*x_0$ and (iv) is proven.

Now, let us prove (iii), i.e., the existence of a unique fixed point of T in $Sym_{-}(n;N_Q)$. If, for absurd, another fixed point H_1 would exist then we have $H_1 \leq H^*$. Therefore T would map $[H_1, H^*]$ into itself. Therefore T would result a contraction but this leads to an absurd.

Point (v) is a direct consequence of Lemma 10. It should be noted that optimal solutions exist since $\liminf_{N\to\infty} \beta^N x_N' H^* x_N \leq 0$ (see Stokey et Alii [7]). ■

Proof of Theorem 2 It will be sufficient to prove that the matrices A and $A-BC^+B'$ are comparable so that the assert will be a direct consequence of the Corollary 9.

In fact, $A \leq A - BC^+B'$. In addition, by Lemma 5 it follows that $\gamma A - BC^+B'$ must be negative semidefinite and hence $A - BC^+B' \leq (1-\gamma)A$. Therefore the matrix A and $A - BC^+B'$ are comparable and this concludes the proof. ■

Proof of Proposition 3. Use the linear translation $(x,y) = (\xi,\eta) + (h,h)$. The (4) becomes

$$(\xi,\eta)'Q(\xi,\eta) + (2Ah + Bh + a)'\xi + (2B'h + 2Ch + b)'\eta + \gamma \tag{8}$$

where $\gamma = q(h,h)$. Since the Euler equation (6) turns into:

$$(2B'\bar{x} + 2C\bar{x} + b) + \beta(2A\bar{x} + 2B\bar{x} + a) = 0,$$

if we define $u = 2A\bar{x} + 2B\bar{x} + a$, then $2B'\bar{x} + 2C\bar{x} + b = -\beta u$. Therefore, if the translation is made with $h = \bar{x}$, the (8) becomes $(\xi,\eta)'Q(\xi,\eta) + u'\xi - \beta u'\eta + \bar{\gamma} - \beta\bar{\gamma}$ where $\gamma = \bar{\gamma} - \beta\bar{\gamma}$, that is $\bar{\gamma} = \gamma/(1-\beta)$.

Now, let $\hat{H}$ be a fixed point of the T_Q Bellman operator (5) that is, $\xi'\hat{H}\xi = \sup_{\eta\in\Re^n}\left[(\xi,\eta)'Q(\xi,\eta)+\beta\eta'\hat{H}\eta\right]$. By adding $u'\xi+\bar{\gamma}$ to the left and the right-hand sides we obtain

$$\xi'\hat{H}\xi+u'\xi+\bar{\gamma} = \sup_{\eta\in\Re^n}\left[(\xi,\eta)'Q(\xi,\eta)+u'\xi-\beta u'\eta+\bar{\gamma}-\beta\bar{\gamma}+\beta(\eta'\hat{H}\eta+u'\eta+\bar{\gamma})\right].$$

Hence the quadratic function $\xi'\hat{H}\xi+u'\xi+\bar{\gamma}$ turns out to be a fixed point of T which becomes $(x-\bar{x})'\hat{H}(x-\bar{x})+u'(x-\bar{x})+\frac{1}{1-\beta}\gamma$, coming back to the old variables (x,y). This completes the proof. ■

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