A Dynamic Heterogeneous Beliefs CAPM

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1 Plan of Talk

- Motivation
- A Static Heterogeneous Beliefs CAPM
  - Heterogeneous Beliefs and Consensus Belief
  - Equilibrium Return Relation, Betas, and Equilibrium Prices
- A Dynamic Heterogeneous Beliefs CAPM
  - Market fractions
  - Equilibrium Return Relation, Dynamic Beta, and Equilibrium Price
- A numerical example
  - Fundamentalists, trend followers and noise traders
  - Equilibrium returns, dynamic betas and Sharpe ratios
- Some Conclusions

2 Motivation

- Heterogeneous agent literature becoming well developed recently (e.g. Brock and Hommes, Lux and Marchesi)
  - Typically one risky/ one risk free asset framework.
  - Focus on patterns of price and return dynamics.
- Much less work on multiple assets and portfolio considerations
  - See Böhm and Wenzelburger; Chiarella, Dieci and He; Chiarella, Dieci and Gardini
- The effect of heterogeneity on CAPM little studied
  - See Lintner(1969)
- Aim of this paper is to study effect of heterogeneity on CAPM, taking dynamic feedback into consideration.

3 Heterogeneous Beliefs CAPM—A Static Model

3.1 Heterogeneous Beliefs and Consensus Belief

- **Market**: one frisk-free asset (r_f) and N risky assets ($\widetilde{r}_j, j = 1, 2, \cdots, N$).
- **Heterogeneous Beliefs**
  - Some of the ideas go back to Lintner (1969).
  - Assume $\widetilde{r}_j \sim MVN$
  - Heterogeneous beliefs $\mathcal{B}_i$ defined by $\mathcal{B}_i(\widetilde{\mathbf{r}}) = (\mathbb{E}_i(\widetilde{\mathbf{r}}), \Omega_i = Cov_i(\widetilde{r}_k, \widetilde{r}_l))$.
- **Optimal Portfolio**:
  - Investor i has a concave utility of wealth function $u_i(\cdot)$.
  - Portfolio wealth: $\widetilde{W}_i = W_0^i(1 + r_f + w^T(\tilde{r} - r_f 1))$

– The global absolute risk aversion:

$$\theta_i := -E_i\left[u_i''(\widetilde{W}_i)\right] / E_i\left[u_i'(\widetilde{W}_i)\right]$$

– The optimal portfolio of investor i:

$$\mathbf{w}_i = \frac{\theta_i^{-1}}{W_0^i}\Omega_i^{-1}E_i\left[\widetilde{\mathbf{r}} - r_f\mathbf{1}\right].$$

- **Aggregation**:
  – Aggregate wealth

$$\sum_{i=1}^{I} W_0^i \mathbf{w}_i = \sum_{i=1}^{I} \theta_i^{-1}\Omega_i^{-1}E_i\left[\widetilde{\mathbf{r}} - r_f\mathbf{1}\right]$$

  – The vector of the *aggregate wealth proportions* in the risky assets

$$\mathbf{w}_a = \frac{1}{W_{m0}} \sum_{i=1}^{I} W_0^i \mathbf{w}_i = \frac{1}{W_{m0}} \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} E_i \left[\tilde{\mathbf{r}} - r_f \mathbf{1}\right]$$

- **Consensus Belief**: $\mathcal{B}_a = \{\mathbb{E}_a(\tilde{\mathbf{r}}), \Omega_a\}$
  - Aggregate risk aversion: $\Theta := \left(\sum_{i=1}^{I} \theta_i^{-1}\right)^{-1}$.
  - An "aggregate" variance/covariance matrix Ω_a can be defined as

$$\Omega_a^{-1} = \Theta \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1}.$$

  - The "aggregate" expected returns on the risky assets $E_a(\tilde{\mathbf{r}})$:

$$E_a(\tilde{\mathbf{r}}) = \Theta \Omega_a \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} E_i(\tilde{\mathbf{r}})$$

3.2 Equilibrium CAPM

- **Market Portfolio**:
  - We define the random return $\widetilde{r}_m$ on the market

$$\widetilde{W}_m := \sum_{i=1}^{I} \widetilde{W}_i = W_{m0}(1 + \widetilde{r}_m) \quad \Rightarrow \quad \boxed{\widetilde{r}_m = \frac{\widetilde{W}_m}{W_{m0}} - 1}$$

  - In terms of aggregate wealth proportions

$$\widetilde{r}_m := r_f + \mathrm{w}_a^\top (\widetilde{\mathrm{r}} - r_f \mathbf{1})$$

  - the aggregate ‘consensus’ variance belief:

$$\sigma_{a,m}^2 := \mathrm{w}_a^\top \Omega_a \mathrm{w}_a$$

– Then the aggregate expected market return

$$E_a(\widetilde{r}_m) := r_f + \mathrm{w}_a^\top \left(E_a\left(\widetilde{\mathrm{r}}\right) - r_f \mathbf{1}\right)$$

– Aggregate variance of market portfolio becomes

$$\sigma^2_{a,m} = \frac{1}{\Theta W_{m0}} \left\{\mathrm{w}_a^\top \left[E_a\left(\widetilde{\mathrm{r}}\right) - r_f \mathbf{1}\right]\right\}$$

- **Return Relation**
  – The aggregate expected market risk premium is proportional to the aggregate relative risk aversion of the economy:

$$[E_a(\widetilde{r}_m) - r_f] = \Theta W_{m0} \sigma^2_{a,m}$$

  – Aggregate excess return

$$\Omega_a \mathrm{w}_a = \frac{1}{\Theta W_{m0}} [E_a(\tilde{r}) - r_f.\mathbf{1}]$$

– The CAPM Equilibrium relation under the heterogeneous beliefs:

$$[E_a(\tilde{\mathbf{r}}) - r_f\mathbf{1}] = \frac{1}{\sigma^2_{a,m}}\Omega_a \mathbf{w}_a [E_a(\tilde{r}_m) - r_f].$$

- **Heterogeneous beta**:

$$\beta_{a,m} = \frac{\Omega_a \mathbf{w}_a}{\sigma^2_{a,m}} = \frac{[E_a(\tilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \Omega_a^{-1}\mathbf{1}}{[E_a(\tilde{\mathbf{r}}) - r_f\mathbf{1}]^\top \Omega_a^{-1}[E_a(\tilde{\mathbf{r}}) - r_f\mathbf{1}]}[E_a(\tilde{\mathbf{r}}) - r_f\mathbf{1}]$$

3.3 Equilibrium Prices

- Assume that agents have CARA utility $\Rightarrow \theta_i$ = constant.
- In this case we obtain explicitly the optimal demands

$$\mathbf{w}_i = \frac{1}{W_0^i}\theta_i^{-1}\Omega_i^{-1}E_i\left[\widetilde{\mathbf{r}} - r_f\mathbf{1}\right]$$

- The equilibrium price

$$\mathbf{p}_0 = \mathbf{Z}^{-1}\sum_{i=1}^{I}\theta_i^{-1}\Omega_i^{-1}E_i\left[\widetilde{\mathbf{r}} - r_f\mathbf{1}\right]$$

where $\mathbf{z} := [z_1, z_2, ..., z_N]^T$ the supply vector and $\mathbf{Z} := diag[z_1, z_2, ..., z_N]$.

- The betas can also be expressed in terms of market clearing prices:-

$$\beta_{a,m} = \frac{\mathbf{p}_0^\top\mathbf{z}}{\mathbf{p}_0^\top\mathbf{Z}\Omega_a\mathbf{Z}\mathbf{p}_0}\Omega_a\mathbf{Z}\mathbf{p}_0$$

4 Heterogeneous CAPM—A Dynamic Model

4.1 Market Fractions and Consensus Belief

- Incorporate into a dynamic setup into the CAPM-like return relationships in the static framework.
- Group the I investors into a finite number of agent-types $h \in H$
  - $I_h, h \in H$, the number of investors in group h.
  - $n_h := I_h / I$ the fraction of agents of type h.
- Supply: $\mathbf{s} := (1/I)\mathbf{z}$ the supply of shares per investor.
- Define the "average" risk aversion: $\theta_a := \left(\sum_{h \in H} n_h \theta_h^{-1} \right)^{-1}$

- The aggregate beliefs can be rewritten,

$$\Omega_a = \theta_a^{-1} \left(\sum_{h \in H} n_h \theta_h^{-1} \Omega_h^{-1} \right)^{-1}$$

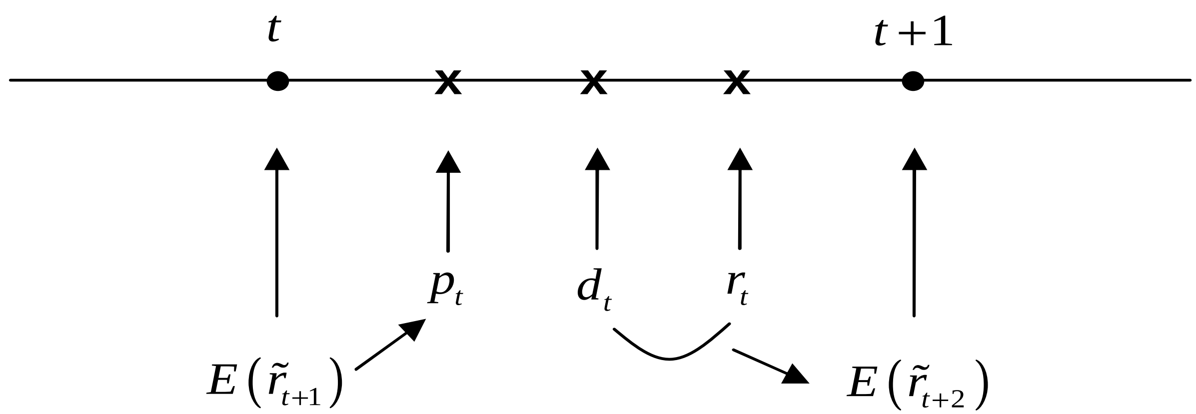
$$E_a(\tilde{\mathbf{r}}) = \theta_a \Omega_a \sum_{h \in H} n_h \theta_h^{-1} \Omega_h^{-1} E_h(\tilde{\mathbf{r}})$$

- the equilibrium prices are rewritten as

$$\mathbf{p}_0 = \mathbf{S}^{-1} \sum_{h \in H} n_h \theta_h^{-1} \Omega_h^{-1} \left[E_h \left(\tilde{\mathbf{r}} \right) - r_f \mathbf{1} \right]$$

4.2 Heterogeneous Beliefs

- Assume one-period ahead utility maximization
- From time t to time $t + 1$.



- Heterogeneous agents' assessments about $\widetilde{\mathbf{r}}_{t+1}$ are functions of the information up to time $t - 1$.

- For belief-type $h \in H$

$$\Omega_{h,t} := [Cov_{h,t}(\widetilde{r}_{j,t+1}, \widetilde{r}_{k,t+1})] = \Omega_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...)$$

$$E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) = \mathbf{f}_h(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, ...)$$

- Similarly for the aggregate beliefs $\Omega_{a,t}$ and $E_{a,t}(\widetilde{\mathbf{r}}_{t+1})$.

4.3 Dynamic Equilibrium and Beta

- The market clearing **prices** at time t become

$$\mathbf{p}_t = \mathbf{S}^{-1} \sum_{h \in H} n_h \theta_h^{-1} \Omega_{h,t}^{-1} [E_{h,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]$$

- The realized **returns** can be written

$$\mathbf{r}_t = \mathbf{F}(\mathbf{r}_{t-1}, \mathbf{r}_{t-2}, \ldots; \widetilde{\mathbf{d}}_t)$$

- The random return on the market portfolio is

$$\widetilde{r}_{m,t+1} = \frac{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]^\top \Omega_{a,t}^{-1} \widetilde{\mathbf{r}}_{t+1}}{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]^\top \Omega_{a,t}^{-1} \mathbf{1}}$$

- At the beginning of $(t, t+1)$ the aggregate beliefs about returns (based on information up to time $t-1$) satisfy

$$E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1} = \beta_{a,mt}[E_{a,t}(\widetilde{r}_{m,t+1}) - r_f]$$

- The "aggregate" beta coefficients are

$$\beta_{a,mt} = \frac{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]^\top \Omega_{a,t}^{-1} \mathbf{1}}{[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]^\top \Omega_{a,t}^{-1} [E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]} [E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f \mathbf{1}]$$

- The "aggregate" betas are time varying due to time varying beliefs about both the second moment and the first moment of the returns distribution.

5 An example

- Consider a specific example of interaction of different beliefs types
- Two types of agents, ***fundamentalists***, and ***trend followers*.**
  - **Fundamentalists**:

$$E_{f,t}(\widetilde{\mathbf{r}}_{t+1}) = \rho_f, \qquad \Omega_{f,t} = \overline{\Omega}_f.$$

  - **Trend Followers**:

$$\begin{aligned} E_{c,t}(\widetilde{\mathbf{r}}_{t+1}) &= \rho_c + \gamma(\mathbf{r}_{t-1} - \mathbf{u}_{t-1}), \\ \mathbf{u}_{t-1} &= \delta \mathbf{u}_{t-2} + (1-\delta)\mathbf{r}_{t-1} \\ \Omega_{c,t} &= \overline{\Omega}_c + \lambda \mathbf{V}_{t-1}, \\ \mathbf{V}_{t-1} &= \delta \mathbf{V}_{t-2} + \delta(1-\delta)(\mathbf{r}_{t-1} - \mathbf{u}_{t-2})(\mathbf{r}_{t-1} - \mathbf{u}_{t-2})^\top \end{aligned}$$

- In addition, we consider - ***noise traders*** - whose demand for each risky asset is an exogenous random disturbance.
- θ_f and θ_c the risk aversion coefficients of the two agent-types
- n_f and $n_c = 1 - n_f$ their market fractions
- $\theta_a = \left(n_f\theta_f^{-1} + n_c\theta_c^{-1}\right)^{-1}$ the average risk aversion.
- The aggregate variances/covariances and expected excess returns are given, by

$$\Omega_{a,t} = \left(\frac{n_f}{\theta_f} + \frac{n_c}{\theta_c}\right)\left(\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1} + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}\right)^{-1}$$

$$E_{a,t}(\widetilde{\mathrm{r}}_{t+1}) \quad = \quad \theta_a\Omega_{a,t}\left[\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1}E_{f,t}(\widetilde{\mathrm{r}}_{t+1}) + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}E_{c,t}(\widetilde{\mathrm{r}}_{t+1})\right]$$

- The dynamic model becomes the noisy nonlinear dynamical system

$$\mathbf{p}_t = \mathbf{S}^{-1}\left\{\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1}\rho_f + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}[\rho_c + \gamma(\mathbf{r}_{t-1} - \mathbf{u}_{t-1})] - \left(\frac{n_f}{\theta_f}\overline{\Omega}_f^{-1} + \frac{n_c}{\theta_c}\Omega_{c,t}^{-1}\right) r_f \mathbf{1}\right\}$$

$$\mathbf{r}_t = \mathbf{P}_{t-1}^{-1}(\mathbf{p}_t + \widetilde{\mathbf{d}}_t) - \mathbf{1}$$

where

$$\mathbf{P}_{t-1} = \mathbf{diag}(p_{1,t-1}, p_{2,t-1}, \cdots, p_{N,t-1}).$$

- The effect of noise traders:
  - The risky asset demand from the noise traders is described by the random vector $\widetilde{\xi}_t := [\widetilde{\xi}_{1,t}, \widetilde{\xi}_{2,t}, ..., \widetilde{\xi}_{N,t}]^\top$, $\widetilde{\xi}_{j,t}$
    * i.i.d. with $E(\widetilde{\xi}_{j,t}) = 0$,
    * $Var(\widetilde{\xi}_{j,t}) = q^2 s_j^2$,
    * $E(\widetilde{\xi}_{j,t}, \widetilde{\xi}_{k,t}) = 0, j, k = 1, 2, ..., N$.
    * $\widetilde{\Xi}_t := diag(\widetilde{\xi}_{1,t}, \widetilde{\xi}_{2,t}, ..., \widetilde{\xi}_{N,t})$.

– The market clearing conditions in the presence of noise traders:-

$$\theta_a^{-1}\boldsymbol{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}] + \widetilde{\boldsymbol{\Xi}}_t\mathbf{p}_t = \mathbf{S}\mathbf{p}_t$$

– The market clearing prices thus become

$$\mathbf{p}_t = (\mathbf{S}-\widetilde{\boldsymbol{\Xi}}_t)^{-1}\theta_a^{-1}\boldsymbol{\Omega}_{a,t}^{-1}[E_{a,t}(\widetilde{\mathbf{r}}_{t+1}) - r_f\mathbf{1}] \qquad (5.1)$$

– Note that the introduction of noise traders is formally equivalent to assuming a noisy supply vector $\widetilde{\mathbf{s}}_t = \mathbf{s}-\widetilde{\boldsymbol{\xi}}_t$.

6 Some numerical experiments

- Constant homogeneous beliefs: *fundamentalists* (and *noise traders*)
- Time-varying heterogeneous beliefs about expected returns: *fundamentalists*, *trend followers* (and *noise traders*)
- Role of 'market fraction'
- Effect of updating second moment beliefs
- Focus on
  - 'Market portfolio' weights
  - Asset returns and market return (realized)
  - Beta coefficients
  - Market price of risk (in terms of aggregate 'consensus' beliefs)

3 risky assets

one risk-free asset

asset 1

asset 2

asset 3

market portfolio

Base parameter selection

$$\rho_f = \rho_c = \begin{bmatrix} 9\% & 11\% & 15\% \end{bmatrix}'$$

$$\gamma = 0.05, \quad \delta = 0.95, \quad \lambda = 0$$

$$\theta_f = \theta_c = 1$$

$$\bar{\Omega}_f = \bar{\Omega}_c = \begin{bmatrix} 0.16^2 & 0 & 0 \\ 0 & 0.20^2 & 0 \\ 0 & 0 & 0.24^2 \end{bmatrix}$$

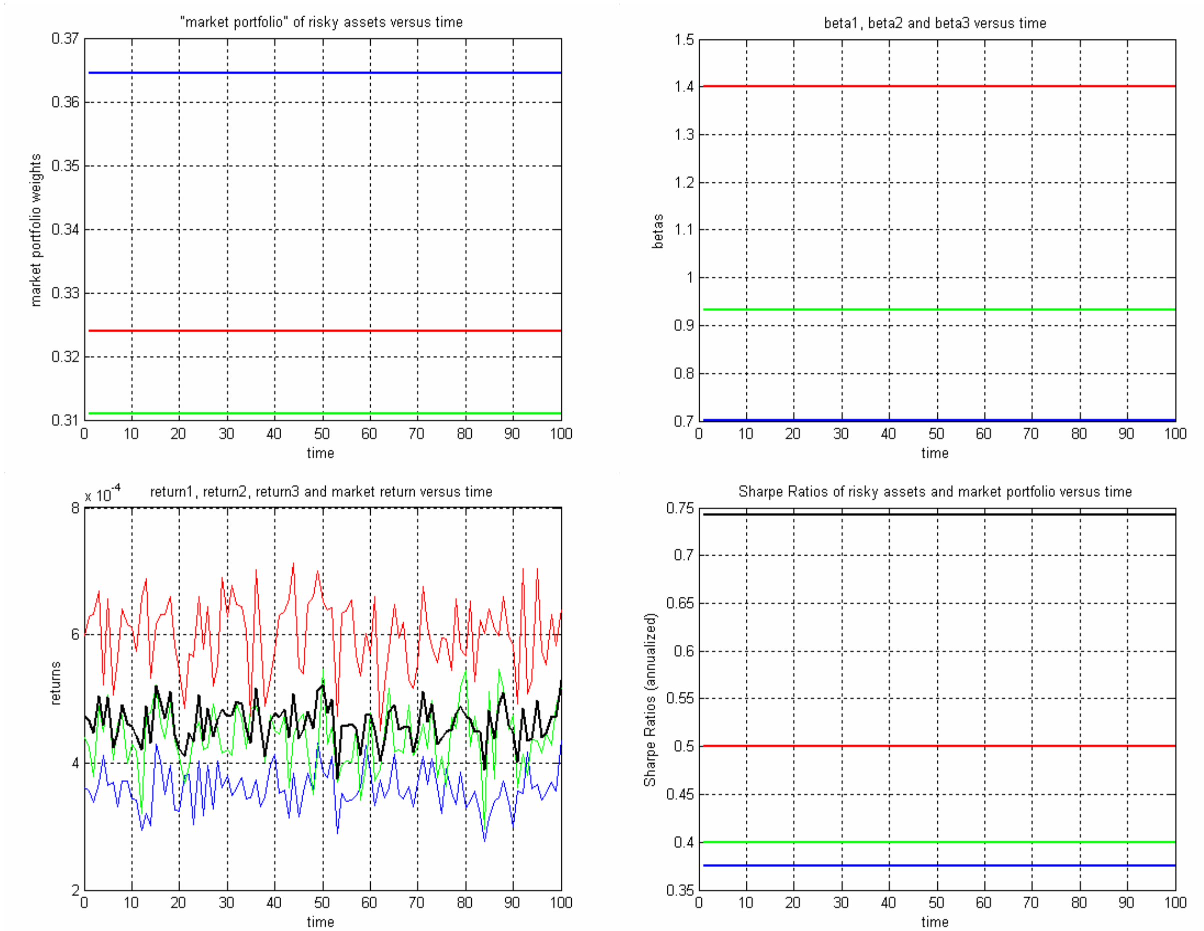
$$\bar{d} = \begin{bmatrix} 210 & 220 & 310 \end{bmatrix}'$$

dividend st.dev.:10% of average dividend

$$r_f = 3\%$$

$$s = \begin{bmatrix} 0.001 & 0.001 & 0.001 \end{bmatrix}'$$

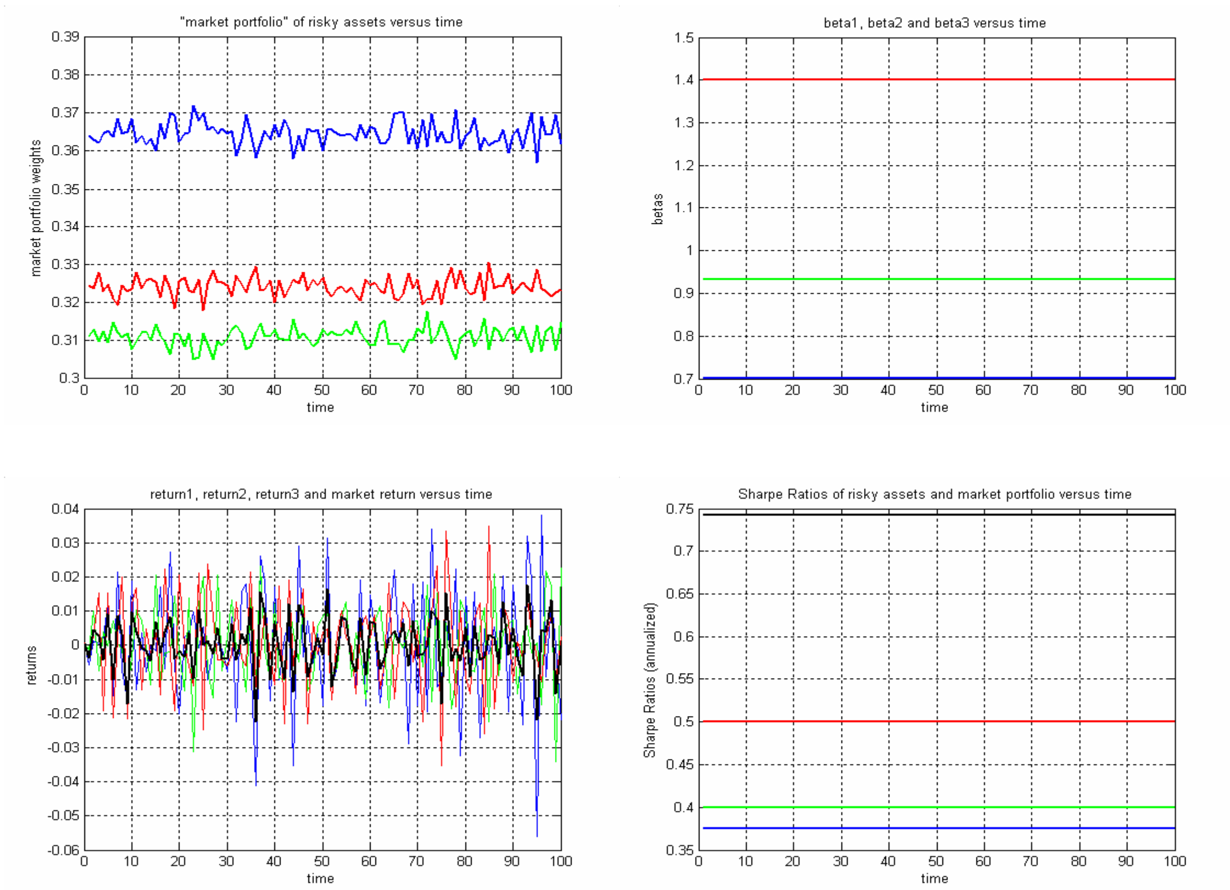
- **Fundamentalists with constant beliefs - no noise traders**
  (noise from dividend process)



- **Fundamentalists with constant beliefs - noise traders**

  (noise trading st.dev.: 1% of supply)

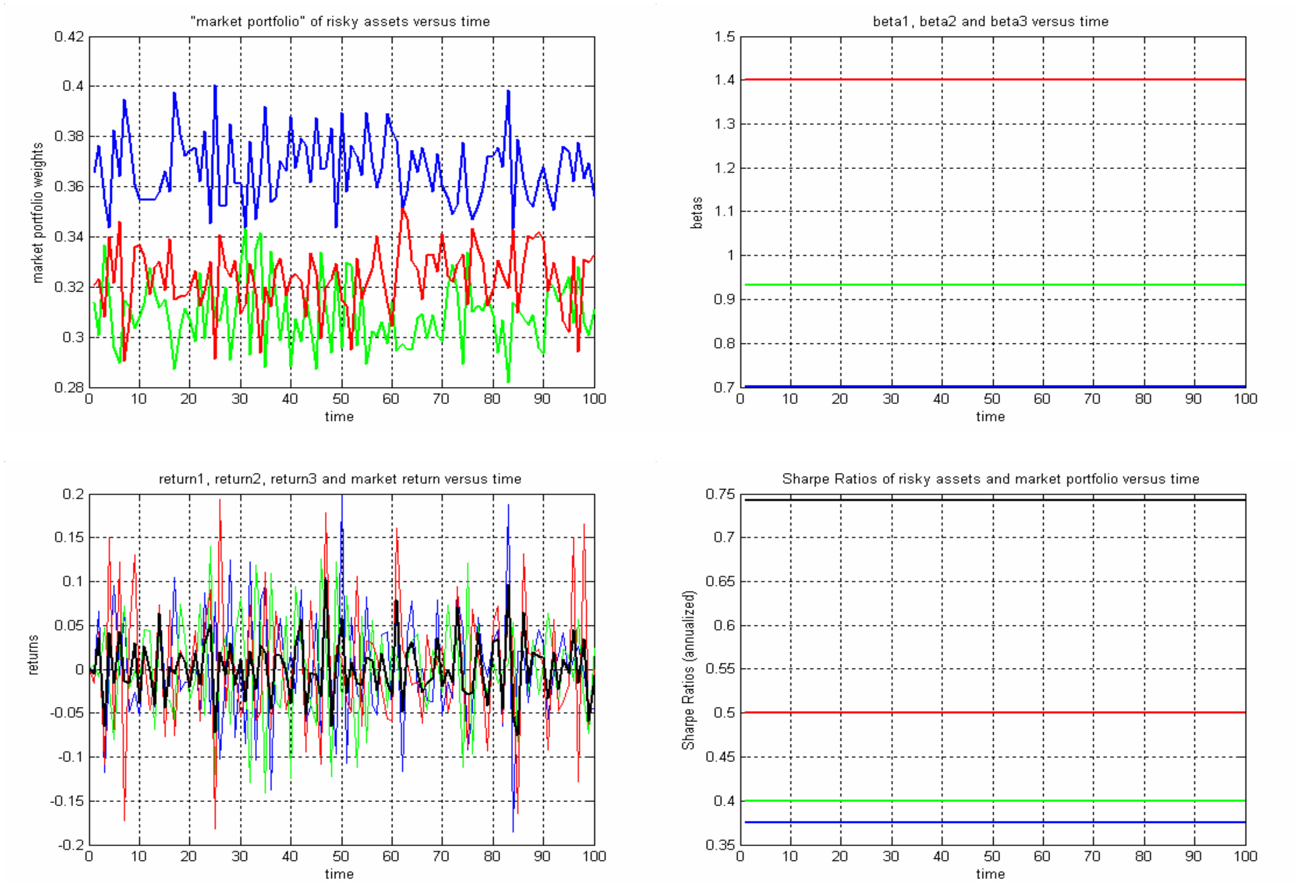
  (noise from dividend process and noise trading)



- **Fundamentalists with constant beliefs - noise traders**

  (noise trading st.dev.: 5% of supply)

  (noise from dividend process and noise trading)

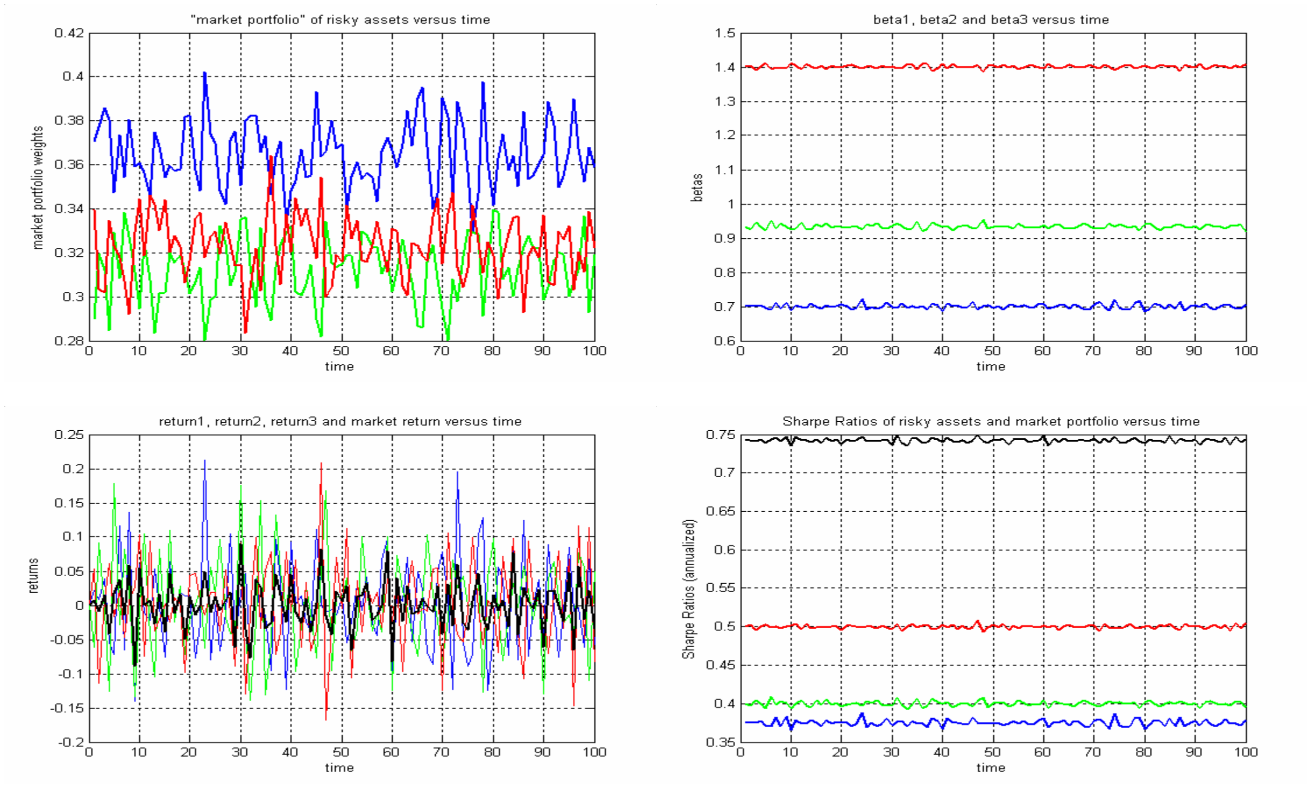


- **Noisy returns feed back into aggregate beliefs and beta coefficients**

20% trend followers with time varying expectations (extrapolated from past returns)

80% fundamentalists with constant beliefs, noise traders

(noise trading st.dev.= 5% of supply)

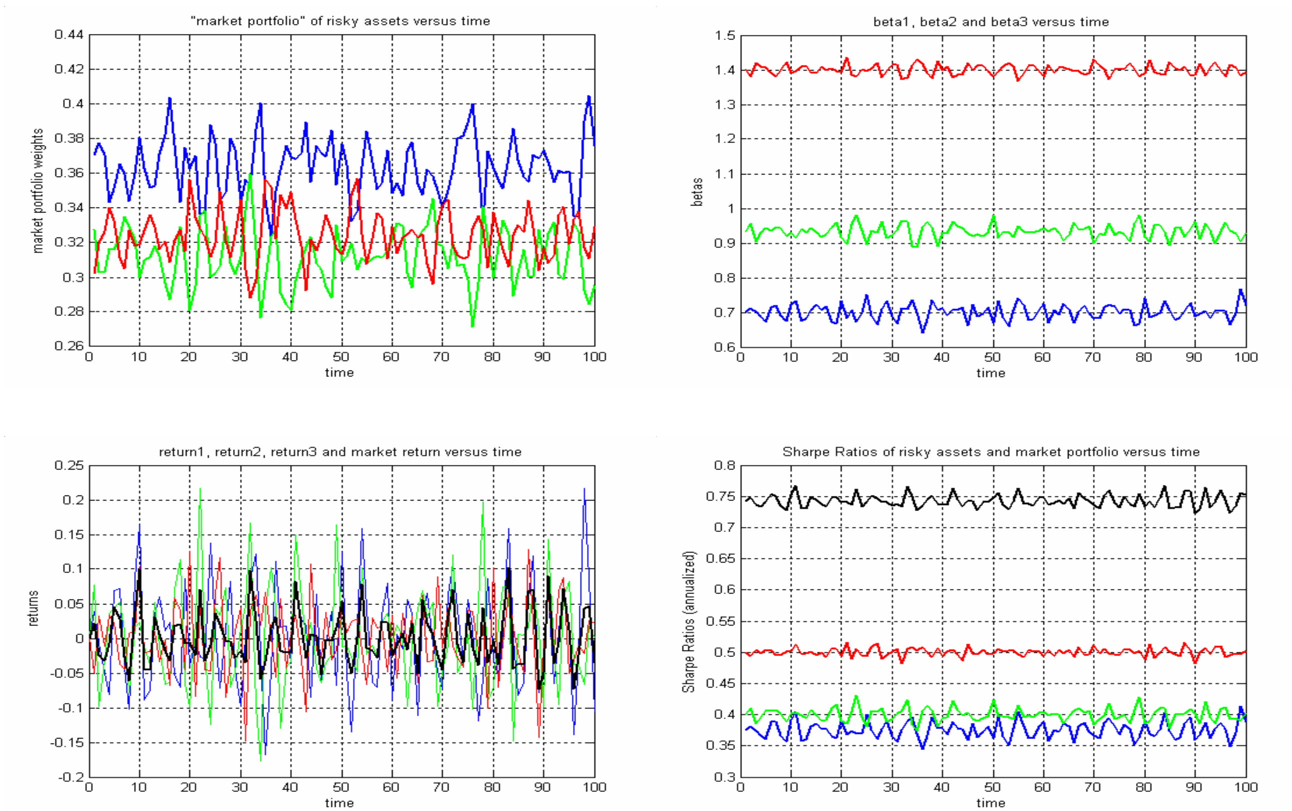


- **Noisy returns feed back into aggregate beliefs and beta coefficients**

40% trend followers with time varying expectations (extrapolated from past returns)

60% fundamentalists with constant beliefs, noise traders

(noise trading st.dev.= 5% of supply)

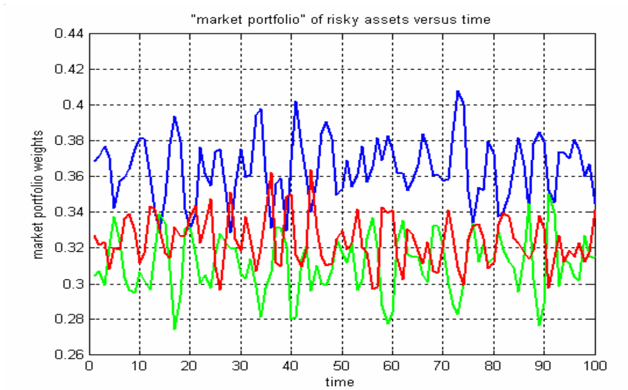


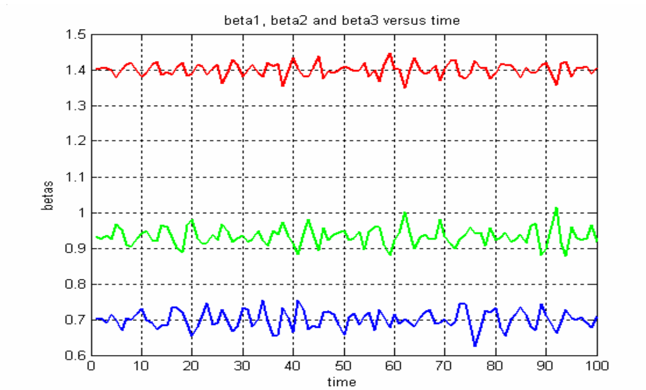
- **Noisy returns feed back into aggregate beliefs and beta coefficients**

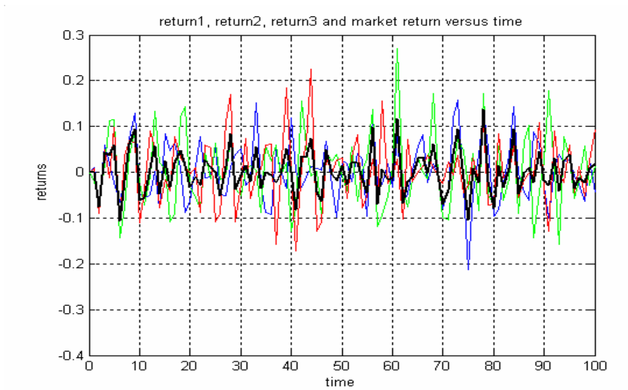
75% trend followers with time varying expectations (extrapolated from past returns)

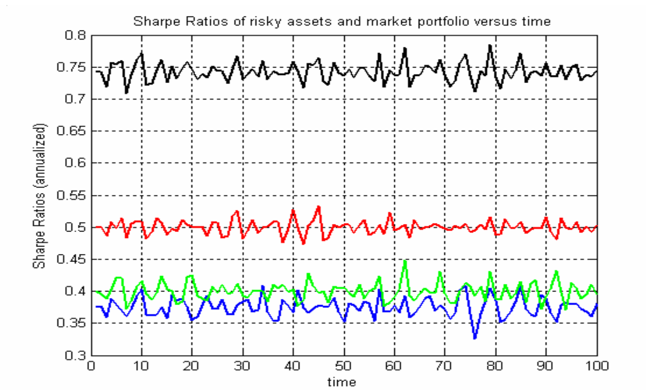
25% fundamentalists with constant beliefs,

noise traders (noise trading st.dev.= 5% of supply)





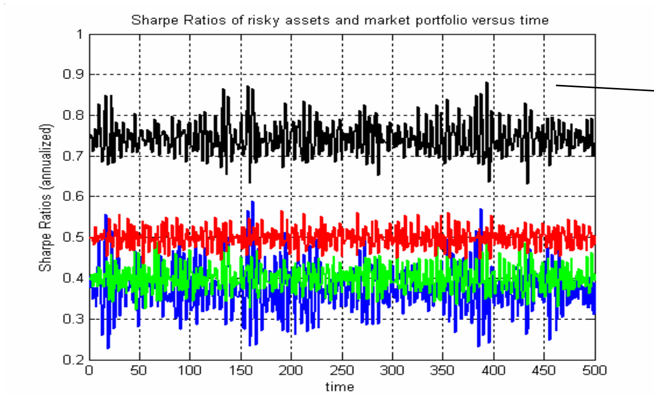


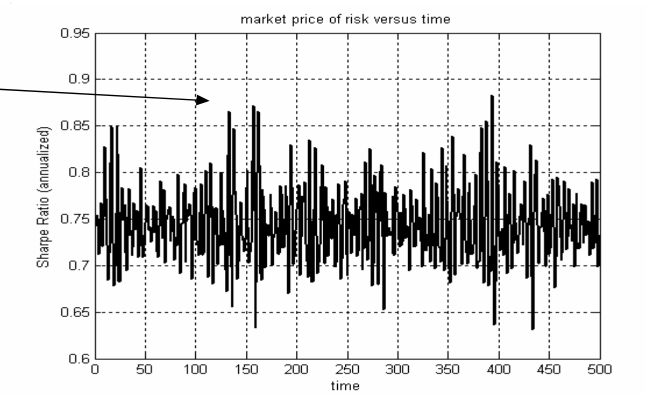


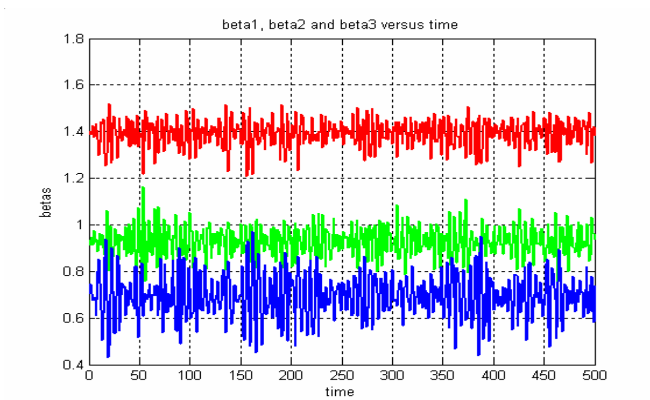
- **Effect of stronger trend extrapolation ($\gamma = 0.065$)**

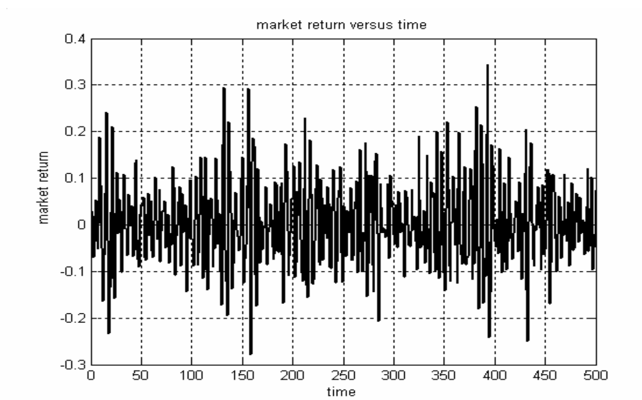
75% trend followers, 25% fundamentalists

(noise trading st.dev.= 5% of supply)









- **Effect of updating second moment beliefs**

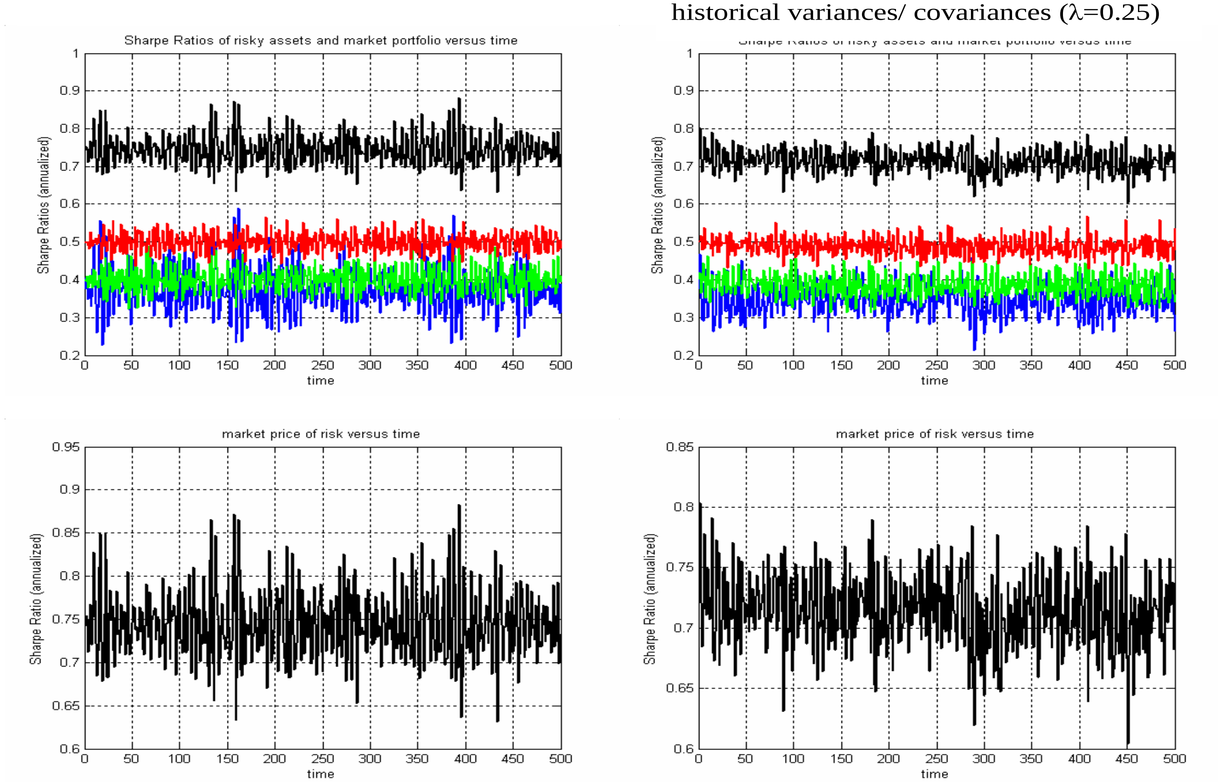
75% trend followers, 25% fundamentalists,

strong trend extrapolation and noise trading

($\gamma = \mathbf{0.0625}$, noise trading st.dev.= 5% of supply).

No updating of second moment beliefs (λ =0)

Second moment beliefs updated based on historical variances/ covariances (λ=0.25)



7 Conclusion

- Formulate a heterogeneous agent CAPM
- Rediscovered - in a different notation - some early neglected work of Lintner
- Set up a dynamic framework that incorporates expectations feedback
- Time varying beta driven by expectations feedback
- Looked at the simple fundamentalists/ trend followers/ noise traders set-up as one example of an updating scheme
- Future work will focus on
  - Further simulations to incorporate correlation structure
  - Broader class of agent types
  - Properties of the time-variation of beta