Effort Dynamics in Supervised Workgroups

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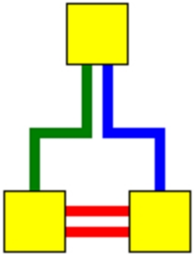
Literature and Motivation

- Holmstrom (1982), *Moral hazard in teams.* BJE.
- Adams (1965), *Inequity in social exchange.* AESP.
- Dal Forno & Merlone (2007), *Incentives in supervised teams: an experimental and computational approach.* JSC.
- Dal Forno & Merlone (2008), *Individual incentive in workgroups. From human subject experiments to agent-based simulation* IJIEM.
- Dal Forno & Merlone (2008), *Individual-based versus group-based incentives in supervised workgroups. The role of individual motivation.* Submitted.

The Model

The Supervised Workgroup



- a supervisor
- two subordinates

The Model

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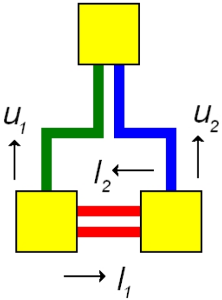
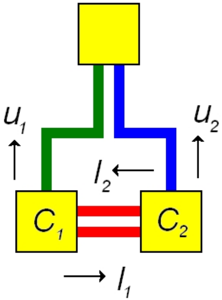


- a supervisor
- two subordinates

- u_i effort with the supervisor
- l_i effort with the partner

The Model

The Supervised Workgroup



- a supervisor
- two subordinates

- u_i effort with the supervisor
- l_i effort with the partner

- $u_i + l_i \leq c_i$

The Model

The production function

The production function is

$$(u_1 + u_2)^{\alpha} (l_1 + l_2)^{\beta}$$

where

- α: output elasticity with respect to the joint effort with the supervisor
- β: output elasticity with respect to the joint effort with the partner
- $0 \leq \alpha \leq 1$ and $\beta = 1 - \alpha$

We assume that the production is sold at unitary price

The supervisor's problem

Agents' compensation is:

$$w_i = s + b_i u_i + b_t (u_1 + u_2)^{\alpha} (l_1 + l_2)^{\beta}$$

where:

- s is a base salary sufficient to meet the participation constraint of the agent
- b_i is the incentive given to subordinate i for its individual effort with supervisor
- b_t is the incentive given both for team output.

We assume that:

- the supervisor declares the bonuses
- the subordinates decide their efforts in order to maximize their wage.

The Model

The Supervisor's Problem

The supervisor can only observe u_i :
She must design a linear compensation scheme (b_t, b_1, b_2) to maximize net production (bilevel programming problem)

$$\max_{b_t, b_1, b_2} (1 - 2b_t)(u_1 + u_2)^{\alpha}(l_1 + l_2)^{\beta} - b_1 u_1 - b_2 u_2$$

s.t. given b_t, b_1, b_2 the subordinates solve:

$$\max_{u_1, l_1} \quad w + b_t (u_1 + u_2)^{\alpha}(l_1 + l_2)^{\beta} + b_1 u_1$$
$$\max_{u_2, l_2} \quad w + b_t (u_1 + u_2)^{\alpha}(l_1 + l_2)^{\beta} + b_2 u_2$$

The Agent's Problem

Assume agents maximize the gross production

$$\max_{u_1, u_2, l_1, l_2} \quad (u_1 + u_2)^\alpha (l_1 + l_2)^\beta \quad \texttt{sub} \quad u_i + l_i \le c_i, \quad i = 1, 2$$

There is a continuum of solutions

$$\begin{cases} u_1 + u_2 = \frac{\alpha}{\alpha+\beta}(c_1 + c_2) \\ l_1 + l_2 = \frac{\beta}{\alpha+\beta}(c_1 + c_2) \end{cases}$$

a rather natural effort allocation is

$$(u_i, l_i) = \left(\frac{\alpha}{\alpha + \beta} c_i, \frac{\beta}{\alpha + \beta} c_i\right), \quad i = 1, 2$$

which is focal in the sense of Schelling (1960)

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The Supervisor's Problem

With fully rational agents the solution is obvious

$$\begin{cases} b_t = \varepsilon > 0 \\ b_1 = 0 \\ b_2 = 0 \end{cases}$$

The Effort Dynamics

Formalization

The simpler dynamics: the rational case ⇒ focal equilibrium

$$\begin{cases} l_1^* = \frac{\beta c_1}{\alpha+\beta} \\ \\ l_2^* = \frac{\beta c_2}{\alpha+\beta} \end{cases}$$

This equilibrium cannot hold in the long run when the subordinates have different capacities: individuals with different capacity but same reward may experience inequity (Adams, 1965):

“ *Inequity exists for Person whenever he perceives that the ratio of his outcomes to the inputs and the ratio of Other’s outcomes to Other’s input are unequal.* ”

$$\frac{O_P}{I_P} \neq \frac{O_a}{I_a}$$

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The Effort Dynamics

Formalization

Two-dimensional map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(l_1, l_2) : \begin{cases} l_1' = r_1(l_2) \\ \\ l_2' = r_2(l_1) \end{cases}$$

Reaction functions: $r_1 : L_2 \to L_1$ and $r_2 : L_1 \to L_2$

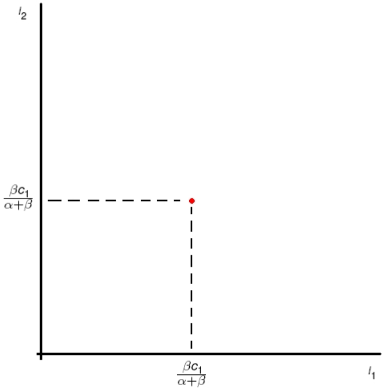
Strategy sets: $L_1 = [0, c_1] \subseteq \mathbb{R}$ and $L_2 = [0, c_2] \subseteq \mathbb{R}$

Trajectory: given an initial condition $\left(l_1^0, l_2^0\right) \in L_1 \times L_2$

$$\forall t \geq 0, \qquad \left\{l_1^t, l_2^t\right\} = T^t\left(l_1^0, l_2^0\right)$$

The Effort Dynamics

Formalization



The Effort Dynamics

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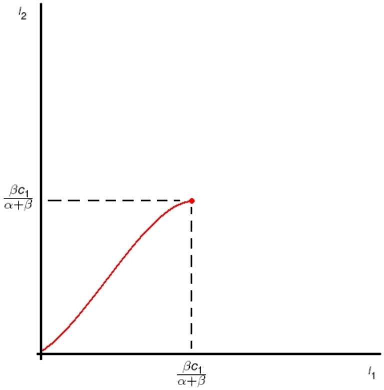

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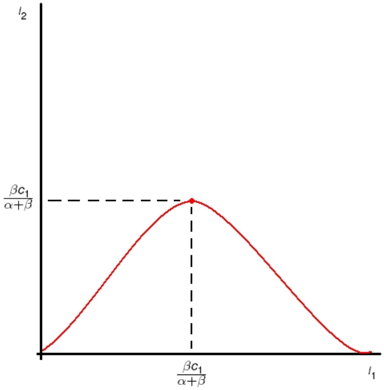

The Effort Dynamics

Formalization

Family of feasible functions:

$$\begin{cases} l_1^{t+1} = \lambda_1 \left(\frac{l_2^t}{\theta_1}\right)^{k_1-1} e^{-\frac{l_2^t}{\theta_1}} \\ l_2^{t+1} = \lambda_2 \left(\frac{l_1^t}{\theta_2}\right)^{k_2-1} e^{-\frac{l_1^t}{\theta_2}} \end{cases}$$

Conditions on parameters:

$$\lambda_i = \frac{\beta c_i}{\alpha+\beta}\left(\frac{e}{k_i-1}\right)^{k_i-1}$$

$$\theta_i = \frac{\beta c_i}{(\alpha+\beta)(k_i-1)}$$

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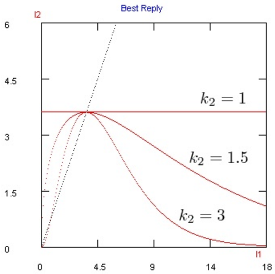
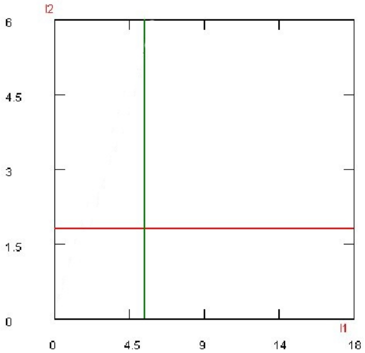


The Effort Dynamics

$\boxed{k_1 = k_2 = 1}$ → Tolerant agents

$$\begin{cases} l_1^{t+1} = \frac{\beta c_1}{\alpha+\beta} \\ l_2^{t+1} = \frac{\beta c_2}{\alpha+\beta} \end{cases}$$



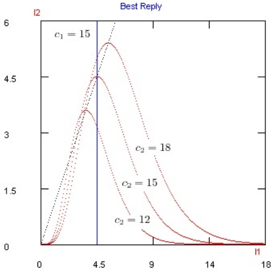
Unique (stable) fixed point.
The production is maximized.

The Effort Dynamics

$\boxed{k_1 = 1 \ , \ k_2 > 1}$ → Only one tolerant agent

$$\begin{cases} l_1^{t+1} = \frac{\beta c_1}{\alpha+\beta} \\ l_2^{t+1} = \frac{\beta c_2}{\alpha+\beta} \left(\frac{(\alpha+\beta) l_1^t}{\beta c_2} \right)^{k_2 - 1} e^{(k_2-1)\left(1 - \frac{(\alpha+\beta) l_1^t}{\beta c_2}\right)} \end{cases}$$

Unique (stable) fixed point.
If $c_1 = c_2$ then the production is maximized.



The Effort Dynamics

Efficiency

Proposition

Assume that one subordinate is tolerant and the other is not:

$$k_1 = 1, k_2 > 1.$$

Then:

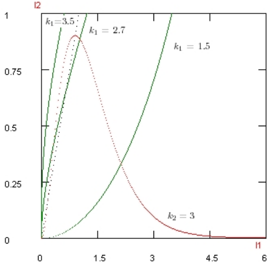
- *the production is maximized when their capacities are identical.*
- *for any fixed capacity gap, the intolerant agent reduces the effort with the colleague to a greater extent if his capacity is the largest; yet, in this case, the production variation is not necessarily the greatest.*

The Effort Dynamics

$\boxed{k_1 > 1 \; , \; k_2 > 1} \rightarrow$ No tolerant agents

$$\begin{cases} l_1^{t+1} = \frac{\beta c_1}{\alpha+\beta} \left(\frac{(\alpha+\beta) l_2^t}{\beta c_1} \right)^{k_1 - 1} e^{(k_1-1)\left(1 - \frac{(\alpha+\beta) l_2^t}{\beta c_1}\right)} \\ l_2^{t+1} = \frac{\beta c_2}{\alpha+\beta} \left(\frac{(\alpha+\beta) l_1^t}{\beta c_2} \right)^{k_2 - 1} e^{(k_2-1)\left(1 - \frac{(\alpha+\beta) l_1^t}{\beta c_2}\right)} \end{cases}$$

One, two, or three fixed points.



The Effort Dynamics

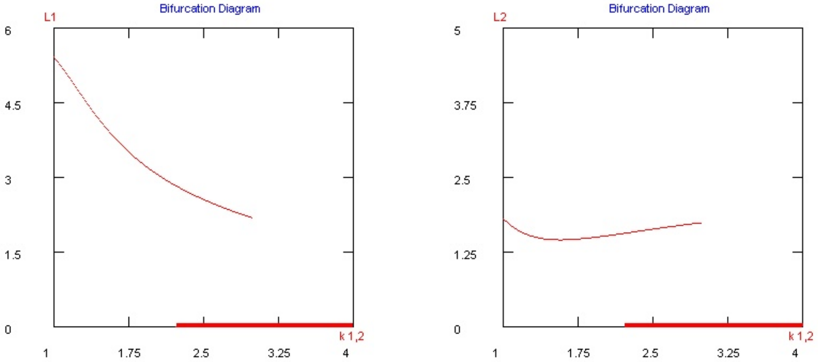
Eigenvalues:

$$\lambda_1 = -\sqrt{e^{\frac{\beta c_1 - l_1}{\beta c_1}(k_1-1)+\frac{\beta c_2 - l_2}{\beta c_2}(k_2-1)} \left(\frac{l_1}{\beta c_1}\right)^{k_1-2} (k_1-1)\,\frac{\beta c_1 - l_1}{\beta c_1} \left(\frac{l_2}{\beta c_2}\right)^{k_2-2} (k_2-1)\,\frac{\beta c_2 - l_2}{\beta c_2}}$$

$$\lambda_2 = \sqrt{e^{\frac{\beta c_2 - l_2}{\beta c_2}(k_2-1)+\frac{\beta c_1 - l_1}{\beta c_1}(k_1-1)} \left(\frac{l_2}{\beta c_2}\right)^{k_2-2} (k_2-1)\,\frac{\beta c_2 - l_2}{\beta c_2} \left(\frac{l_1}{\beta c_1}\right)^{k_1-2} (k_1-1)\,\frac{\beta c_1 - l_1}{\beta c_1}}$$

The Effort Dynamics

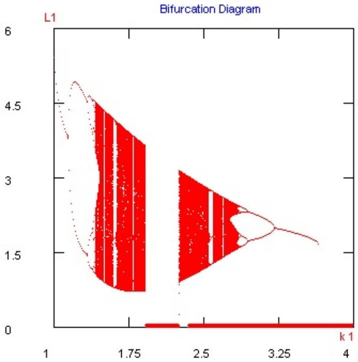
$\boxed{k_1 = k_2 = k}$ $\rightarrow$ bifurcation diagram



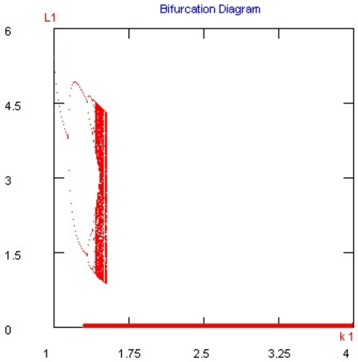
$$\left(l_1^0, l_2^0\right) = (0.7, 0.5)$$

The Effort Dynamics

$\boxed{k_1 \neq k_2 \quad , \quad k_1, k_2 > 1}$ → bifurcation diagram



$\left(l_1^0, l_2^0\right) = (0.1, 0.1)$

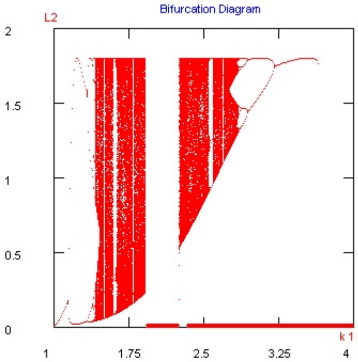


$\left(l_1^0, l_2^0\right) = (0.5, 0.5)$

$k_2 = 7.5 \quad , \quad c_1 = 18 \quad , \quad c_2 = 6$

The Effort Dynamics

$\boxed{k_1 \neq k_2 \quad , \quad k_1, k_2 > 1}$ → bifurcation diagram



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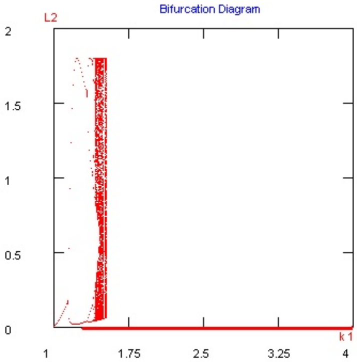

$\left(l_1^0, l_2^0\right) = (0.5, 0.5)$

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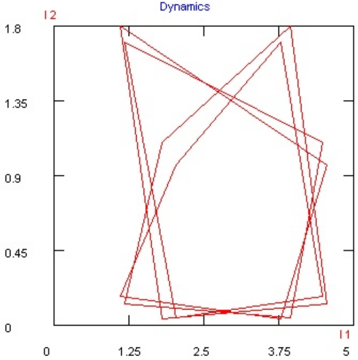
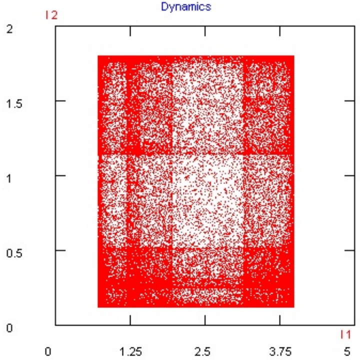
The Effort Dynamics

$\boxed{k_1 \neq k_2 \quad , \quad k_1, k_2 > 1}$ → Cycles and Chaos



$k_1 = 1.39$

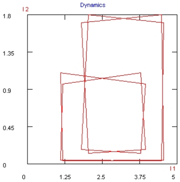


$k_1 = 1.7$

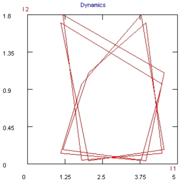
$\left(l_1^0, l_2^0\right) = (1.12, 0.17) \quad , \quad k_2 = 7.5 \quad , \quad c_1 = 18 \quad , \quad c_2 = 6$

The Effort Dynamics

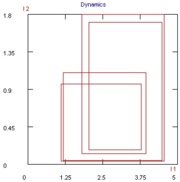
$$k_1 = 1.39$$
$$k_2 = 7.5$$
$$c_1 = 18$$
$$c_2 = 6$$



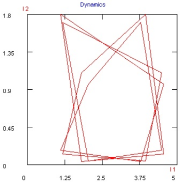
(4.56, 0.04)



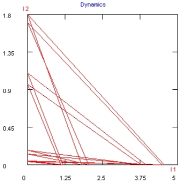
(4.56, 0.17)



(4.49, 1.70)

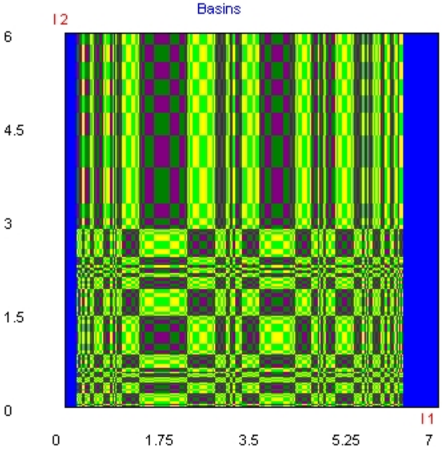


(1.12, 0.17)



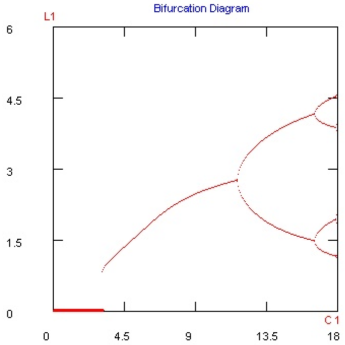
(1.81, 0.00)

The Effort Dynamics

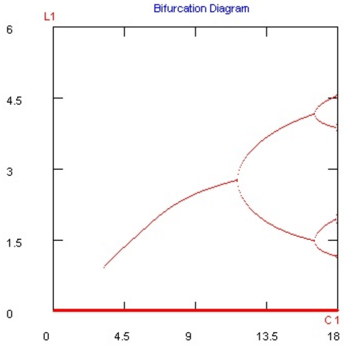


The Effort Dynamics

Other results: bifurcation on the capacity c_1



(0.1, 0.5)



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The Effort Dynamics

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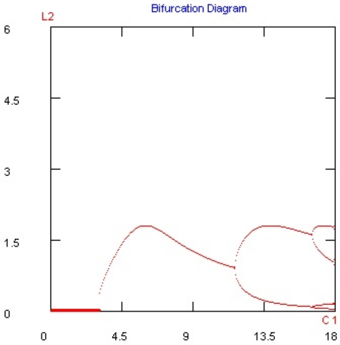
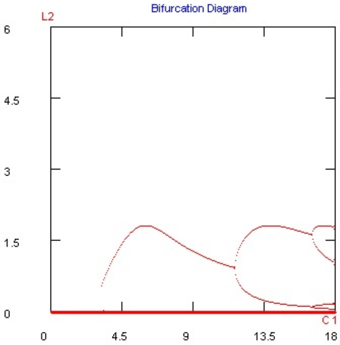


(0.1, 0.5)

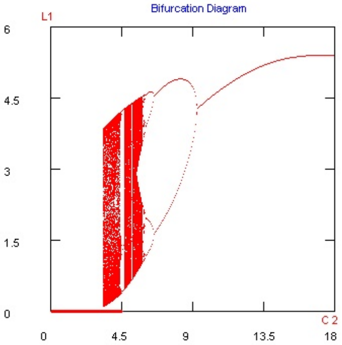


(0.5, 0.5)

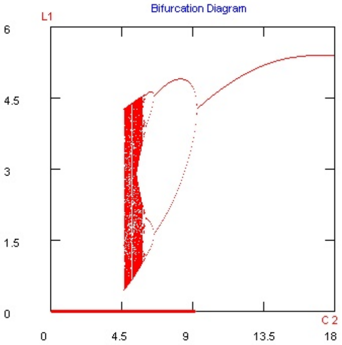
$k_1 = 1.39 \quad , \quad k_2 = 7.5 \quad , \quad c_2 = 6$

The Effort Dynamics

Other results: bifurcation on the capacity c_2



(0.1, 0.5)

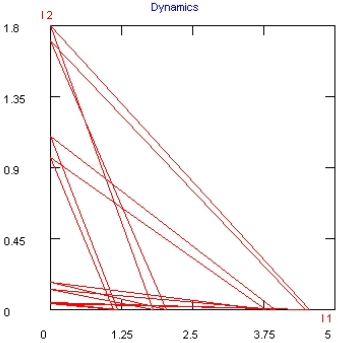


(0.5, 0.5)

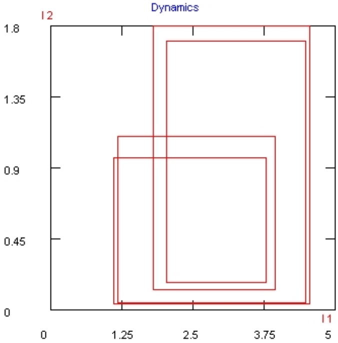
$k_1 = 1.39 \quad , \quad k_2 = 7.5 \quad , \quad c_1 = 18$

Global analysis

Coexistence of finite period attractors



(1.81, 0.00)

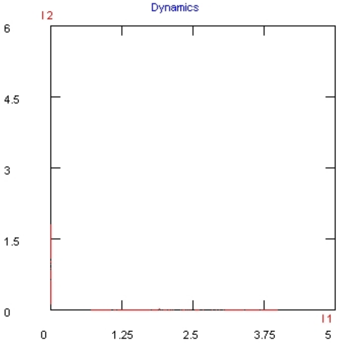


(4.49, 1.70)

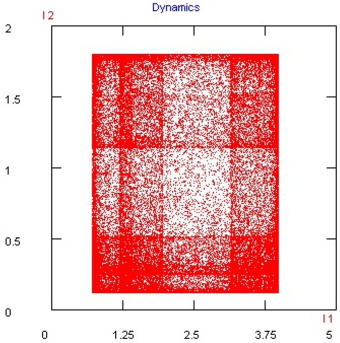
$k_1 = 1.39$, $k_2 = 7.5$, $c_1 = 18$, $c_2 = 6$

Global analysis

Coexistence of chaotic attractors



(0.1, 0.5)

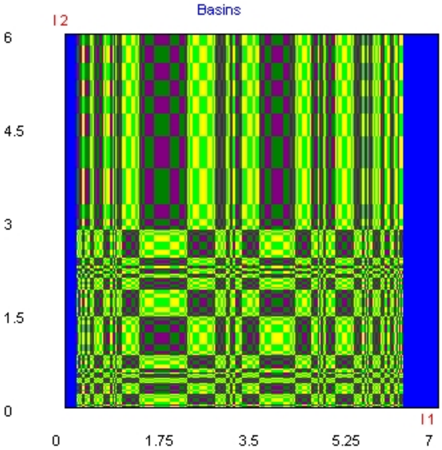


(0.7, 0.7)

$k_1 = 1.7$, $k_2 = 7.5$, $c_1 = 18$, $c_2 = 6$

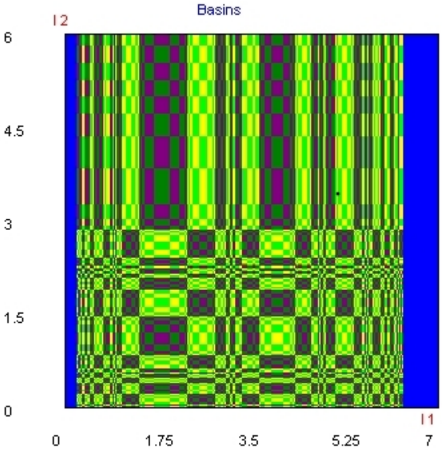
Global analysis

Basin of attraction of the origin



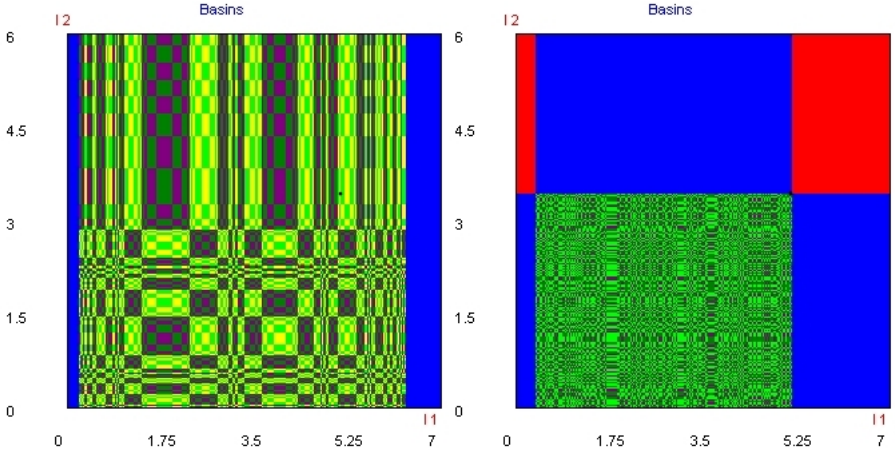
Global analysis

Basin of attraction of the origin



Global analysis

Basin of attraction of the origin



Conclusion

$c_1 = c_2$

- $k_1 = k_2 = 1 \rightarrow$ rational workgroup, efficiency
- $k_1 = 1, k_2 > 1 \rightarrow$ efficiency
- $k_1, k_2 > 1 \rightarrow$ coexistence of attractors (with retaliation)

$c_1 \neq c_2$

- $k_1 = k_2 = 1 \rightarrow$ rational workgroup, efficiency
- $k_1 = 1, k_2 > 1 \rightarrow$ loss of efficiency, but no retaliation
- $k_1, k_2 > 1 \rightarrow$ coexistence of cycles (with retaliation), chaos, expansion of the (non connected) basin of the origin

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