A mathematical model for the diffusion of a new technology

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Urbino, September 25, 2008
MDEF 2008

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1. The Theory of Innovation Diffusion
2. A model for the diffusion of a new technology
3. Asymptotic behavior of the solutions
4. Conclusions
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Part I

The Theory of Innovation Diffusion

The diffusion of an innovation is the process by which an innovation is communicated through certain channels over time among the members of a social system (Rogers, 2005)

The adoption process

- ◇ *Knowledge*;
- ◇ *Persuasion*;
- ◇ *Decision or evaluation*;
- ◇ *Implementation*;
- ◇ *Confirmation or adoption*.

} Communication channels

Historical distinction
between
technological breakthroughs (innovations)
and
engineering refinements (improvements)
(De Solla Price, 1985)

Marketing problems

1960:
To develop marketing strategy in order to promote the adoption of a new product and penetrate the market.

The Bass model, 1969

$$\frac{dA(t)}{dt} = p(m - A(t)) + \frac{q}{m}A(t)(m - A(t)) \tag{1}$$

p is a parameter that takes into account as the new adopters joint the market as a result of external influences: activities of firms in the market, advertising, attractiveness of the innovation
q refers to the magnitude of influence of another single adopter

Time delay model

Lag between the discovery of a new technology and its adoption:

- Models of uncertain profitability: a firm has an incentive to delay adoption because it can gather information as time passes, and thus perhaps avoid adopting an unprofitable technology
- Models of technological uncertainty: uncertainty generated by further technological developments. If there is a rapid technological progress, then there is very little chance to a firm can recover its investment once it has installed a new technology

DSEMS

Stage structure model

Time delay models with stage structure for describing the adoption process of a new product

W. Wang, and P. Fergola, and S. Lombardo, and G. Mulone, 2006

Beretta (2001)

Part II

A model for the diffusion of a new technology

Aim

We will propose a *time delay model* to simulate the adoption process when individuals in the social system are influenced by external factors (government incentives and production costs) and internal factors (interpersonal communication). We will find a final level of adopters and we will carry out a qualitative analysis to study the *stability* of model equilibrium solution.

The model

$$A(t) + P(t) = C$$
$$\Downarrow$$

$$\frac{dA(t)}{dt} = [a + \alpha A(t-\tau)]\,(C - A(t-\tau))k - \delta A(t) - \gamma A(t)A(t-\tau), \qquad \forall \tau > 0 \qquad (2)$$

$$a = e^{\eta(i-c)}$$
$$k = e^{-\rho\tau}$$

The model

- ✓ The total population C consists of the adopters of the new technology, A, and of the potential adopters, P.
- ✓ i is a government incentive and c represents the production costs, $e^{\eta(i-c)} = a$ is an external factor of influence.
- ✓ τ is the average time for an individual to evaluate whether to adopt the technology or not. Then, the knowledge and the awareness of technology occur at time $t-\tau$ and in the interval $[t-\tau, t]$ the individual decides whether to adopt the technology or not.

The model

- ✓ ρ is the percentage of individuals that don't adopt the technology after the test period, $e^{-\rho\tau} = k$ is the fraction of individuals that are still interested in the adoption after τ.
- ✓ α is the valid contact rate of adopters with potential adopters, δ the discontinuance rate of adopters and γ a valid contact rate between the adopters at time t and those at time $t - \tau$, it represents the intra-specific competition coefficient.

Part III

Asymptotic behavior of solutions

The equilibrium

$$A^* = \frac{(\alpha Ck - ak - \delta) + \sqrt{(\alpha Ck - ak - \delta)^2 + 4(\gamma + \alpha k)akC}}{2(\gamma + \alpha k)}$$

Unique positive equilibrium

We found an unique positive equilibrium!

Stability

Let

$$x(t) = A(t) - A^*.$$

Then

$$\frac{dx(t)}{dt} = a_0 x(t) + b_0 x(t-\tau) + f(x(t-\tau), x(t))$$

where

$$\begin{aligned} a_0 &= -(\gamma A^* + \delta), \\ b_0 &= (\alpha k C - ak - 2\alpha k A^* - \gamma A^*), \\ f(x(t-\tau), x(t)) &= -\gamma x(t) x(t-\tau) - \alpha k x^2(t-\tau). \end{aligned}$$

Stability

The characteristic equation of the delayed differential equation linear part is

$$-\lambda\tau e^{\lambda\tau} + a_0\tau e^{\lambda\tau} + b_0\tau = 0$$

$$\Downarrow$$

Hayes Theorem

Hayes Theorem

Theorem

All roots of the equation $pe^z + q - ze^z = 0$, where p and q are real numbers, have negative real parts if and only if $p < 1$, $p < -q$ and $\sqrt{\theta^2 + p^2} > -q$, where θ is the root of $\theta = p \tan \theta$, such that $0 < \theta < \pi$ (If $p = 0$, then $\theta = \pi/2$).

Stability

Theorem

The equilibrium is asymptotically stable if

$$\tau^2 < \frac{\theta^2}{[\phi(\alpha,\delta,\gamma,a,k,C)]^2 - 2(\gamma A^* + \delta)\phi(\alpha,\delta,\gamma,a,k,C)}$$

where $\phi(\alpha,\delta,\gamma,a,k,C) = \sqrt{(\alpha kC - \delta - ak)^2 + 4akC(\gamma + \alpha k)}$ *and* θ *is the root of* $\theta = -(\gamma A^* + \delta)\tau \tan\theta$*, such that* $\frac{\pi}{2} < \theta < \pi$

Global stability

We apply the methods proposed by X. Wang, L Liao (2004) to demonstrate the global stability of the solution of the time delay differential equation.

Recall the delay differential equation

$$\frac{dx(t)}{dt} = a_0 x(t) + b_0 x(t-\tau) + f(x(t-\tau), x(t))$$

where

$$\begin{aligned} a_0 &= -(\gamma A^* + \delta), \\ b_0 &= (\alpha k C - ak - 2\alpha k A^* - \gamma A^*), \\ f(x(t-\tau), x(t)) &= -\gamma x(t) x(t-\tau) - \alpha k x^2(t-\tau). \end{aligned}$$

Aim

Prove the global stability of the zero solution

Set $V(t) = x^2(t)$ $\qquad v^2 = \limsup_{t\to\infty} V(t)$

Procedure

1.

Prove that $v^2 < \infty$.

If $v^2 = +\infty$, then $\exists t^* > \tau$ such that $V'(t^*) \geq 0$ and $V(t^*) \geq V(t)$, $0 < t < t^*$

$$\begin{aligned} V'(t^*) &= 2x(t^*)X'(t^*) \\ &= 2x(t^*)[a_0 x(t^*) + b_0 x(t^* - \tau) - \gamma x(t^*)x(t^* - \tau) - \alpha k x^2(t^* - \tau)] \\ &\quad \text{if } b_0 \text{ is negative, we finish the proof, if it is positive we obtain} \\ &\leq 2x(t^*)X'(t^*) = 2x(t^*)[(a_0 + b_0)x(t^*) - \gamma x(t^*)x(t^* - \tau) - \alpha k x^2(t^* - \tau) \\ &< 0 \end{aligned}$$

Contradiction! $\Rightarrow v^2 < \infty$

Procedure

2.

Prove that $v = 0$.

If $v > 0$, $\forall \epsilon > 0$, $\exists T > 0$ such that $x^2(t) < v^2 + \epsilon$, $\forall t \geq T$

$\forall \epsilon > 0$, $\exists T + \tau > 0$ and $\exists t^* > T$ such that $v^2 - \epsilon < x^2(t^*)$

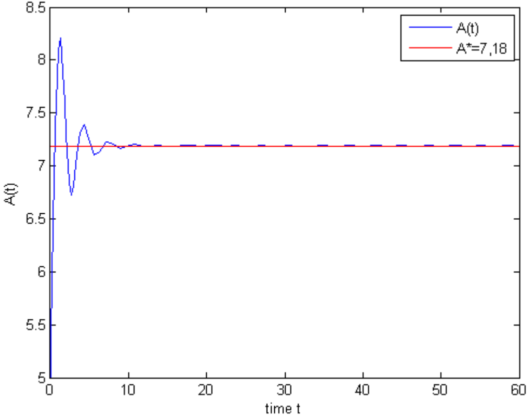
$$\begin{aligned}
V'(t^*) &= 2x(t^*)X'(t^*) \\
&= 2x(t^*)[a_0 x(t^*) + b_0 x(t^* - \tau) - \gamma x(t^*)x(t^* - \tau) - \alpha k x^2(t^* - \tau)] \\
& \quad \text{if } b_0 \text{ is positive} \\
&\leq 2x(t^*)[a_0(v^2 - \epsilon) + b_0(v^2 + \epsilon) - \gamma x(t^*)x(t^* - \tau) - \alpha k x^2(t^* - \tau)] \\
&\leq 2x(t^*)[a_0(v^2 - \epsilon) + b_0(v^2 + \epsilon)] \\
&= 2x(t^*)[(a_0 + b_0)v^2 + \epsilon(b_0 - a_0)] \\
& \quad \text{recall that } (a_0 + b_0) < 0 \\
&< 0
\end{aligned}$$

Contradiction! $\Rightarrow v = 0$

THE SOLUTION IS GLOBALLY ASYMPTOTICALLY STABLE

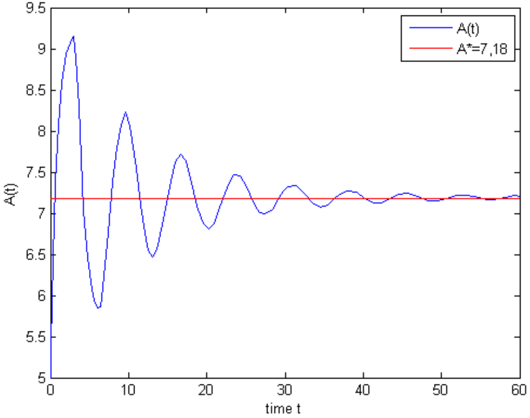
Simulation results

$\alpha = 0.1 \quad a = 0.5 \quad C = 20 \quad k = 0.8 \quad \delta = 0.3 \quad \gamma = 0.2 \quad A(0) = 5 \quad \tau = 1$



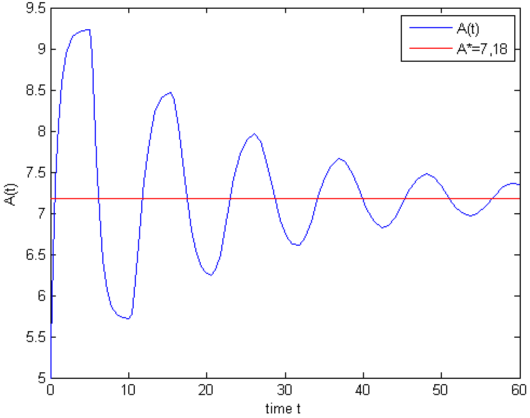
Simulation results

$A(0) = 5 \quad \tau = 3$



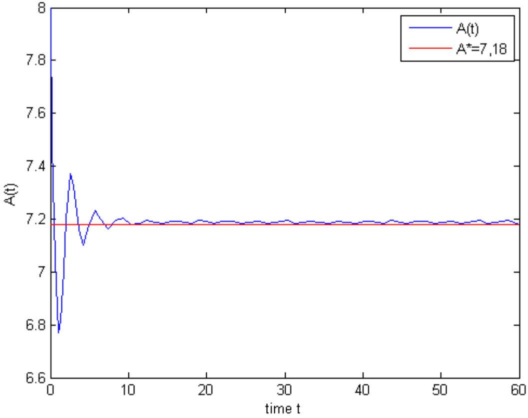
Simulation results

$A(0) = 5 \qquad \tau = 5$



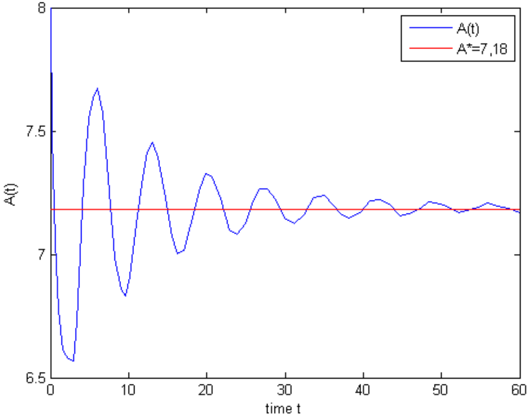
Simulation results

$A(0) = 8 \qquad \tau = 1$



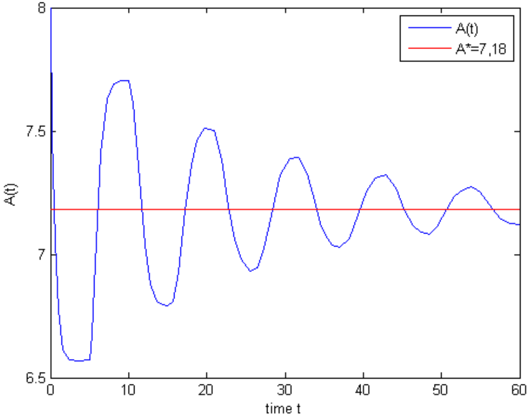
Simulation results

$A(0) = 8 \qquad \tau = 3$

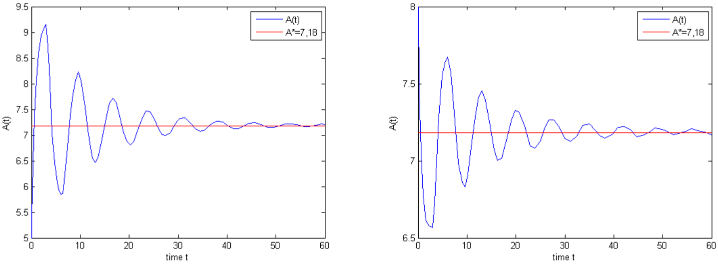


Simulation results

$A(0) = 8 \qquad \tau = 5$

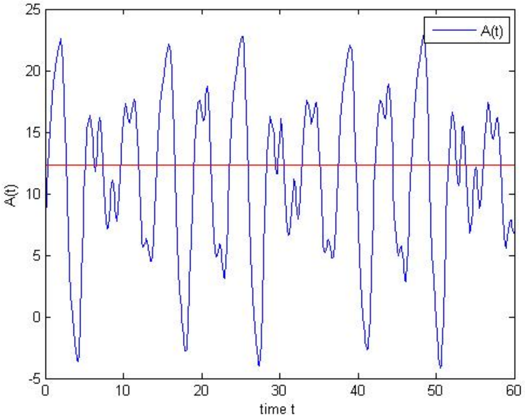


Simulation results



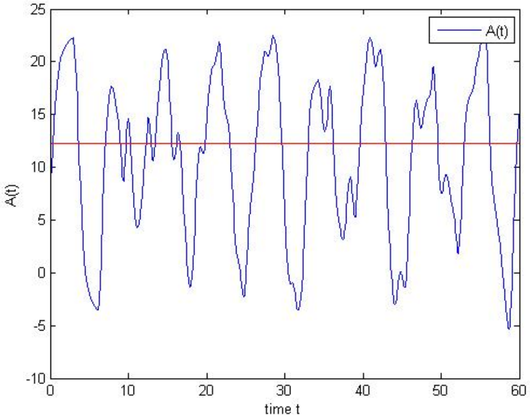
Simulation results

$$\alpha = 0.4 \quad a = 0.6 \quad C = 20 \quad k = 0.8 \quad \delta = 0.3 \quad \gamma = 0.2 \quad \tau = 2$$



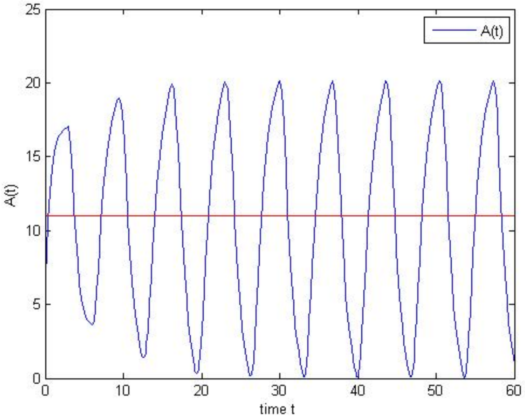
Simulation results

$\alpha = 0.4 \quad a = 0.6 \quad C = 20 \quad k = 0.8 \quad \delta = 0.3 \quad \gamma = 0.2 \quad \tau = 3$



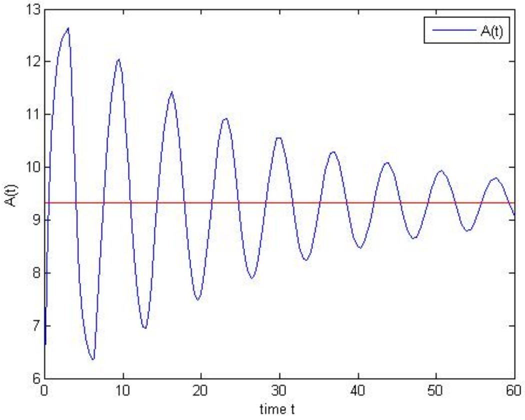
Simulation results

$\alpha = 0.3 \quad a = 0.5 \quad C = 20 \quad k = 0.8 \quad \delta = 0.3 \quad \gamma = 0.2 \quad \tau = 3$



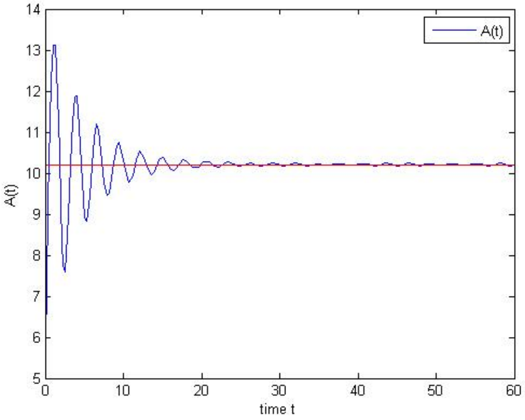
Simulation results

$\alpha = 0.2 \quad a = 0.5 \quad C = 20 \quad k = 0.8 \quad \delta = 0.3 \quad \gamma = 0.2 \quad \tau = 3$



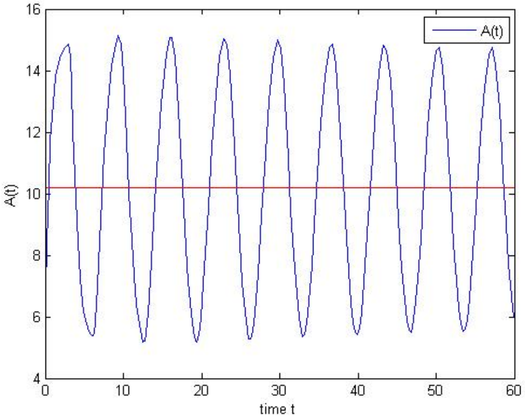
Simulation results

$\alpha = 0.25 \quad a = 0.5 \quad C = 20 \quad k = 0.8 \quad \delta = 0.3 \quad \gamma = 0.2 \quad \tau = 1$



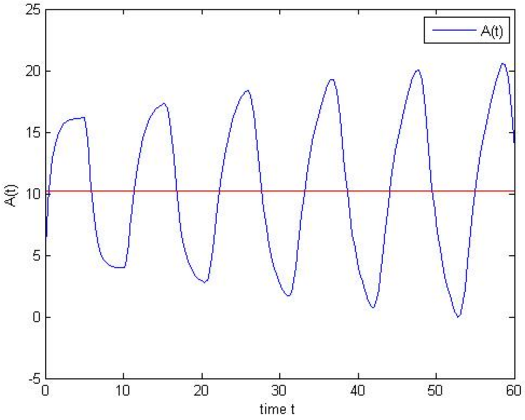
Simulation results

$\alpha = 0.25 \quad a = 0.5 \quad C = 20 \quad k = 0.8 \quad \delta = 0.3 \quad \gamma = 0.2 \quad \tau = 3$



Simulation results

$\alpha = 0.25 \quad a = 0.5 \quad C = 20 \quad k = 0.8 \quad \delta = 0.3 \quad \gamma = 0.2 \quad \tau = 5$



Part IV

Conclusions

Conclusions I

Conclusions

- We have proposed a mathematical model with stage structure to simulate adoption processes. The model includes the awareness stage, the evaluation stage and the decision-making stage.
- We have found the final level of adopters for certain parameters and, from studying the local stability of the equilibrium, we have found that the final level of adopters is unchanged under small perturbations (if the evaluation delay is small).
- We have proved that the solution is also globally stable.

Part V

Future Research

Future Research I

First Solutions of a delay differential equation usually are periodic and we wish to investigate the existence of periodic solutions of our model and to study their stability.

Second Since there exist stochastic factors in the environment as well as in the interior system and the action of these factors can cause a random adoption pattern of the new technology, we wish to investigate the effect of stochastic perturbations for the model.

References I

F.M. Bass.
A new product growth model for consumer durables.
Management Sci., 15:215–227, January 1969.

R. Bellman and K.L. Cooke.
Differential-difference equations.
Academic Press, New York, 1963.

E. Beretta and Y. Kuang.
Geometric stability switch criteria in delay differential system with delay-dependent parameters.
SIAM J. Math.Anal., 33:1144–1165, 2001.

D. de Solla Price.
The science/technology relationship, the craft of experimental science, and policy for the improvement of high technology innovation.
Research Policy, 13:3–20, 1984.

References II

J.K. Hale and S.M.V. Lunel.
Introduction to Functional Differential Equations.
Springer-Verlag, New York, 1993.

V. Mahajan, E. Muller, and F.M. Bass.
New-Product Diffusion Models.
in Eliashberg, J, Lilien, G.L (Eds),Handbooks in Operations Research and Management Science, Elsevier Science Publishers, New York, NY, 1993.

E.M. Rogers.
Diffusion of Innovations.
The Free Press, New York, 4th edition, 1995.

W. Wang, P. Fergola, S. Lombardo, and G. Mulone.
Mathematical models of innovation diffusion with stage structure.
Applied Mathematical Modelling, 30:129–146, 2006.

References III

X. Wang and L-Liao.
Asymptotic behavior of solution of delay logistic differential equation with negative instantaneously terms.
Applied Math. and Comp., 153:68–74, 2004.