Zero-beta CAPM under Heterogenous Beliefs

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Plan of Talk

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1 Literature and Motivation

- **Sharpe-Lintner-Mossin Capital Asset Pricing Model (CAPM)**:
  - Plays a central role in finance theory
  - Paradigm of homogeneous beliefs and a rational representative agent.
  - Criticisms from theoretic and empirical points
- **Literature on heterogeneous beliefs**:
  - **Heterogeneous beliefs can affect aggregate market returns**.
  - Typically the **heterogeneous beliefs** reflect
    * Difference of risk attitude —Huang and Litzenberger (1988)
    * Difference of opinion among the agents—Lintner (1969), Miller (1977), Mayshar (1982), Varian (1985), Abel (1989, 2002), Cecchetti *et al.*(2000).
    * Difference of information upon which agents are trying to learn —Williams (1977), Detemple and Murthy (1994) and Zapatero (1998).

- **Focus on** the heterogeneous in the risk preferences and expected payoffs or returns of risky assets, rather than the variances and co-variances, except Lintner (1969) and Chiarella, Dieci and He (2006).

- **Empirical Studies**:
  - Divergence of opinion and stock price:
    - Miller's hypothesis (1977): Market clearing price of stocks with divergence of opinion will be higher.
    - Diether et al. (2002) and Ang *et al.* (2006) provide empirical evidence for Miller's hypothesis.
  - Equity risk premium puzzle and risk-free rate puzzle;
  - Managed funds under-perform the market.

- **Mean-Variance Analysis and Heterogeneity**
  - Heterogeneity in both mean and var/covariance matrix: Lintner (1969)
  - Main obstacle: the complexity and heavy notation involved when the number of assets and the dimension of the heterogeneity increase.
- **Questions trying to understand**:
  - Do the two fund (without risk free asset) theorems still hold?
  - Does the geometric relationship between frontiers with and without risk-free asset still hold?
  - What is the impact of the heterogeneity on the market equilibrium price and mean variance frontier?
  - Measured by the Sharpe ratio, can managed fund beat the market?
  - Does heterogeneity improve the market performance?
  - Can the heterogeneity be used to explain some empirical stylized facts?

2 Payoff Setup

2.1 Market and Heterogeneous Beliefs

- **Market**: Assume a one-period economy with N risky assets, indexed by $j, k = 1, 2, \cdots, N$ and I investors indexed by $i = 1, 2, \cdots, I$.
- **Assets**: $\tilde{\mathrm{x}} = (\tilde{x_1}, \cdots, \tilde{x_N})^T$ is the random payoff vector of the risky assets.
- **Heterogeneous beliefs**: $\mathcal{B}_i = (\mathbb{E}_i(\tilde{\mathrm{x}}), \Omega_i)$,

$$\mathrm{y}_i = \mathbb{E}_i(\tilde{\mathrm{x}}) = (y_{i,1}, y_{i,2}, \cdots, y_{i,N})^T, \qquad \Omega_i = (\sigma_{i,jk})_{N \times N}.$$

where

$$y_{i,j} = \mathbb{E}_i[\tilde{x}_j], \quad \sigma_{i,jk} = Cov_i(\tilde{x}_j, \tilde{x}_k)$$

for $1 \leq i \leq I$ and $1 \leq j, k \leq N$.

- **Portfolio selection problem**:
  - Assume $U_i(w) = -e^{-\theta_i w}$, θ_i being investor i's ARA;
  - Investor i's end-of-period wealth $\tilde{W}_i$ is normally distributed.
  - Investor i optimal portfolio measured in number of shares $\mathbf{z}_i$ is determined by $\max_{\mathbf{z}_i} Q_i(\mathbf{z}_i), Q_i := \mathbf{y}_i^T \mathbf{z}_i - \frac{\theta_i}{2}\mathbf{z}_i \Omega_i \mathbf{z}_i$ subject to the wealth constraint $\mathbf{p}_0^T \mathbf{z}_i = W_0^i$,
  - W_0^i is the initial wealth and $\mathbf{p}_0$ is the vector of equilibrium prices.
- **Optimal Portfolios**: For given market price vector $\mathbf{p}_0$ of risky assets, the optimal risky portfolio $\mathbf{z}_i^*$ of investor i is uniquely given by

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}[\mathbf{y}_i - \lambda_i^* \mathbf{p}_0], \qquad \lambda_i^* = \frac{\mathbf{p}_0^T \Omega_i^{-1} \mathbf{y}_i - \theta_i W_0^i}{\mathbf{p}_0^T \Omega_i^{-1} \mathbf{p}_0}.$$

$\lambda_i^* = \frac{1}{p_{oj}} \frac{\partial Q_i(\mathbf{z}^*)}{\partial z_{ij}}$ measures *investor i's optimal marginal certainty equivalent wealth per unit of asset j relative to its market price.*

2.2 Consensus Belief and Equilibrium Asset Prices

- **Market clearing condition**: $\sum_i^I \mathbf{z}_i^* = \sum_{i=1}^I \bar{\mathbf{z}}_i := \mathbf{z}_m$,
- **Equilibrium price**:

$$\mathbf{p}_0 = \Big(\sum_{i=1}^{I} \theta_i^{-1} \lambda_i^* \Omega_i^{-1}\Big)^{-1} \Big[\Big(\sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} \mathbf{y}_i\Big) - \mathbf{z}_m\Big]. \qquad (1)$$

- **Consensus Beliefs**: $\mathcal{B}_a$,

$$\theta_a := \Big(\frac{1}{I}\sum_{i=1}^{I} \theta_i^{-1}\Big)^{-1}, \qquad \lambda_a^* := \frac{1}{I}\theta_a \sum_{i=1}^{I} \theta_i^{-1}\lambda_i^*.$$

$$\Omega_a = \theta_a^{-1}\lambda_a^* \Big(\frac{1}{I}\sum_{i=1}^{I} \lambda_i^* \theta_i^{-1} \Omega_i^{-1}\Big)^{-1},$$

$$\mathbf{y}_a = \mathbb{E}_a(\tilde{\mathbf{x}}) = \theta_a \Omega_a \left(\frac{1}{I} \sum_{i=1}^{I} \theta_i^{-1} \Omega_i^{-1} \mathbb{E}_i(\tilde{\mathbf{x}}) \right);$$

- the market equilibrium price $\mathbf{p}_o$

$$\mathbf{p}_0 = \frac{1}{\lambda_a^*}[\mathbf{y}_a - \frac{1}{I}\theta_a \Omega_a \mathbf{z}_m];$$

- the equilibrium optimal portfolio of agent i

$$\mathbf{z}_i^* = \theta_i^{-1}\Omega_i^{-1}\left[(\mathbf{y}_i - \frac{\lambda_i^*}{\lambda_a^*}\mathbf{y}_a) + \frac{\lambda_i^*}{I\lambda_a^*}\theta_a \Omega_a \mathbf{z}_m \right].$$

- Weighted average behaviour: Let $\tau_i = 1/\theta_i$ and $\tau_a = \sum_{i=1}^{I} \tau_i$.

$$\lambda_a^* = \sum_{i=1}^{I} \frac{\tau_i}{\tau_a}\lambda_i^*, \qquad \mathbf{p}_o = \frac{1}{\tau_a}\Omega_a \sum_{i=1}^{I} \frac{\lambda_i^*}{\lambda_a^*}\mathbf{p}_{i,o}.$$

2.3 Zero-beta Heterogeneous CAPM (ZHCAPM)

$$\mathbb{E}_a[\tilde{\mathbf{r}}] - (\lambda_a^* - 1)\mathbf{1} = \beta[\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1)],$$
$$\mathbb{E}_a(\tilde{r}_m) - (\lambda_a^* - 1) = \frac{\theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}} > 0.$$

where

$$\lambda_a^* = \frac{\mathbf{z}_m^T \mathbf{y}_a - \theta_a \mathbf{z}_m^T \Omega_a \mathbf{z}_m / I}{W_{m0}}, \qquad W_{m0} := \mathbf{z}_m^T \mathbf{p}_0 = \sum_{i=1}^{I} W_0^i,$$

and $\beta = (\beta_1, \beta_2, \cdots, \beta_N)^T$,

$$\beta_j = \frac{Cov_a(\tilde{r}_m, \tilde{r}_j)}{\sigma_a^2(\tilde{r}_m)} = \frac{W_{mo}}{p_{oj}} \frac{Cov_a(\tilde{x}_j, \tilde{W}_m)}{\sigma_{a,m}^2}, \qquad j = 1, \cdots, N.$$

2.4 The Impact of Heterogeneity on the Market

- The aggregation property does not hold;
- If an asset is more risky than the market ($\beta_j > 1$), an increase in ARA for any investor increases asset's price and decreases the expected future returns, and vice versa for a less risky asset.
- When the risk aversion coefficients becomes more divergent with the average unchanged, the aggregation results in lower (higher) equilibrium price and higher (lower) expected return.
- Miller's hypothesis holds conditionally: $(1 + \mathbb{E}_a(\tilde{r}_j))(1 - \alpha_j) > \lambda_a^*$ and optimism and risk aversion is negatively correlated.
- Two fund theorem does not hold in general, but the optimal portfolios of investors can be very close to the market frontier.

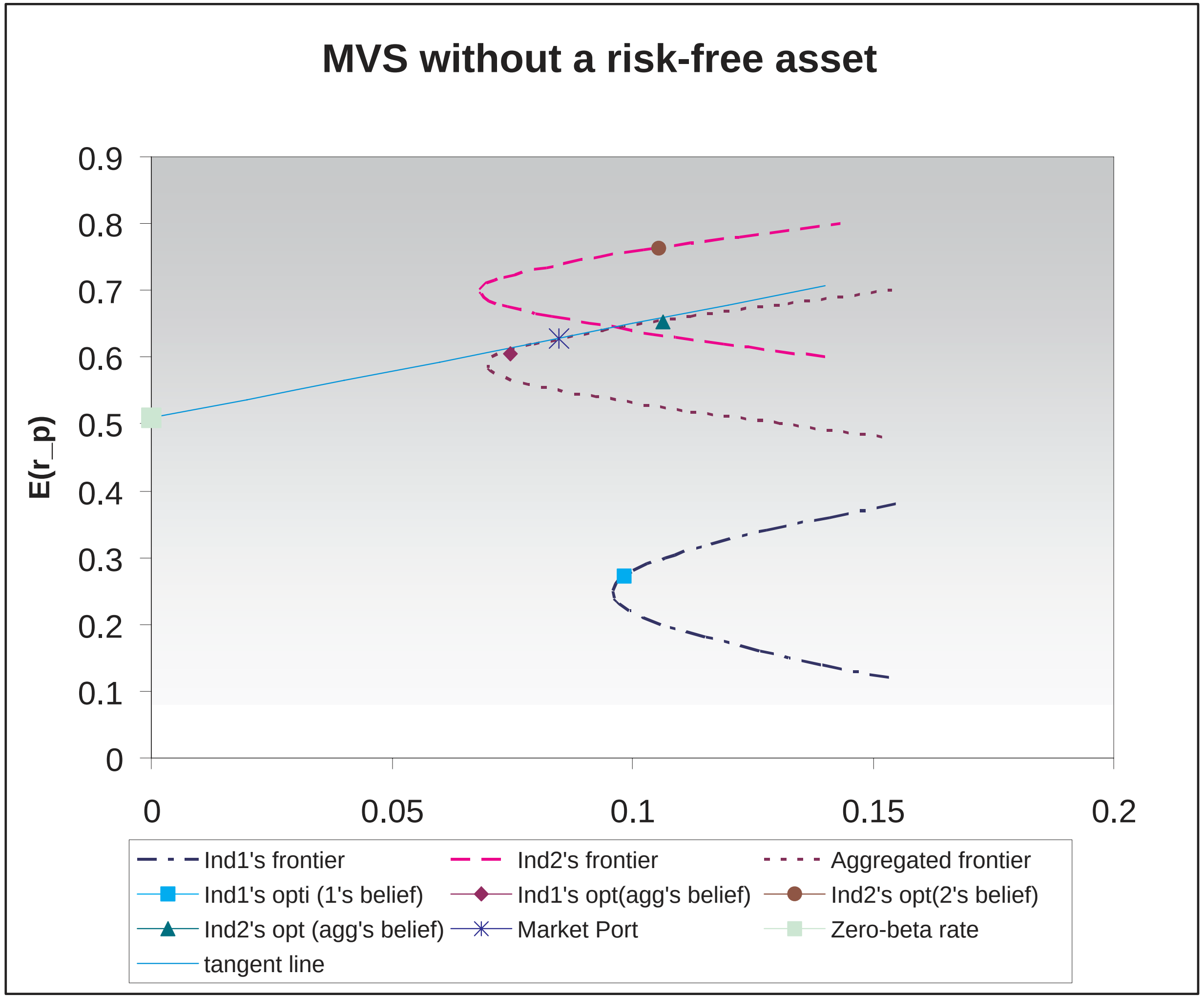
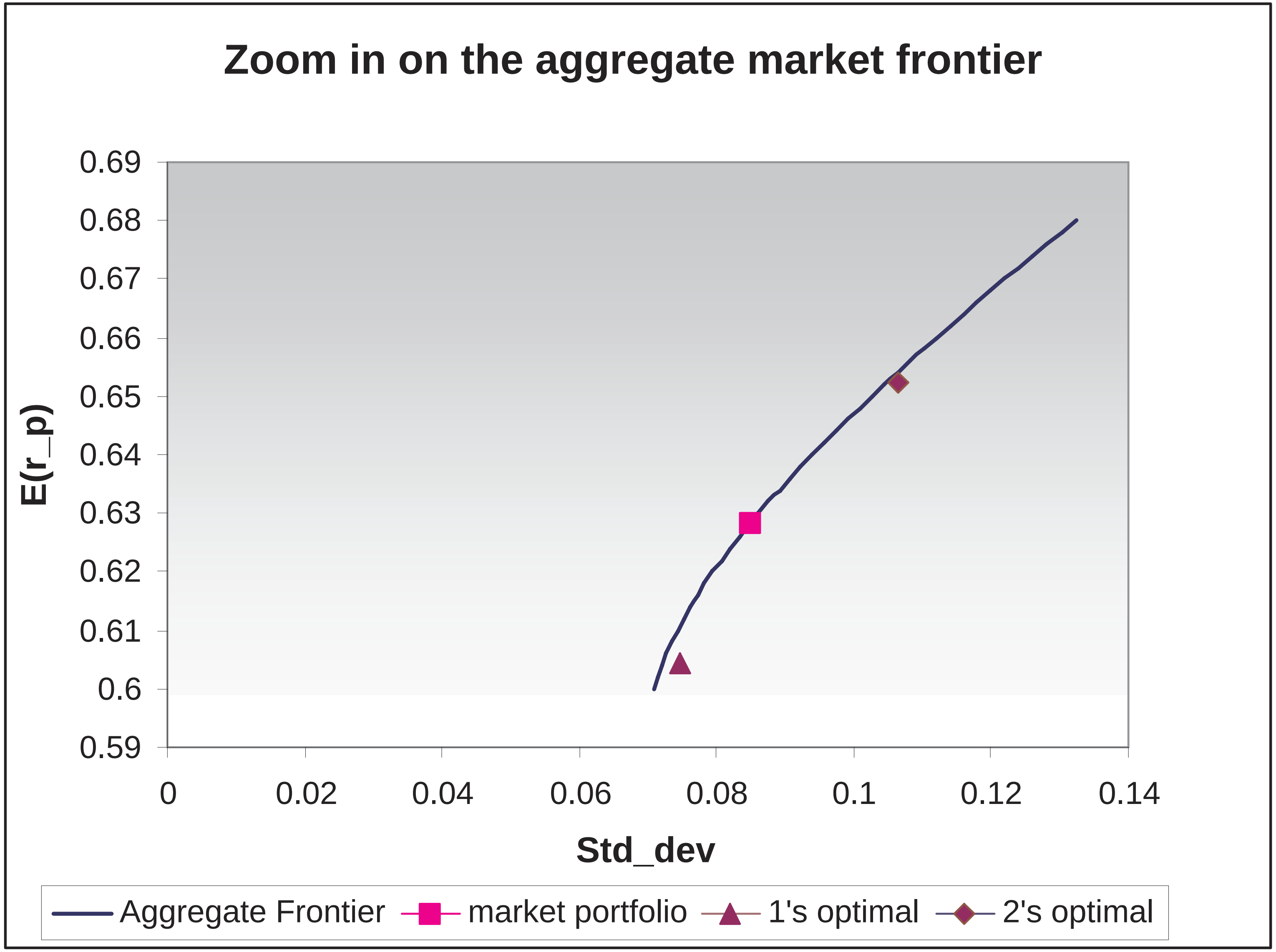


Figure 1: $\theta_1 > \theta_2, \mathbf{y_1} < \mathbf{y_2}, \Omega_1 > \Omega_2$

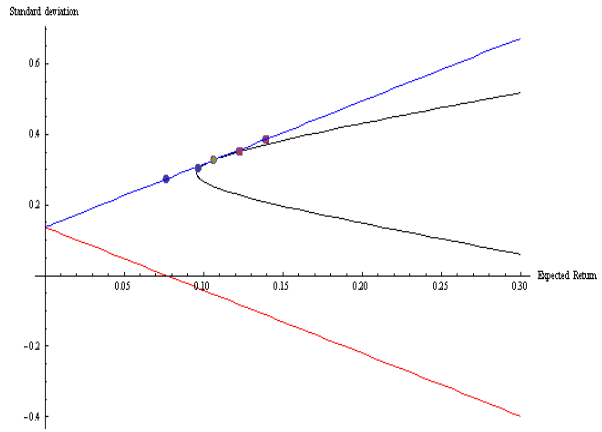

Zoom in on the aggregate market frontier
0.69
0.68
0.67
0.66
0.65
0.64
0.63
0.62
0.61
0.6
0.59
E(r_p)
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
Std_dev
Aggregate Frontier
market portfolio
1's optimal
2's optimal

- Example:

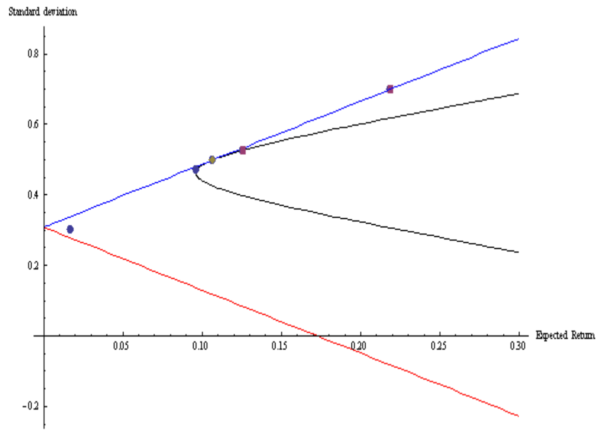
$$\boldsymbol{\theta_1 = 5, \qquad \theta_2 = 1, \qquad \mathbf{y}_i = (1 + \delta_i)\mathbf{y}_o,}$$

$$\boldsymbol{\Omega_i = D_i C D_i, \qquad D_i = (1 + \epsilon_i) D_o.}$$

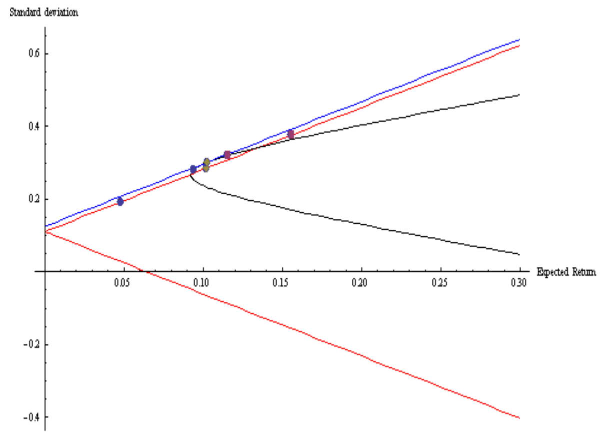
- Tangency relation holds with heterogeneous expected payoffs only—(a1)-(a2);
- Tangency relation breaks down with heterogeneous variances—(a3)-(a4);
- Adding a risk-free asset not necessarily improve the performance of the market.



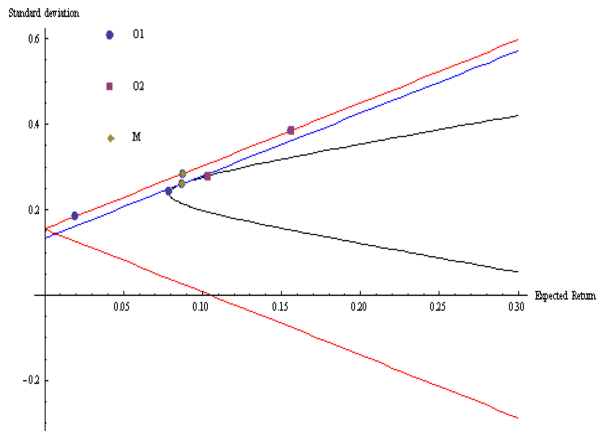
(a1) $(\delta_1, \delta_2) = (0.2, 0)$



(a2) $(\delta_1, \delta_2) = (0, 0.2)$



(a3) $(\epsilon_1, \epsilon_2) = (-0.2, 0)$



(a4) $(\epsilon_1, \epsilon_2) = (0, -0.2)$

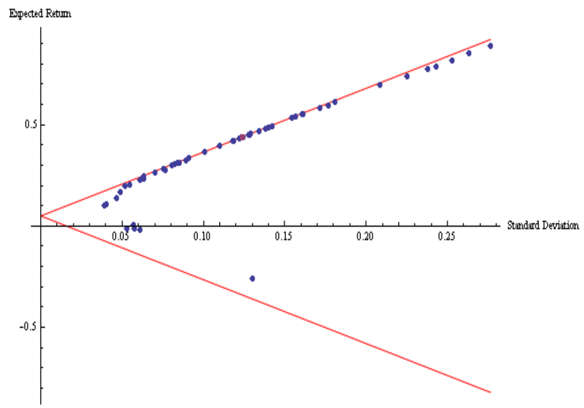
Robust for many investors with additive or multiplicative distribution of beliefs: for $i = 1, \cdots, 50$,

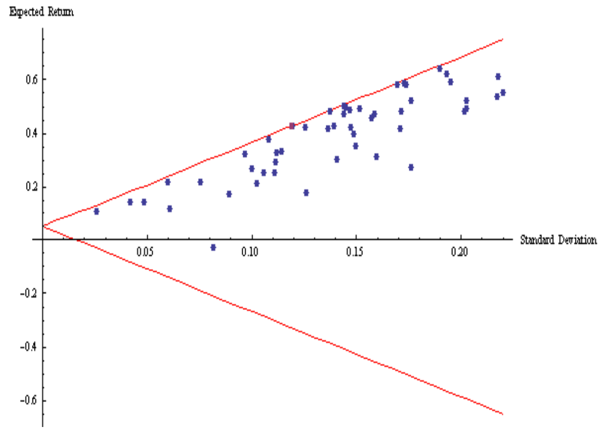
(i). $\mathbf{y}_i = (1 + \delta_i)\mathbf{y}_o, \Omega_i = D_i C D_i, D_i = (1 + \epsilon_i) D_o, \delta_i \sim \mathcal{N}(0, \sigma^2_{\delta_i})$ and $\epsilon_i \sim \mathcal{N}(0, \sigma^2_{\epsilon_i})$;

(ii). $\mathbf{y}_i = \delta_i + \mathbf{y}_o, \Omega_i = D_i C D_i$, $D_i = Diag[\epsilon_i + (0.7933, 0.8770, 1.4622)^T], \delta_i \sim \mathcal{MN}(0, \Sigma_{\delta_i})$ and $\epsilon_i \sim \mathcal{MN}(0, \Sigma_{\epsilon_i})$ and $\Sigma_{\delta_i} = \sigma_{\delta_i} Diag[1]$ and $\Sigma_{\epsilon_i} = \sigma_{\epsilon_i} Diag[1]$.

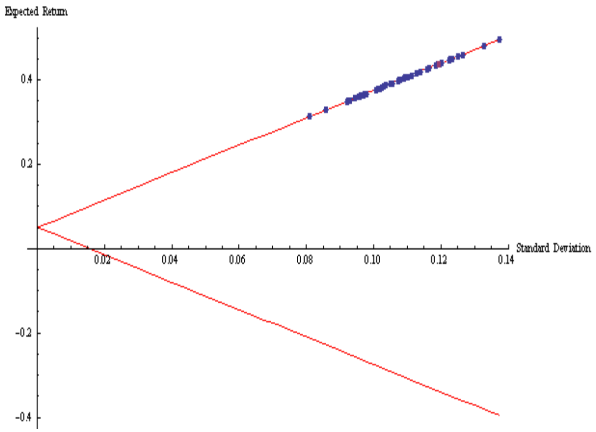
- Observation: heterogeneity in expected payoff has more significant impact on the optimal portfolios of investors than the heterogeneity in variances. Also, when the belief dispersions are additive rather than proportional, the heterogeneity impacts more significantly, in particular the heterogeneity in the expected payoffs.

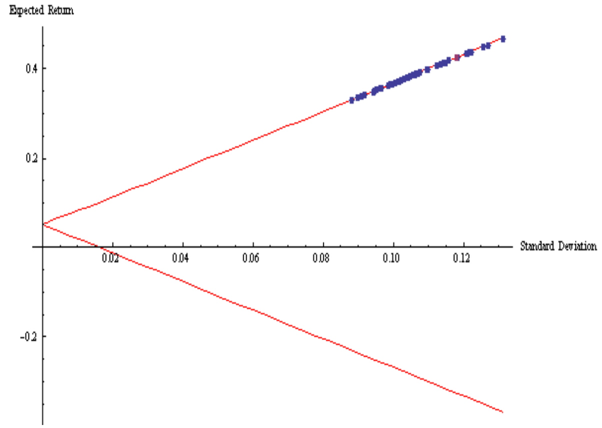
Univariate dispersion   Multivariate dispersion





(c1) $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0.2, 0)$ for all i

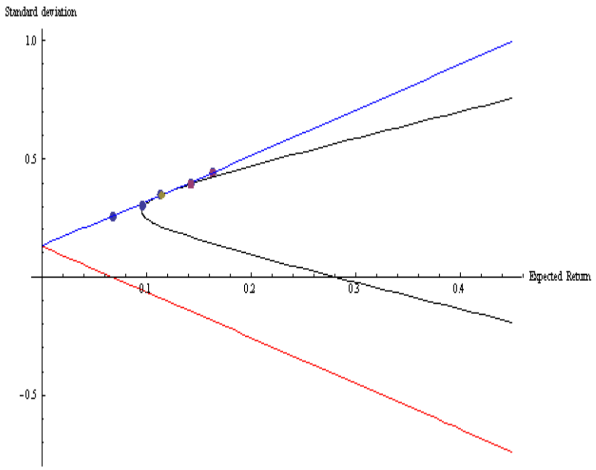




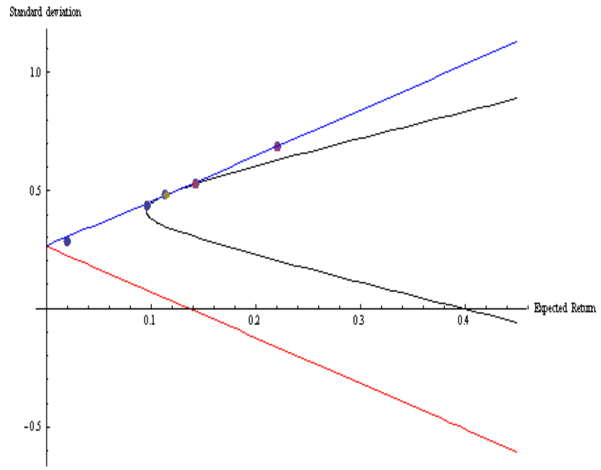
(c2) $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0, 0.03)$ for all i

3 Return Setup

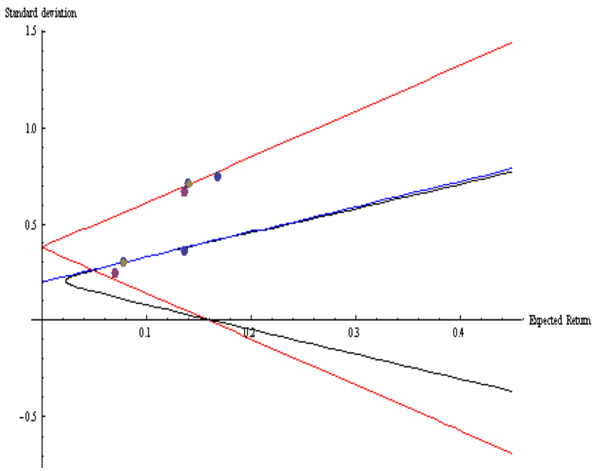
- Investors form their beliefs in terms of rate of returns rather than payoffs: $\mathcal{B}(\mathbb{E}_i(V_i, \tilde{\mathrm{r}}))$;
- Both equilibrium and consensus belief still exist and the consensus belief is of the same form as in the previous payoff setup.
- However, impact of heterogeneity is significantly different.
- In this case, market can gain much more mean-variance efficiency through the existence of a risk-free asset—**quasi one fund theorem**.
- In the case with many investors, when the noise is multivariate, dispersion in the belief of the variance of asset returns can actually cause investors' optimal portfolio to deviate from the CML.



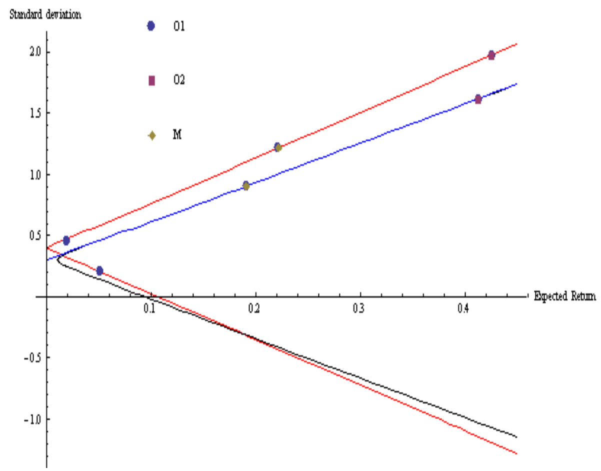
(b1) $(\delta_1, \delta_2) = (0.2, 0)$



(b2) $(\delta_1, \delta_2) = (0, 0.2)$



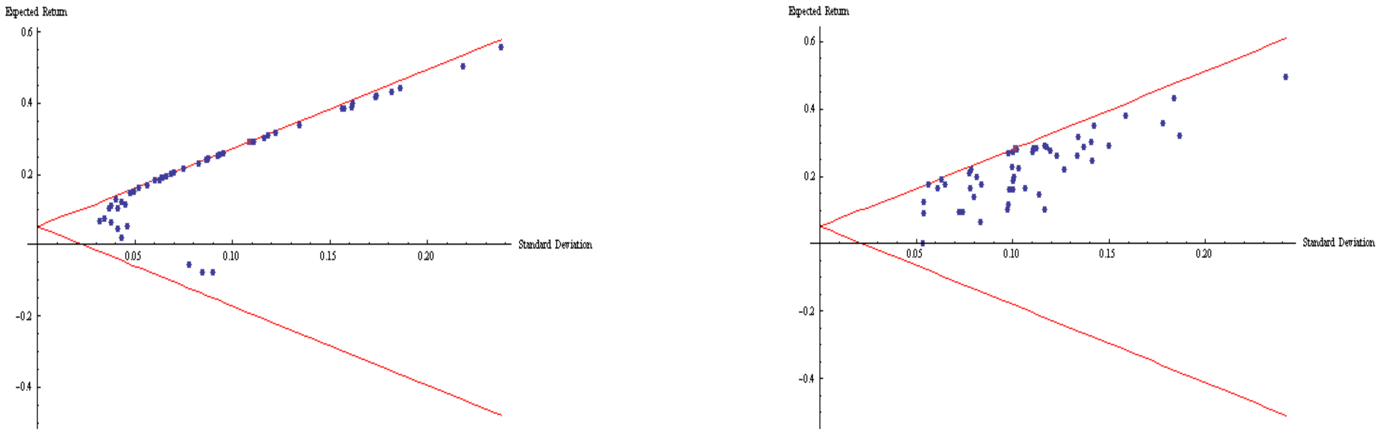
(b3) $(\epsilon_1, \epsilon_2) = (-0.2, 0)$



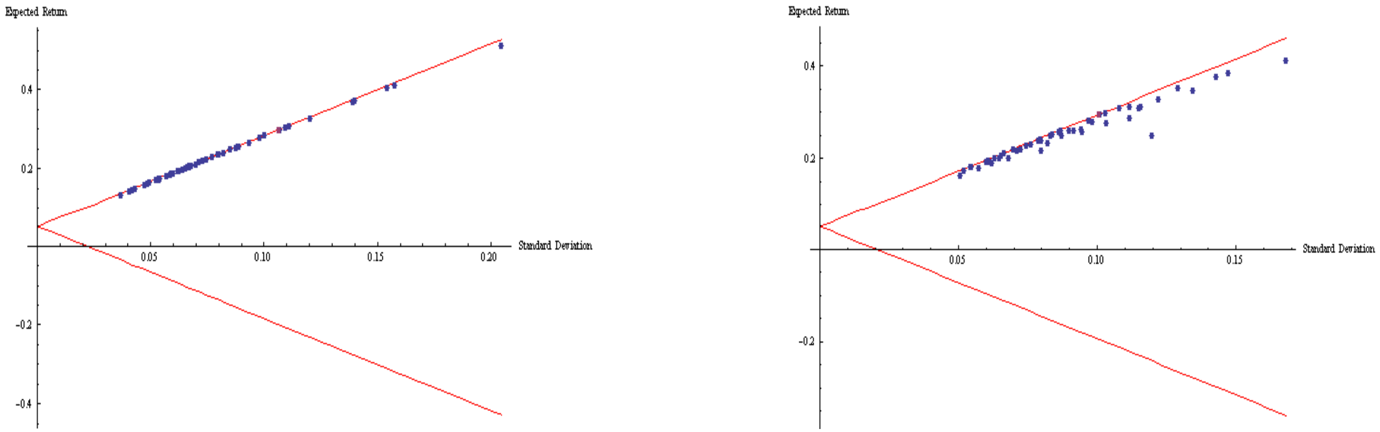
(b4) $(\epsilon_1, \epsilon_2) = (0, -0.2)$

Figure 2: Return setup.

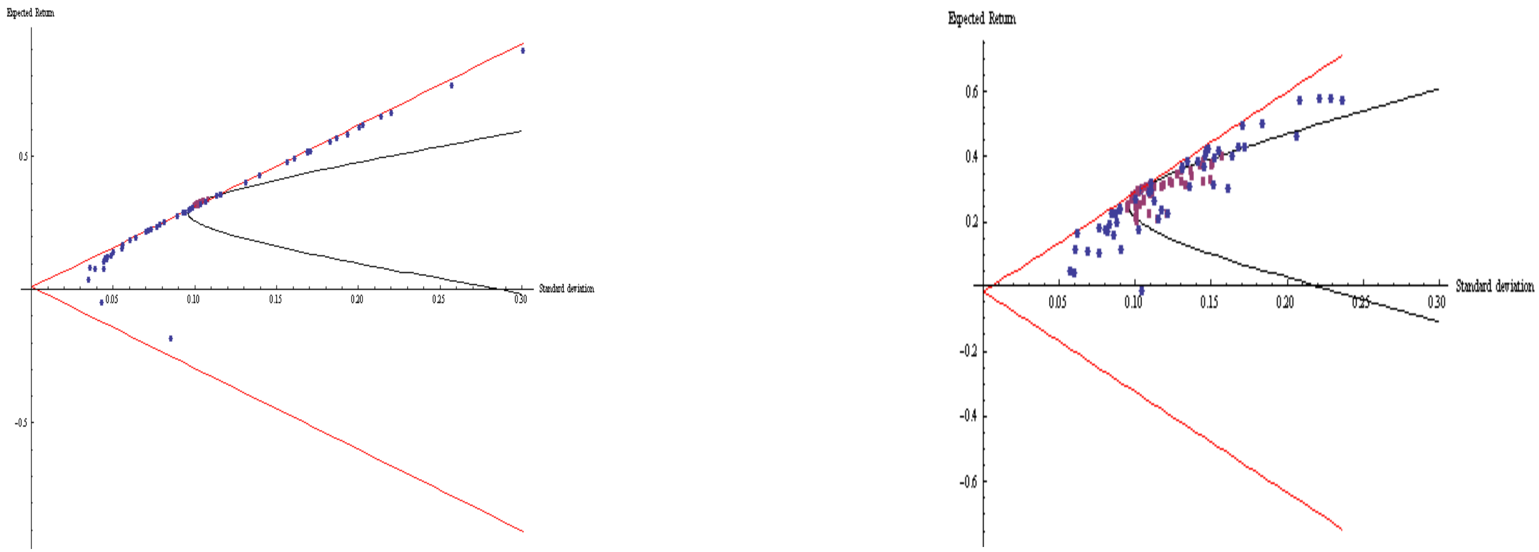
Univariate dispersion　　　　Multivariate dispersion



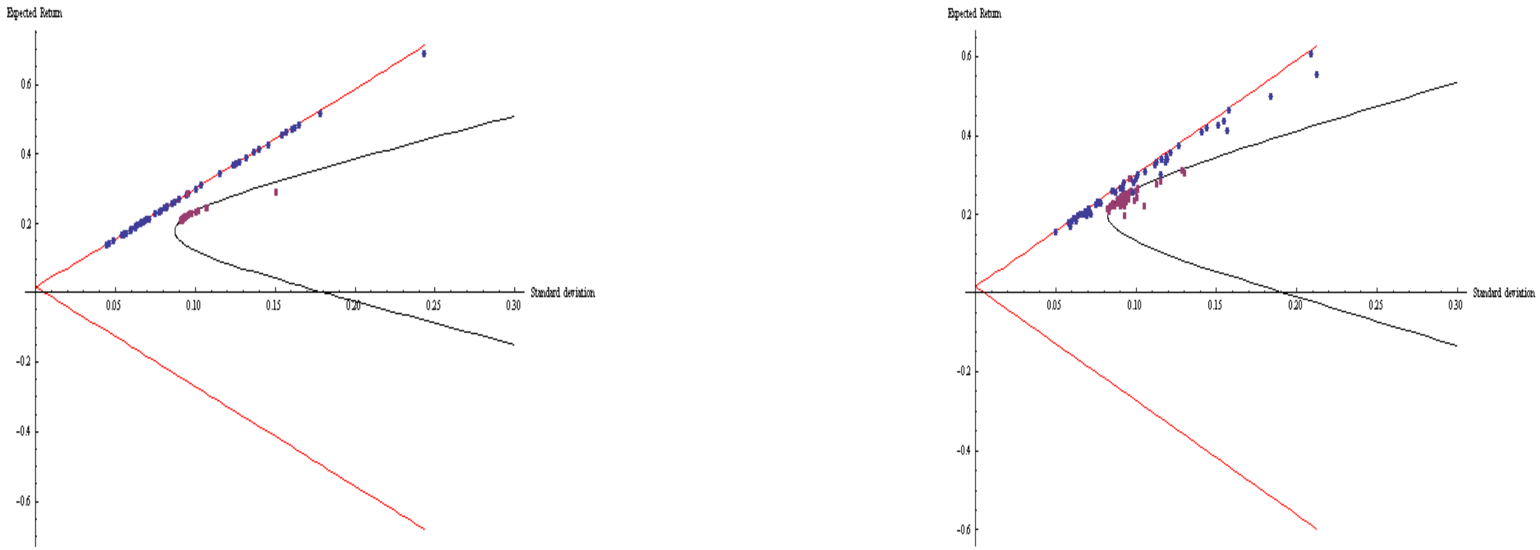
(c1) Heterogeneity in expected returns $(\boldsymbol{\sigma_{\delta_i}, \sigma_{\epsilon_i}}) = (\mathbf{0.2, 0})$



(c2) Heterogeneity in covariance $(\boldsymbol{\sigma_{\delta_i}, \sigma_{\epsilon_i}}) = (\mathbf{0, 0.03})$.



(e1) Heterogeneity in the expected returns $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0.2, 0)$.



(e2) Heterogeneity in the variances $(\sigma_{\delta_i}, \sigma_{\epsilon_i}) = (0, 0.03)$.

4 Conclusions

- The (Zero-Beta) CAPM-like relations in both price and returns hold with heterogeneous beliefs.
- The market aggregation behaviors, including aggregate variance/covariance matrix, the market expected payoff/return and the equilibrium price, are weighted average of heterogeneous individual behavior.
- The standard two fund theorem under homogeneous belief does not hold under heterogeneous beliefs and the optimal portfolios become mean-variance inefficient in market equilibrium belief, BUT can lead to almost perfect rationality;
- The geometric tangency relationship breaks down.
- Provides theoretical justifications on why individual investors with heterogeneous beliefs should under-perform the market.

- Existence of a risk-free asset may improve or reduce market efficiency.
- When many investors have dispersion in their beliefs, some of their portfolio may become significantly inefficient;
- The results can be used to explained some empirical results.
- Future work: Extension to a dynamic setting to including learning and expectation feedback.

A modification of Sharpe's *Index Fund Premise*(IFP)

IFPa Few of us are as smart as all of us.

IFPb Few of us are as smart as all of us, and it is hard to identify such people in advance.

IFPc Few of us are as smart as all of us, and it is hard to identify such people in advance, and they **definitely**[a] charge more than they are worth.

[a] It reads "may" in Sharpe's book