Discontinuous Dynamics in Binary Choice Models with Social Influence

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Schelling's 1973 paper

Hockey Helmets, Concealed Weapons, and Daylight Saving, *JCR*

- Should I wear the helmet or not during the hockey match?
- Should I carry a weapon or going unarmed?
- Switch watch to daylight saving time or stay on standard time?
- Should I go to the department weekly conference or not?
- Should I get annual flu vaccination or not ?
- Should I spray the insecticide in my garden or not?
- Should I go to vote for my favourite party or not?
- Should I take the car or not ?
- Should I dress elegant or not at the annual meeting of my society?

Schelling’s 1973 paper

Hockey Helmets, Concealed Weapons, and Daylight Saving, *JCR*

- Binary choices
- Externalities: situations where choices affect the whole population
- *n*-player games
- unspecified whether continuous or discrete time scale

Schelling's 1973 paper

Schelling's assumptions

Main simplifications

1. each player has a purely binary choice (e.g. L or R, 0 or 1)
2. the interaction is impersonal:
   number matters, not identity
3. x is the fraction of population choosing R

A dynamic adjustment is implicitly assumed:

1. $R(x) > L(x) \Rightarrow x \nearrow$
   $L(x) > R(x) \Rightarrow x \searrow$
2. equilibria $x = x^*$ where :
   - interior equilibria: $R(x^*) = L(x^*)$
   - boundary equilibria:
     - $x^* = 0$ provided that $R(0) < L(0)$
     - $x^* = 1$ provided that $R(1) > L(1)$.

The formal model

Time scale

- Schelling (1978, ch.3) describes several situations where individuals make repeated binary choices, with an evident discrete time scale according to the collective behavior observed in the previous period
- Schelling (1978, ch. 3) writes: "The phenomenon of overshooting is a familiar one at the level of individual ... consequently "Numerous social phenomena display cyclical behavior".

Discrete time adjustment.

The formal model

Time scale

Discrete time adjustment.

In economic and social systems changes over time are usually related to decisions that cannot be continuously revised.
If one takes a decision at a given time, very rarely such decision can be modified after an infinitesimal time

- Examples
  - voting a proposal
  - boycott a particular country
  - riding a bicycle to work
  - spraying an insecticide
  - getting vaccinated

The formal model

Our formalization

Formal model

- n player population
- $x \in [0, 1]$: fraction of players choosing R
  - $x = 0$ means all choose L,
  - $x = 1$ means all choose R
- Payoffs are functions $R(x)$ and $L(x)$ defined in $[0, 1]$ Each player decides according by comparing payoff functions

At time t, x_t players are playing strategy R

- If $R(x_t) > L(x_t)$ a fraction of the $(1 - x_t)$ players that are playing L switch to strategy R
- If $R(x_t) < L(x_t)$ a fraction of the x_t players that are playing R switch to strategy L

$$x_{t+1} = f(x_t) =$$

$$\begin{cases} x_t + \delta_R g\left[\lambda\left(R\left(x_t\right) - L\left(x_t\right)\right)\right]\left(1 - x_t\right) & \text{if } R\left(x_t\right) \geq L\left(x_t\right) \\ x_t - \delta_L g\left[\lambda\left(L\left(x_t\right) - R\left(x_t\right)\right)\right] x_t & \text{if } R\left(x_t\right) < L\left(x_t\right) \end{cases} \tag{1}$$

where

- $\delta_R, \delta_L \in [0, 1]$ maximum values of switching fractions
- $\lambda > 0$ switching propensity (or speed of reaction)
- $g : \mathbb{R}^+ \rightarrow [0, 1]$ is a continuous and increasing function

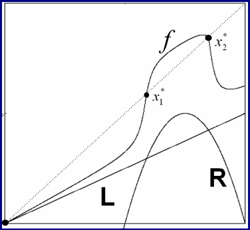
$$g(0) = 0 \quad \lim_{z \to \infty} g(z) = 1$$

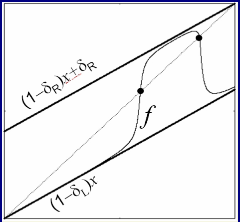
The formal model

Our formalization

- If the payoff functions are continuous, then the map f is continuous
- Even if $L(x)$ and $R(x)$ are smooth functions, the map f is not smooth where $R(x) = L(x)$
- The graph of f is contained in the strip bounded by two lines

$$(1-\delta_L)\,x \leq f\,(x) = (1-\delta_R)\,x + \delta_R$$





Some Payoff considered by Schelling

One intersection

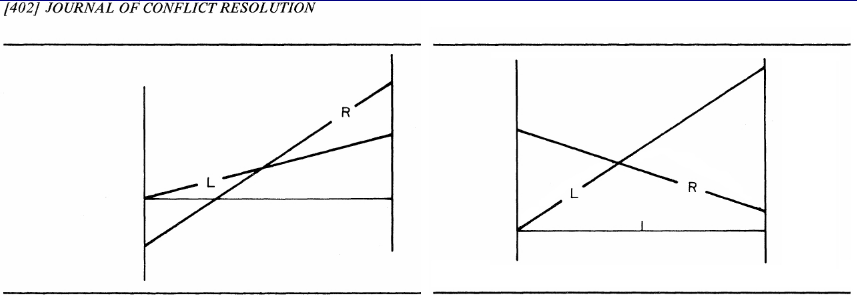


Figure 8.

Figure 9.

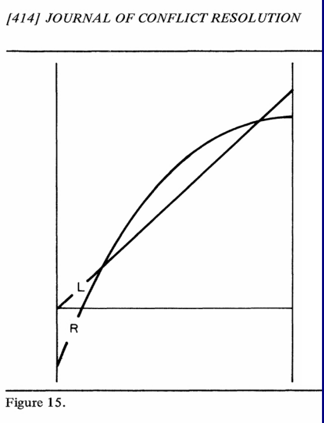
R going unarmed
L carrying a visible weapon

R take the car
L stay home

Some Payoff considered by Schelling

Two intersections

R spraying insecticide
L not spraying insecticide



Some results

(Bischi and Merlone, 2008)

- some unexpected results coming from using the formalism of discrete time dynamical systems represented by iterated noninvertible maps.
- a global dynamic analysis, based on the geometric properties of noninvertible one-dimensional maps, allow us to get some insight into the complexity of attracting sets and complex structure of the basins of attraction.
- The overshooting effects is not an artificial effect, rather, as stressed by Schelling (1978), overshooting and over-reaction arise quite naturally in social systems, due to emotional attitude, excess of prudence or lack of information.

The limit case

Definitions

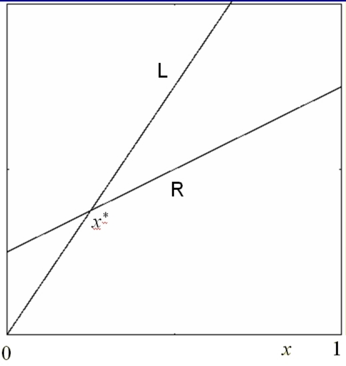
The limit case is obtained for $\lambda \to +\infty$:

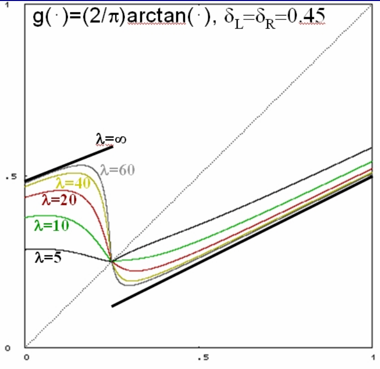
$$x_{t+1} = f_\infty(x_t) = \begin{cases} (1-\delta_R)\,x_t + \delta_R & \text{if} \quad R(x_t) > L(x_t) \\ x_t & \text{if} \quad R(x_t) = L(x_t) \\ (1-\delta_L)\,x_t & \text{if} \quad R(x_t) < L(x_t) \end{cases}$$

Equivalent to considering $g(\cdot) = 1$
i.e. the switching rate only depends on the sign of the difference between payoffs no matter how much they differ.

The limit case

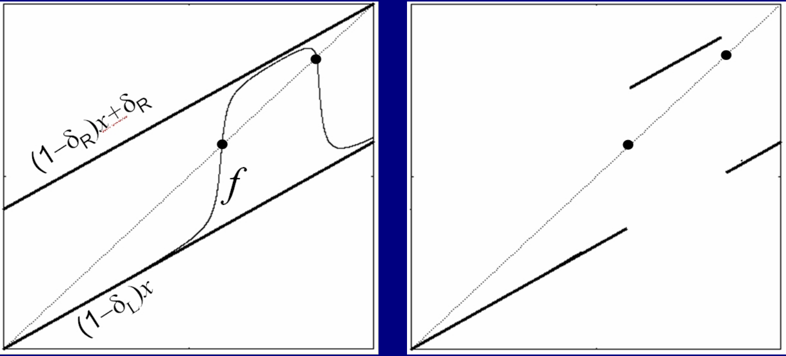
One intersection





The limit case

Two intersections



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The limit case (one discontinuity)

Definitions

Curves L and R intersect ad d

Two cases:

$$x' = T_1(x) = \begin{cases} (1-\delta_R)\,x & \text{if} \quad x < d \\ (1-\delta_L)\,x + \delta_L & \text{if} \quad x > d \end{cases} \qquad (2)$$

$$x' = T_2(x) = \begin{cases} (1-\delta_R)\,x + \delta_R & \text{if} \quad x < d \\ (1-\delta_L)\,x & \text{if} \quad x > d \end{cases} \qquad (3)$$

with constraints

$$0 < \delta_L < 1, \quad 0 < \delta_R < 1$$

and

$$x \in [0, 1]$$

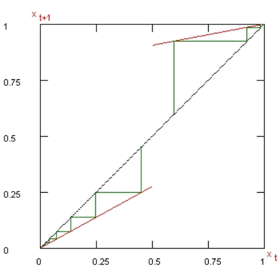
The limit case (one discontinuity)

Simple dynamics

$$x' = T_1(x) = \begin{cases} (1-\delta_R)\,x & \text{if} \quad x < d \\ (1-\delta_L)\,x + \delta_L & \text{if} \quad x > d \end{cases}$$

$\delta_R = .81$

$\delta_L = .45$



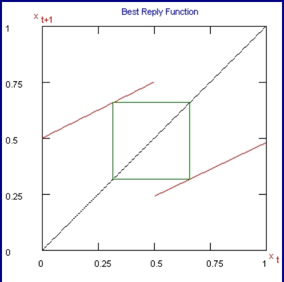
The limit case (one discontinuity)

Simple dynamics

$$x' = T_2(x) = \begin{cases} (1-\delta_R)\,x + \delta_R & \text{if} \quad x < d \\ (1-\delta_L)\,x & \text{if} \quad x > d \end{cases}$$

$\delta_R = .5$

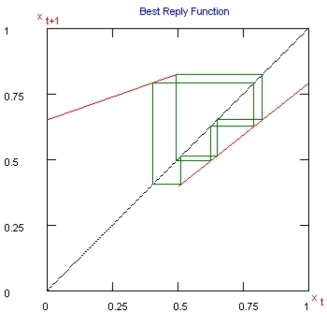
$\delta_L = .52$



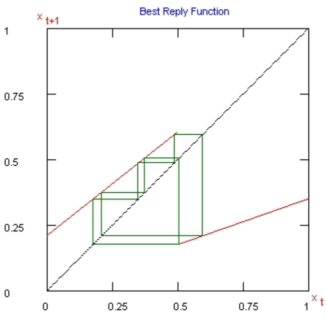
The limit case (one discontinuity)

Simple dynamics

but also



$\delta_R = .65 \quad \delta_L = .21$

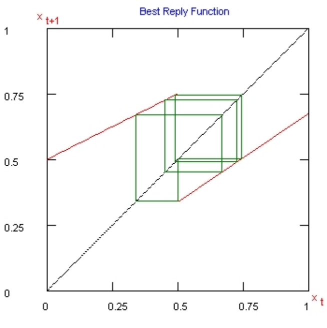


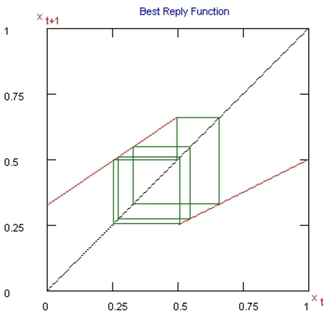
$\delta_R = .21 \quad \delta_L = .65$

The limit case (one discontinuity)

Simple dynamics

and also





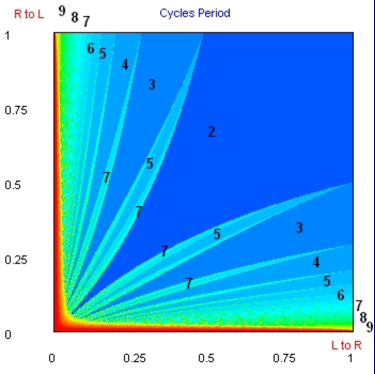
$\delta_R = .5 \quad \delta_L = .325$

$\delta_R = .325 \quad \delta_L = .5$

The limit case (one discontinuity)

Map attractors

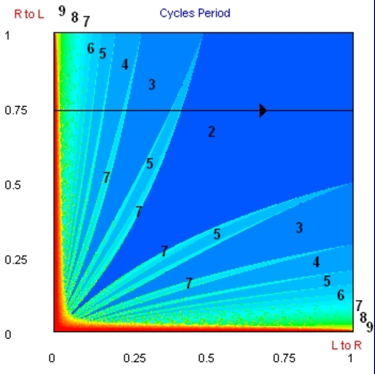
Fixed δ_L and δ_R the map has only one attractor

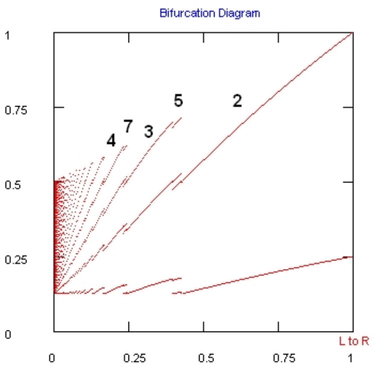


a stable cycle of period k

The limit case (one discontinuity)

Map attractors



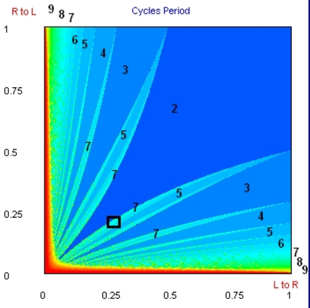


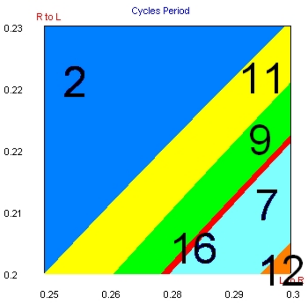
Fixing $\delta_L = .75$

The limit case (one discontinuity)

Map attractors

Increasing the scale...





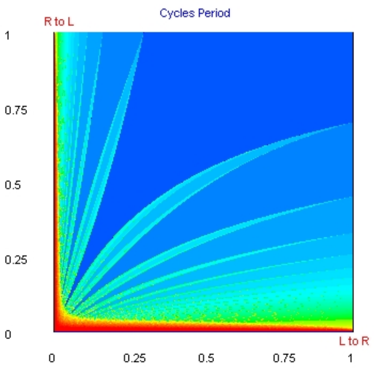
between the regions of the 2-cycle and the 3-cycle, there exist infinitely many other intervals of existence of cycles of period

$$2n + 3m \qquad \forall n \geq 1, \forall m \geq 1.$$

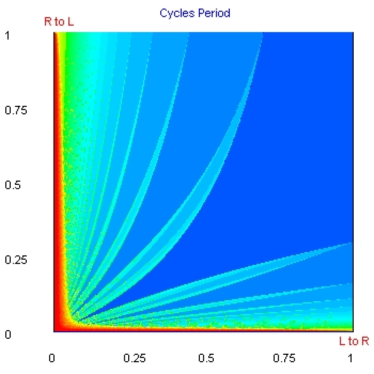
The limit case (one discontinuity)

Map attractors

Discontinuity point position...only qualitative changes



$d = .3$

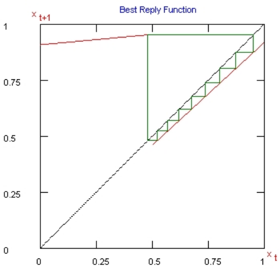


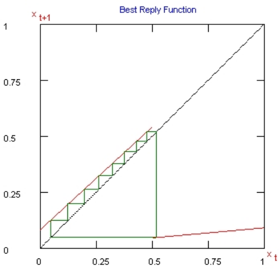
$d = .7$

The limit case (one discontinuity)

First degree tongues

Consider principal tongues, or main tongues (Banergjee, Feigen, Di Bernardo, Maistrenko)
or tongues of first degree (Leonov, Mira):
cycles of period k having one point on one side of the discontinuity point and $(k-1)$ points on the other side





The limit case (one discontinuity)

First degree tongues

Consider

$$x' = T_2(x) = \begin{cases} (1-\delta_R)\,x + \delta_R & \text{if} \quad x < d \\ (1-\delta_L)\,x & \text{if} \quad x > d \end{cases}$$

written as

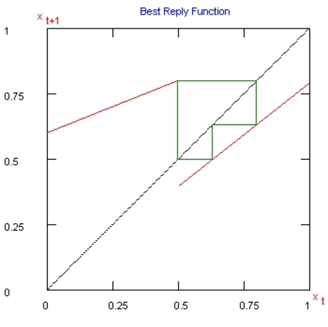
$$x' = T_2(x) = \begin{cases} T_L(x) = m_1 x + (1-m_1) & \text{if} \quad x < d \\ T_R(x) = m_2 x & \text{if} \quad x > d \end{cases}$$

where

$$m_1 = 1 - \delta_R, \qquad m_2 = 1 - \delta_L$$

The limit case (one discontinuity)

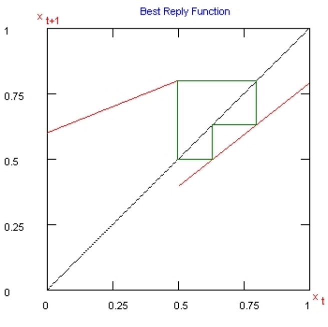
First degree tongues

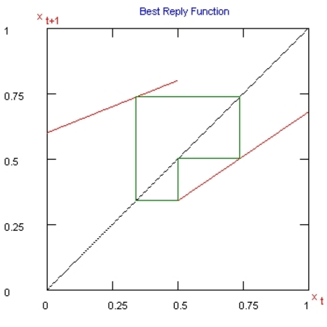


$$T_R \cdot T_R \cdot T_L\,(d) = d$$

The limit case (one discontinuity)

First degree tongues





$$T_R \cdot T_R \cdot T_L(d) = d$$

$$T_R \cdot T_L \cdot T_R(d) = d$$

The limit case (one discontinuity)

First degree tongues

k-cycle with periodic points $x_1, \ldots, x_k$,

$$T_L, T_R, \ldots, T_R \quad \longrightarrow \quad m_1 = \frac{m_2^{(k-1)} - d}{(1-d)m_2^{(k-1)}}$$

$$T_R, T_L, T_R, \ldots, T_R \quad \longrightarrow \quad m_1 = \frac{m_2^{(k-2)} - d}{(1-m_2 d)m_2^{(k-2)}}$$

The limit case (one discontinuity)

First degree tongues

k-cycle with periodic points $x_1, \dots, x_k$,

$$m_{1i} := \underset{\displaystyle \frac{m_2^{(k-1)} - d}{(1-d)m_2^{(k-1)}}}{\overset{T_L, T_R, \dots, T_R}{\downarrow}} \leq m_1 \leq \underset{\displaystyle \frac{m_2^{(k-2)} - d}{(1-m_2 d)m_2^{(k-2)}}}{\overset{T_R, T_L, T_R, \dots, T_R}{\downarrow}} =: m_{1f}$$

The limit case (one discontinuity)

First degree tongues

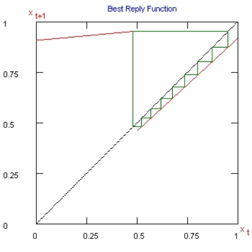
k-cycle $(x_1^*, x_2^*, ..., x_k^*)$ with $x_1^* < d$ and $x_i^* > d$ for $i > 1$

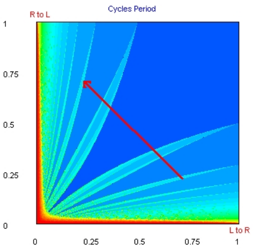
$$
\begin{aligned}
x_1^* &= \frac{m_2^{(k-1)}(1-m_1)}{1-m_1 m_2^{(k-1)}} \\
x_2^* &= m_1 x_1^* + 1 - m_1 \\
x_3^* &= m_2(m_1 x_1^* + 1 - m_1) \\
x_4^* &= m_2^2(m_1 x_1^* + 1 - m_1) \\
&\vdots \\
x_k^* &= m_2^{(k-2)}(m_1 x_1^* + 1 - m_1)
\end{aligned}
$$

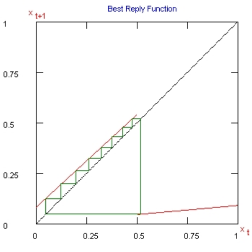
The limit case (one discontinuity)

First degree tongues

The other kind of cycles are symmetrical







The limit case (one discontinuity)

First degree tongues

k-cycle with periodic points $x_1, \ldots, x_k$,

$$T_R, T_L, \ldots, T_L \quad\quad T_L, T_R, T_L, \ldots, T_L$$

$$\downarrow \quad\quad \downarrow$$

$$m_2 = \frac{d-1+m_1^{(k-1)}}{dm_1^{(k-1)}} \quad\quad m_2 = \frac{d-1+m_1^{(k-2)}}{m_1^{(k-2)}(m_1 d+1-m_1)}$$

The limit case (one discontinuity)

First degree tongues

k-cycle with periodic points $x_1, \ldots, x_k$,

$$
\begin{array}{ccccc}
 & T_R, T_L, \ldots, T_L & & T_L, T_R, T_L, \ldots, T_L & \\
 & \downarrow & & \downarrow & \\
m_{2i} := & \dfrac{d-1+m_1^{(k-1)}}{dm_1^{(k-1)}} & \leq m_2 \leq & \dfrac{d-1+m_1^{(k-2)}}{m_1^{(k-2)}(m_1 d+1-m_1)} & =: m_{2f}
\end{array}
$$

The limit case (one discontinuity)

First degree tongues

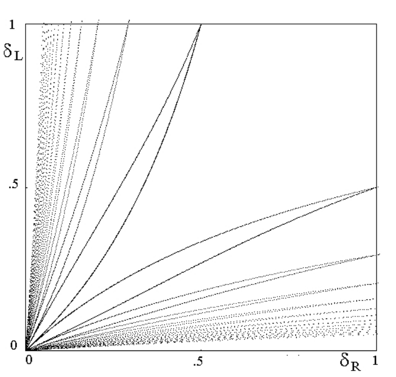
k-cycle $(x_1^*, x_2^*, ..., x_k^*)$ with $x_1^* > d$ and $x_i^* < d$ for $i > 1$

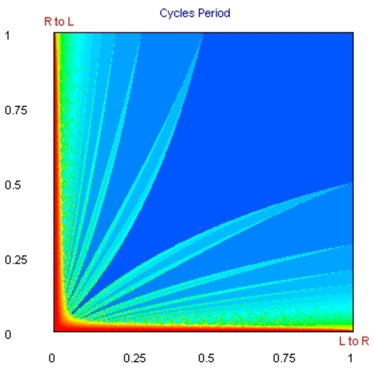
$$
\begin{aligned}
x_1^* &= \frac{1 - m_1^{(k-1)}}{1 - m_2 m_1^{(k-1)}} \\
x_2^* &= m_2 x_1^* \\
x_3^* &= m_1 m_2 x_1^* + 1 - m_1 \\
x_4^* &= m_1^2 m_2 x_1^* + m_1(1 - m_1) + (1 - m_1) \\
&\vdots \\
x_k^* &= m_1^{(k-2)} m_2 x_1^* + (1 - m_1^{(k-2)})
\end{aligned}
$$

The limit case (one discontinuity)

First degree tongues

The bifurcation formulae depend also from the discontinuity point *d*



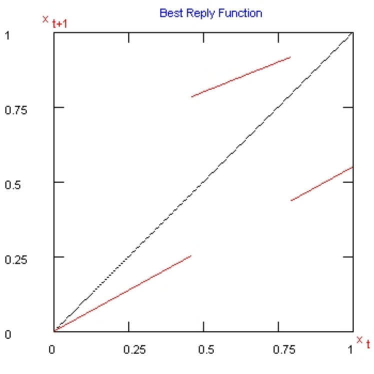


The limit case (two discontinuities)

Simple dynamics

We assume

- an increasing step
- a decreasing step



since the other case can be considered in a similar way

The limit case (two discontinuities)

Simple dynamics

The formalization is

$$x' = T_1(x) = \begin{cases} (1-\delta_R)\,x & \text{if} \quad x < d_1 \\ (1-\delta_L)\,x + \delta_L & \text{if} \quad d_1 < x < d_2 \\ (1-\delta_R)\,x & \text{if} \quad x > d_2 \end{cases}$$

with the constraints

1. $0 < d_1 < d_2 < 1$
2. $0 < \delta_R < 1$
3. $0 < \delta_l < 1$

$$x \in [0,1]$$

The limit case (two discontinuities)

Simple dynamics

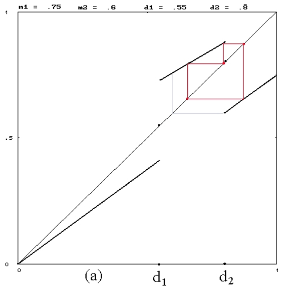
Defining

- $m_1 = 1 - \delta_R$
- $m_2 = 1 - \delta_L$

$$x' = T(x) = \begin{cases} T_O(x) = m_1 x & \text{if} \quad x < d_1 \\ T_I(x) = m_2 x + 1 - m_2 & \text{if} \quad d_1 < x < d_2 \\ T_O(x) = m_1 x & \text{if} \quad x > d_2 \end{cases}$$

The limit case (two discontinuities)

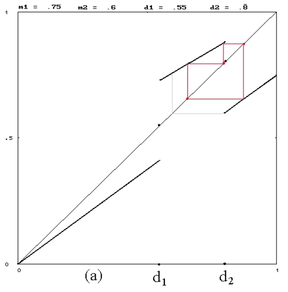
Possible attractors

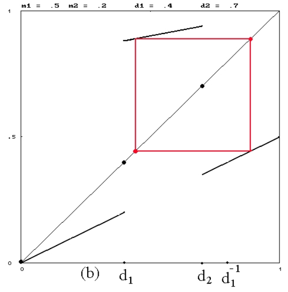


- the origin
- period 3 cycle

The limit case (two discontinuities)

Possible attractors





- the origin
- period 3 cycle

- the origin
- period 2 cycle

The limit case (two discontinuities)

Possible attractors

Depending on the parameters

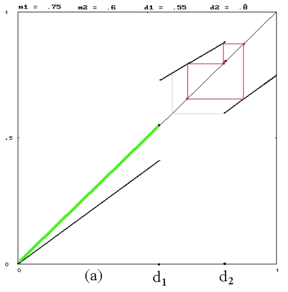
- d_1
- d_2
- $m_1 \leftrightarrow \delta_1$
- $m_2 \leftrightarrow \delta_2$

we can have

- origin as the unique attractor
- coexistence with a k-period cycle

The limit case (two discontinuities)

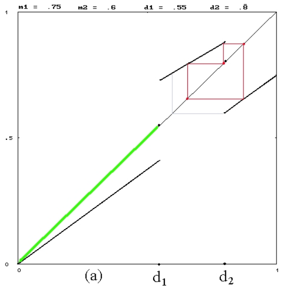
Basins

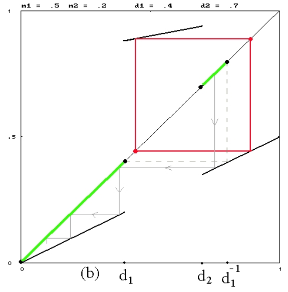


- $B(O) = [0, d_1[$
- $B(C) =]d_1, 1]$

The limit case (two discontinuities)

Basins



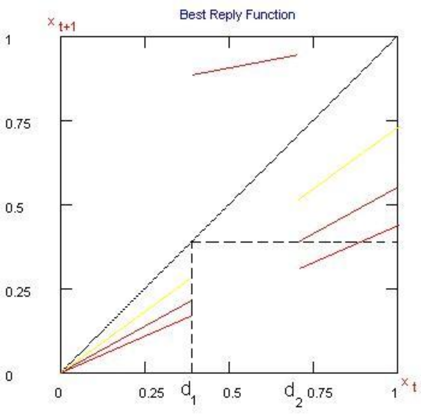


- $B(O) = [0, d_1[$
- $B(C) =]d_1, 1]$

- $B(O) = [0, d_1[\bigcup]d_2, \xi[$
- $B(C) =]d_1, d_2[\bigcup]\xi, 1]$

The limit case (two discontinuities)

Border collision bifurcation



$$m_1 d_2 = d_1 \Rightarrow \delta_R^* = \frac{d_2 - d_1}{d_2}$$

The limit case (two discontinuities)

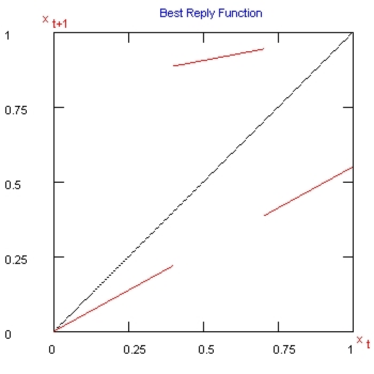
Basins

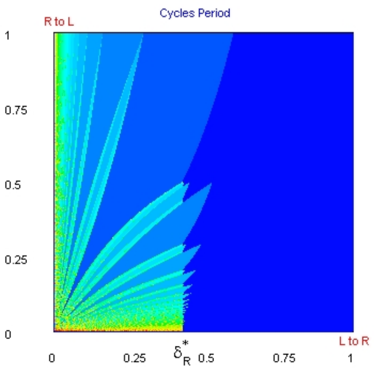
Proposition

1. *When* $\delta_R < \delta_R^*$ *then there are two coexistent attractors (the origin and a* $k-$ *cycle* C *for some integer* k*) and the two basins are made up of only one interval:* $B(O) = [0, d_1[$ *and* $B(C) =]d_1, 1]$.
2. *When* $\delta_R > \delta_R^*$ *then either the origin is the only attractor in the whole interval* $[0, 1]$ *or there are two coexistent attractors (the origin and a* $k-$ *cycle* C *for some integer* k*) and the two basins are made up of two pieces:* $B(O) = [0, d_1[\cup]d_2, \xi[$ *and* $B(C) =]d_1, d_2[\cup]\xi, 1]$, *where* $\xi = \frac{d_1}{m_1}$.

The limit case (two discontinuities)

Map attractors

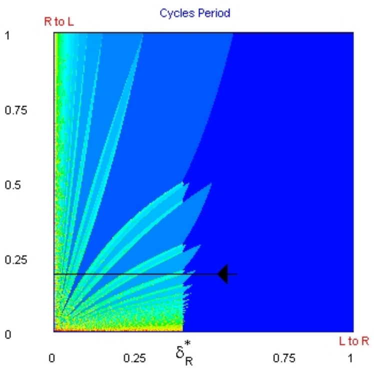


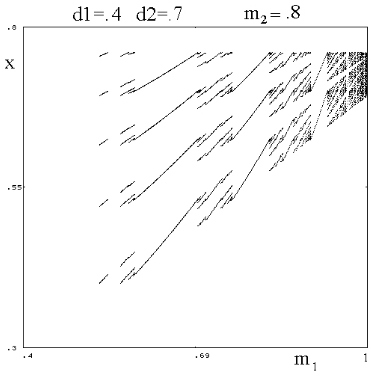


Fixing $d_1 = .4$ and $d_2 = .7$

The limit case (two discontinuities)

Map attractors

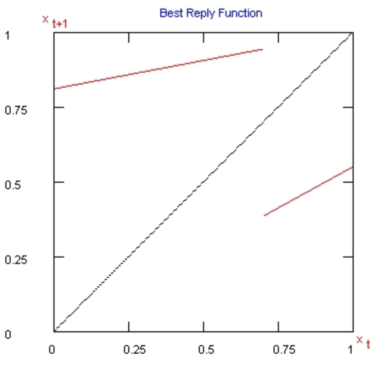


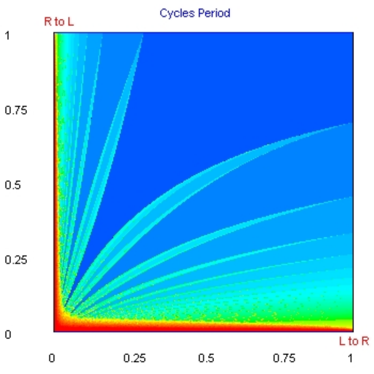


Fixing $\delta_L = .2$

The limit case (two discontinuities)

Map attractors



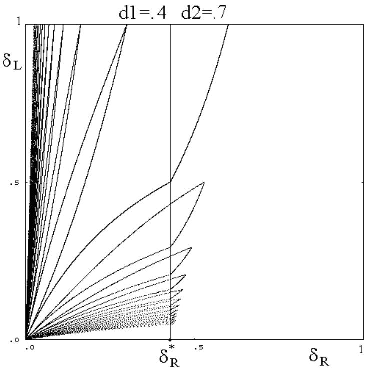


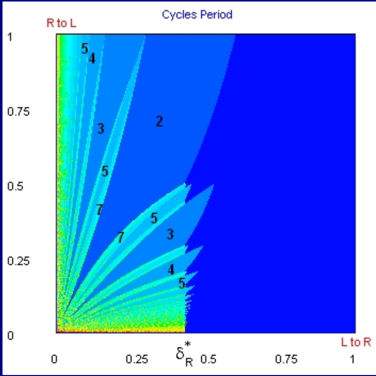
Fixing $d_1 = 0.0$ and $d_2 = .7$

The limit case (two discontinuities)

First degree tongues

We can drawn the bifurcation curves of the principal *k*-cycles for $k = 2, 1, 15$





The limit case

Some results

We remark that

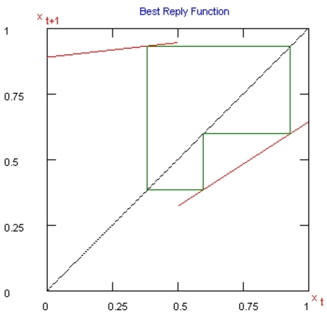
1. all the tongues are disjoint, i.e. we can have only a single attractor at each fixed pair of values of the two parameters
2. All the cycle, when existing, are asymptotically stable. *This follows since, the slope (or eigenvalue) of the function* $T_2^k = T_2 \circ ... \circ T_2$ *(k times) in the periodic points of the cycle, is given by*

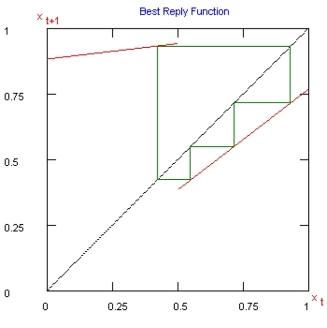
$$\underbrace{m_1^p}_{p \text{ points on the left}} \cdot \underbrace{m_2^{(k-p)}}_{k-p \text{ points on the right}} < 1$$

The limit case

Second degree tongues

Consider any pair of contiguous first degree tongues, e.g.:





with rotation numbers

$$\frac{1}{k_1} = \frac{1}{3} \qquad \frac{1}{k_1+1} = \frac{1}{4}$$

The limit case

Second degree tongues

...then it is possible to two infinite families of periodic tongues:

- $\frac{1}{k_1} \oplus \frac{1}{k_1+1} = \frac{2}{2k_1+1}, \quad \frac{2}{2k_1+1} \oplus \frac{1}{k_1} = \frac{3}{3k_1+1}, \quad \cdots$

$$\frac{n}{nk_1+1} \qquad \forall n > 1$$

- $\frac{1}{k_1} \oplus \frac{1}{k_1+1} = \frac{2}{2k_1+1}, \quad \frac{2}{2k_1+1} \oplus \frac{1}{k_1+1} = \frac{3}{3k_1+2}, \quad \cdots$

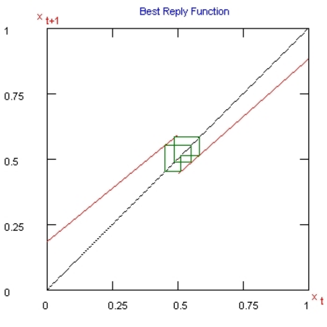
$$\frac{n}{nk_1+n-1} \qquad \forall n > 1$$

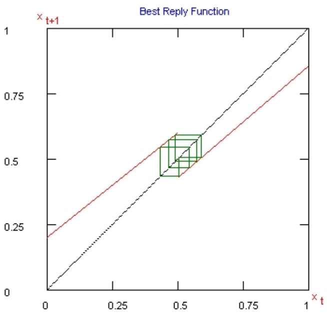
called second degree tongues.

The limit case

Higher degree tongues

Iteratively, consider any pair of contiguous second degree tongues, with rotation numbers





$$\frac{n}{nk_1+1} = \frac{2}{5}$$

$$\frac{n+1}{(n+1)k_1+1} = \frac{3}{7}$$

The limit case

Higher degree tongues (rotation numbers)

...we can construct two infinite families of periodicity tongues, called third degree tongues.

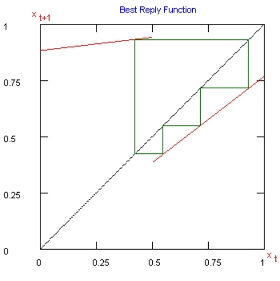
And so on:

all the rational numbers can be obtained this way,
giving
all the infinite existing periodicity tongues.

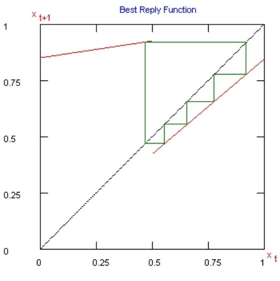
The limit case

Higher degree tongues (symbolic sequences)

Associating symbol sequences to cycles



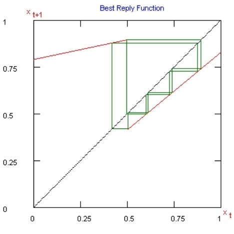
LRRR



LRRRR

The limit case

Higher degree tongues (symbolic sequences)



$$LRRR \oplus LRRRR = LRRRLRRRRR$$

More generally,

$$\exists \quad \sigma \quad \wedge \quad \exists \quad \tau$$
$$\Downarrow$$
$$\exists \qquad \sigma \oplus \tau = \sigma\tau$$

Conclusion

- limit case, modeling impulsive agents
- one intersection
  - first degree tongues, analytical expression
  - k-period cycles
- two intersections
  - basins of attraction
  - first degree tongues, analytical expression
  - k-period cycles
- rotation numbers
- symbolic sequences

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