Growing Through Chaotic Intervals

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- description of the model;
- fixed points, 2-cycles, chaotic intervals; critical flip bif. and homoclinic bif.;
- classificaiton of border-collision bifurcations.

The growth model (Matsuyama,1999)

Piecewise smooth unimodal map $x_{t+1} = \phi(x_t)$, where

$$\phi(x) = \begin{cases} f(x) = Gx^{1-1/\sigma}, \ 0 < x < 1 \ (Solow\ regime) \\ g(x) = \frac{Gx}{1+\theta(x-1)}, \quad x > 1 \ (Romer\ regime) \end{cases}$$

$$\theta = (1 - 1/\sigma)^{1-\sigma}, \ \sigma > 1.$$

$$x_t = K_t/N_t\sigma F\theta;$$

K_t - capital;

N_t - number of type of intermediate goods introduced up to time t;

F - some constant.

The output Y_t is related to K_t and N_s, $0 < s < t$, through a production function.

A const. proportion of Y_t is left to be used as capital in the next period.

σ denotes the demand elasticity of the intermediate good.

Matsuyama, 1999:

- the model may have stable equilibria or unstable ones;
- the dynamics may oscillate alternatively between the Solow regime and the Romer regime, when there is a stable 2-cycle;
- the 2-cycle may loose its stability, leading to different dynamic behaviors, when the parameter G belongs to the range $(1, (\theta - 1))$;
- complex dynamic behaviors may occur, although a 3-cycle cannot exist.

Mitra, 2001:

- chaos may occur, at least when the parameter σ is quite high ($\sigma = 50$);
- The suf. cond.: the third iterate of the maximum is a point below the fixed point of the Romer regime (exactly the cond. for which the fix. p. has homoclinic trajectories).

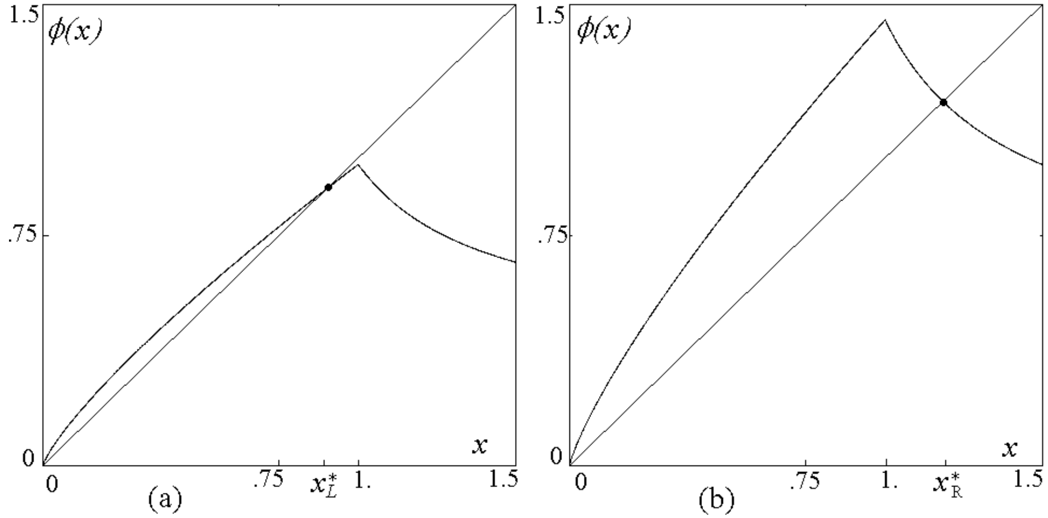
Mukherjy, 2005:

- the cond. for chaos may be satisfied also at lower values of σ, $\sigma = 22$;
- the transition to chaos may occur via the per.-doubling bif. sequence (while this is not possible).

Attacting fixed points, absorbing interval

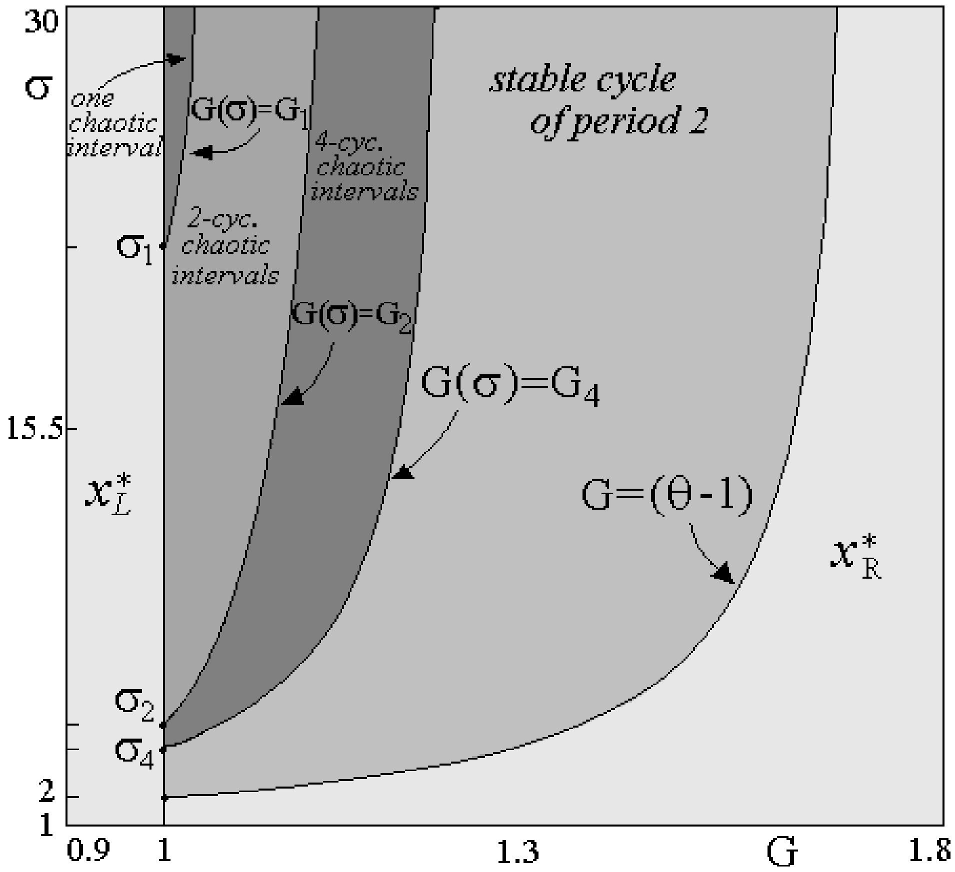
$f(x)$ has a unique fix. p. $x_L^* = G^\sigma$ which exists ($x < 1$) for $G < 1,$ and when it exists, it is always globally attracting.

For $G > (1-\theta)$ the fix. p. $x_R^* = 1 + \frac{G-1}{\theta}$ in the Romer regime is globally asymptotically stable ($x > 1$).



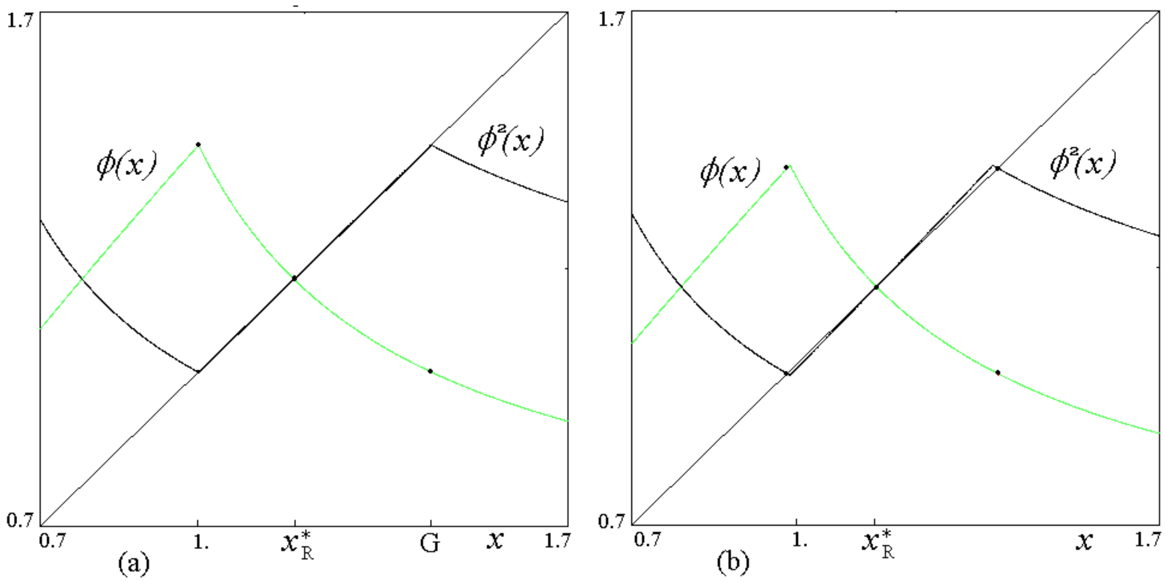
$\exists$ absorbing interval $[g(G), G] : \phi([g(G), G]) \subseteq [g(G), G].$

Two-dimensional bifurcation diagram in the (G, σ)-parameter plane



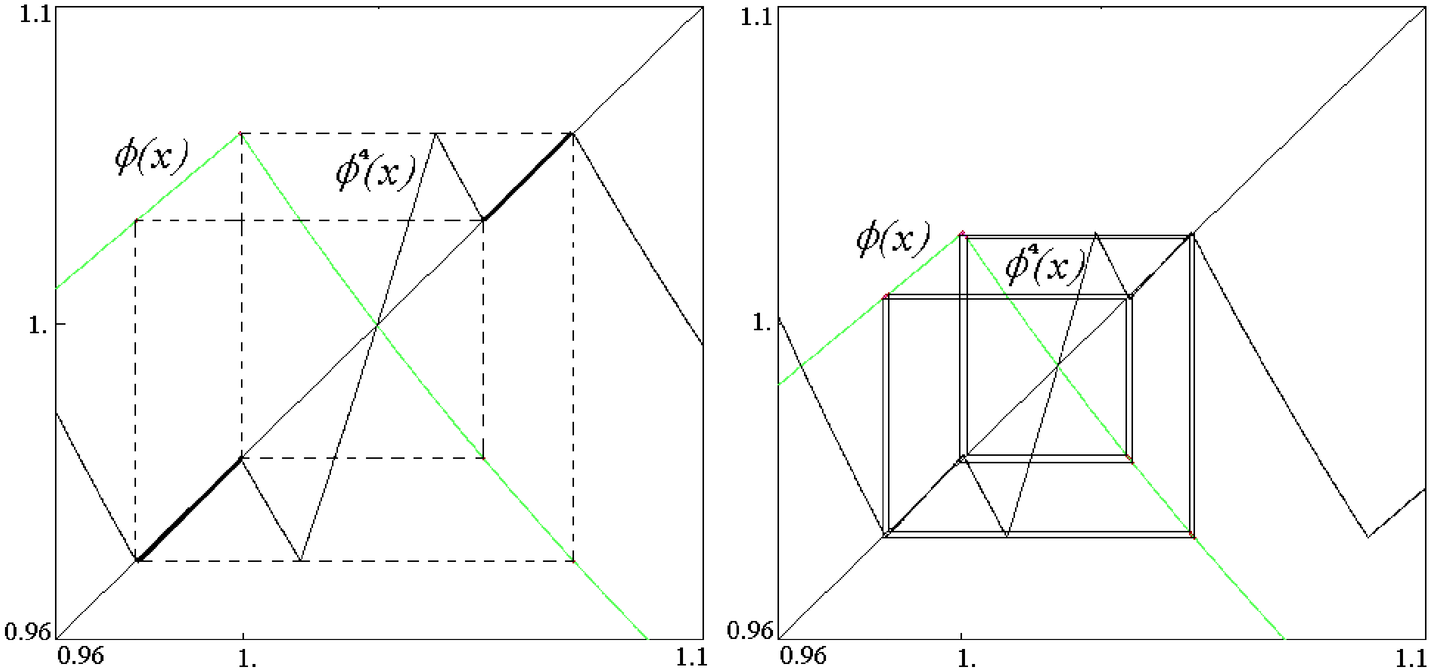
Cycle of period 2

Let $\sigma > 2$ be fixed and G decrease, starting from some $G > (\theta - 1)$ for which x_R^* is stable. Then the loose of stability of x_R^* occurs via a critical bifurcation: At $G = (\theta - 1)$ all the points of a segment are 2-cycles (in particular, $\{1, G\}$).



A unique 2-cycle exists, after the bifurcation, for $G < (\theta - 1)$, say $\{x_L, x_R\}$: $x_L < 1$ and $x_R > 1$.

The 2-cycle becomes unstable as G decreases reaching the value $G = G_4$.



(a) (b)

Proposition 1. The stability region of the 2-cycle for any fixed value $\sigma > 2$, is bounded by the curves of implicit equations $g^2(1) = 1$ (which corresponds explicitly to $G = (\theta - 1))$ and $g \circ f \circ g^2(1) = 1$ (implicit equation for $G(\sigma) = G_4)$.

Chaotic intervals

Cycles of period three cannot exist, but the chaotic regimes exists anyhow. The sufficient condition stated by Mitra can be enforced in terms of homoclinic trajectories: (Devaney 1987, Gardini 1994):

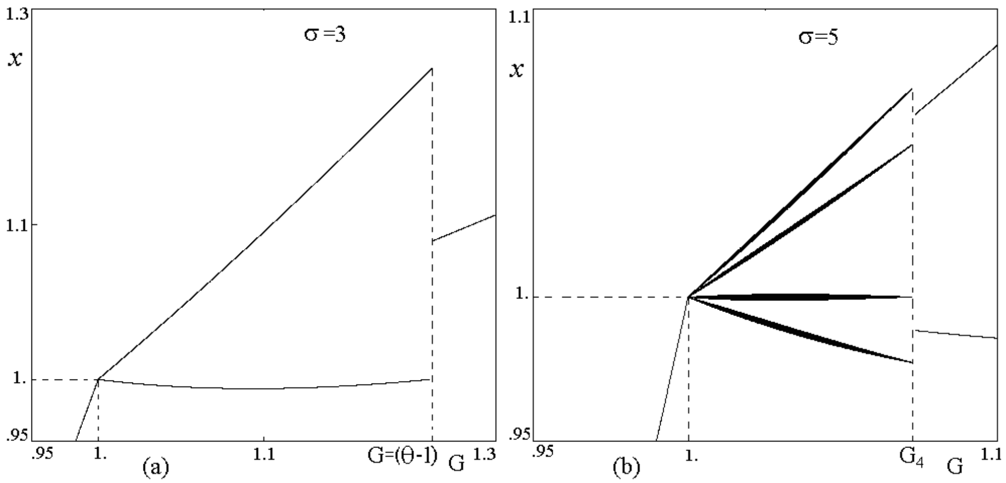
Proposition 2. Let m be the unique critical point of a continuous piecewise smooth unimodal map of an interval into itself, say $F : I \to I,\ F(m)$ maximum (resp. minimum), with a unique unstable fixed point $x^*,$ and a sequence of preimages of m tends to $x^*.$ Then the first homoclinic orbits (all critical) of the fixed point x^* occur when the critical point satisfies $F^3(m) = x^*$. When the critical point is a local maximum (resp. minimum) then for $F^3(m) < x^*$ (resp. $F^3(m) > x^*)$ infinitely many (noncritical) homoclinic orbits of the fixed point exist, and thus there is a closed invariant set $X \subseteq [F^2(m), F(m)]$ (resp. $X \subseteq [F(m), F^2(m)]$) on which the map is topologically conjugate to the shift automorphism, and thus F is chaotic, in the sense of topological chaos (with positive topological entropy).

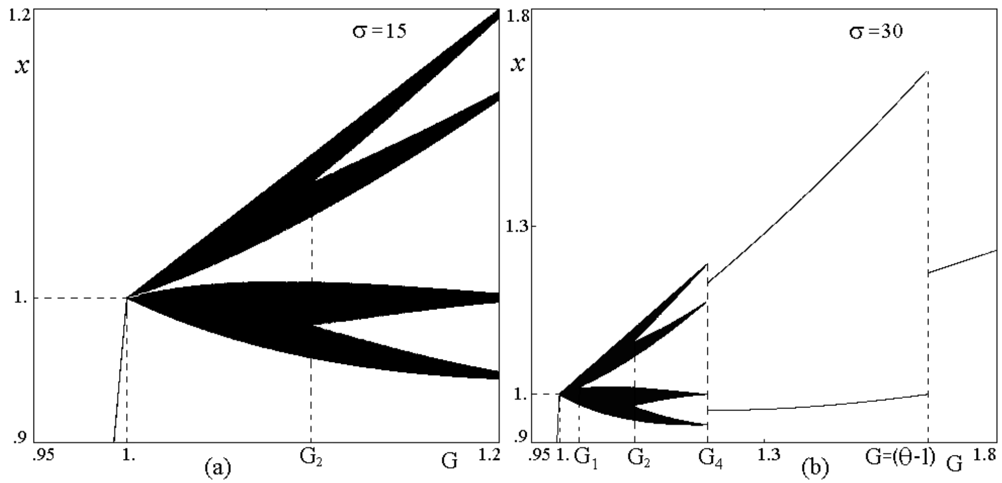
Proposition 3. For any fixed value $\sigma > 2$ when the fixed point and the 2-cycle of the map are unstable, the attractors are full measure chaotic intervals.

The bifurcation $I_4 \Rightarrow I_2$ is the homoclinic bifurcation of the repelling 2-cycle. The condition to detect this homoclinic bifurcation (Proposition 2 applied to ϕ^2) is $\phi^5(1) = x_R,$ which corresponds to $g^2 \circ f \circ g^2(1) = x_R.$

The bifurcation $I_2 \Rightarrow I_1$ occurs when the fix. p. in the Romer regime becomes homoclinic (by Proposition 2): $\phi^3(1) = x_R^*$ that corresponds to $f \circ g^2(1) = x_R^*,$ or more explicitly reads as follows:

$$G\left(\frac{G^2}{1+\theta(G-1)}\right)^{\left(1-\frac{1}{\sigma}\right)} = 1 + \frac{G-1}{\theta}.$$


σ =3
x
1.3
1.1
1.
.95
.95
1.
1.1
G=(θ-1)
G
1.3
(a)
σ=5
x
1.1
1.
.95
.95
1.
G4
G
1.1
(b)


σ =15
x
1.2
1.
.9
.95
1.
G2
G
1.2
(a)
σ =30
x
1.8
1.3
1.
.9
.95
1.
G1
G2
G4
1.3
G=(θ-1)
G
1.8
(b)

Border-collision bifurcation at $G = 1$

Theorem. The border-collision bifurcation of the fixed point $x^* = 1$ of the map ϕ, occurring at $G = 1$ for any $\sigma > 1$, gives rise to

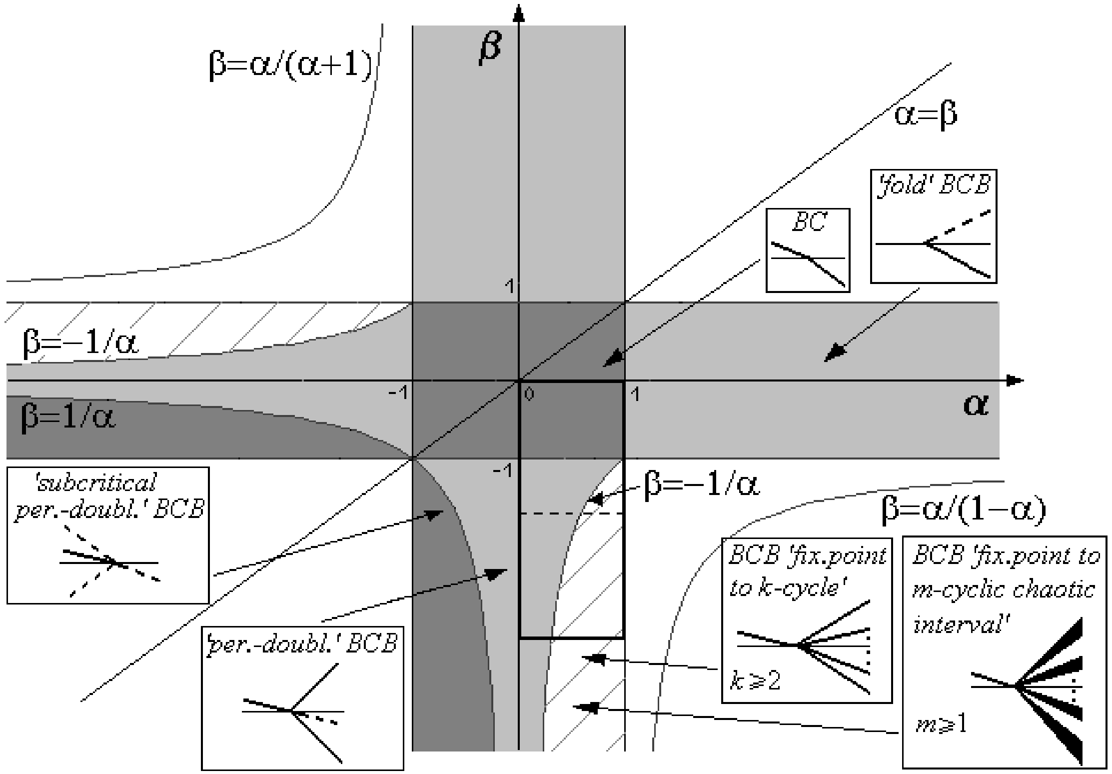
- an attracting fixed point x_R^* if $1 < \sigma < 2$;
- an attracting cycle of period 2 if $2 < \sigma < \sigma_4 \simeq 3.825$;
- attracting 4-cyclical chaotic intervals if $\sigma_4 < \sigma < \sigma_2 \simeq 6.123$;
- attracting 2-cyclical chaotic intervals if $\sigma_2 < \sigma < \sigma_1 \simeq 21.231$;
- an attracting chaotic interval if $\sigma > \sigma_1$.

Proof. The result of the border-collision bifurcation of the fixed point depends on the left and right side derivatives of $\phi(x)$ evaluated at $x = 1$ for $G = 1,$ here denoted α and $\beta,$ respectively:

$$\alpha = \lim_{x \to 1_-} \frac{d}{dx} \phi(x), \ \beta = \lim_{x \to 1_+} \frac{d}{dx} \phi(x).$$

The related normal form is given by the skew-tent map $\psi : y \mapsto \psi(y)$ defined by the function

$$\psi(y) = \begin{cases} \alpha y + \varepsilon, & y \leq 0, \\ \beta y + \varepsilon, & y \geq 0. \end{cases}$$



The coefficients of the normal form in terms of the parameter σ :

$\alpha = (1 - \frac{1}{\sigma}),\ \beta = (1 - \theta),\ (0 < \alpha < 1,\ 1 - e < \beta < 0)$

The border-collision curve B of $x^* = 1$ in terms of α and β : $\beta = 1 - \alpha^{\alpha/(\alpha-1)}$.

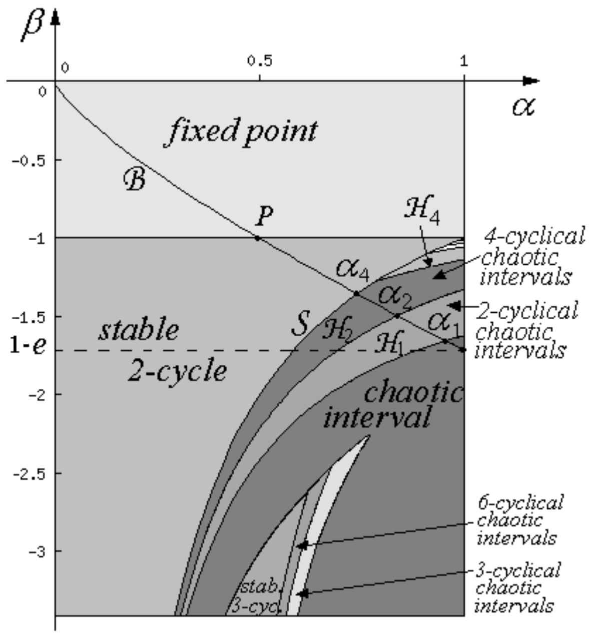
B intersects

1) the straight line $\beta = -1$ (critical flip bif. of y^*);

2) S : $\beta = -1/\alpha$, (critical flip bif. of the 2-cycle);

3) H_2 : $\alpha = (-1 - \sqrt{1 + 4\beta^4})/2\beta^3$, (hom. bif. of the 2-cycle);

4) H_1 : $\beta = (-1 + \sqrt{1 + 4\alpha^2})/2\alpha$, (hom. bif. of y^*).



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