Discontinuous maps in a duopoly model with capacity limit plants

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Presentation Plan

1. Introduction
2. The model
   - Demand and cost functions
   - Marginal Cost and Marginal Revenue
   - The reaction curve and the dynamic system
3. The map
   - Step 1: The one discontinuity subcase
   - Step 2: Periodicity of the new tongues
   - Step 3: Position of the periodic cycles
   - Step 4: Starting and closing conditions
4. References

Discontinuous Maps

- Recently, in many applied works, piecewise linear or piecewise smooth maps (both continuous and discontinuous) are used;
- We are interested in discontinuous maps with one or more discontinuity points;
- This class of maps is characterized by the so called *border-collision bifurcations*, i.e. contacts between an invariant set of a map with the border of its region of definition;

Border-collision bifurcations in the literature

- Even if it is still mainly unknown, Leonov (1959,1962) has given a recurrence relation to find the analytic expression of families of bifurcations occurring in a one-dimensional piecewise linear map with one discontinuity point;
- Mira (1978,1987), Maistrenko et al. (1993,1995,1998) and Feigen[1] also analyzed this kind of bifurcation;
- The name "*border-collision bifurcations*" is due to Nusse and Yorke (1992,1995);
- The most recent contributes are those of Avrutin and Schanz (2006), Avrutin et al. (2006) and Banerjee et al. (2000a,b).

[1] The works of Feigen have been recently re-published in Di Bernardo et al. (1999)

Piecewise linear or smooth maps in Economics

- Since the pioneering works by Richard Day (see for example Day (1982,1994)) focussed on piecewise linear maps, several applications to Economics ultimately lead to models which are described by piecewise linear or piecewise smooth maps both continuous and discontinuous;

- Examples of piecewise linear (or smooth) maps that are continuous are: Hommes (1991,1995), Hommes and Nusse (1991), Hommes et al. (1995), Gallegati et al. (2003), Puu and Sushko (2002,2006), Sushko et al. (2003,2005,2006), Gardini et al. (2006a, 2006b), Gardini et al. (2008);

Piecewise linear or smooth maps in Economics

- In Puu (2007), Puu et al. (2002,2005) and Sushko et al. (2004) the evolution of the state variable is given by discontinuous maps with one or more discontinuity point;
- A discontinuous map arises also in the Bohm and Kaas (2000) version of the standard one-sector Kaldor-Pasinetti model of economic growth;
- In particular, our study moves from the emergence of a map with more than one discontinuity points in a duopoly model where firms are endowed with capacity limit plants (Tramontana et al. 2008).

- The demand function is **isoelastic**

$$P(Q) = \frac{1}{Q} = \frac{1}{x+y} \tag{1}$$

CES production function ⇒ Cost function

$$x^{-\rho} = k_{x,i}^{-\rho} + l^{-\rho}$$

$$\text{Se } \rho = 1 \qquad \Rightarrow \qquad l = \frac{k_{x,i}x}{k_{x,i}+x} \tag{2}$$

Using (2) in the cost function $C_i = ck_i + rl$ we obtain:

$$C_i(x) = rk_{x,i} + \omega \frac{k_{x,i}x}{k_{x,i}-x} \tag{3}$$

The parameter $k_{x,i}$ represents the capacity limit, in fact:

$$\lim_{x \to k_{x,i}} C = \infty$$

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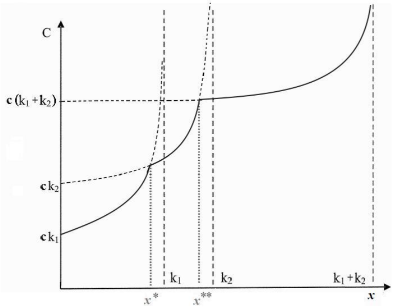
The cost function with two plants



- Every firms has three alternatives: to use one plant or to combine them
- Combining both the plants, firms follow the principle of the same marginal cost

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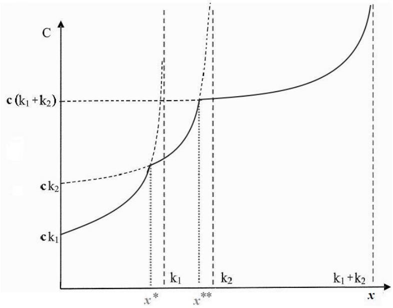

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- Combining both the plants, firms follow the principle of the same marginal cost

We obtain the marginal cost:

$$MC_x = \frac{dC_x}{dx} = w\frac{k_{i,x}^2}{(k_{i,x} - x)^2} \qquad \text{for } i = 1, 2, 3 \qquad (4)$$

where $k_{i,x}$ represents the technology adopted by the firm.

Using the isoelastic demand function (1), we obtain the total revenue and marginal revenue functions:

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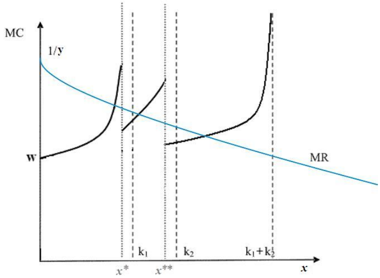
Marginal Costs and Marginal Revenues



- The branches of the cost functions are fixed by the available technologies, but the marginal revenue curve can change shape and position at each time period.
- It is possible to find 3 points where $RM_x = CM_x$

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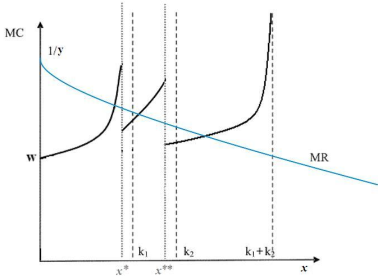

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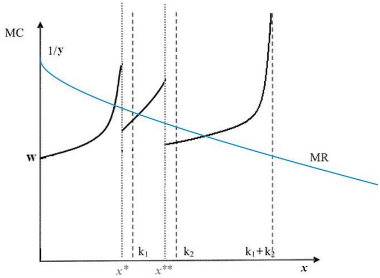

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Points of local and global maximum profit

- The production level to which corresponds the intersection between marginal costs and marginal revenues for the technology $i-$th is given by:

$$x^* = k_{x,i} \frac{\sqrt{\frac{y}{w}} - y}{k_{x,i} + \sqrt{\frac{y}{w}}} \qquad \text{for } i = 1, 2, 3 \tag{6}$$

- We are not sure that the three intersections are all on the *active* branches of the marginal cost curve
- If more than one local maximum points coexist, then the firm chooses the one to which corresponds the maximum profit (If it is higher than the fixed costs)

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Reaction curves

Let us consider the case in which the firms can use the same two technologies (*symmetric case*). In this case:

$$k_{x,i} = k_{y,i} = k_i \qquad \text{for } i = 1, 2, 3$$

The firm does not know what will be the competitor's choice. We suppose the adoption of static (naive) expectations:

$$y^e_{t+1} = y_t$$

We obtain this reaction curve:

$$x_{t+1} = r(y_t) \Rightarrow \begin{cases} k_i \frac{\sqrt{\frac{y_t}{w}} - y_t}{k_i + \sqrt{\frac{y_t}{w}}} & \text{if } y_t < \frac{1}{w} \\ 0 & \text{if } y_t \geq \frac{1}{w} \end{cases} \tag{7}$$

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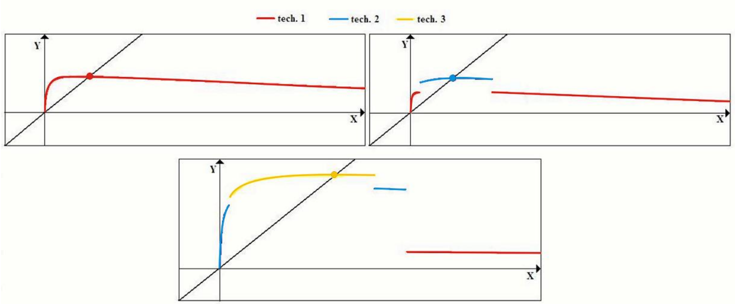
Reaction curve (examples)



- The reaction curve could be formed *branches* corresponding to all the technological alternatives
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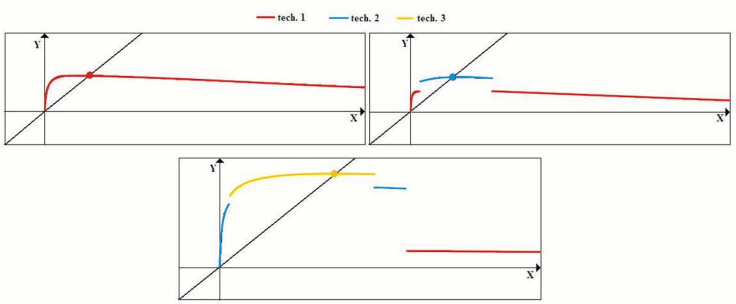

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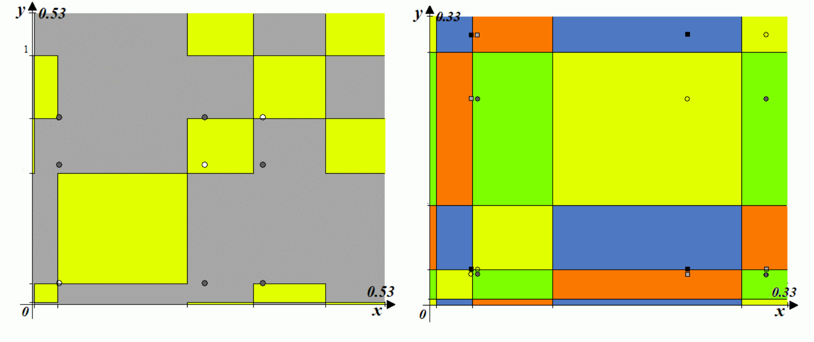
Multistability



- Multistability is present for almost all parameters' configurations

The simplest two discontinuities map

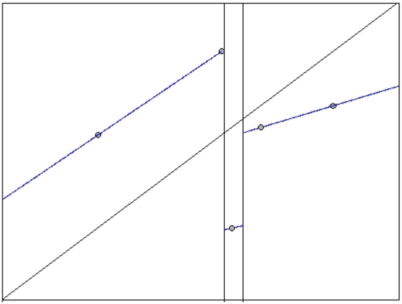
As a first step to understand maps with more than one discontinuity points, we start analyzing the linear two discontinuities case:

$$x' = T(x) = \begin{cases} f_1(x) = ax + 1 \\ f_2(x) = bx + p \\ f_3(x) = cx + q \end{cases} \tag{8}$$

with $a, b, c \in (0,1)$

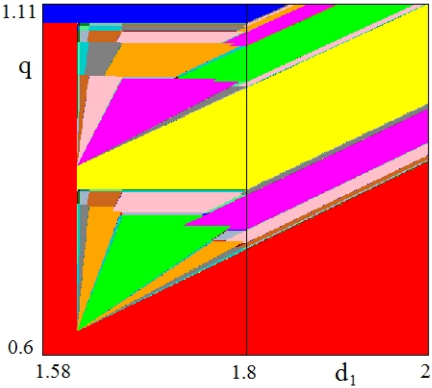
Absence of unstable cycles

- All the possible cycles of period $k > 1$ are stable because the slopes are all higher than 1 in absolute value;
- If the cycle have l points on the left side of the discontinuity point, r points on the right side and $k - l - r$ in the middle, then the eigenvalue is given by $a^l b^{(k-l-r)} c^r$ which by assumption is positive and less than 1;



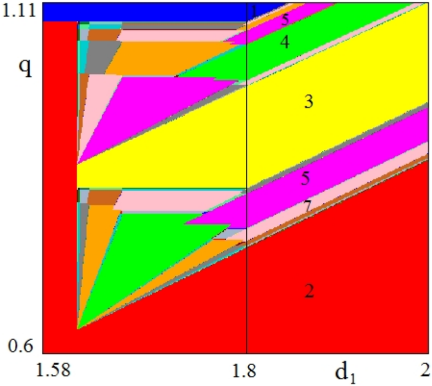
Interesting situation

- An interesting situation is obtained using $a = 0.9, b = 0.3, c = 0.4, p = 0.2, d_2 = 1.8$ and let the parameters d_1 and q vary:



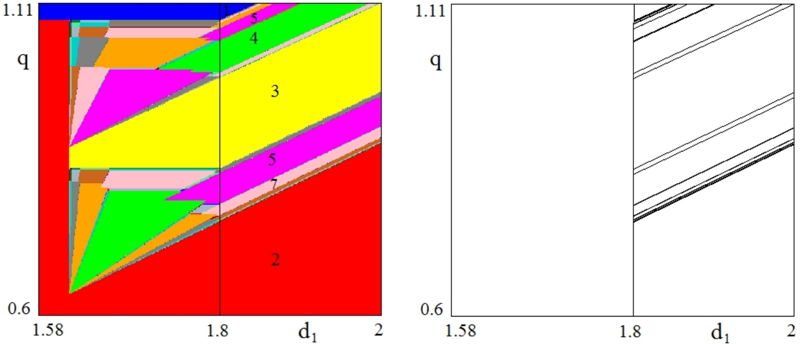
Interesting situation

- For $d_1 = d_2$ the middle piece disappears and for $d_1 > d_2$ the map reduces to a one discontinuity map:

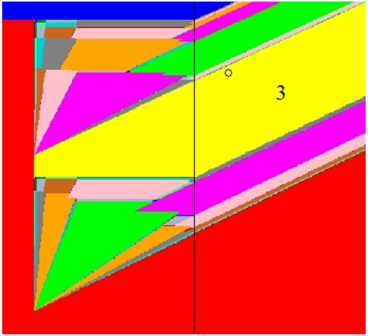


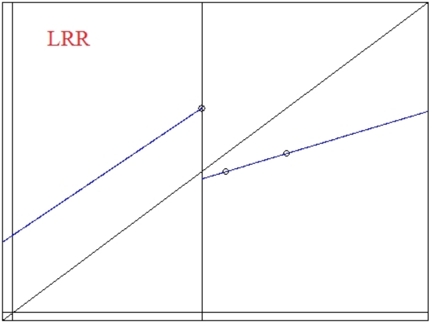
Interesting situation

- The direct sum and the boxes in files structure permit us to explain the periodic tongues and the bifurcation curves:



- Let us consider the main tongue with one point on the left and two on the right side
- For $d_1 > d_2$ the starting condition is LRR applied to the d_1



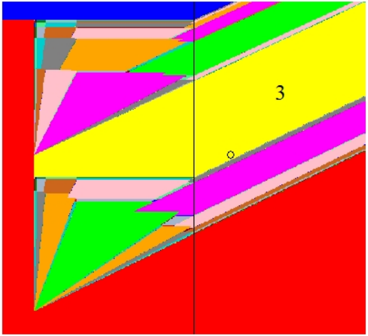


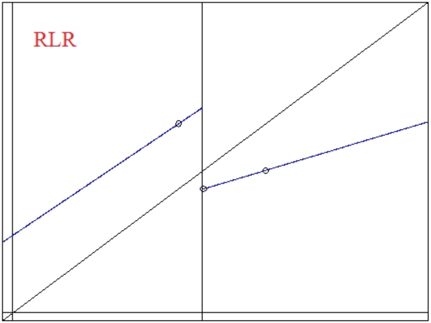
- The analytical expression of the curve is obtained in this way:

$$\begin{array}{ll} & d_1 \\ L & ad_1 + 1 \\ R & c(ad_1+1)+q \\ R & c^2(ad_1+1)+cq+q \end{array} \tag{9}$$

$$d_1 = c^2(ad_1+1)+cq+q \quad \Rightarrow \quad q = \frac{d_1(1-ac^2)-c^2}{1+c} \tag{10}$$

- Consider now the closing condition which is RLR applied again to the d_1



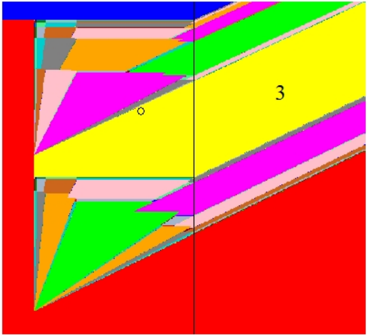


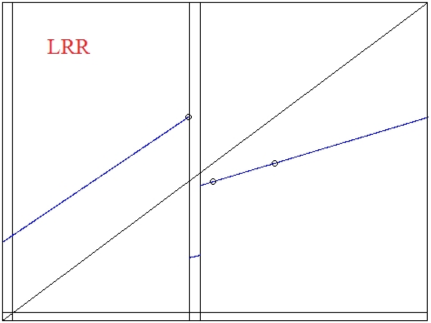
- The analytical expression of the curve is obtained in this way:

$$\begin{array}{ll} & d_1 \\ R & cd_1 + q \\ L & a(cd_1 + q) + 1 \\ R & ac(cd_1 + q) + c + q \end{array} \tag{11}$$

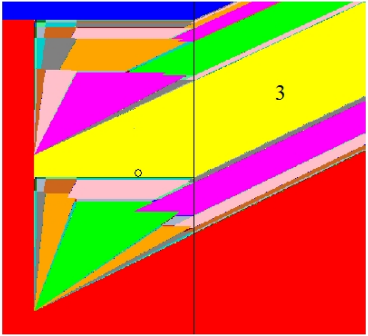
$$d_1 = ac(cd_1 + q) + c + q \quad \Rightarrow \quad q = \frac{d_1(1 - ac^2) - c}{1 + ac} \tag{12}$$

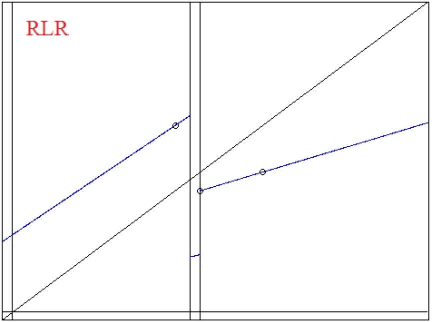
- For $d_1 < d_2$ the starting condition is the same, i.e. LRR applied to the d_1





- The closing condition is now RLR applied to $\overline{d_2}$





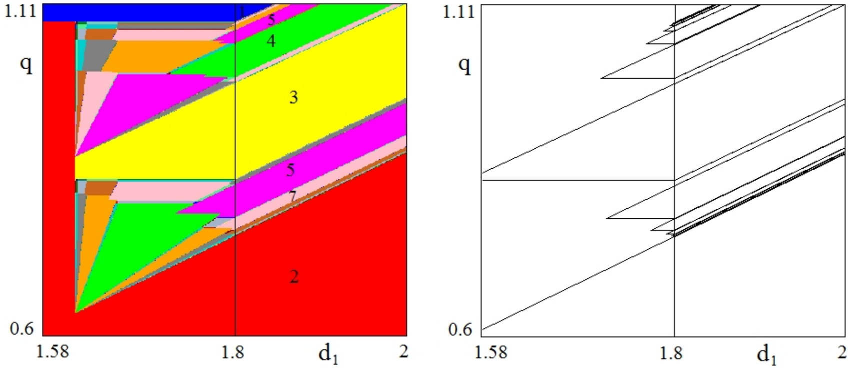
- The analytical expression of the curve does not depend on d_1, so in the plane (d_1, q) is represented by an horizontal line:

$$\begin{array}{ll} & \overline{d_2} \\ R & c\overline{d_2} + q \\ L & a(c\overline{d_2} + q) + 1 \\ R & ac(c\overline{d_2} + q) + c + q \end{array} \tag{13}$$

$$d_1 = ac(c\overline{d_2} + q) + c + q \quad \Rightarrow \quad q = \frac{\overline{d_2}(1 - ac^2) - c}{1 + ac} \tag{14}$$

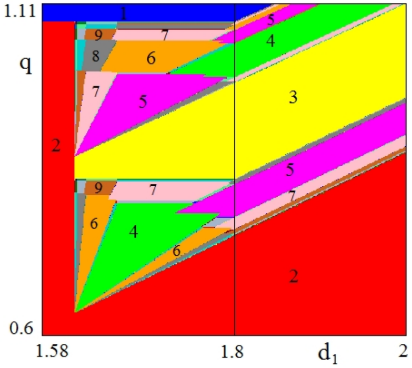
Situation after step 1

- We are now able to explain these portions of the bifurcation curves:



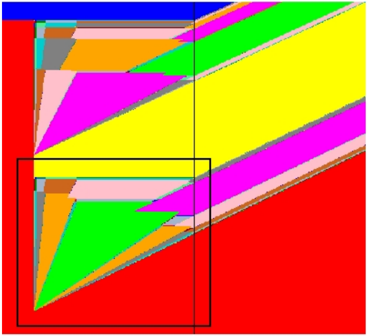
The direct sum

The direct sum still works...



Boxes in files

Paying attention to the symbolic order the boxes in files structure remains also with two discontinuity points



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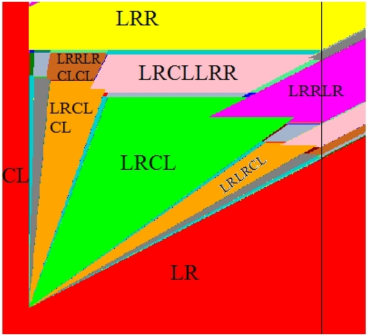
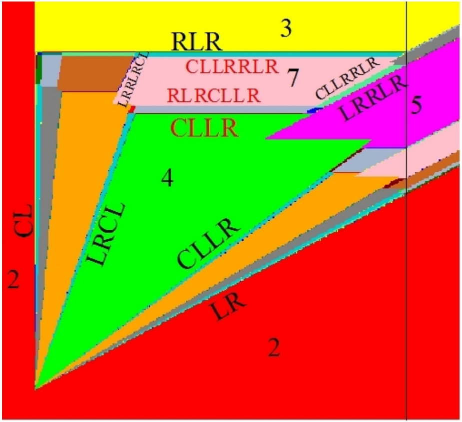
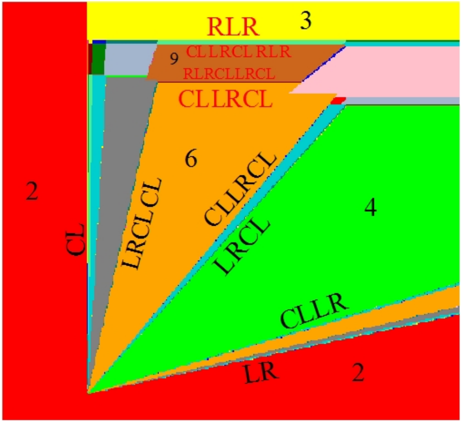


- In order to obtain the bifurcation curves, we need to know the right sequence of maps and apply them on d_1 or d_2
- Let us call neighbours of a periodicity tongue the two tongues whose periodicity gives (by direct sum) the periodicity of the tongue we consider
- The rules are:
    - ◇ The right (resp. left) bif.curve of the left (resp. right) neighbour gives the order for the right (resp. left) bif. curve of the tongue;
    - ◇ The lower (resp. higher) bif.curve of the high (resp. low) neighbour gives the order for the lower (resp. higher) bif. curve of the tongue.

Examples


3
RLR
CLLRRLR
7
RLRCLLR
LRRLRCL
CLLRRLR
LRRLR
5
CLLR
4
LRCL
CLLR
CL
2
LR
2

Examples


RLR
3
9
CLLRCLRLR
RLRCLLRCL
CLLRCL
6
2
CL
LRCLCL
CLLRCL
LRCL
4
CLLR
LR
2

- We can subdivide curves in families
- An example is given by the bif. curves of the form $LR(CL)^n$

$$
\begin{array}{ll}
 & d_1 \\
L: & ad_1+1 \\
R: & c(ad_1+1)+q \\
C: & bc(ad_1+1)+bq+p \\
L: & abc(ad_1+1)+abq+ap+1 \\
C: & ab^2c(ad_1+1)+ab^2q+abp+b+p \\
L: & a^2b^2c(ad_1+1)+a^2b^2q+a^2bp+ab+ap+1 \\
\dots &
\end{array}
\tag{15}
$$

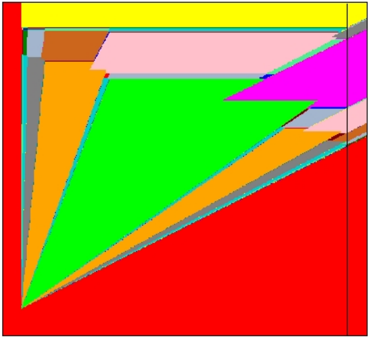
after $2n$ applications we have:

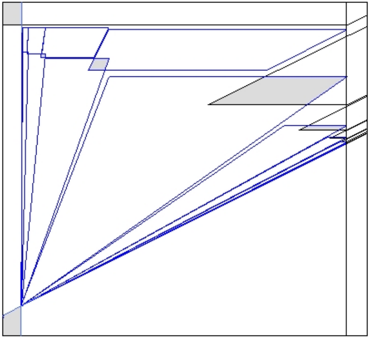
$$
\phi(n)=(ab)^{n-1}[c(ad_1+1)+q]+\sum_{k=0}^{n-2}(ab)^n+\frac{p}{b}\sum_{k=1}^{n-1}(ab)^k \tag{16}
$$

- The curves are obtained with the condition $d_1 = \phi(n)$ for $n = 1, 2, 3...$

$$d_1 = \frac{(ab)^{n-1}(c+q) + \frac{1-(ab)^{n-1}}{1-ab} + \frac{p}{b}\frac{ab-(ab)^n}{1-ab}}{1-(ab)^{n-1}ac} \quad (17)$$

- Now, following an analogous mechanism for the other curves...





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