Updating wealth in a dynamic asset pricing model

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Introduction

- **Traditional approach in Economics and Finance**

1) <u>Representative agent</u>

Agents are homogeneous

2) <u>Rational Expectation hypothesis</u>

Agents are rational, i.e. they have perfect knowledge about the economic environment and use it to form expectations

$$\Rightarrow X_{t+1}^{e} = E(X_{t+1} \,|\, F_t) \equiv E_t(X_{t+1})$$

- **Alternative approach to rational expectations**

1) Shift from a representative agent to heterogeneous agent systems

(shift from simple analytically tractable models to a more complicated framework)

2) Shift from full rationality to *<u>bounded rationality</u>*

Related literature

Constant Absolute Risk Aversion **(CARA)** utility function:

Asset Pricing Model Under Heterogeneous Expectations

- *Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model*

Brock and Hommes (1998)

Constant Relative Risk Aversion **(CRRA)** utility function:

Asset Pricing Model with Heterogeneous Expectations and *Wealth Dynamics*

- *An Adaptive Model on Asset Pricing and Wealth Dynamics with Heterogeneous Trading Strategies*

Chiarella and He (2002)

- *Asset price and wealth dynamics in a financial market with heterogeneous agents*

Chiarella, Dieci and Gardini (2006)

The Model: ingredients

✓ An asset pricing model with two alternatives

- risk free asset: $R=1+r_f$ gross return

- risky asset: p_t price per share of the risky asset, $\{y_t\}$ IID dividend process

✓ Dynamics of wealth

ALL AGENTS BELONGING TO THE SAME GROUP AGREE TO SHARE THEIR WEALTH WHENEVER AN AGENT GETS IN THE GROUP (OR LEAVES IT)

$\overline{w}_{h,t}$ is the average wealth of agent type h at time t given by

the total wealth of group h in the fraction of agents belonging to this group

Dynamics of wealth of investor of type h

$$\boxed{W_{h,t+1} = (1 - z_{h,t})\overline{w}_{h,t} R + z_{h,t}\overline{w}_{h,t}(1 + \rho_{t+1})}$$

$$\rho_t = \frac{p_t + y_t - p_{t-1}}{p_{t-1}}$$

return of the risky asset

Fraction of wealth invested in the risky asset:

$$\max_{z_{h,t}} E_{h,t}[u_h(W_{h,t+1})] \Rightarrow z_{h,t} = \frac{E_{h,t}[\rho_{t+1} - r]}{\lambda_h Var_{h,t}[\rho_{t+1} - r]}$$

optimal (approximate) solution under the assumption of CRRA utility function (Chiarella and He (2001))

✓ updated fractions n_{ht} formed on the bases of discrete choice probability

$$n_{h,t} = \frac{\exp[\beta(\phi_{h,t-1} - C_h)]}{Z_t}, Z_t = \sum_h \exp[\beta(\phi_{h,t-1} - C_h)]$$

$$\beta \text{ intensity of choice, } C_h \text{ information costs}$$

Adaptive Belief System Different types of agents have different beliefs about future variables and prediction selection is based upon a ***performance measure***

Performance measure

At time t+1 agent h measures his realized performance and chooses whether to stay in his group or to switch to another one

We measure the past performance as the personal wealth coming from the investment in the risky asset with respect to the average wealth:

$$\phi_{h,t} = (\rho_{t+1} - r) z_{h,t}$$

Market populated by two groups of agents: h=1,2

At each time agents can move from one group to another while both movements are not simultaneously possible

Define the difference in fractions of agents of type h from time t to time t+1

$$\Delta n_{h,t+1} = n_{h,t+1} - n_{h,t} \Rightarrow \Delta n_{1,t+1} = -\Delta n_{2,t+1}$$

Define the difference in fractions of agents at time t

$$m_t = n_{1,t} - n_{2,t}$$

$$\Delta n_{1,t+1} \geq 0 \ (i.e.\ m_{t+1} \geq m_t)$$

$$\Rightarrow \ \Delta n_{1,t+1} \ fraction\ of\ agents\ moves\ from\ group\ 2\ to\ group\ 1$$

$$\Delta n_{1,t+1} < 0 \ (i.e.\ m_{t+1} < m_t)$$

$$\Rightarrow \ \Delta n_{1,t+1} \ fraction\ of\ agents\ moves\ from\ group\ 1\ to\ group\ 2$$

Dynamics of wealth of group h

DEFINE the ***"weighted average wealth of group h"*** $\widetilde{W}_{h,t} = \overline{w}_{h,t} n_{h,t}$

which represents the share of wealth that group h produces to the total wealth

Wealth of group 1

$$\tilde{W}_{1,t+1} = \begin{cases} n_{1,t}\left(W_{1,t+1} - W_{2,t+1}\right) + n_{1,t+1} W_{2,t+1} & \text{if} \quad m_{t+1} \geq m_t \\ n_{1,t+1} W_{1,t+1} \quad \text{if} \quad m_{t+1} < m_t \end{cases}$$

Wealth of group 2

$$\tilde{W}_{2,t+1} = \begin{cases} n_{2,t+1} W_{2,t+1} \quad \text{if} \quad m_{t+1} \geq m_t \\ n_{2,t}\left(W_{2,t+1} - W_{1,t+1}\right) + n_{2,t+1} W_{1,t+1} & \text{if} \quad m_{t+1} < m_t \end{cases}$$

where $W_{h,t+1} = (1 - z_{h,t})\overline{w}_{h,t} R + z_{h,t}\overline{w}_{h,t}(1 + \rho_{t+1})$

Price setting rule of the market maker

$$p_{t+1} - p_t = E_{t,f}(p^\star_{t+1} - p^\star_t) + p_t H_t(N^D_t - N^S_t).$$

The total demand is:

$$N^D_t = \frac{n_{1,t} z_{1,t} \bar{w}_{1,t} + n_{2,t} z_{2,t} \bar{w}_{2,t}}{p_t}.$$

Supply is fixed, for simplicity:

$$N_t^S = 0, \forall t$$

Under the assumption of i.i.d. dividend process:

$$p^* = \frac{E[y_{t+1}]}{r} = \frac{\bar{y}}{r}$$

and assuming:

$$H_t(\bullet) = \alpha \frac{p_t}{\widetilde{W}_{1,t} + \widetilde{W}_{2,t}}(\bullet), w_{h,t} = \frac{\widetilde{W}_{h,t}}{\widetilde{W}_{1,t} + \widetilde{W}_{2,t}}, w_t = w_{1,t} - w_{2,t}$$

we obtain:

$$\boxed{\frac{p_{t+1} - p_t}{p_t} = \alpha \left(z_{1,t} \frac{1 + w_t}{2} + z_{2,t} \frac{1 - w_t}{2} \right)}$$

Therefore:

$$p_{t+1} = \left[\frac{\alpha}{2}\left(z_{1,t} + z_{2,t} + (z_{1,t} - z_{2,t})w_t\right) + 1\right] p_t \qquad (5.12)$$

$$m_{t+1} = tanh\{\beta/2[(z_{1,t} - z_{2,t})[\alpha/2(z_{1,t} + z_{2,t} + (z_{1,t} - z_{2,t})w_t) + \frac{\bar{y}}{p_t} - r] - C]\} \qquad (5.13)$$

and

$$w_{t+1} = \begin{cases} \frac{F_1}{G} + 1 & if \;\; m_{t+1} \geq m_t \\ \\ \frac{F_2}{G} - 1 & if \;\; m_{t+1} < m_t \end{cases} \qquad (5.14)$$

With:

$$F_1 = \frac{-4(1-w_t)\{R+z_{2,t}[\frac{\alpha}{2}(z_{1,t}+z_{2,t}+(z_{1,t}-z_{2,t})w_t)+\frac{\bar{y}}{p_t}-r]\}}{(1-m_t)[exp\{\beta[(z_{1,t}-z_{2,t})[\frac{\alpha}{2}(z_{1,t}+z_{2,t}+(z_{1,t}-z_{2,t})w_t)+\frac{\bar{y}}{p_t}-r]-C]\}+1]},$$

$$F_2 = \frac{4(1+w_t)\{R+z_{1,t}[\frac{\alpha}{2}(z_{1,t}+z_{2,t}+(z_{1,t}-z_{2,t})w_t)+\frac{\bar{y}}{p_t}-r]\}}{(1+m_t)[exp\{-\beta[(z_{1,t}-z_{2,t})[\frac{\alpha}{2}(z_{1,t}+z_{2,t}+(z_{1,t}-z_{2,t})w_t)+\frac{\bar{y}}{p_t}-r]-C]\}+1]},$$

$$G = 2R+[z_{1,t}+z_{2,t}+(z_{1,t}-z_{2,t})w_t]\cdot[\frac{\alpha}{2}(z_{1,t}+z_{2,t}+(z_{1,t}-z_{2,t})w_t)+\frac{\bar{y}}{p_t}-r].$$

Two important belief types

Fundamentalists versus chartists

- ***Fundamentalists***

$$E_{1,t}(p_{t+1}) = p^*$$ Prices return to their fundamental value

- ***Chartists or technical analysts***

$$E_{2,t}(p_{t+1}) = ap_t, \quad a \in \mathbb{R}$$ Prediction selection based upon simple trading rules

Resulting three-dimensional dynamical system

$$\boxed{x_t = \frac{p^*}{p_t},\ w_{h,t} = \frac{\widetilde{W}_{h,t}}{\widetilde{W}_{1,t} + \widetilde{W}_{2,t}},\ w_t = w_{1,t} - w_{2,t}}$$

$$x_{t+1} = f_1(x_t, w_t) = \frac{x_t}{\frac{\alpha}{2\lambda\sigma^2}\left(x_t - 2 + a + 2r(x_t - 1) + (x_t - a)w_t\right) + 1} \tag{5.24}$$

$$m_{t+1} = f_2(x_t, w_t)$$
$$= tanh\left\{\beta/2\left[\tfrac{1}{\lambda\sigma^2}(x_t - a)\left[\tfrac{\alpha}{2\lambda\sigma^2}(x_t - 2 + a + 2r(x_t - 1) + (x_t - a)w_t) + r(x_t - 1)\right] - C\right]\right\} \tag{5.25}$$

$$w_{t+1} = f_3(x_t, m_t, w_t) = \begin{cases} \frac{F_1}{G} + 1 & if \quad m_{t+1} \geq m_t \\ \\ \frac{F_2}{G} - 1 & if \quad m_{t+1} < m_t \end{cases} \tag{5.26}$$

$C = C_1 - C_2$ is the difference between information costs

Where:

$$F_1 = \frac{-4(1-w_t)\{R+\frac{1}{\lambda\sigma^2}[a-1+r(x_t-1)][\frac{\alpha}{2\lambda\sigma^2}(x_t-2+a+2r(x_t-1)+(x_t-a)w_t)+r(x_t-1)]\}}{(1-m_t)[exp\{\beta[\frac{1}{\lambda\sigma^2}(x_t-a)[\frac{\alpha}{2\lambda\sigma^2}(x_t-2+a+2r(x_t-1)+(x_t-a)w_t)+r(x_t-1)]-C]\}+1]},$$

$$F_2 = \frac{4(1+w_t)\{R+\frac{1}{\lambda\sigma^2}(1+r)(x_t-1)[\frac{\alpha}{2\lambda\sigma^2}(x_t-2+a+2r(x_t-1)+(x_t-a)w_t)+r(x_t-1)]\}}{(1+m_t)[exp\{-\beta[\frac{1}{\lambda\sigma^2}(x_t-a)[\frac{\alpha}{2\lambda\sigma^2}(x_t-2+a+2r(x_t-1)+(x_t-a)w_t)+r(x_t-1)]-C]\}+1]},$$

$$G = 2R+\frac{1}{\lambda\sigma^2}[x_t-2+a+2r(x_t-1)+(x_t-a)w_t]\cdot[\frac{\alpha}{2\lambda\sigma^2}(x_t-2+a+2r(x_t-1)+(x_t-a)w_t)+r(x_t-1)].$$

Steady states

The map has two types of steady states:

- ***fundamental steady state*** characterized by the price being at the fundamental value
- ***non fundamental steady states*** which coexist with the fundamental one

Fundamental steady state E_f

$$x_f = 1,\ m_f = \tanh\left\{-\frac{\beta C}{2}\right\},\ w_f = 1$$

Non fundamental steady states $E_{nf1},\ E_{nf2},\ E_{nf3},\ E_{nf4}$

$$x^1{}_{nf} = \frac{1-a}{r}+1,\ m^1{}_{nf} = \tanh\left\{-\frac{\beta}{2}\left[\frac{1}{\lambda\sigma^2}\frac{1+r}{r}(1-a)^2 - C\right]\right\},\ w^1{}_{nf} = -1$$

$$x^2{}_{nf} = 0,\ m^2{}_{nf} = \tanh\left\{-\frac{\beta}{2}\left[\frac{a}{\lambda\sigma^2}\left(\frac{\alpha}{\lambda\sigma^2}(1+r)+r\right)-C\right]\right\},\ w^2{}_{nf} = 1$$

$$x^3{}_{nf} = 0,\ m^3{}_{nf} = \tanh\left\{-\frac{\beta}{2}\left[\frac{a}{\lambda\sigma^2}\left(\frac{\alpha}{\lambda\sigma^2}(1+r-a)+r\right)-C\right]\right\},\ w^3{}_{nf} = -1$$

$$x^4{}_{nf} = 0,\ m^4{}_{nf} = \tanh\left\{-\frac{\beta}{2}C\right\},\ w^4{}_{nf} = 1-2\frac{r+1}{a}-2r\frac{\lambda\sigma^2}{\alpha a}$$

Trapping regions and stability analysis

$$\boxed{X : T(X) \subseteq X}$$

Proposition 5.4.1. *There exists* $\bar{a} \in \mathbb{R}$ *such that* $\forall a, \alpha, r, \lambda, \sigma^2$ *with* $a > \bar{a}$ *and* $\frac{\alpha(1+r)}{\lambda\sigma^2} \leq 1$, *the set* $X = \{(x_t, m_t, w_t) : x_t \geq 1, w_t = 1\}$ *is trapping for any initial condition* (x_0, m_0, w_0) *with* $x_0 \geq 1$ *and* $m_0 = -1 + \epsilon$ $(\epsilon \geq 0$ *small enough).*

The ***fundamental steady state*** is ***locally asymptotically stable*** for high values of *a* and for any initial condition with *x* greater than or equal to one and m small enough

Proposition 5.4.2. *For any* $\alpha, r, \lambda, \sigma^2$, *there exists* $\bar{a} = \frac{\frac{\alpha}{\lambda\sigma^2} + r(\frac{\alpha}{\lambda\sigma^2}+1)}{\frac{\alpha}{\lambda\sigma^2} - r(\frac{\alpha}{\lambda\sigma^2}+1)}$ *such that for* $\frac{\alpha}{\lambda\sigma^2} > \frac{r}{1-r}$ *and* $a > \bar{a}$, *the set* $Y = \{(x_t, m_t, w_t) : w_t = -1\}$ *is trapping for any initial condition* (x_0, m_0, w_0) *with* $m_0 = 1 - \epsilon$ $(\epsilon \geq 0$ *small enough).*

$$E_{nf1}$$

$$x^1_{nf} = \frac{1-a}{r} + 1,\; m^1_{nf} = \tanh\left\{-\frac{\beta}{2}\left[\frac{1}{\lambda\sigma^2}\frac{1+r}{r}(1-a)^2 - C\right]\right\},\; w^1_{nf} = -1$$

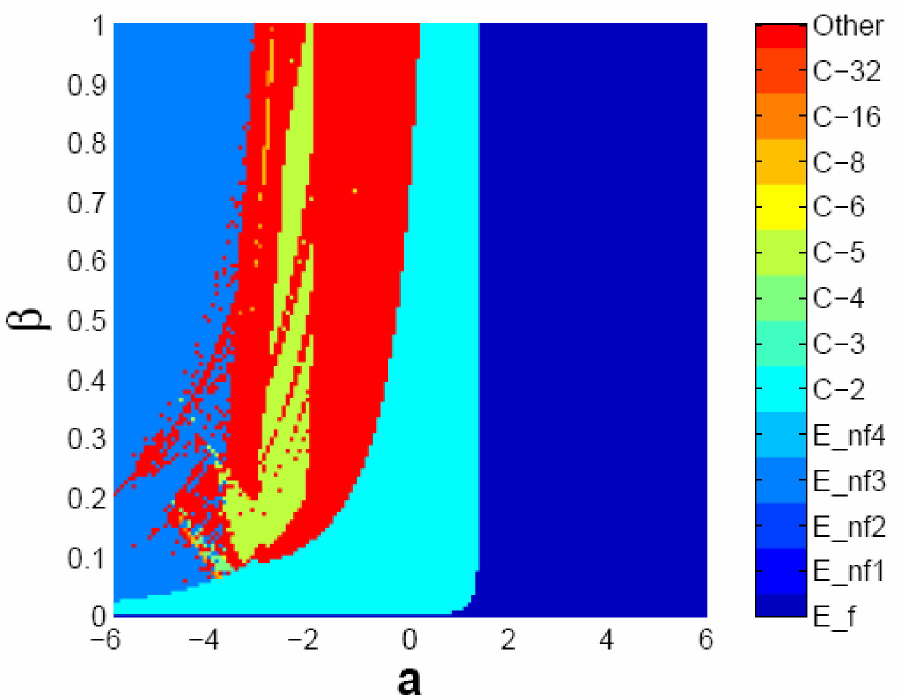
does exist if a < 1+r (or a=1+r) and it is ***locally unstable***

$$E_{nf3}$$

$$x^3_{nf} = 0,\; m^3_{nf} = \tanh\left\{-\frac{\beta}{2}\left[\frac{a}{\lambda\sigma^2}\left(\frac{\alpha}{\lambda\sigma^2}(1+r-a)+r\right) - C\right]\right\},\; w^3_{nf} = -1$$

is ***locally asymptotically stable***

Numerical Simulations



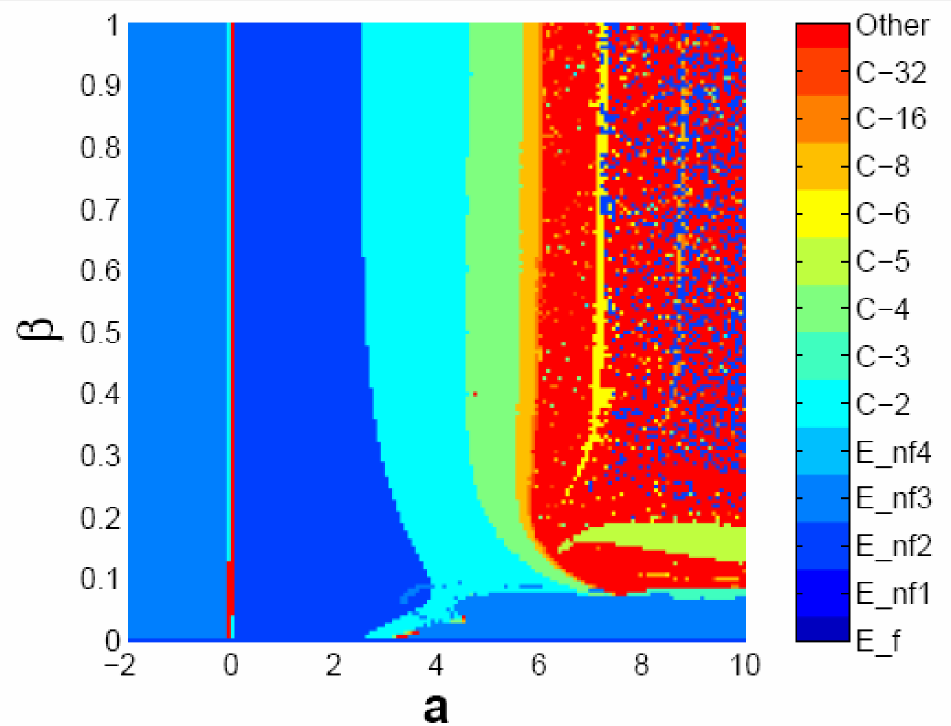
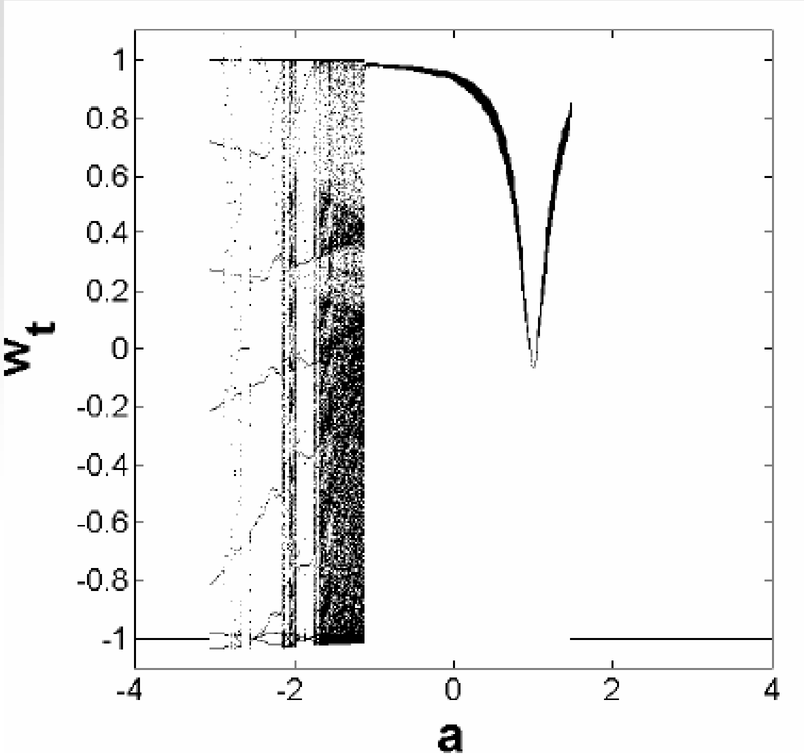


Figura 1: Two-dimensional bifurcation diagram of the map T in the (a, β) parameter plane for the initial condition $x_0 = 1.3$, $m_0 = -1$ and $w_0 = 1$ and parameter values $\alpha = 0.5$, $\lambda = 1$, $\sigma^2 = 1$, $r = 0.02$, $C = 0.5$. If a is great enough the fundamental steady state is asymptotically stable.

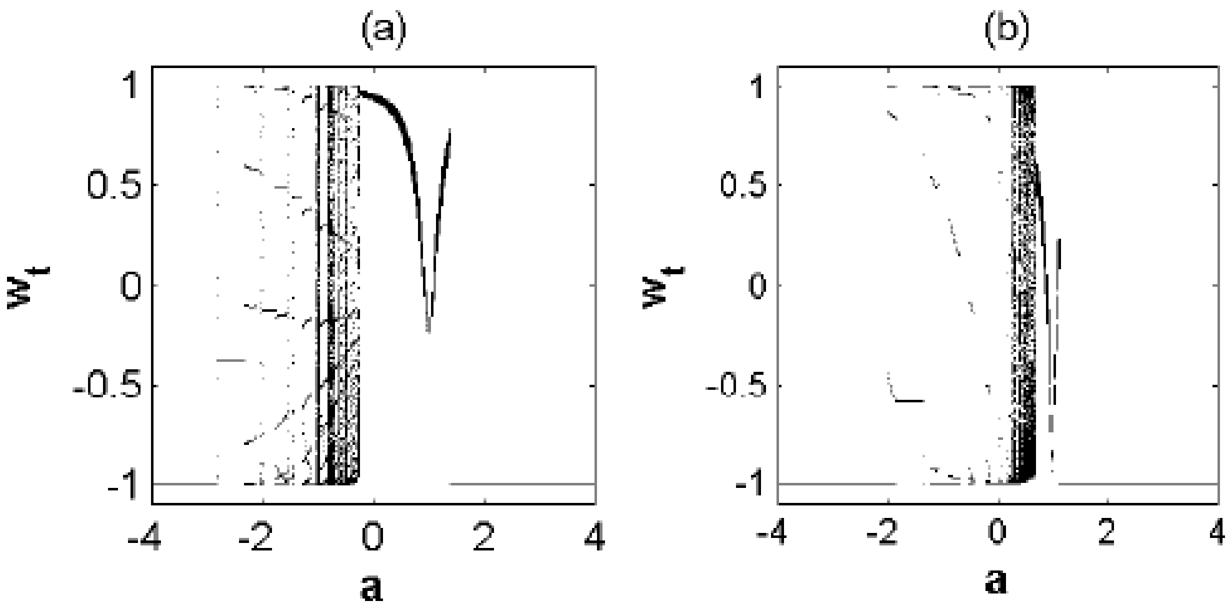
Figura 2: Two-dimensional bifurcation diagram of the map T in the (a, β) parameter plane for the initial condition $x_0 = 1.3$, $m_0 = -1$ and $w_0 = 1$ and parameter values $\alpha = 4$, $\lambda = 1$, $\sigma^2 = 1$, $r = 0.02$, $C = 0.5$. The regions corresponding to different attracting cycles are shown by different colors.

One-dimensional bifurcation diagram of w w.r.t. a

$$x_0 = 1.2, m_0 = -0.8, w_0 = -0.8, \alpha = 0.2,$$
$$\lambda = 1, \sigma^2 = 1, r = 0.02, C = 0.5$$



$$\beta = 0.3$$



$$(a) \quad \beta = 1$$
$$(b) \quad \beta = 7$$

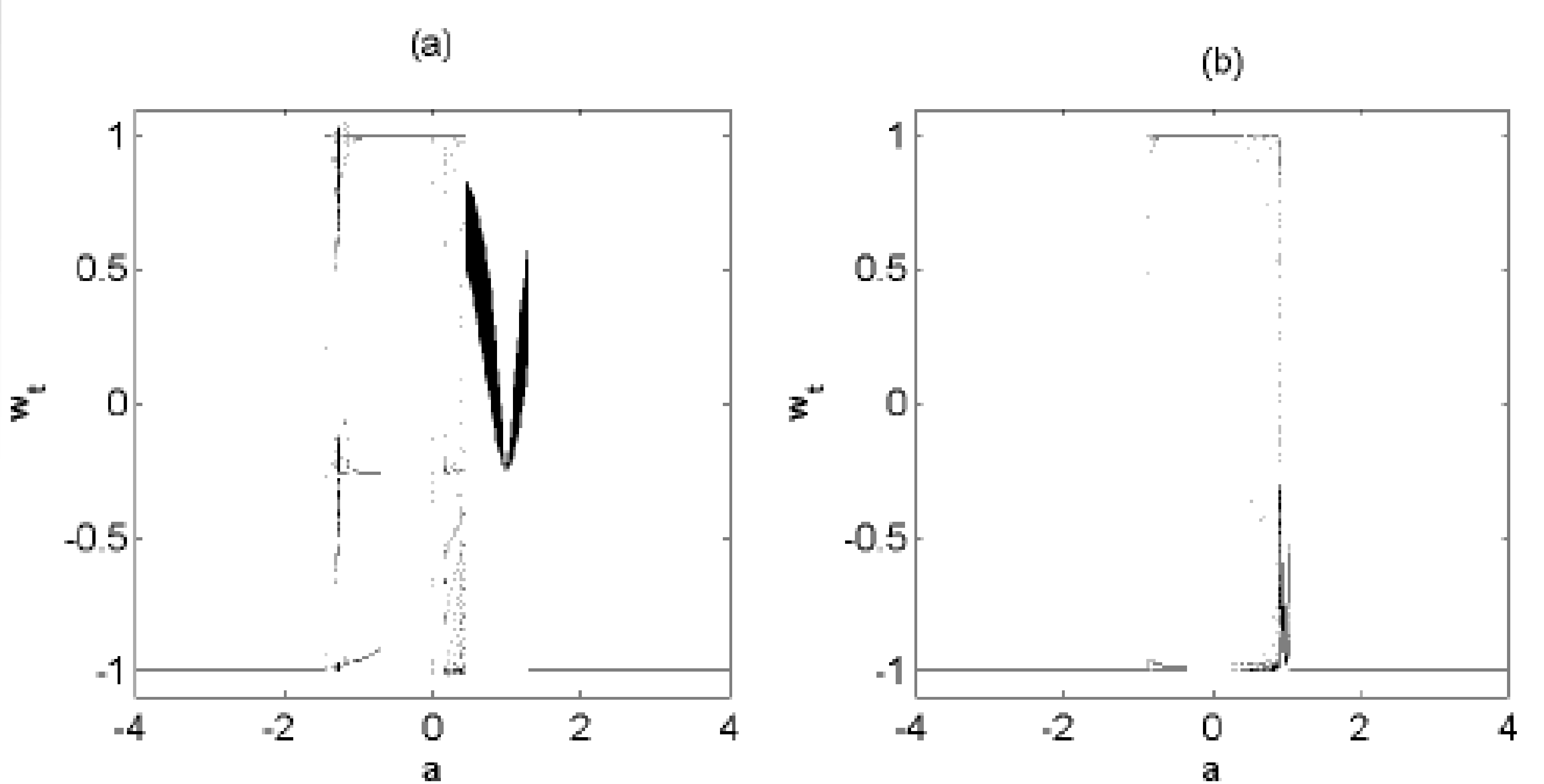


Figura 5: One-dimensional bifurcation diagram of the state variable w_t with respect to a for the initial condition $x_0 = 0.9$, $m_0 = 0.8$ and $w_0 = 0.8$ and parameter values $\alpha = 1$, $\lambda = 1$, $\sigma^2 = 1$, $r = 0.02$, $C = 0.5$. In panel (a) $\beta = 1$ while in panel (b) $\beta = 9$.

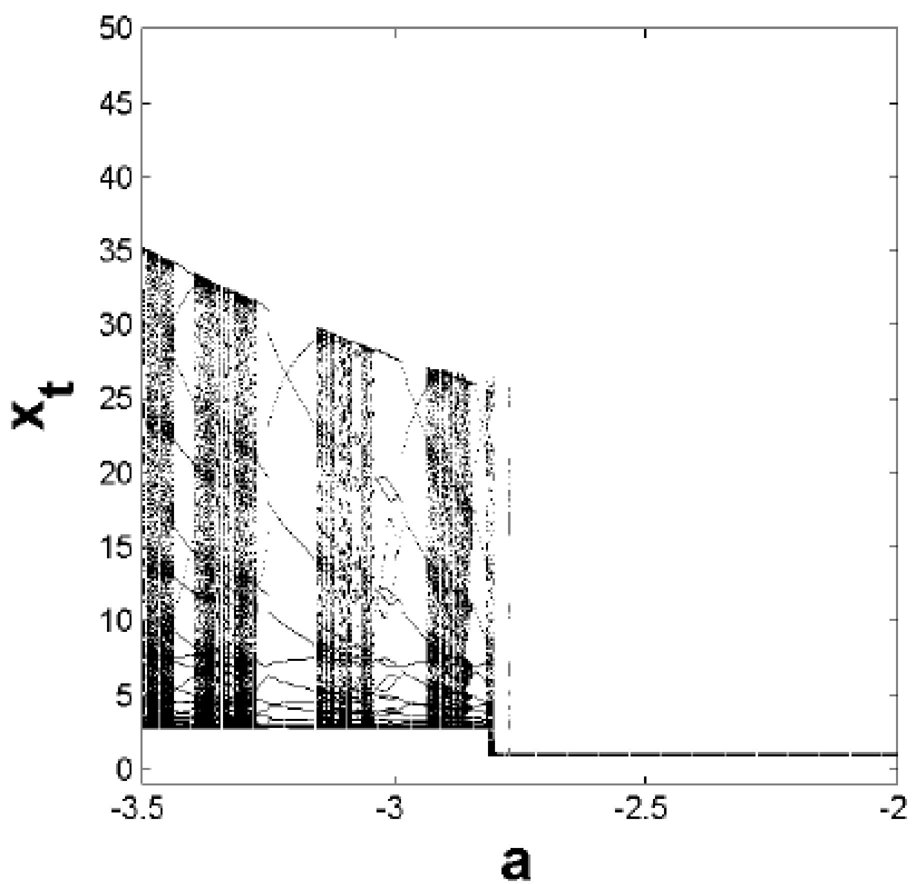


Figura 6: One-dimensional bifurcation diagram of the state variable x_t with respect to a for the initial condition $x_0 = 0.8$, $m_0 = 0$ and $w_0 = 0$ and parameter values $\alpha = 0.1$, $\lambda = 1$, $\sigma^2 = 1$, $r = 0.02$, $C = 0.5$, $\beta = 0.6$.

Basins of attraction

The role of the parameter a

As *a* increases the structure becomes complicated

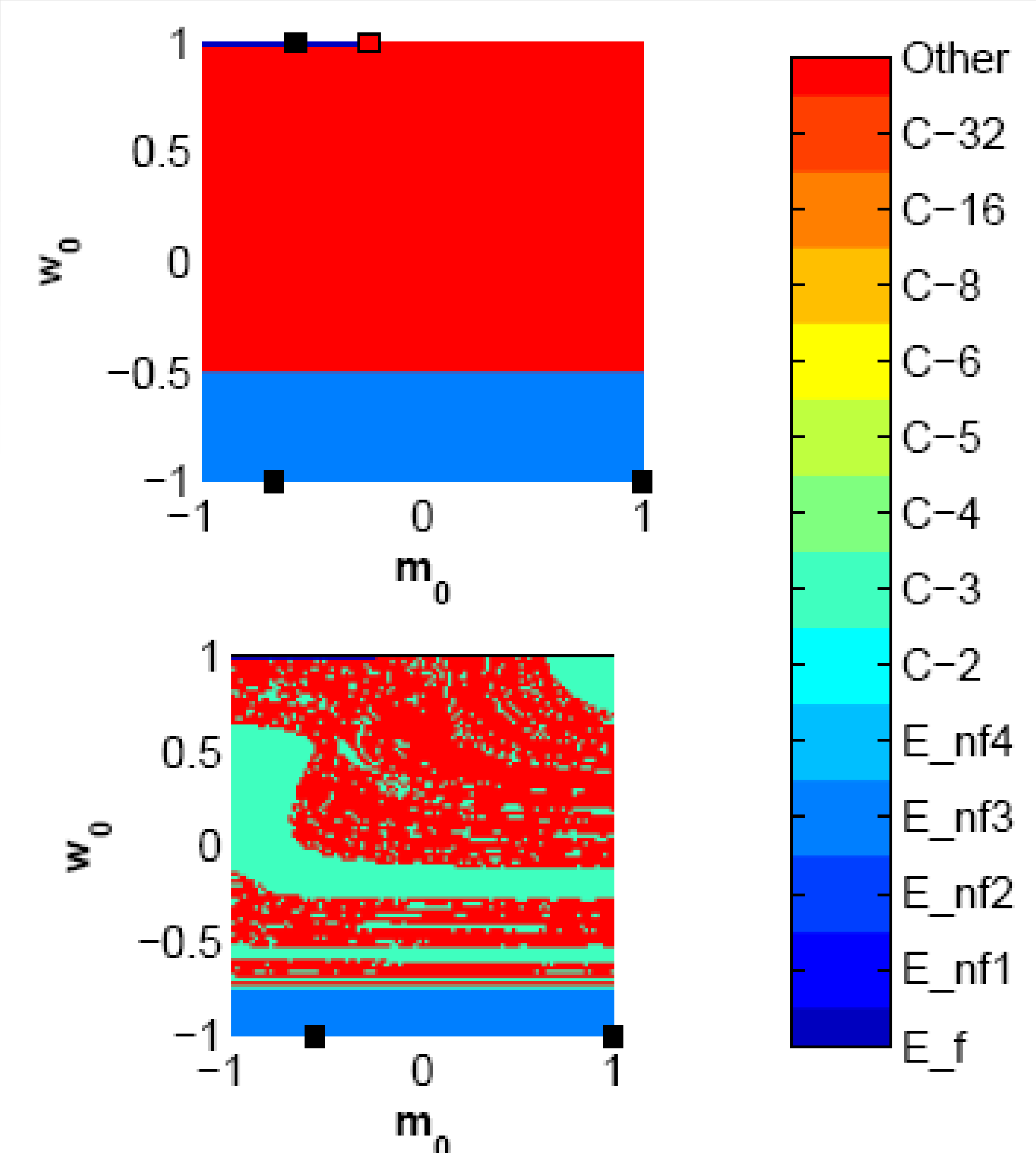


Figura 7: Basin of the attractors for the initial condition $x_0 = 1$ and parameter values $\alpha = 2$, $\lambda = 1$, $\sigma^2 = 1$, $r = 0.02$, $C = 0.5$, $\beta = 1$. In panel (a) $a = -0.4$, in panel (b) $a = -0.3$.

Conclusions

We have developed a new model of asset price and wealth dynamics in which:

the dynamics of wealth takes into account
the wealth coming from a different group

The presence of fundamentalists and chartists leads the system to have two kinds of steady states which coexist in the phase space

The map has been restricted to particular trapping sets in order to perform the stability analysis, this is due to the particular form of our system