Consumption, money holdings and constraints: a critical implication for the Euler equation.[*]

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Abstract

In this paper we develop a simple analytical solution for studying optimal consumption with financing constraints and uncertain income. We show that when utility depends on money holdings, financing constraints do not invalidate the Euler Equation up to the bound. This happens because household selects at once the consumption path that

[*] I wish to thank GianItalo Bischi, Laura Gardini, Paolo Liberati, Annamaria Lusardi, Enrico Saltari and Davide Ticchi for some useful comments and discussions.

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assures the optimality of his intertemporal decisions even when the constraint binds. Of course, the behavior of such a consumer di®ers markedly from the standard consumption model with constraints: the main result of the present analysis is that the Euler equation is always respected not only in the unconstrained status, but also in the constrained one.

JEL classi¯cation:D11;D81;D91.

Keywords: Consumption; Uncertainty; Monetary resources; Financing constraints.

1 Introduction

Mathematical convenience, rather than innate plausibility, has always been the main rationale of the standard consumption model (SCM) in which utility solely depends on consumption. However, a great deal of empirical evidence suggests that this framework may be inadequate, in practice, to capture the main characteristics of the consumption-saving behavior. For example, tests of the stochastic intertemporal Euler equation have typically produced strong statistical rejections, leading researchers to look for explanations of these failures (Flavin, 1981; Campbell, 1987; Jappelli, 1990; Runkle, 1991; Deaton, 1992; Jappelli and Pagano, 1994; Carroll, 1994; Browning and Lusardi, 1996; Attanasio, 1999; Carroll, 2001). One of the most common answer to this breakdown asserts that financing constraints can induce a one-side violation of the Euler equation, causing the failure of the intertemporal optimal consumption condition (Flavin, 1985; Hubbard and Judd, 1986; Hayashi, 1985). This strand of research recognizes that financing constraints can affect consumption in two different ways. Firstly, corner solutions can inform the allocation of consumption when constraints are actually binding. In this scenario, the Euler equation is violated because the consumer cannot anticipate the services of future labor income in order to equate the weighted marginal utilities of consumption over time. Then, as Hayashi (1987), Zeldes (1989) and Deaton (1991) have emphasized, current consumption can be a®ected by future restrictions even when such ¯nancing constraints are actually slack. The basic idea of all these papers is the following. The fact that the ¯nancial constraint is (or has some probability of being) binding in a future period should not cause a violation of the intertemporal ¯rst order condition because as long as the consumer is not up again the restriction, it is possible to reallocate total resources, period by period, in order to satisfy the Euler equation. Thus, in the intermediate phases the intertemporal ¯rst-order conditions hold regardless of the presence of liquidity constraints. But, when the constraint is currently binding, this process is interrupted, and the upper bound leads to a violation of the Euler equation.[1] We call this approach `standard expectation view'.[2]

Now, although this anticipative behavior seems to capture all the inter-

[1] The main implication of this dynamic process is that when constraint is ine®ective, so that the Euler equation looks like the standard one, optimal behavior will not generally be the same as for an agent who will never be constrained.

[2] A second, related issue is stressed by Ja®ee and Stiglitz (1990). These authors observe that one of the main limitations of models with borrowing constraints is the use of comparative statics to analyze the relationship between aggregate demand and ¯nancial resources. This approach, they argue, makes it di±cult to focus on inter-period issues affecting the investment process. More speci¯cally, Ja®ee e Stiglitz asserts that anticipated future credit rationing can have e®ects on current aggregate demand, \even when there is no credit rationing at present. Thus the impact of the credit rationing can not be assessed just by looking at those periods in which there is direct evidence for its presence" (p.874).

period features a®ecting the optimal consumption plan, it fails to attach su±cient importance to the way in which expectations concerning future monetary resources can a®ect current consumption decisions. In this perspective, one would expect that consumers should anticipate at the current time the future constraint, selecting immediately the optimizing expected consumption path which provides the maximized value function, given the uncertain income and the loosest constraint. In other words, the forward-looking consumer who wants to smooth his consumption should change his policy at the initial time, choosing at once the optimal trajectory that assures the respect of the Euler equation, even when the constraint is binding.[3]

One way to formalize this `augmented expectation view' is to relax the restrictive assumption on preferences, thereby allowing the marginal utility of consumption to change not only with consumption but also with total money holdings. More precisely, we build up a model where the marginal utility of current consumption is a®ected by both the level of consumption and monetary resources. If, in some future period, consumption is constrained because

[3] Coherent with this view is the analysis of Rabault (2002) who sustains that \despite the attention that income °uctuation problem have received in the past, and despite the empirical role of the Euler equation, conditions under which the optimal policy of such models leads the agent to exhaust his borrowing capacity have not been throughly explored" (p.218).

of limited ¯nancial resources, the household will anticipate this restriction immediately selecting the path that assures the optimality of consumption policy at any time. According to this latter arrangement, we introduce a `state' variable such as monetary resources in the utility function. This state variable a®ects the utility function in two di®erent ways: it is assumed that money yields both a direct marginal utility to the consumer and an indirect utility which is determined by the relationship of money with consumption, through the intertemporal budget constraint.

Using this framework, we demonstrate that ¯nancing constraints do not invalidate the Euler Equation up to the bound because the household selects at once the consumption-saving plan that assures the optimality of his intertemporal decisions even when the constraint binds. Of course, the behavior of such a consumer di®ers markedly from the standard consumption model with constraints: the main result that comes out of the present analysis is that the Euler equation is always respected not only in the unconstrained status but also in the constrained one.

We start from a simple stochastic process for labor income. As in Hayashi (1985), Zeldes (1989) and Deaton (1991), a signi¯cant result is that ¯nancial constraints do not need to be currently binding in order to a®ect current consumption. But, in contrast with them, the constraint does not invalidate the intertemporal ¯rst order condition when the constraint binds. From this point of view, our model is coherent with the problem discussed by Rabault (2002) where in a discrete time context, and in presence of latent borrowing constraints, the consumer might systematically avoid exhausting his borrowing capacity. It is, however, di®erent from Rabault's problem because our set-up is developed in a continuous time framework with an explicit state variable (monetary resources) in the utility function.

Our analysis has important implications also to the many empirical studies which have used the Euler equation for testing the constrained behavior of agents. First, since in our framework the Euler equation characterizes the optimal dynamic behavior of both constrained and unconstrained consumers, the empirical analyses on consumption and ¯nancing restrictions employing this relationship may be biased, because this structural equation cannot discriminate among individuals with di®erent ¯nancing constraints.

Second, this forward-looking behavior allows the consumer to optimize his sequential choices along the entire time horizon, selecting the optimal consumption trajectory even when the constraints are eventually binding. Hence, the present model of consumption with imperfect capital markets and uncertainty is consistent with the SCM where the rational agents attempt to keep the marginal utility of consumption constant over time (Browning-Crossley, 2001).

The paper is organized as follows. In the next section we explain the assumptions of the model. In section 3 we solve the intertemporal consumption problem with financing constraints, discussing the properties of the Euler equation. Section 4 concludes.

2 The assumptions

In this section, we discuss the assumptions characterizing our model of constrained consumption under uncertainty. It has the following features:

(1) Household acts in an imperfect capital market, where financing constraints are simple quantity restrictions.

(2) Consumption in each period is a function of both financial assets and current income. Indeed, since money is fungible between financial assets and income, consumption will be a function of only their sum.

(3) The evolution of labor income follows a continuous-time random walk. Of course, this dynamics affects the evolution of the monetary resources.

(4) The household derives his expected utility from both consumption and monetary resources. The intertemporal co-evolution of them is the result of the interrelation between consumption decisions and budget constraint.

(5) Utility function is assumed to be quadratic in both the arguments.

(6) The individual rate of time preference is bigger than the real interest rate, that is $\delta > r$: For simplicity we assume that r is constant over time.

2.1 Financing constraints

The label `financing constraints' includes both borrowing and liquidity restrictions. Indeed, for consumers who cannot borrow, or can only do so at penal rates of interest, consumption expenditures are likely to be closely tied to current income receipts. So, the first step in the integration of financing constraints in the present model must be their exact definition. We make the assumption that net wealth w_t cannot be smaller than a lower constant bound, that is:

$$w_t \geq -D \qquad (1)$$

and, further, that labor income can °uctuate randomly until it reaches respectively either the lower bound y or the upper bound Y . This means that:

$$y \quad y_t \quad Y \tag{2}$$

Equation (1) is the borrowing constraint. If D = 0; the consumer is fully rationed in credit market; whereas if D > 0 the consumer is partially rationed and the net wealth can be negative. Equation (2) identi¯es the liquidity constraint. The set of restrictions on equations (1) and (2) constitutes what we refer to as ¯nancing constraints throughout this paper. Note that the inability to borrow does not imply inability to save. In fact, liquidity constrained consumer can have good reasons to shift consumption forward in time by saving.

Obviously, the sum of wealth and labor income gives the maximum amount of money holds by the agent in each period:

$$x_t = w_t + y_t \tag{3}$$

Following Deaton (1991), we can call x_t as cash in hand. It identi¯es the total resources that, in any period t; the household can use for consumption.

Finally, to simplify algebra we assume that $D = 0$ and $y = 0$. Hence, the dynamics of x_t is comprised in the interval:

$$0 \quad x_t \quad \beta \tag{4}$$

where β is the higher attainable level of monetary resources. Its value is given at any time by the sum of the accumulated wealth with current income.

2.2 Uncertainty

To incorporate uncertainty in our model it is sufficient to assume that the stochastic evolution of labor income, under a free float, is described by the continuous-time random walk:

$$dy_t = \sigma dz \tag{5}$$

where σ is the (constant) variance parameter. The term dz is the increment of the standard Wiener process, with mean $E(dz) = 0$; and variance $E(dz)^2 = dt$: Now, since the evolution of wealth can be written as:

$$dw_t = r\left(w_t + y_y - c_t\right) dt \tag{6}$$

where c_t is the istantaneous consumption, substituting expression (3) in (6) we get the intertemporal budget constraint for cash in hand:

$$dx_t = r(x_t - c_t)\,dt + dy_t \tag{7}$$

because $dw_t = dx_t - dy_t$. We impose that consumption be positive at all times, to avoid that consumer can choose a negative level of consumption to cover his debt in period t. In this perspective the condition $x_t \geq c_t$ can be seen as the financing constraint of the model. Then, substituting for (5) in (7) we obtain the stochastic dynamics of x_t:

$$dx_t = r(x_t - c_t)\,dt + \sigma dz \tag{8}$$

which describes the unpredictable pattern of the cash in hand, given income uncertainty. If $\sigma = 0$ then equation (8) becomes:

$$dx_t = r(x_t - c_t)\,dt$$

meaning that, with certainty and free float the accumulation saving rate depends only on control variable c_t. However, since we are considering the case ¾ > 0 with ¯nancing constraints, uncertainty on monetary resources must a®ect the evolution of consumption over time: within the interval, de¯ned by the ¯nancing constraint (4), the variable x_t can change freely; once, however, it has reached one of the two boundary values, its dynamics changes in an unpredictable manner a®ecting consumption.

2.3 Utility function

The integration of consumption goods and monetary resources in the utility function is the most signi¯cant novelty of the model. Traditionally, we refer to Pigou (1941) and Patinkin (1965) as the ¯rst authors introducing real money balances in the felicity function. According to Patinkin, real balances can provide some precautionary services to consumers, implying that monetary resources can have their own utility, and, thus, must be introduced as an explicit argument in the utility function. Of course, this way to treat monetary resources can have important economic implications. For example, in Patinkin's model the channel of transmission between real and monetary markets is determined, among other factors, by the so called real balance effect. Then, other important contributions within the in¯nite horizon model with real balances in utility function are in Sidrausky (1967), Brock (1974), Fisher (1979), and more recently in Obstfeld and Rogo® (1983, 1985), and Obstfeld (1984).

The use of monetary resources in the utility function has been, however, criticized by Clower (1967) who argued that, to study the role of money in carrying out real transactions, one should introduce a sort of `transaction technology' for money in the consumption process. He modeled this technology as a cash-in-advance constraint. But, recently this criticism appears to have been muted because of an important paper by Feenstra (1986). He showed that under certain regularity conditions, the maximization problem with money, modeled using a cash-in-advance constraint with liquidity costs, is equivalent to a maximization problem with monetary resources in the utility function. This result is derived from conventional model of money demand, such as the transaction and the precautionary models. Thus, the procedure of introducing monetary resources in the utility function seems to be generally viewed as being as an acceptable approximation.

In what follows we introduce monetary resources together with real expenditures in the utility function. It will be shown that, in a continuous-time model, the relationship between money and consumption has important implications for the Euler equation in the constrained scenario. In particular, we employ the following quadratic utility function:

$$u(c_t; x_t) = a\left(c_t - \frac{1}{2}c_t^2\right) + b\left(x_t - \frac{1}{2}x_t^2\right) \tag{9}$$

where a and b are parameters that measure the relative importance that household assigns to consumption, c_t; and cash in hand, x_t: The function $u(c_t; x_t)$ is concave and continuously differentiable in both the arguments. We require that household's resources are such that consumption is always in the range where the marginal utility is positive. Equation (9) is non-standard, but it has a pregnant meaning: according to investment models, we are assuming that the value function $u(c_t; x_t)$ is affected by both the control variable c_t and the state variable x_t: From this perspective, equation (9) is a more general specification of the traditional quadratic form which depends only on control c_t.[4]

Of course, the assumption of quadratic utility function is stringent, but it is required for deriving a closed form solution.[5] We have, however, two

[4]It is important to stress that this kind of felicity function is often used in generalized consumption model. For example, Bernanke (1985) employes a quadratic utility function on both durables and non-durable goods with adjustment costs to show that with non-separability in utility the transactionn costs may affect the time series properties of both components of expenditure.

[5]As usual, we can interpret the quadratic function as a local approximation of the underlying utility function.

main reasons to employ this specification. First of all, using this formulation we can compare our results with those of the SCM which are often based on a quadratic form (Hall, 1978; Flavin, 1981; Deaton 1992). Second, and in common with Besley (1995) and Romer (1996) using this quadratic specification we can induce local risk-aversion that is a precautionary saving behavior that traditionally is incompatible with a linear marginal utility. As it will be explained, this behavior is caused by the anticipation of the future bound when the financing constraint is currently slack. This forward-looking behavior produces a non linear dynamics for consumption over time, so that, for a given value of consumption, an increase in uncertainty about future monetary resources may cause a decrease in current consumption, that we can interpret as precautionary demand for saving.

Finally, for financing constraints to be relevant the household must be inpatient enough to want to bring money from the future to the present to smooth consumption. For this reason, we require that $\delta > r$. In this scenario, financing constraints interact with precautionary motives because "the inability to borrow when times are bad provides an additional motive for accumulating assets when times are goods, even for inpatient consumer" (Deaton, 1991, p.1222).

3 The optimality criterion and the Euler equation

To study the optimal behavior of potentially constrained households we employ a partial equilibrium approach, where the interest rate r is given, and where time horizon is infinite. We assume that consumer maximizes his expected utility. To solve such optimization problem we employ a two stage procedure: we solve the unconstrained problem; then, we impose the boundary values of the state variable in the solution to obtain the constrained allocation.[6]

The consumer has to find the solution to the following problem:

$$\max_{c_t} E_t(U_0) = E_t \int_0^{\infty} \left[a\left(c_t - \frac{1}{2}c_t^2\right) + b\left(x_t - \frac{1}{2}x_t^2\right) \right] e^{-\delta t} dt$$

[6] This is a standard way to solve problems of dynamic optimization under uncertainty and exogenous constraints. A similar procedure has been used, for example, in Krugman (1991), and Froot and Obstfeld (1991) for problems of exchange rate dynamics with target zones; the case of optimal consumption and portfolio rules is discussed in a series of papers by Merton (1990). Bertola (1994) provides an excellent discussion of these techniques, in the presence of certainty and uncertainty. Finally, this kind of methodology has also been used in models of labor demand with costs for firing and hiring (Bentolila and Bertola (1990). Recently, Saltari and Travaglini (2001) employ this procedure to study the interrelation between investment decisions and financing constraints.

subject to the intertemporal budget constraint:

$$dx_t = r\,(x_t - c_t)\,dt + \sigma dz$$

where δ is the individual rate of time preference.

To solve this problem we can set up the corresponding current Hamiltonian:

$$H_t = E_t \left\{ a \left(c_t - \frac{1}{2}c_t^2 \right) + b \left(x_t - \frac{1}{2}x_t^2 \right) + \lambda_t \left[r\,(x_t - c_t) + \sigma \frac{dz}{dt} \right] \right\} e^{-\delta t} \tag{10}$$

where λ_t is the costate variable, that is the actualized shadow value of future resources. From H_t we obtain the first order conditions:

$$a\,(1 - c_t) = \lambda_t r \tag{10.1}$$

$$r\,(x_t - c_t) + \sigma \frac{dz}{dt} = \frac{dx}{dt} \tag{10.2}$$

$$E_t \left(\frac{d\lambda}{dt} \right) - \lambda_t \delta = -b\,(1 - x_t) - \lambda_t r \tag{10.3}$$

Equations (10.1) is the first order condition for consumption. In the unconstrained case, it states that discounted value of the marginal utility for consumption is equal to the marginal utility λ_t of monetary resources at time t. Then, the maximum principle involves two equations of motion. Equation (10.2) is nothing but a restatement of the equation for the stochastic budget constraint.

In turn, equation (10.3) is the Euler equation which states that the intertemporal maximization problem (10) implies that the marginal utility of cash in hand is kept constant over time. To see this implication rearrange (10.3) in the form:

$$\lambda_t = \frac{b(1-x_t)}{(\delta-r)} + \frac{E_t\left(\frac{d\lambda}{dt}\right)}{(\delta-r)} \tag{11}$$

Expression (11) is an intertemporal equilibrium condition. The left-hand side denotes the shadow value of x_t over time. This equation requires that λ_t is equal in magnitude to the sum of the two terms on the right hand side of (11). The first of these, $\frac{b(1-x_t)}{(\delta-r)}$, represents the direct marginal contribution of the actualized cash in hand to current utility, whereas the second $\frac{E_t(d\lambda/dt)}{(\delta-r)}$ identifies the marginal contribution of x_t to the enhancement of future wealth. Of course, this latter factor measures the indirect effect of monetary resources on consumption through the intertemporal budget constraint. Note that by assumption $(\delta - r)$ is positive: For then, substituting by λ_t; the first order condition can be written as:

$$c_t = 1 - \frac{r}{a}\left[\frac{b(1 - x_t)}{\delta - r} + \frac{E_t\left(\frac{d\lambda}{dt}\right)}{\delta - r}\right] \tag{12}$$

This expression says that the forward-looking behavior on $E_t\left(\frac{d\lambda}{dt}\right)$ is a necessary condition to drive the consumption c_t along the optimal path. This suggests that households need to form expectations on future financing resources in making his current consumption decisions. This latter statement can be, further, clarified differentiating the first order condition (10.1) with respect to time:

$$E_t\left(\frac{d\lambda}{dt}\right) = -\frac{a}{r}E_t\left(\frac{dc}{dt}\right)$$

and substituting this expression in (12) to obtain the Euler equation for the level of consumption:

$$c_t = \mu + \gamma x_t + \rho E_t \frac{dc(x_t)}{dt} \tag{13}$$

To simplify the notation we write $\gamma = \frac{rb}{a(\delta - r)}$; $\mu = 1 - \gamma$; and $\rho = \frac{1}{(\delta - r)}$.

Using this ¯rst order di®erential equation for c_t; we can re°ect on the implicit assumption of the model. The household controls c_t to maximize his expected utility. Nonetheless, because of the intertemporal relation (13), there is the need for the household to take into account the whole planning problem. In fact, equation (13) implies that at each point in time the level of c_t is determined by the istantaneous cash in hand x_t plus the expected consumption $E_t \frac{dc(x_t)}{dt}$, which is itself function of future monetary resources. Thus, the maximization of the felicity function requires to select initially the optimal consumption-saving plan which satis¯es the ¯rst order condition (10.1). It is clear from equations (13) that consumers cannot plan optimally without knowing the entire expected path of the future monetary resources.

Hence, the key of this model is that consumption decisions are based not only on the contribution of monetary resources, at a point in time, but also on the expectations of the consumer in order to avoid large adjustments in current consumption which would violate the intertemporal Euler equation (13). Given this expectations view, how can the future ¯nancing constraints a®ect current consumption?

3.1 The optimal expected consumption with financing constraints

To analyze the relationship between current consumption and future constraints assume that x_t follows the stochastic dynamics defined by (8).

Given that from the Euler equation (13) c_t is a function of t, the dynamic relationship between c_t and t can be expressed through the variable x_t; that is $c = c(x)$: Applying Ito's Lemma to c(x) we obtain an explicit expression for $E_t \frac{dc(x)}{dt}$:

$$dc(x) = c_x (dx) + \frac{1}{2} c_{xx} (dx)^2$$

Substituting by (8) and taking expectations we get:

$$E_t \frac{dc(x)}{dt} = [r(x - c)] c_x + \frac{1}{2} \sigma^2 c_{xx}$$

Hence, at any time, we can express changes in c as a function of x:

$$c(x) = \mu + \gamma x + \rho [r(x - c)] c_x + \frac{1}{2} \rho \sigma^2 c_{xx} \qquad (14)$$

Equation (14) has no general analytical solution because of the stochastic drift $\rho[r(x - c)]$, but we can solve analytically in particular cases and characterize the form of the solution elsewhere.

3.1.1 Case 1: $c = x$

As first point, note that if $c = x$; appropriate if household has a propensity to spend money equal to one, the drift is equal to zero. For coherence, this condition requires also that $\gamma = 1$ and $\theta = 0$: In this scenario, the effective dynamics of current consumption is determined by the unexpected evolution of stochastic income. Indeed, since equation (14) reduce to:

$$c(x) = x + \frac{1}{2}\rho\sigma^2 c_{xx} \tag{15}$$

the optimal consumption rule would say that if cash in hand is uncertain, it will be optimal to take immediately into account its variance. Indeed, variability (σ^2) on future resources affects current consumption, and, as equation (15) illustrates, the correlation between current consumption and σ^2 depends on the sign of the second derivative of $c(x)$: For example, only if $c_{xx} < 0$ we have precautionary saving.

To characterize the solution note that equation (15) is a second order homogeneous differential equation in $c(x)$, and its general solution can be expressed as a linear combination of any two independent components. We try the guess $c(x) = Ae^{\pi x}$ to obtain:

$$\frac{1}{2}\left(\rho\sigma^2\right)\pi^2 - 1 = 0$$

This is the characteristic equation with roots:

$$\pi_{1,2} = \pm\left(\frac{2}{\rho\sigma^2}\right)^{\frac{1}{2}}$$

with $\pi_1 > 0$ and $\pi_2 = -\pi_1 < 0$. Hence, the general solution of this differential equation is:

$$c(x) = x + A_1 \exp(\pi_1 x) + A_2 \exp(\pi_2 x) \qquad (16)$$

As this expression illustrates $c(x)$ has two components: the fundamental x; and the complementary solution. Of course, if x can change randomly without constraints, then the constants must be set equal to zero to avoid mispriced consumption strategies. In this case the consumption rule is:

$$c(x) = x$$

which simply states that consumption cannot be greater than x (the present budget constraint). In this scenario, consumption follows a random walk rule (Hall, 1978).

But, if household operates in imperfect capital markets, the two constants A_1 and A_2 exert their a®ects on current consumption. Consequently, to have an optimal decision rule the consumption pro¯le must change to take into account the future constrained values of x: Once the upper (lower) bound has been reached; the value of x can only decrease (increase) randomly, implying a decrease (increase) in c(x). To focus on restrictions, let equation (4) de¯nes the critical range. As long as x lies within the interval, its evolution is described by equation (8). When, on the other hand, x reaches one of the two boundary values, the evolution of dx becomes a modi¯cation of the process (8).

As x tends to ¯, c(x) tends to its own maximum level C; for the same reason, when cash in hand tends to the lower level zero; then c(x) tends to the minimum value c. This implies that when the constraints are binding it must be veri¯ed that:

$$c_x(0) = 0 = c_x(¯) \tag{17}$$

This expression is sometimes called smooth pasting condition, and it is a sufficient condition to allow the calculation of the constants A_1 and A_2: But, what is remarkable here is that the boundary condition (17) implies that over the range $[0;\beta]$ the consumption path will be optimal only if the trajectory is continuous and smooth for any value of x: It is only when $c(x)$ is continuous and smooth that its first derivative $\frac{dc(x)}{dt}$ exists and it is continuous. Consequently, the imposition of (17) guarantees the respect of the Euler equation (15), or more generally (14), not only when the constraints are slack, but also when the state variable x is up to the bounds.

Applying (17) to (16) we obtain the explicit solution

$$
\begin{aligned}
c(x) \quad = \quad & \frac{1}{\pi_1}\left(\frac{\exp(\pi_2\beta) - \exp(\pi_2 y)}{\exp(\pi_1\beta)\exp(\pi_2 y) - \exp(\pi_2\beta)\exp(\pi_1 y)}\right)\exp(\pi_1 x) + \\
& + \frac{1}{\pi_2}\left(\frac{\exp(\pi_1 y) - \exp(\pi_1\beta)}{\exp(\pi_1\beta)\exp(\pi_2 y) - \exp(\pi_2\beta)\exp(\pi_1 y)}\right)\exp(\pi_2 x) + x
\end{aligned}
$$

Notice that only when the two barriers become infinitely distant the consumption rule is a liner function of the fundamental.

The dynamics of $c(x)$ can be defined for all x in the interval $[0;\beta]$. Hence, the function $c(x)$ can be interpreted as representing the increased level of consumption for a consumer whose cash in hand changes from a situation marked by credit rationing and scarce income, to a situation in which monetary resources are increased, but to a level where no further expansion is possible.

Figure 1 shows the linear function of the fundamental $c = x$, and the S-shaped locus representing the non-linear functional form $c(x)$, which is tangent at 0 and β with values $c(0) = c$ and $c(\beta) = C$.[7] It meets the boundaries smoothly to first order, but the optimal boundary values for consumption, c and C, lie, respectively, below β and above 0. This happens because the state variable x can never exceed the interval $[0, \beta]$, and with $\sigma > 0$ and financing constraints, the consumption will surely falls randomly below 0 (above β) after reaching it.[8]

[7] The optimal consumption path in figure 1 is drawn for particular values of parameters. More precisely, we have supposed that $\pi_1 = -\pi_2 = 2$, $\theta = 0$, $\gamma = 1$, and that the upper bound for cash in hand is equal to $\beta = 3$. For equation (4) the lower bound is, in turn, $x = 0$. Note that for this value the level of minimum consumption is positive and equal to $c = 0.497$, implying dissaving. In other words, to assure consumption (when income is equal to zero) the consumer uses his wealth accumulated in previous periods when $y > 0$. The value of saving is graphically represented by the distance of the S-curve from the $c = x$ straight line. Note that in this model the consumption curve is simmetric with respect to value $x^* = 1.5$, which identifies the inflection point along the S-shaped locus. Finally note that, since consumption is a function of x, not of y, and $c(x)$ can be greater than less than or equal to y, there is no explicit information about the level of saving in the range $[\alpha, \beta]$.

[8] This point is further discussed in the next sections.

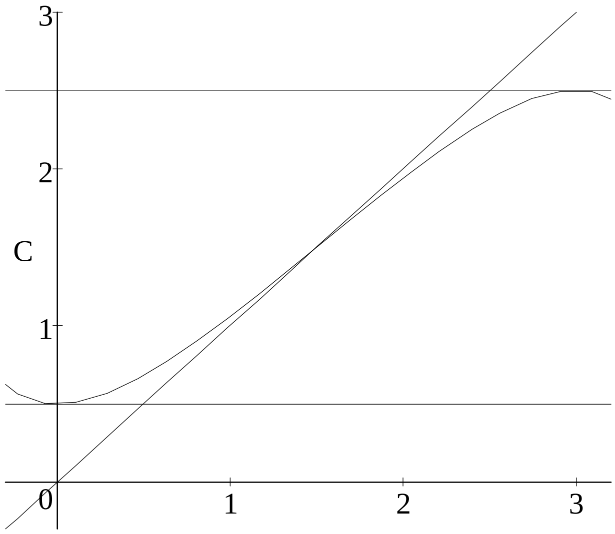

Figure 1. The optimal pro¯le of consumption when x = c

More precisely, when x is higher than the lower level 0; the value of the consumption rises, following a non linear increasing relationship. Once, however, x passes the in°ection point, where $c(x^{¤}) = x^{¤}$; the curve becomes concave: the consumer anticipates the e®ect of the upper constraint at the current time. In other words, the household perceives closeness to the upper bound as an exacerbation of the ¯nancing constraint. This contributes to slowing down expected consumption $\frac{dc_t}{dt}$; which for x = ¯ is equal to zero (in our example ¯ = 3). Hence, the consumer that appears to be more ¯nancially constrained, takes into account the uncertainty over future cash in hand and it is more reluctant to consume.[9] In turn, the consumer less ¯nancially constrained exhibits a greater positive sensitivity to x.

Two are the main consequences of this behavior. As ¯rst point, observe that to have a precautionary saving the value of the cash in hand must be higher than the critical value x¤: For example, let us suppose that x evolves in such a way as to realize value x_1: In a perfect capital market the value of consumption would then be c_1 on the straight line $c = x$: The foorward-looking consumer anticipates, however, that the cash in hand will never move outside the range [0; ¯], and that the closer the upper ¯nancing constraint ¯; the higher is the probability that x will be lower in the future. This bearish expectation a®ects current consumption, reducing its level along the S-curve, generating a precautionary saving. On the other hand, for values of x smaller than x¤ the path is convex and the increases of uncertainty has a positive e®ect on current consumption because, for small value of x, a higher ¾^2 implies a higher probability to gain a higher income.

This interpretation of the S-curve makes it possible to reach a ¯rst conclusion: namely that latent ¯nancing constraints can a®ect the consumer's behavior even when the bounds are currently slack. This is the consequence

[9] Similar graphic representation but for the only case of upper bound are in Heller and Starr (1979), Helpman (1981) and Deaton (1991).

of entering money in the felicity function, and of the forward-looking behavior by consumers that anticipate the possibility of future constraints at the current time. It is this anticipative behavior which assures the respect of the Euler equation when the constraints bind.

3.1.2 Case 2: $x - c = k$

In this case we assume that the consumer has a constant propensity to save equal to k. Hence, the differential evaluation equation for $c(x)$ can be written as

$$c(x) = \theta + \gamma x + \varphi c_x + \frac{1}{2}\rho\sigma^2 c_{xx} \tag{18}$$

where $\gamma < 1$ measures the marginal propensity to consume, and where the component of the expected drift $\varphi = \rho r k$ is, now, constant. Equation (18) can be solved using the standard method. With respect to the previous example, the solution differs for the fundamental

$$c(x) = (\theta + \gamma x)$$

which appears to be a linear keynesian consumption rule, and for the values of the roots which are given by

$$\pi_{1,2} = -\frac{rk}{\sigma^2} \pm \left(\frac{r^2k^2}{\sigma^4} - \frac{2}{\rho\sigma^2} \right)^{\frac{1}{2}}$$

with $\pi_1 > 0$ and $\pi_2 < 0$. Then, as in the previous case, consumption path has a S-shaped trajectory.

3.1.3 Case 3: $x - c$ stochastic

When $(x - c)$ is stochastic, the same boundary conditions (17) determine the solution, though it cannot be obtained explicitly. We can still derive key qualitative features of the solution from the consideration of the stochastic evaluation equation. To prove this point, consider the differential equation (14):

$$c(x) = \theta + \gamma x + \rho \left[r\,(x - c) \right] c_x + \frac{1}{2}\rho\sigma^2 c_{xx}$$

and the range $[\alpha, \beta]$, where now α represents the lower bound. Evaluating this equation for $x = \alpha$ and $x = \beta$, and using the boundary conditions (17) that must hold at α and β, we find:

$$
\begin{aligned}
c(\beta) &= \theta + \gamma\beta + \frac{1}{2}\rho\sigma^2 c_{xx} \\
c(\alpha) &= \theta + \gamma\alpha + \frac{1}{2}\rho\sigma^2 c_{xx}
\end{aligned}
$$

The two expressions show that in the neighborhood of the bounds the evaluation equation (14) does not depend on the stochastic drift $\rho r(x - c)$. But, close to the upper bound β the derivative $c_{xx} < 0$, signifying that along the S-curve the threshold value $c(\beta)$ is below the straight line; that is $c(\beta) < \theta + \gamma\beta$. In turn, close to the boundary value α the derivative $c_{xx} > 0$, implying that $c(\alpha) > \theta + \gamma\alpha$. These are the same properties that the value function had in the previous two cases. Consequently, the smooth pasting condition (17) is sufficient to have an S-shaped path for potentially constrained consumption.

3.1.4 Optimal constrained behavior and the Euler equation

We can now use all this information to explain the optimal trajectory of $c(x)$. Looking at equation (14) we see that the value of consumption at any time is an intertemporal equilibrium relationship, which depends on both cash in hand and future consumption. Given an initial value for x, any future increase (or decrease) in expected monetary resources implies a corresponding increase (or decrease) in future consumption and consequently, given the arbitrage relationship (14), in current consumption. If, at some particular time, the household anticipates that with some positive probability the future cash in hand will be no higher than the upper level β; he comprehends that in the long run the consumption will not grow beyond the maximum trigger value C. This information affects the rate of change of consumption. As β draws closer, the upper value of the monetary resources exerts an ever stronger influence on current consumption, and after a certain value (x^*), the level c_t becomes a concave function of x. In other words, as x approaches the upper barrier the consumer realizes that future consumption plans will be constrained by the availability of monetary resources. A forward-looking consumer will anticipate this trend in the fundamental; as a result future constraints will be reflected in the household's current decisions. In these circumstances, it is not surprising that as x tends to β consumption converges smoothly to C; becoming tangent at the trigger value C; in such a way as to satisfy the intertemporal Euler equation:

It should be remarked, once more, that the non-linear dynamics of the c (x) has an important implication: the e®ect of increasing uncertainty, ¾^2, on saving depends on the initial value of x. In the convex part of the c (x) curve increases in ¾^2 rises the value of the current consumption. For larger values of x; on the other hand; consumption path is concave and the consumption-uncertainty correlation is reversed. This latter property distinguishes the present model from the Deaton's model (1991) where the combination uncertainty with ¯nancing constraints can only generate a precautionary saving behavior. Thus, in this simple model the interaction between current consumption and future monetary resources can provide a more complex consumption behavior, and the household can smooth consumption even when the constraints are binding.

4 Conclusions

In this paper we have shown that when utility depends on both consumption and monetary resources the presence of ¯nancing constraints and uncertain income does not invalidate the Euler equation up to the bound. This can happen because in a forward-looking environment the presence of a slack ¯-nancing restriction in°uences the behavior of a rational consumer well away from the point at which the constraint binds. To obtain this result, it is required a shifting in the current level of consumption, for ensuring that the Euler equation changes, period by period, and it is satisfied even when the constraint binds. This means that households which face rationing in the future, but which are free from constraints at the present time, will make sequential decisions to achieve a coherent and optimal consumption plan using not only currently available information, but also expectations on future monetary resources. As a consequence, an optimizing consumption-saving path implies that the Euler equation must be continuous at the boundary values. In the presence of financing constraints it is only this continuity condition which guarantees the respect of the intertemporal Euler conditions, that is the optimality of the consumption strategy.

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