A Conjectural Cooperative Equilibrium for Strategic Form Games

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Abstract

This paper proposes a concept of cooperative equilibrium for games in strategic form, denoted *Conjectural Cooperative Equilibrium* (CCE). This concept is based on the simple conjecture that joint deviations from any strategy profile are followed by an optimal and noncooperative reaction of non deviators. We show that a CCE exists in all symmetric supermodular games. Furthermore, we discuss the existence of a CCE also in specific submodular games associated with environmental economies.

Keywords: Strong Nash Equilibrium, Cooperative Games, Public Goods.

JEL Classification: C7

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1 Introduction

Intuitively a *cooperative equilibrium* is a collective decision adopted by a group of individuals that can be viewed as *stable* (i.e., an equilibrium) against all feasible deviations by single members or by proper subgroups. While modelling the possibilities of cooperation may not pose the social scientist particular problems, at least once an appropriate economic or social situation is clearly outlined, the definition of *stability* may be a more demanding task for the modeler. This because the outcome, and the profitability, of players' deviations heavily depends on the *conjectures* they make over the reaction of other players. As an example, a neighborhood rule to keep a common garden clean possesses different stability properties whether the conjectured reactions in the event of shirking is, in turn, that the garden would be kept clean anyway or, say, that the common garden would be abandoned as a result. Similarly, countries participating to an international environmental agreement will possess different incentive to comply with the prescribed pollution abatements whether defecting countries expect the other partners to be inactive or to retaliate.

The main focus of the present paper are cooperative equilibria of games in strategic form. A cooperative equilibrium of a game in strategic form can be defined as a *strategy profile* such that no subgroup of players can "make effective" - by means of alternative strategy profiles - utility levels higher for its members than those obtained at the equilibrium. As expressed in the example above, the content of the equilibrium concept depends very much on the utility levels that each coalition can potentially make effective and this, in turn, depends on the *conjectures* over the reactions induced by deviations.

In line with the idea of Nash equilibrium, one important cooperative equilibrium proposed by Aumann (1959) extends the assumption of "zero conjectures" to every coalitional deviation. Accordingly, a *Strong Nash Equilibrium* is defined as a strategy profile that no group of players can profitably object, given that remaining players are expected not to change their strategies. Strong Nash Equilibria are at the same time Pareto Optima and Nash Equilibria; in addition they satisfy the Nash stability requirement for each possible coalition. As a consequence, the set of Strong Nash Equilibria is often empty, preventing the used of this otherwise appealing concept in most economic problems of strategic interaction.

In this paper we propose a cooperative equilibrium for games in strategic form, based on the assumption that players deviating from an arbitrary strategy profile have *non zero conjectures* on the reaction of the remaining players. More precisely, the *conjectural cooperative equilibrium* we propose assumes that these remaining players are expected to optimally and independently react according to their best response map. The assumption of a "best response" conjecture introduces a very natural rationality requirement in the equilibrium concept. Furthermore, the coalitional incentives to object are considerably weakened with respect to the Strong Nash Equilibrium, thus ensuring the existence of a cooperative conjectural equilibrium in all symmetric games in which players' actions are strategic complements in the sense of Bulow et al. (1985), i.e., in all supermodular games (see Topkis (1998)).

The mechanics underlying such a result can be illustrated by means of the following symmetric 3x3 matrix game.

	A	**B**	**C**
A	x, x	d, h	a, c
B	h, d	b, b	e, f
C	c, a	f, e	y, y

Suppose that in the game (b, b) is an efficient outcome, i.e, such to maximize the sum of players' payoffs. To be a cooperative equilibrium, such an outcome has to be immune from each player switching his own strategy, given his own expectation on the rival's optimal response to the switch. When players actions are strategic substitutes (and the game submodular), each player's reaction map is downward sloped, implying that any move from (b, b) by one player would generate a predicted outcome on the *asymmetric* diagonal of the matrix. In this case, if we let, in the example, $a > b > c > h$, and $b > \frac{a+c}{2}$, the efficient outcome (b, b) will not certainly be a conjectural cooperative equilibrium, for player 1 will profitably deviate from it (from B to A), conjecturing that her rival's best reply will go in the opposite direction (from B to C), and getting a payoff of $a > b$. The same will happen if $c > b > a > e$, in which case player 2 will deviate by switching from B to C. In contrast, suppose that the game above is supermodular, with the associated increasing reaction maps. In this case, the conjectured outcomes in case of deviations from outcome (b, b) are only (x, x) and (y, y). As a result, if either player finds it profitable to switch either to A or to B ($x > b$ and $y > b$, respectively) then the assumption that (b, b) is an efficient outcome is contradicted. We can conclude that (b, b) is a conjectural cooperative equilibrium of the symmetric game described above whenever supermodularity holds. Note that in our example, if $d > b$, the efficient outcome (b, b) is a conjectural cooperative equilibrium although it is neither a Strong Nash Equilibrium nor a Nash Equilibrium.

The above example, although providing a clear insight of how supermodularity and symmetry work in favour of the existence of an equilibrium, contains two substantial simplifications: the presence of only two players, ruling out existence problems related to the formation of coalitions, as well as the restriction to 3 strategies, thus forcing the increasing best replies to generate symmetric outcomes, from which, the fact that (B,B) is an equilibrium, directly follows. However, in the paper we are able to show that the existence result holds for any number of players and strategies, provided a symmetry assumption on the effect of players' own strategies on the payoff of rivals is fulfilled.

Related Literature. The problem of defining cooperative equilibrium concepts have been centered on the formulation of *conjectures* ever since the pioneering work of von Neumann and Morgenstern's (1944). The concepts of α and β core, formally studied by Aumann (1967), are based on their early proposal of representing the worth of a coalition as the aggregate payoff that it can guarantee its members in the game being played. Formally obtained as the minmax and maxmin payoff imputations for the coalition in the game played against its complement, the α and β characteristic functions express the behaviour of extremely risk averse coalitions, acting *as if* they expected their rivals to minimize their payoff. Although fulfilling a rationality requirement in zero sum games, α and β-assumptions do not seem justifiable in most economic settings. Moreover, the little profitability of coalitional objections usually yield very large set of solutions (e.g., large cores).

Other approaches have looked at the choice of forming coalitions as a strategy in well defined games of coalition formation (see Bloch (1997) for a survey). Among others, the gamma and delta games in Hurt and Kurz (1985).[1] The gamma game, in particular, is related to the present analysis, since it predicts that if the grand coalition N is objected by a subcoalition S, the complementary set of players splits and act as a noncooperative fringe. On the same behavioural assumption is based the concept of γ core, introduced by Chander and Tulkens (1997) in the analysis of environmental agreements, where a characteristic function is obtained as the Nash equilibrium between the forming coalition and all individual players in its complement. As in the present approach, based on deviations in the underlying strategic form game, the γ core assumes that the forming coalition expects outside players to move along their (individual) reaction functions. Differently from our approach, however, there the forming coalition forms before choosing its Nash equilibrium strategy in the game against its rivals, while here deviating coalitions directly switch to new strategies in the underlying game, expecting their rivals to react in the same manner as followers in a Stackelberg game. In applying our concept to the analysis of stability of environmental coalitions, we may interpret these differences as the description of different structures in the process of deviation. While the γ core seem to describe settings in which the formation of a deviating coalition is publicly observed before the choice of strategies, our approach best fits situations in which deviating coalitions can implement their new strategies before their formation is monitored, enjoying a

[1] More precisely, Hurt and Kurz (19) present endogenous coalition formation games and look at the Strong Nash of these games. Other related papers (i.e., Chander and Tulkens (1998), Yi (1998)) look at the Nash equilibrium taking as given the gamma and delta rule of coalition formation.

positional advantage.

The paper is organized as follows. The next section introduce the conjectural cooperative equilibrium in a standard setup of strategic form games. Section 3 presents the main paper result: for a well defined class of games, symmetric supermodular games, a conjectural cooperative equilibrium always exists. Section 4 discuss in detail the meaning of the result and the role of the slope of players' reaction maps. This is done through a descriptive example of environmental economy whose cooperative conjectural equilibrium exists depending on individuals' preferences. Section 5 concludes.

2 Set Up

We consider a **game in strategic form** $G = \left(N, (X_i, u_i)_{i \in N}\right)$, in which $N = \{1, ..., i, ..., n\}$ is the set of players, X_i is the set of strategies for player i, with generic element x_i, and $u_i : X_1 \times ... \times X_n \rightarrow R_+$ is the payoff function of player i. We denote by $S \subset N$ any coalition of players, and by $\bar{S}$ its complement with respect to N. For each coalition S, we denote by $x_S \in X_S \equiv \prod_{i \in S} X_i$ a profile of strategies for the players in S, and use the notation $x = x_N$.

A **Pareto Optimum (PO)** for G is a strategy profile such that there exists no alternative profile which is preferred by all players to and it is strictly preferred by at least one player. The Pareto Optimum x^e is **efficient** if it maximizes the sum of payoffs of all players in N.

A Nash Equilibrium (NE) for G is defined as a strategy profile $\bar{x} \in X_N$ such that no player has an incentive to change his own strategy, i.e., such that

$$u_i(\bar{x}) \geq u_i(x_i, \bar{x}_{N \setminus i}) \quad \forall i \in N \text{ and } \forall x_i \in X_i.$$

Nash equilibria are stable with respect to individual deviations, given that the effect of such deviations is evaluated keeping the strategies played by the other players fixed at the equilibrium levels.

In trying to formulate equilibrium concepts that allow coalitions of players to coordinate in the choice of their strategies, a natural extension of the Nash equilibrium is given by the concept of **Strong Nash equilibrium (SNE)**, a strategy profile that no coalition of players can improve upon given that the effect of deviations is, again, evaluated keeping the strategies of other players fixed at the equilibrium levels. So, $\hat{x} \in X_N$ is a SNE for G if there exists no $S \subset N$ and $x_S \in X_S$ such that

$$\begin{aligned} u_i(x_S, \hat{x}_{\bar{S}}) &\geq u_i(\hat{x}) \;\; \forall i \in S; \\ u_h(x_S, \hat{x}_{\bar{S}}) &> u_h(\hat{x}) \;\; \text{for some } h \in S. \end{aligned}$$

Obviously, all SNE of G are both Nash Equilibria of G and Pareto Optima. As a result, SNE fails to exist in many economic problems, and in particular, whenever Nash Equilibria fail to be Optimal. Although the lack of existence of SNE can be interpreted as a poor specification of the game theoretic model, it precludes the use of this otherwise appealing concept of a cooperative equilibrium in many important applications.

In this paper we propose a concept of cooperative equilibrium for G based on the introduction of non-zero conjectures in the evaluation of the profitability of coalitional deviations. The concept we propose captures the idea that players in $\bar{S}$ are expected to react to the deviation of S by making optimal choices (contingent on the strategy profile played by S) as independent and noncooperative players. The conjectured optimizing reactions of outside players are formally described by the map $r_{\bar{S}} : X_S \to X_{\bar{S}}$ defined as follows:[2]

$$r_j\left(x_S\right) = \arg\max_{x_j} u_j\left(x_j, r_{\bar{S}\setminus j}(x_S), x_S\right) \quad \forall j \in \overline{S},$$

where r_j denotes the j-th coordinate of $r_{\bar{S}}$. Note that $r_{\overline{S}}\left(x_S\right)$ is the Nash equilibrium of the restricted game obtained from G by considering the set of players $\overline{S}$ and by parameterizing their payoff function by x_S. We obtain the following definition.

Definition 1 *A* ***Conjectural Cooperative Equilibrium (CCE)*** *is a strategy profile* x^* *such that there exists no coalition* S *and strategy profile* $x_S \in X_S$ *for which*

$$\begin{aligned} u_i(x_S, r_{\bar{S}}(x_S)) &\geq u_i(x^*) \;\; \forall i \in S; \\ u_h(x_S, r_{\bar{S}}(x_S)) &> u_h(x^*) \;\; \textit{for some } h \in S. \end{aligned}$$

Note that all CCEs are Pareto Optima, but need not be NE of the underlying game. In a 2x2 Prisoner's Dilemma, for instance, although no SNE exists, the efficient strategy profile is a CCE.

3 On the existence of conjectural cooperative equilibria in supermodular games

This section contains our main result, showing that if the payoff functions u_i are supermodular for all $i \in N$ and if the game G satisfies two symmetry requirements, then every G admits a conjectural cooperative equilibrium.

[2]The following definition implicitely refers to a singlevalued map $r_{\bar{S}}$. We will deal with the possibility of multivalues reaction maps in the section of supermodular games, for which a natural selection is adopted.

3.1 Supermodularity

We start by defining the concept of supermodularity. For any two strategy profiles x, y in X_N, we define the join element $(x \wedge y)$ and the meet element $(x \vee y)$ as follows:

$$\begin{aligned}(x \wedge y) &= (\min\{x_1, y_1\}, ..., \min\{x_n, y_n\});\\ (x \vee y) &= (\max\{x_1, y_1\}, ..., \max\{x_n, y_n\}).\end{aligned}$$

Definition 2 *The set X_N is a lattice if $(x \vee y) \in X_N$ and $(x \wedge y) \in X_N$ for all $x, y \in X_N$.*

Definition 3 *The function $u_i : X_N \to R$ is supermodular if for all $x, y \in X_N$:*

$$u_i(x \vee y) + u_i(x \wedge y) \geq u_i(x) + u_i(y).$$

The function u_i is strictly supermodular if the inequality is strict for all unordered $x, y \in X_N$.

Definition 4 *The function $u_i : X_N \to R$ has increasing differences in (x_i, x_j) if the term $u_i(x_{N\setminus\{i,j\}}, x_i, x_j) - u_i(x_{N\setminus\{i,j\}}, x_i, x'_j)$ is increasing in x_i for all $x_j > x'_j$.*

The following lemmas apply some results in the theory of supermodular games to the present setting. When omitted, we refer to Topkis (1998) for proofs.

Lemma 1 *If u_i is supermodular on X_N, then u_i has increasing differences in $(x_S, x_{\bar{S}})$ for all $S \subset N$.*

Lemma 2 *For all x_S the set $r_{\bar{S}}(x_S)$ has an upper and least element.*

Proof. Direct application of lemma 7 in the appendix. ■

We denote by $r^u_{\bar{S}}$ the selection of the map $r_{\bar{S}}$ obtained by considering its upper element, and by $r^l_{\bar{S}}$ the selection obtained by considering its least elements.

Lemma 3 *The maps $r^u_{\bar{S}}$ and $r^l_{\bar{S}}$ are non decreasing in x_S.*

Proof. Direct application of lemma 8 in the appendix. ■

3.2 Assumptions

We now list the assumptions needed for our main result.

Assumption 1 *X_i is an ordered compact lattice, for all $i \in N$.*

Assumption 2 *u_i is supermodular on X_N and upper hemicontinuous in x_i for all $x_{N\backslash i} \in X_{N\backslash i}$.*

Assumption 3 *(Symmetric Players)**:** For all $x \in X$ and all pairwise permutations $p : N \rightarrow N$:*

$$u_{p(i)}\left(x_{p(1)}, ..., x_{p(n)}\right) = u_i\left(x_1, ..., x_n\right).$$

Assumption 4 *(Symmetric Externalities)**:** One of the following two cases must hold:*

1. *Positive externalities: $u_i(x)$ strictly increasing in $x_{N\backslash i}$ for all i and all $x \in X_N$;*
2. *Negative externalities: $u_i(x)$ strictly decreasing in $x_{N\backslash i}$ for all i and all $x \in X_N$.*

3.3 Results

We start by a preparatory lemma, allowing to deal with the case of multivalued maps $r_{\bar{S}}$.

Lemma 4 *If the payoff functions exhibit positive (negative) externalities, then for all x_S the element $r^u_{\bar{S}}(x_S)$ $(r^l_{\bar{S}}(x_S))$ Pareto dominates all other elements in $r_{\bar{S}}$.*

Proof. Directly implies by assumption 4. ■

Motivated by Lemma 4, we will assume that in the case of positive externalities, the upper element of the set of Nash equilibrium best response is played, while the least element is played in the case of negative externalities. We can justify this assumption by arguing that if some pre-play communication is available for players in $\overline{S}$ then the Pareto optimal Nash equilibrium should be expected.

The next lemma establishes a results about the ordering of the strategies played by S and those played by the complementary coalition $\overline{S}$, when the former chooses a Pareto Optimal profile given the reaction map $r_{\bar{S}}$ defined as a $x^* \in X_N$ such that $x^*_{\bar{S}} = r_{\overline{S}}(x^*_S)$ and such that there exists no $x'_S \in X_S$ for which $u_i\left(x'_S, r_{\bar{S}}(x'_S)\right) \geq u_i(x^*)$ $\forall i \in S$ and $u_i\left(x'_S, r_{\bar{S}}(x'_S)\right) > u_i(x^*)$ for at least one $i \in S$. This is without loss of generality since, if S cannot improve upon a given strategy profile by means of a PO profile for S, then it cannot by means of any non optimal strategy profile.

Lemma 5 *Let $i \in S$ and $j \in \overline{S}$, and denote by $x \in X$ and $y \in X$ the strategies of player $i \in S$ and player $j \in \overline{S}$, respectively, at x^* Then:*
i) positive externalities imply $x \geq y$;
ii) negative externalities imply $x \geq y$.

Proof. Let $U_i(x,y) \equiv u_i\left(x^*_{S\backslash i}, x, x^*_{N\backslash S\backslash i}, y\right)$, and similarly let $U_j(x,y) = u_j\left(x^*_{S\backslash i}, x, x^*_{N\backslash S\backslash i}, y\right)$. We use supermodularity of u_i to write:

$$U_i(y,y) + U_i(x,x) \geq U_i(x,y) + U_i(y,x). \tag{1}$$

By the properties of x^*,

$$U_j(x,y) \geq U_j(x,x), \tag{2}$$

implying by symmetry that

$$U_i(y,x) \geq U_i(x,x). \tag{3}$$

Using (1) and (3) we obtain

$$U_i(y,y) \geq U_i(x,y) = u_i(x^*). \tag{4}$$

Now suppose that $y > x$ and assume that the game has positive externalities. By lemma 3, the equilibrium best response map has non decreasing upper element, so that

$$y > x \Rightarrow r^u(x^*_{S\backslash i}, y) \geq r^u(x^*_S) = x^*_{\bar{S}}. \tag{5}$$

By positive externalities

$$u_i(x^*_{S\backslash i}, y, r(x^*_{S\backslash i}, y)) > u_i(x^*_{S\backslash i}, y, r^u(x^*_S)) = U_i(y,y). \tag{6}$$

Equations (4) and (6) imply

$$u_i\left(x^*_{S\backslash i}, y, r^u(x^*_{S\backslash i}, y)\right) > u_i(x^*), \tag{7}$$

a contradiction.

Suppose now that $y < x$ and assume that the game has negative externalities. Supermodularity of u_i and u_j imply

$$y < x \Rightarrow r^l(x^*_{S\backslash i}, y) \leq r^u(x^*_S) = x^*_{\bar{S}}. \tag{8}$$

By negative externalities

$$u_i(x^*_{S\backslash i}, y, r^l(x^*_{S\backslash i}, y)) \geq u_i(x^*_{S\backslash i}, y, r^l(x^*_S)) = U_i(y,y). \tag{9}$$

Again, equations (4) and (9) imply

$$u_i(x^*_{S\backslash i}, y, r(x^*_{S\backslash i}, y)) > u_i(x^*). \tag{10}$$

The proof is completed by the observation that since equations (7) and (10) hold for all $i \in S$, the assumption that x^* is a Pareto Optimum is contradicted. ■

The next result shows that at x^*, members of S cannot be better off than members of $\overline{S}$. This result generalizes to the present setting of coalitional actions a well known property of the subgame perfect equilibrium in two player symmetric supermodular games, in which the "leader" is weakly worse off than the "follower". This generalization will be shown in theorem 1 to directly imply the existence of conjectural cooperative equilibria.

Lemma 6 *Let $x^* \in X_N$ be defined as in lemma 5. Let $i \in S$ and $j \in \bar{S}$. Then $u_j(x^*) \geq u_i(x^*)$.*

Proof. The following inequalities hold:

$$u_j\left(x_S^*, x_{\bar{S}}^*\right) \geq u_j\left(x_S^*, x_{\bar{S}\backslash j}^*, x_i^*\right) \geq u_j\left(x_{S\backslash i}^*, x_j^*, x_{\bar{S}\backslash j}^*, x_i^*\right). \tag{11}$$

The first part is implied by the conditions defining the profile x^*; the second part follows from lemma 5 and assumption 4. By the assumption of symmetric players, we also have

$$u_j\left(x_{S\backslash i}^*, x_j^*, x_{\bar{S}\backslash j}^*, x_i^*\right) = u_i\left(x_S^*, x_{\bar{S}}^*\right). \tag{12}$$

Inequalities (11) and (12) imply

$$u_j\left(x^*\right) \geq u_i\left(x^*\right),$$

which proves the result. ■

Now, we prove the main result of this section, that a CCE always exists in all symmetric supermodular games. The proof of the theorem is constructive in that, it shows that the efficient strategy profile x^e satisfies the requirements of a conjectural equilibrium.

Theorem 1 *Let the game G be a symmetric supermodular game. Then, G admits a conjectural cooperative equilibrium.*

Proof. Let x^e be an efficient strategy profile for G, that is, a strategy profile that maximizes the aggregate payoff of N. Suppose there exists a coalition $S \subset N$ and joint strategy $x_S \in X_S$ such that for all $i \in S$:

$$u_i(x_S, r_{\bar{S}}(x_S)) > u_i(x^e). \tag{13}$$

Note that by lemma 6, it must be that

$$\frac{\sum_{i \in S} u_i(x_S, r_{\bar{S}}(x_S))}{s} \leq \frac{\sum_{j \in N\backslash S} u_j\left(x_S, r_{\bar{S}}(x_S)\right)}{n-s}, \tag{14}$$

otherwise there would exist $h \in S$ and $k \in \bar{S}$ for which

$$u_h\left(x_S, r_{\bar{S}}(x_S)\right) > u_k\left(x_S, r_{\bar{S}}(x_S)\right).$$

By condition (14) we obtain the following implication:

$$\frac{\sum_{i \in S} u_i(x_S, r_{\bar{S}}(x_S))}{s} > u_i(x^e) \Rightarrow \frac{\sum_{j \in \bar{S}} u_j\left(x_S, r_{\bar{S}}(x_S)\right)}{n-s} > u_i(x^e). \tag{15}$$

We conclude that if $u_i(x_S, r_{\bar{S}}(x_S)) > u_i(x^e)$ for all $i \in S$ then, using (13) and (15), we obtain

$$s\frac{\sum_{i \in S} u_i(x_S, r_{\bar{S}}(x_S))}{s} + (n-s)\frac{\sum_{j \in N \setminus S} u_j\left(x_S, r_{\bar{S}}(x_S)\right)}{n-s} > s\frac{\sum_{i \in S} u_i(x^e)}{s} + (n-s)\frac{\sum_{j \in \bar{S}} u_j(x^e)}{n-s} \tag{16}$$

or,

$$\sum_{i \in N} u_i(x_S, r_{\bar{S}}(x_S)) > \sum_{i \in N} u_i(x^e) \tag{17}$$

which contradicts efficiency of x^e. ■

4 On the Existence of Equilibria in Submodular Games

4.1 The Role of the Slope of the Reaction Map

Theorem 1 establishes sufficient conditions for the existence of a conjectural cooperative equilibrium of the game G. The crucial condition, strategic complementarity in the sense of Bulow et al. (1985), generates non decreasing best replies; in particular, the supermodularity of payoff functions implies that the Nash responses of players outside a deviating coalition are a non decreasing function of the strategies of coalitional members. This feature ensures that each players outside S is better off than each coalitional member of S when deviating. Deviations by proper subcoalitions of players are therefore little profitable, while the grand coalition, not affected by this "deviator's curse", produces a sufficiently big aggregate payoff for a stable cooperative outcomes to exist.

In this section we show how the same mechanics responsible for our existence result on the class of supermodular games, provide useful insight for the analysis of games with strategic substitutes, as, for instance, environmental and public goods games. We will use as an illustration an environmental Cobb-Douglas economy to show that *as long as best replies are not "too" decreasing* (thereby providing deviating coalitions with a not "too" big positional advantage), stable cooperative outcomes exist.

4.2 An illustration using a Cobb-Douglas environmental economy

We consider an economy with set of agents $N = \{1, ..i, .., n\}$, in which $z \geq 0$ is the environmental quality enjoyed by agents, $x_i \geq 0$ is a private good, $p_i \geq 0$ is a polluting emission originated as a by-product of the production of x_i. We assume that for each i in N preferences are represented by the Cobb-Douglas utility function

$$u_i\left(z, x_i\right) = z^{\alpha} x^{\beta},$$

technology is described by the production function

$$x_i = p_i^{\gamma},$$

and emissions accumulate according to the additive law

$$z\left(p\right) = A - \sum_{i \in N} p_i \tag{18}$$

where A is a constant expressing the quality of a pollution-free environment. We will assume that γ, α and β are all positive and $\gamma \leq 1$.

We associate with this economy the game G_e with players set N, strategy space $\left[0, p_i^0\right]$ for each i, with $\sum_{i \in N} p_i^0 < A$, and payoffs $U_i(p_1, ..., p_n) = z^{\alpha} p_i^{\delta}$, where $\delta = \beta\gamma$. Using this (symmetric) setup, we can express the maximal per-capita payoff of each coalition S in the event of a deviation from an arbitrary strategy profile in G as follows:

$$u_i\left(S\right) = s^{-\delta} A^{\alpha+\delta} \alpha^{2\alpha} \left(\alpha + \delta\right)^{-\alpha-\delta} \left(\alpha + \delta\left(n - s\right)\right)^{-\alpha} \delta^{\delta}. \tag{19}$$

This simple setup of an environmental economy can be used to illustrate how CCE exist when best replies are not *too* decreasing or, in other terms, when strategies are not too substitute. This in turn requires that players' utilities does not decrease *too much* with other players' choice, a property mainly depending on the level of log-concavity of the term $z\left(p\right)^{\alpha}$. We prove this analytically for the case $\delta = 1$, while we rely on numerical simulation for the general case.

Note that $z\left(p\right)^{\alpha}$ is log-concave (and the game is not log-supermodular) for $\alpha > 0$, and best replies are decreasing. The environmental game admits a unique Nash equilibrium $\bar{p}$ with $\overline{p}_i = \frac{A}{\alpha+n}$ for every $i \in N$, and a unique efficient profile p^e (by efficient we mean "aggregate welfare maximizer"). Simple algebra yields the following expression:

$$u_i\left(S\right) = s^{-1} A \alpha^{\alpha+1} \alpha^{2\alpha} \left(\alpha + 1\right)^{-\alpha-1} \left(\alpha + n - s\right)^{-\alpha}.$$

The profitability of individual deviation from the efficient strategy profile p^e is evaluated as follows:

$$u_i(p^e) - u_i(S) = \alpha^\alpha (\alpha + n - 1)^{-\alpha} n - 1 < 0 \Leftrightarrow \alpha < 1.$$

It follows that when the function $z(p)^\alpha$ is strictly concave ($\alpha < 1$), then no CCE exists. However, when $\alpha = 1$, the CCE is unique, and equal to p^e. It is also easy to show that for $\alpha > 1$ ($z(.)^\alpha$ convex) the strategy profile p^e is still a CCE. We conclude that the existence of a CCE only requires a not too strong log-concavity of $z(.)^\alpha$. This ensures that the marginal utility of each consumer does not decrease too much with the rivals' private consumption and hence, a deviating coalition, by expanding its pollution (and private consumption) does not exploit too much its advantage against complementary players. When this is the case, although the environmental game is a natural "strategic substitute" game, the CCE exists.

It is interesting to relate the existence of a stable cooperative (and efficient) solution with the relative magnitude of the parameters α, β and γ, expressing the intensity of preferences for the environment and for private consumption, and the characteristics of technology. It turns out that in order for an agreement on emissions to be reached, agents must put enough weight on the environment in their preferences (high enough α), and emissions must not be too "productive" according to the available technology. In other words, this conclusion rephrases the common intuition that a clean environment is sustainable only if agents care enough for ambient quality.

As we said, the analysis of existence of a CCE for the general case (that is, removing the assumption $\delta = 1$) is not possible in analytical terms. In what follows we show by means of computations that the set of CCEa of the game Γ_e can be characterized with respect to three possible configurations of the parameter α, β and γ of the economy: the case $\alpha = \beta\gamma$, in which the CCE is unique and assigning to each player the payoff $u_i(p^e)$ (for this case we provide an analytical proof); the case $\alpha > \beta\gamma$, in which the set of CCEa strictly includes the profile p^e; the case $\alpha < \beta\gamma$, in which the set of CCE is empty.

Proposition 1 *If $\alpha = \beta\gamma$ the unique CCE is the efficient profile p^e.*

Proof. We first show that no profile $p \neq p^e$ can be a CCE. By (19) we obtain

$$u_i(p^e) - u_i(\{i\}) = \frac{\alpha^\alpha A^{\alpha+\delta} (\alpha + \delta)^{-\alpha-\delta} \delta^\delta \left[(\alpha + \delta (n-1))^\alpha - \alpha^\alpha n^\delta \right]}{n^\delta (\alpha + \delta (n-1))^\alpha}$$

from which

$$u_i(p^e) - u_i(\{i\}) = 0 \Longleftrightarrow \left[(\alpha + \delta (n-1))^\alpha - \alpha^\alpha n^\delta \right] = 0;$$

Using the fact that $\delta = \beta\gamma$ we get

$$\left[(\alpha + \delta(n-1))^{\alpha} - \alpha^{\alpha} n^{\delta}\right] = [\alpha + \alpha(n-1)]^{\alpha} - (\alpha n)^{\alpha} = 0$$

from which

$$u_i(p^e) = u_i(\{i\}).$$

To show that p^e is a CCE, it suffices to show that $u_i(S) \leq u_i(p^e)$ for all coalitions S such that $s > 1$. Using (19) we obtain

$$u_i(p^e) - u_i(S) \geq 0 \Longleftrightarrow \left[s^{\delta}(\alpha + \delta(n-s))^{\alpha} - \alpha^{\alpha} n^{\delta}\right] \geq 0$$

which, using again the fact that $\delta = \beta\gamma$ reduces to

$$u_i(p^e) - u_i(S) \geq 0 \Longleftrightarrow [s(\alpha + \alpha(n-s))]^{\alpha} \geq (\alpha n)^{\alpha}.$$

The last condition can be rewritten as

$$u_i(p^e) - u_i(S) \geq 0 \Longleftrightarrow s + (n-s)s + s^2 \geq n + s^2$$

which is always satisfied since $s \geq 1$. ■

Proposition 2 *If $\alpha > \beta\gamma$ then p^e is a CCE.*

Proof. We proceed by numerical simulations. Our aim is to show that whenever $\alpha > \beta\gamma$ the difference $u_i(p^e) - u_i(S)$ is positive for every s. We first consider the case $s = 1$. We plot the graph of

$$f_i(\alpha, n) \equiv \max\{(u_i(p^e) - u_i(\{i\})), 0\}$$

for the fixed value of $\delta = 0.5$. The domains are taken to be $(1, 10000)$ for n and $(0, 1)$ for α.

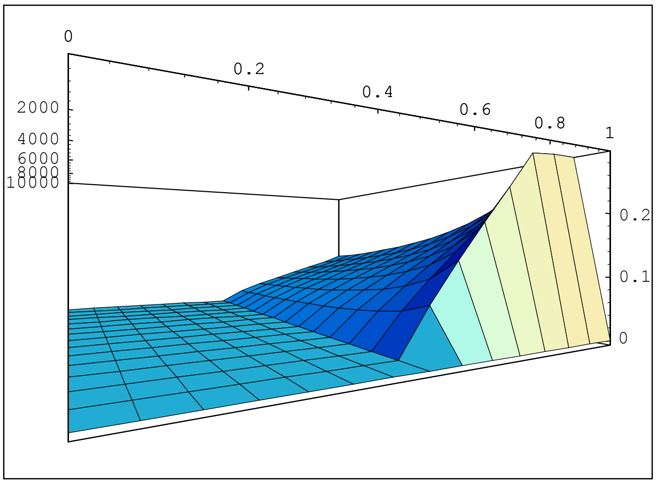

Figure 1: $f_i(\alpha, n)$ for the case $\delta = 0.5$.

From Figure 1 it is evident that $u_i(p^e) > u_i(\{i\})$ whenever $\alpha > 0.5 = \delta$. Similar graph (not provided here) are obtained for other values of δ in the range $(0,1)$.

We perform the same exercise for coalition of size $s > 1$. We plot the function

$$f(\alpha, s) \equiv \max\{(u_i(p^e) - u_i(\{S\})), 0\}$$

for fix values of n and δ. The domains are taken to be $(\delta, 1)$ for α and $(1, n]$ for s. For the case $n = 1000$ and $\delta = 0.2$ we obtain the following graph:

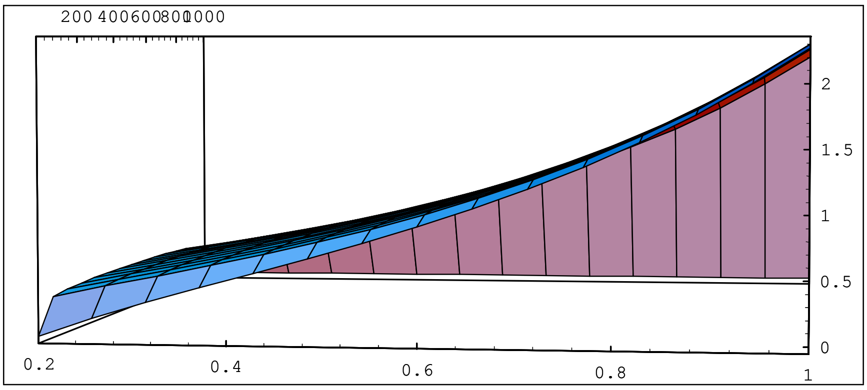

Figure 2: $f(\alpha, s)$ for the case $\delta = 0.2$.

In Figure 2 the graph of $f(\alpha, s)$ all lies above the zero plane for all values of $s \in (1, n]$ and of $\alpha \in (\delta, 1)$. Summing up, whenever $\alpha > \delta$ we found that $u_i(p^e) > u_i\{i\}$ for $s \geq 1$; we thus conclude that whenever $\alpha > \delta$ then p^e is a CCE. ■

Proposition 3 *If $\alpha > \beta\gamma$ there exists no CCE.*

Proof. We again proceed by numerical simulations and evaluate the function

$$\hat{f}_i(\alpha, n) \equiv \min\{(u_i(p^e) - u_i(\{i\})), 0\}$$

for an arbitrary player $i \in N$ and a fixed value of δ. The domains are taken to be $(0, 1)$ for α and $[1, 10000]$ for n. Figure 3 depicts the graph of $\hat{f}_i(\alpha, n)$ for the case $\delta = 0.5$ (different values of δ are reported in the appendix):

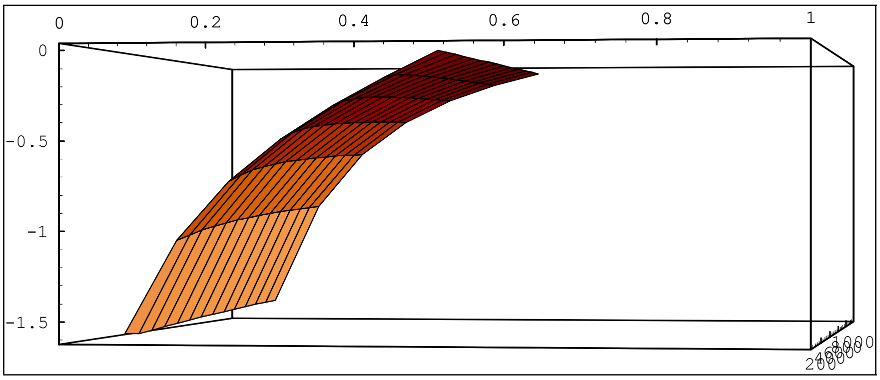

Figure 3: $\hat{f}_i(\alpha, n)$ for the case $\delta = 0.5$.

It is evident from Figure 3 (and from numerical evaluations around the point $\alpha = 0.5$) that for any value of n in the selected range, $u_i(p^e) < u_i(\{i\})$ for the whole range of values of $\alpha < \delta$. We thus conclude that for such values there is no CCE. ■

The above results can be usefully summarized by plotting the value of the difference $[u_i(p^e) - u_i\{i\}]$ as a function of the parameter α for fixed values of δ, n and for $s = 1$.

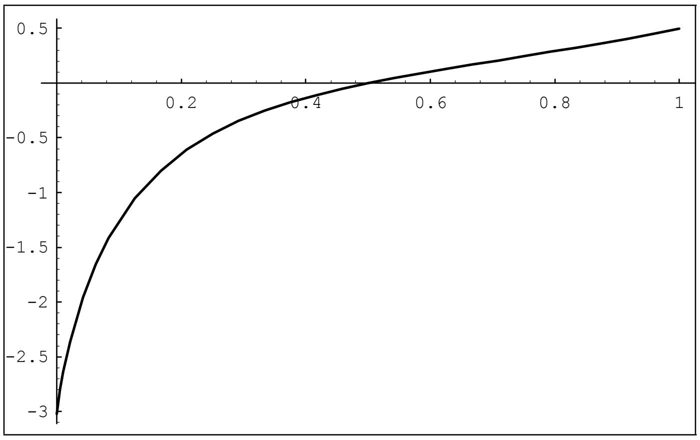

Figure 4: The difference $u_i(p^e) - u_i\{i\}$ for the case $\delta = 0.5$ and $n = 10000$.

5 Concluding remarks

In this paper we have proposed a new cooperative equilibrium concept for games in strategic forms, based on the assumption that deviators expect other players to react optimally and independently in accordance to their best response map. We have employed the properties of reaction maps in supermodular games to show that equilibria exist on this class of games under some additional symmetry axioms. We have also discussed existence of equilibria in submodular games, and in particular, in the case of a specific Cobb-Douglas environmental economy. In particular, we have shown how the degree of submodularity of the associated game, and the existence of an equilibrium, is closely related to the intensity of preferences for the environmental quality and for private consumption. This example formalizes the intuitive insight that if agents care "enough" about the environmental quality, then an efficient agreement on pollution emissions and on cost sharing can be achieved.

APPENDIX

Lemma 7 *If A is a compact sublattice of R^n and f is upper hemi continuous on A, then* $\arg\max_{x\in A} f(x)$ *is a nonempty compact sublattice of R^n, and has a greatest and least element. If $f(x)$ is strictly supermodular, then* $\arg\max_{x\in X} f(x)$ *is a chain.*

Proof. See Topkis (1998). ■

Lemma 8 *Let T be a lattice, and let $(G_t)_{t\in T}$ be a collection of strategic form games. Let the set of strategies X_i^t be constant over T, and let the payoff functions $u_i^t(x_i, x_{-i})$ be upper hemicontinuous in x_i for all $x_i \in X_i$ and $x_{-i} \in X_{-i}$ and have increasing differences in (x_i, x_{-i}) on X. Then there exist a greatest and a least Nash equilibrium elements for each game G_t, and these elements are increasing in t.*

Proof. See Topkis (1998). ■

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