Dividends and Equity Prices: The Variance Trade Off

Margaret Bray
London School of Economics
Houghton Street,
London WC2A 2AE
UK
m.m.bray@lse.ac.uk

Giovanni Marseguerra
Università Cattolica di Milano,
Largo A. Gemelli 1
20123 Milano,
Italy
gmarseg@mi.unicatt.it

April 12, 2002

Abstract

This paper shows that standard corporate finance theory implies that there is potentially a trade off between the variances of dividends and equity prices. We show how the trade off works in a stochastic difference equation model of dividend policy, demonstrating that the solution may be unstable for plausible parameter values. At the boundary of the feasible set of price and dividend variances, prices and dividends are perfectly correlated and both follow an AR(1) process. We calculate explicit formulae for the variances, and show that firms could it principle make prices completely predictable, by immediately incorporating all news about the present value of earnings into dividends. By choosing to smooth dividends firms increase the variance of prices, and may also increase the variance of dividends. We show how this can easily result in sample variances which violate variance bounds inequalities.

Acknowledgement. *We are grateful for comment from Charles Goodhart, Erzo Luttmer, PierCarlo Nicola and participants in the Financial Markets Group Workshop at LSE.*

1 Introduction

Dividends famously pose a puzzle (Black, 1976), or more accurately puzzles, which involve both asset pricing and corporate finance. Indeed the simplest theories in these two areas generate an apparent paradox. Asset pricing theory says that the value of equity is the expected present value of future dividends. Corporate finance theory (Modigliani Miller, 1958, 1961) says that dividends are irrelevant, the total value of a firm does not depend on financial policy. In a firm which issues debt and equity, the value of equity is equal to the value of future earnings minus the value of debt, regardless of dividend policy. The resolution of the paradox is that dividend policy is ultimately constrained by capital structure, in the end the capital markets will not allow firms to finance high dividends by selling shares or issuing debt if earnings do not warrant it. This suggests modelling the dividend process and capital structure at the same time. This paper aims at developing such an approach (see Bray and Marseguerra, 1997, and Marseguerra, 1998, for preliminary results in this line of research). This takes us outside the usual framework for the analysis of dividend policy, which takes the capital structure and net dividend (dividends plus share repurchases minus share issues) as given, and looks at the decomposition of the net dividend into its three components. The classic question here is Black's (1976) original dividend puzzle, why do companies pay dividends given the tax disadvantages of doing so (see Allen and Michaely, 1995, for a survey of the vast literature on dividend policy, and Allen Bernardo and Welch, 2000, for a recent explanation of the puzzle based on tax clienteles).[1]

We in contrast take models of dividend setting as determining the net dividend, and look at their implications for the stochastic processes of earnings, equity values, debt and dividends within a simple and very orthodox framework, in which assets values are expected present discounted values with a constant discount rate, and the Modigliani Miller theorem holds. This is conceptually a simple exercise, although algebraically burdensome in places. Why is it worth doing? Because we believe the results are surprising, and go some way towards explaining why it has been so hard to resolve some aspects of the dividend puzzle.

We start with a very simple and orthodox model of the firm in section 2, and show that whatever the dividend policy

$$p_t + d_t = (1 + r)p_{t-1} + e_t \tag{1.1}$$

where, r is the expected rate of return, and (this is the substance of the proposition) e_t is the innovation at date t in the expected present discounted value of current and future earnings from date t onwards. This is not a deep or surprising result, but it does highlight an important aspect of dividend policy; how do dividends respond to the shock e_t in the expected present discounted value of earnings? The stylised facts of dividend smoothing (Lintner, 1956) suggests

[1] See also Bagwell and Shoven,1989, and Fama and French, 2001, for changes in the size and composition of net dividends.

that the answer is not much, in which case (1.1) implies that the shock is largely manifested in the equity price, so taking the variance of earnings shocks as given smooth dividends would seem to imply bumpy prices. We explore this further in Section 3, showing that if detrended equity prices, dividends and earnings shocks are stationary, then equation (1.1) implies bounds on the variances of dividends and equity prices which are stated in Proposition 2 and graphed in Figure 2. The feasible set has two important features. Firstly it confirms the possibility of a trade off between the variances of prices and dividends. Secondly at the boundaries of the feasible set prices and dividends are perfectly correlated, something we come back to in Section 6 where we show that this implies that detrended prices and dividends follow an AR(1) process.

Further progress requires modelling of dividends and earnings which we provide in Section 4 in which dividends are a linear function of lagged dividends and capital structure variables (the expected present value of earnings, and the value of debt and equity), earnings and earnings shocks. The model is sufficiently general to have as special cases both the major hypotheses on dividend policy. One is the partial adjustment hypothesis, originally due to Lintner (1956), which states that companies have a target dividend-earnings ratio, but smooth their dividends by only adjusting to it partially. The alternative permanent earnings hypothesis makes dividends a function of some measure of permanent earnings, one such measure being the equity price (as in Marsh and Merton, 1986). The empirical literature has sought to distinguish between the two hypotheses (see for example Fama and Babiak, 1968), but (as we argue in Section 4) this may be difficult to achieve because dividend smoothing can make the two hypotheses observationally equivalent.

There is a long standing debate on whether dividends have a signalling role, whether dividends convey any information which is not already incorporated in earnings, and whether there is a conflict between dividend signalling and smoothing. There are a number of theoretical models of dividend signalling (e.g., Bhattacharya, 1979, Miller and Rock, 1985, John and Williams, 1985). These are all models of one round of dividend setting. We have not addressed the daunting technical difficulties in deriving a signalling model based on optimising assumptions which can extend over an unlimited number of dividend decisions. However our model of dividend setting allows but does not force dividends to convey information about future earnings, which may or may not provide information additonal to that in current earnings.

We model earnings in Section 4 as the sum of a process which can be AR(1) or a random walk, and a white noise process. This is the simplest way of allowing both persistent and transient shocks to earnings. There are two basic approaches to tackling the non-stationarity (see, e.g., Hamilton, 1994). One is to difference the data, for example finding that the first difference of data is stationary. An alternative approach is to estimate an exponential trend, which is used to detrend the data. We allow here for both possibilities. The model gives a set of three simultaneous linear stochastic difference equations (4.7) and (4.8).

The first questions to ask about such a set of equations is whether the resulting solution is stable, and if not stable whether the variables are cointegrated. This is addressed in Proposition 3 and illustrated in Figures 3 and 4. The result depends upon the stability of the earnings process, and the coefficients of lagged dividends and lagged equity prices in the dividend equation, which determine the eigenvalues of the matrix in the difference equation system. We show that plausible values of the coefficients generate eigenvalues which are close to the boundary of the unit circle, so it is not surprising that as the empirical literature suggests (DeJong and Whitheman, 1991), the stationarity and cointegration of prices and dividends is a delicate issue. If dividends are not affected by the balance sheet in the form of lagged equity prices or debt one or more of the roots lies outside the unit circle. If dividends are unaffected by both lagged equity prices and lagged dividends, for example if the company never pays any dividends, both roots lie outside the unit circle.

We solve the model in Section 5. Proposition 4 gives formulae for the time path of prices and dividends generated by the difference equations. These allow us to see the relationship between the partial adjustment and permanent earnings hypotheses on dividend determination, the information conveyed by dividends and the resulting price, dividend and earnings processes. The results make it straightforward to specify conditions under which some or all of prices, dividends and earnings are perfectly correlated. We show that if under these conditions detrended equity prices and dividends are stationary, the variances of prices and dividends lie on the boundary of the feasible set defined in Section 3, and the partial adjustment and permanent income hypotheses of dividend setting are observationally equivalent.

We can also look at the response of dividends and equity prices to current and lagged permanent and transient shocks in earnings. The specification of the dividend equation allows us to follow Lintner's (1956) survey results giving transitory shocks no role in immediate dividend setting. However our model implies that in that case transitory shocks to earnings before interest will inevitably have long terms effects on dividends and equity prices. This effect can work through the effect of dividends on debt, and thus on earnings after interest. In the econometric literature earnings are usually treated as an exogenous variable in explaining dividends (e.g., Lee, 1996, and Chiang et al., 1997), but we suggest this may not be so. This may account for the strong evidence of non-stationarity in earnings (see, e.g., Ali and Zarowin, 1992), so much so that some shocks to earnings appear to last for ever without diminishing. In a world of changing markets and technology this is a somewhat surprising feature of earnings before interest.[2] But if earnings are earnings after interest and therefore endogenous, dividend policy, and in particular dividend smoothing, may account for the non-stationarity of earnings.

In Section 6 we come back to the Section 2 insight that the choice of how much to make dividends move in response to shocks in the expected present

[2] It is however worth remembering, as clearly stated by Cambell and Shiller, that "the precise economic meaning of earnings data is not clearly defined; accounting definitions are complicated and change through time in ways that are not readily documented" (Campbell and Shiller, 1988, p.661).

value of earnings can generate a trade off between the variances of prices and dividends, and derive formulae for these variances which apply when the detrended price and dividend processes are stationary. We show that if one or both of the roots of the equation determining stability lies close to the boundary of the unit circle, something which appears thoroughly plausible, there is in fact little scope for trade offs through this mechanism, and as both roots approach the boundary both variances tend to infinity. We then look at the scope for trade off along the boundary of the feasible set of price and dividend variances. We show that there is enormous scope for trade off here, and indeed it is possible to make prices, but not dividends, follow a deterministic trend, by putting the entire shock to the present value of expected earnings into dividends. We show that by choosing to smooth dividends companies are choosing prices which have large variances both because there is a big initial price response to earnings shocks, and because the shock is persistent.

This is suggestive of the literature discussing Variance Bounds (Shiller (1981) and the subsequent discussion Flavin (1983), Kleidon (1986), Marsh and Merton,(1986)), and the model of this paper does indeed make it easy to see why Variance Bounds estimates can be problematic. In Section 7 we construct a limiting case in which the detrended equity price p_t at date t is close to a random walk $p_0 + \sum_{j=1}^{t} \hat{e}_j$ where $\hat{e}_j$ is the innovation at date j in the expected present discounted value of earnings from detrended earnings from date t onwards. The detrended perfect foresight price p^*_{tT} (the present value of realised dividends from date $t + j$ to T and the equity price at date T) is close to $p_0 + \sum_{j=1}^{T} \hat{e}_j$. (Note that t does not appear in this formula.) At any date t before T the variance bounds inequality is satisfied, $\operatorname{var} p_t < \operatorname{var} p^*_{tT}$. However as the detrended perfect foresight is virtually the same for all t, the expected sample variance of detrended perfect foresight prices is almost zero, whereas the expected sample variance of the detrended prices is bounded away from zero. The sample variances are grossly biased as estimates of variances, and the bias is much greater for perfect foresight prices. Computer simulations generate gross violations of variance bounds.

Section 8 concludes the paper with a discussion of the robustness of the insights gained to less defensible of our assumptions.

2 Dividends and Equity Prices

We start with some very standard asset pricing and corporate finance theory assumptions.

Assumption 1: Asset Pricing

$$E_{t-1}(\widehat{d}_t + \widehat{p}_t) = (1 + r)\widehat{p}_{t-1} \tag{2.1}$$

where $\widehat{d}_t$ is the dividend per share, $\widehat{p}_t$ price per share and r is the expected rate of return.

The substance of this assumption is that the expected rate of return is constant over time. This an unrealistic benchmark (Campbell, Lo and MacKinlay 1997) made for the sake of tractability and simplicity. We believe that the key insights of this paper would hold with a more complicated model of expected returns.

This paper relates dividends and share prices to the earnings and balance sheet of the firm. This requires attention to net dividends d_t defined as the sum of total dividends plus share repurchases minus share issues

$$d_t = n_{t-1}\widehat{d}_t + (n_{t-1} - n_t)\widehat{p}_t \tag{2.2}$$

where n_t is the number of shares outstanding at date t. Dividends are paid at date t on the n_{t-1} shares outstanding at date $t-1$, and $n_{t-1} - n_t$ shares are repurchased at price per share $\widehat{p}_t$. We use notation

$$p_t = n_t\widehat{p}_t. \tag{2.3}$$

so p_t is the total value of equity at date t, immediately after dividend payments and share issues or repurchases. (For the sake of simplicity we assume that these all take place at the same time.) Note from (2.2) and (2.3) that $d_t + p_t = n_{t-1}\left(\hat{d}_t + \hat{p}_t\right)$ so $\frac{d_t+p_t}{p_{t-1}} = \frac{\hat{d}_t+\hat{p}_t}{\hat{p}_{t-1}}$ and thus, from the point of view of returns, we can work with either returns per share using the gross dividend per share, which is the focus of the asset pricing literature, or returns on the total value of equity using the total net dividend which is more natural in a corporate finance context.

Assumption 2: Sources and Uses of Funds

$$d_t + (1+r)B_{t-1} = x_t + B_t \tag{2.4}$$

where B_t is borrowing at date t and x_t is earnings net of investment at date t.

The sources and uses of funds is an accounting identity. However there is economic content in the assumption that the interest rate r in the sources and uses of funds identity (2.4) is the same as the expected return in the asset pricing equation (2.1), a simplifying assumption which is hard to justify. In the context of the CAPM this is equivalent to assuming that the risk premium on debt is the same as the risk premium on equity, a somewhat heroic assumption. Working with a more general assumption would change the interpretation of the model in a way which we will discuss shortly, when we have proved Proposition 1.

Assumption 3: Earnings *The expected present discounted value of future earnings*

$$E_t\left[\sum_{i=1}^{n}\frac{x_{t+i}}{(1+r)^i}\right]$$

tends to a finite limit as n tends to infinity.

This assumption is standard, if it is not satisfied the value of the firm is not well defined.

Assumption 4: No Bubbles *The expected present discounted value of equity n periods hence*

$$E_t\left[\frac{p_{t+n}}{(1+r)^n}\right]$$

tends to 0 *as n tends to infinity.*

Again this is a standard assumption.

Assumption 5: Borrowing Constraint *The expected present discounted value of borrowing n periods hence*

$$E_t\left[\frac{B_{t+n}}{(1+r)^n}\right]$$

tends to 0 *as n tends to infinity.*

This is an assumption on the willingness of creditors to lend to the firm. It rules out for example the possibility that the firm's creditors continually roll over the existing debt without requiring any repayment of capital and interest. It also rules out the possibility that firms hold negative debt, that is a cash pile, which simply grows at the rate of interest. It is not so obvious that firms will be unable to do this, although a free cash pile surely offers even greater temptations to management than a cash pile, and they may be constrained by their shareholders, or the threat or reality of take-over (Jensen, 1986).

The following Proposition is an almost immediate consequence of the assumptions. It states that the unanticipated part of asset returns is the innovation in the expected present value of current and future earnings. This is not surprising, and it is easy to prove. We state it as a Proposition because it is central to the argument of this paper. When firms get news about their current and future earnings they have to make a decision as to how to respond to that news in setting their dividend. If they do not adjust their net dividend by e_t, the entire innovation in the expected present value of current and future earnings, the standard assumptions which we make imply that part that innovation must be reflected in the value of equity p_t. If there are no share issues or repurchases, or decisions on issues and repurchases are announced before gross dividends, any part of e_t which is not immediately incorporated into future earnings is reflected in the equity price.

Proposition 1 (Dividend Shocks) *Suppose that Assumptions 1-5 above hold. Then*

$$p_t + d_t = (1+r)p_{t-1} + e_t \tag{2.5}$$

$$p_t + B_t = \frac{E_t V_{t+1}}{1+r}. \tag{2.6}$$

where

$$V_t = E_t \left[\sum_{i=0}^{\infty} \frac{x_{t+i}}{(1+r)^i} \right]. \tag{2.7}$$

and

$$e_t = E_t V_t - E_{t-1} V_t \tag{2.8}$$

so e_t *is the innovation in* V_t*, the expected present discounted value of current and future earnings at date* t*.*

Proof. Multiplying equation (2.1) by n_{t-1} and using equations (2.2) and (2.3) implies that

$$E_{t-1}(d_t + p_t) = (1+r)p_{t-1}.$$

Iterating this equation gives

$$p_t = E_t \left[\sum_{i=1}^{n} \frac{d_{t+i}}{(1+r)^i} + \frac{p_{t+n}}{(1+r)^n} \right].$$

Using equation (2.4) to eliminate d_{t+i} the equation above from implies that

$$p_t + B_t = E_t \left[\sum_{i=1}^{n} \frac{x_{t+i}}{(1+r)^i} + \frac{(p_{t+n} + B_{t+n})}{(1+r)^n} \right]. \tag{2.9}$$

Assumptions 3-5 then imply that

$$p_t + B_t = E_t \left[\sum_{i=1}^{\infty} \frac{x_{t+i}}{(1+r)^i} \right] \tag{2.10}$$

which establishes (2.6). Also from (2.10) and (2.4)

$$\begin{aligned} & p_t + d_t - (1+r)p_{t-1} \\ = \; & E_t \left[\sum_{i=1}^{\infty} \frac{x_{t+i}}{(1+r)^i} \right] - B_t + [x_t + B_t - (1+r)B_{t-1}] \\ & -(1+r)E_{t-1} \left[\sum_{i=1}^{\infty} \frac{x_{t-1+i}}{(1+r)^i} \right] + (1+r)B_{t-1} \\ = \; & E_t \left[\sum_{i=0}^{\infty} \frac{x_{t+i}}{(1+r)^i} \right] - E_{t-1} \left[\sum_{i=0}^{\infty} \frac{x_{t+i}}{(1+r)^i} \right] = e_t. \end{aligned}$$

■

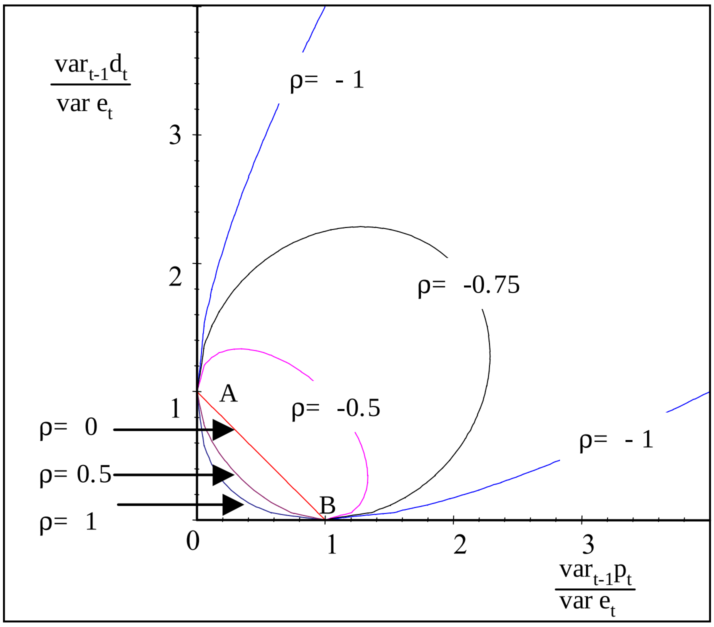

Figure 1: Plots of the conditional variances of prices and dividends for different values of ρ_{t-1}, the conditonal correlation of prices and dividends.

Assumption 2, which implies that the interest rate on debt is constant and the same as the expected return on equity plays an important role here. If we had worked with a richer and more realistic model of debt and returns the equation $p_t+d_t = (1+r)p_{t-1}+e_t$ would still hold, but e_t would be the innovation in the value of equity which would respond to news about earnings, expected returns, and the value of debt. We have assumed for the sake of simplicity that the value of debt does not respond to news because we wish to focus on dividends.

Proposition 1 has immediate consequences for the conditional variances of dividends and prices. Equation (2.5) implies that

$$\operatorname{var}_{t-1} p_t + 2\rho_{t-1}(p_t, d_t) \left(\operatorname{var}_{t-1} p_t\right)^{\frac{1}{2}} \left(\operatorname{var}_{t-1} d_t\right)^{\frac{1}{2}} + \operatorname{var}_{t-1} d_t = \operatorname{var} e_t$$

where the notation var_{t-1} is used for the conditional variance at date $t-1$ and $\rho_{t-1}(p_t, d_t)$ is the conditional correlation so $-1 \leq \rho_{t-1}(p_t, d_t) \leq 1$.

Figure 1 shows the trade off between the conditional variances of p_t and d_t. At point A next period's price is made completely predictable by letting the innovation in the dividend be e_t, the innovation in the present discounted value of earnings given by equations (2.8) and (2.7). At point B next period's dividend is completely predictable, in which case the entire innovation e_t is incorporated into the price. Under the assumptions of this model firms which wished to reduce the conditional variances of both dividends and prices would wish to be on the segment AB for which $\rho = 1$ in Figure 1.

3 The Variance Trade Off

We now make some simple points about the trade off between the variances of prices and dividends under the assumption that there is a number $\gamma \in (0, 1+r)$ such that detrended dividends $\gamma^{-t} d_t$, equity prices $\gamma^{-t} p_t$ and earnings innovations $\gamma^{-t} e_t$ are stationary and have variances. This is not an innocuous assumption, it is not for instance satisfied if dividends and equity prices follow a random walk and are stationary in first differences. We will work with a model of dividends and earnings which, depending on parameter values may or may not satisfy this assumption. There are clearly more general models than the one we use which also satisfy this assumption on variances. However, the specification here assumed does allow us to make a point on the trade offs between dividend smoothing and the volatility of equity prices which is summed up in the next Proposition.

Proposition 2 (The Variance Trade Off) *Suppose that assumptions 1-5 hold, and that dividend policy and the earnings process is such there is a positive constant γ for which detrended prices $\gamma^{-t} p_t$, dividends $\gamma^{-t} d_t$, and the innovation in the expected present value of earnings $\gamma^{-t} e_t$ are stationary. Let $\sigma_d^2 = \operatorname{var} \gamma^{-t} d_t$, $\sigma_p^2 = \operatorname{var} \gamma^{-t} p_t$ and $\sigma_e^2 = \operatorname{var} \gamma^{-t} e_t$. Then if*

$$k = \frac{\gamma}{1+r} \tag{3.1}$$

(and so $0 < k < 1$), the following upper and lower bounds hold:

$$(k^{-2}+1)\sigma_p^2 + \sigma_e^2 + 2\sigma_p\sqrt{k^{-2}\sigma_p^2 + \sigma_e^2} \geq \sigma_d^2 \geq (k^{-2}+1)\sigma_p^2 + \sigma_e^2 - 2\sigma_p\sqrt{k^{-2}\sigma_p^2 + \sigma_e^2}.$$

The upper bound is satisfied as an equality if and only if prices and dividends are perfectly negatively correlated, the lower bound is satisfied as an equality if and only if prices and dividends are perfectly positively correlated. The lower bound on σ_d^2 decreases from a value of σ_e^2 to $\left(1-k^2\right)\sigma_e^2$ as σ_p^2 increases from 0 to $\frac{k^4\sigma_e^2}{(1-k^2)}$ and thereafter increases as σ_p^2 increases. The upper bound on σ_d^2 is an increasing function of σ_p^2.

Proof. Equation (2.5) implies that

$$\sigma_p^2 + 2\rho\sigma_p\sigma_d + \sigma_d^2 = k^{-2}\sigma_p^2 + \sigma_e^2 \tag{3.2}$$

where $\rho = \operatorname{correlation}(p_t, d_t)$. Treating this equation as a quadratic in σ_d

$$\sigma_d = -\rho\sigma_p + \sqrt{(k^{-2} - 1 + \rho^2)\sigma_p^2 + \sigma_e^2}. \tag{3.3}$$

We ignore the negative root because as $k^{-2} > 1$ it makes the standard deviation σ_d negative. From (3.3)

$$\begin{aligned}\frac{\partial \sigma_d}{\partial \rho} &= -\sigma_p + \frac{\rho\sigma_p^2}{\sqrt{(k^{-2}-1+\rho^2)\sigma_p^2+\sigma_e^2}} \\ &= -\sigma_p\left(1-\sqrt{\frac{\rho^2\sigma_p^2}{(k^{-2}-1+\rho^2)\sigma_p^2+\sigma_e^2}}\right).\end{aligned}$$

As by assumption $k^{-2} > 1$, and σ_p is a standard deviation so non-negative, $\frac{\partial \sigma_d}{\partial \rho} \leq 0$. As ρ is a correlation it must lie in $[-1, 1]$. Thus the lower bound on σ_d is derived by setting $\rho = 1$, and the upper bound by setting $\rho = -1$ in (3.3), which gives the bounds on σ_d^2. From (3.3)

$$\frac{\partial \sigma_d}{\partial \sigma_p} = -\rho + (k^{-2}-1+\rho^2)\sqrt{\frac{\sigma_p^2}{(k^{-2}-1+\rho^2)\sigma_p^2+\sigma_e^2}}.$$

If $\rho \leq 0$ this is positive, so for given ρ, k and σ_e^2, as $\sigma_d \geq 0$, σ_d^2 is an increasing function of σ_p^2. In particular if $\rho = -1$, $\frac{\partial \sigma_d}{\partial \sigma_p} > 0$, so the upper bound on σ_d^2 is strictly increasing. If $\rho = 1$

$$\frac{\partial \sigma_d}{\partial \sigma_p} = -1 + k^{-2}\sqrt{\frac{\sigma_p^2}{k^{-2}\sigma_p^2+\sigma_e^2}}$$

which is zero if $\sigma_p^2 = \frac{k^4\sigma_e^2}{(1-k^2)}$, negative for smaller, and positive for larger values of σ_p^2. At the minimum, substituting $\sigma_p^2 = \frac{k^4\sigma_e^2}{(1-k^2)}$, the lower bound on σ_d^2 gives a minimal value of σ_d^2 of $\left(1-k^2\right)\sigma_e^2$. ■

This Proposition establishes that standard assumptions on asset pricing and corporate finance imply that there is potentially a trade-off between the variance of prices and the variance of dividends. This is contrary to the intuition that because prices are the expected present value of dividends it is surprising that prices are bumpy whilst dividends are smooth. We will show how this trade off can be achieved, and argue that all the evidence from Lintner (1956) onwards which suggests that firms do not adjust dividends instantly in response to both permanent and transitory changes in earnings implies that if standard theory applies firms are choosing to make equity prices more volatile than they need be.

Figure 2 illustrates the scope for the variance trade off, plotted for $k = \frac{\gamma}{1+r} = \frac{1}{1.05}$, for which the minimal value of σ_d^2 is $0.093\sigma_e^2$ obtained when $\sigma_p^2 = 8.849\sigma_e^2$. The lower boundary of the feasible set is a decreasing function of σ_p^2 for value below $8.849\sigma_e^2$ and an increasing function for larger values of σ_p^2. However as Figure 2 shows the lower boundary is almost flat once σ_p^2 reaches $3\sigma_e^2$.

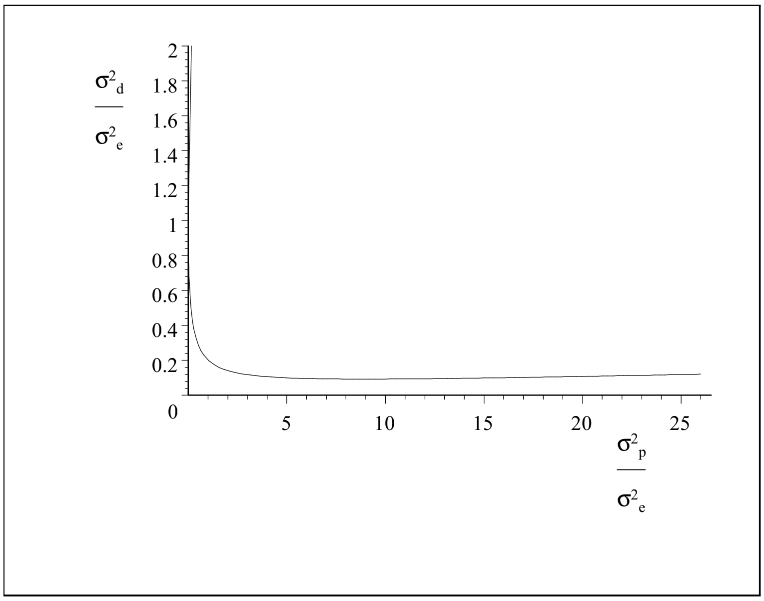

Figure 2: The boundary of the feasible set of price and dividend variances for $k = 1.05^{-1}$.

4 Dividends and Earnings

Our model of earnings is the simplest which allows for growth in earnings, autocorrelation in earnings, and allows us to make a distinction between news which is only relevant to current earning ε_{1t}, and news which also has an impact on expectations of future earnings. ε_{2t}. These are earnings before interest payments, but our model of dividend policy is sufficiently general to allow dividends to depend upon earnings after interest.

Assumption 6: Earnings Process *Earnings x_t are given by*

$$\frac{x_t}{\gamma^t} = \alpha + \frac{(1 + r - \phi\gamma)}{(1 + r)} y_t + \varepsilon_{1t} \tag{4.1}$$

where[3]

$$y_t = \phi y_{t-1} + \varepsilon_{2t} \tag{4.2}$$

[3] The term $\frac{(1+r-\phi\gamma)}{(1+r)}$ in (4.1) may appear strange. It is there to simplify notation by making $\gamma^t(\varepsilon_{1t} + \varepsilon_{2t})$ rather than a more complicated function of ε_{1t} and ε_{2t} the innovation in the expected present value of earnings $V_t = E_t\left[\sum_{i=0}^{\infty} \frac{x_{t+i}}{(1+r)^i}\right]$. To see this note that

$$0 < \gamma < 1 + r$$

$$-\frac{1+r}{\gamma} < \phi < \frac{1+r}{\gamma}.$$

$E\varepsilon_{1t} = E\varepsilon_{2t} = 0$, $\operatorname{var}\varepsilon_{1t} = \sigma^2_{\varepsilon 1}$ and $\operatorname{var}\varepsilon_{2t} = \sigma^2_{\varepsilon 2}$ for all t, $\operatorname{cov}(\varepsilon_{1t}, \varepsilon_{2s}) = 0$ for all s and t, and $\operatorname{cov}(\varepsilon_{1t}, \varepsilon_{1s}) = cov(\varepsilon_{2t}, \varepsilon_{2s}) = 0$ for all $t \neq s$.

This allows for a variety of earnings process including the possibility that earnings follow a random walk ($\alpha = 0$, $\operatorname{var}\varepsilon_{1t} = \sigma^2_{\varepsilon 1} = 0$, $\gamma = \phi = 1$) or a constant plus an AR(1) process ($\operatorname{var}\varepsilon_{1t} = \sigma^2_{\varepsilon 1} = 0$, $\gamma = 1$, $|\phi| < 1$). Thus (4.1) and (4.2) imply $e_t = E_t V_t - E_{t-1} V_t = \gamma^t (\varepsilon_{1t} + \varepsilon_{2t})$ and so from Proposition 1

$$p_t + d_t = (1+r)p_{t-1} + \gamma^t(\varepsilon_{1t} + \varepsilon_{2t}). \tag{4.3}$$

Our dividend model is given by

Assumption 7: Dividend Policy

$$d_t = \gamma^t h + a d_{t-1} + b p_{t-1} + c\gamma^t y_{t-1} + f_1 \gamma^t \varepsilon_{1t} + f_2 \gamma^t \varepsilon_{2t} + f_3 \gamma^t \varepsilon_{3t} \tag{4.4}$$

where ε_{1t} and ε_{2t} are the innovations in the earnings process and ε_{3t} is another white noise process which is included to allow greater generality and is orthogonal to $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$.

As current earnings are given by (4.1) and (4.2), and the expected present value of future earnings by (2.7), this dividend policy allows for the dependence of dividends on lagged dividends, equity prices and current earnings, the expected present value of future earnings, and innovations in current and expected future earnings. From (2.6)

$$p_{t-1} + B_{t-1} = E_{t-1}\left[\sum_{i=1}^{\infty} \frac{x_{t-1+i}}{(1+r)^i}\right] = E_{t-1}\frac{V_t}{1+r} = \frac{\alpha\gamma^t}{1+r-\gamma} + \frac{\phi\gamma^t}{1+r} y_{t-1}$$

so given the inclusion of p_{t-1} and y_{t-1} in the dividend equation, lagged borrowing B_{t-1} can also affect dividends. Thus dividends can be a function of earnings after interest payments on debt. Our specification of dividend policy includes both Lintner's (1956) model which makes dividends adjust partially to

equations (4.1) and (4.2) imply that

$$\begin{aligned} V_t &= E_t\left[\sum_{i=0}^{\infty} \frac{x_{t+i}}{(1+r)^i}\right] = \gamma^t \varepsilon_{1t} + \gamma^t \sum_{i=0}^{\infty}\left(\frac{\alpha\gamma^i}{(1+r)^i} + \frac{1+r-\phi\gamma}{1+r}\frac{\phi^i\gamma^i}{(1+r)^i} y_t\right) \\ &= \gamma^t\left(\varepsilon_{1t} + \frac{\alpha(1+r)}{1+r-\gamma} + y_t\right) = \gamma^t(\varepsilon_{1t} + \varepsilon_{2t}) + \gamma^t \frac{\alpha(1+r)}{1+r-\gamma} + \gamma^t \phi y_{t-1}. \end{aligned}$$

Note the role of the assumptions that $0 < \gamma < 1 + r$ and $-\frac{1+r}{\gamma} < \phi < \frac{1+r}{\gamma}$ in ensuring that the infinite sums converge.

a target fraction of current earnings, and a linear version of Marsh and Merton (1986) which makes dividends proportional to the value of equity. This is not a model of signalling, but the dividend equation does allow for the possibility that both dividends and earnings are informative about future earnings, and thus the value of the firm.

The dividend policy equation can also be written as

$$\left(\frac{d_t}{\gamma^t}\right) = h + \left(\frac{a}{\gamma}\right)\left(\frac{d_{t-1}}{\gamma^{t-1}}\right) + \left(\frac{b}{\gamma}\right)\left(\frac{p_{t-1}}{\gamma^{t-1}}\right) + cy_{t-1} + f_1\varepsilon_{1t} + f_2\varepsilon_{2t} + f_3\varepsilon_{3t}. \tag{4.5}$$

From (4.5) and (4.3)

$$\begin{aligned}\left(\frac{p_t}{\gamma^t}\right) &= -h - \left(\frac{a}{\gamma}\right)\left(\frac{d_{t-1}}{\gamma^{t-1}}\right) + \left(\frac{1+r-b}{\gamma}\right)\left(\frac{p_{t-1}}{\gamma^{t-1}}\right) - cy_{t-1} \\ &\quad +(1-f_1)\varepsilon_{1t} + (1-f_2)\varepsilon_{2t} - f_3\varepsilon_{3t}\end{aligned} \tag{4.6}$$

Equations (4.5), (4.6) and (4.2) give a set of simultaneous linear stochastic difference equations defining the evolution of $(\gamma^{-t}d_t, \gamma^{-t}p_t, y_t)$ which can be written as

$$\begin{bmatrix} \gamma^{-t}d_t \\ \gamma^{-t}p_t \\ y_t \end{bmatrix} = \begin{bmatrix} h \\ -h \\ 0 \end{bmatrix} + M \begin{bmatrix} \gamma^{-(t-1)}d_{t-1} \\ \gamma^{-(t-1)}p_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} f_1\varepsilon_{1t} + f_2\varepsilon_{2t} + f_3\varepsilon_{3t} \\ (1-f_1)\varepsilon_{1t} + (1-f_2)\varepsilon_{2t} - f_3\varepsilon_{3t} \\ \varepsilon_{2t} \end{bmatrix} \tag{4.7}$$

where

$$M = \gamma^{-1}\begin{bmatrix} a & b & \gamma c \\ -a & 1+r-b & -\gamma c \\ 0 & 0 & \gamma\phi \end{bmatrix}. \tag{4.8}$$

Note that mathematically the simple model of prices and dividends defined by the following equations (4.9) and (4.10)

$$\gamma^{-t}d_t = h + \frac{a}{\gamma}\gamma^{-(t-1)}d_{t-1} + \frac{b}{\gamma}\gamma^{-(t-1)}p_{t-1} + f_1\gamma^{-t}e_t \tag{4.9}$$

$$\gamma^{-t}p_t = -h - \frac{a}{\gamma}\gamma^{-(t-1)}d_{t-1} + (k^{-1} - \frac{b}{\gamma})\gamma^{-(t-1)}p_{t-1} + (1-f_1)\,\gamma^{-t}e_t \tag{4.10}$$

is a special case of the model of equation (4.7) in which $c = 0$, $f_1 = f_2$, $f_3 = 0$ and $\gamma^{-t}e_t = \varepsilon_{2t} + \varepsilon_{1t}$. However economically the model of (4.9) and (4.10) is both less restrictive on the earnings process than that of (4.7) as it makes no assumptions on earnings beyond the stationarity of $\gamma^{-t}e_t$, and more restrictive on dividend policy, as it does not allow earnings to enter the dividend equations apart from through e_t.

Standard results on linear stochastic difference equations tell us that the existence of a stable solution of both models depends on whether the eigenvalues of M lie in the unit circle. The eigenvalues are the roots ϕ, $\lambda_1$and λ_2 of the characteristic equation of the matrix M

$$\left[\lambda^2 - \left(\frac{1+r+a-b}{\gamma}\right)\lambda + \frac{a\,(1+r)}{\gamma^2}\right][\phi - \lambda] = 0$$

so

$$\lambda_1 = \frac{1}{2\gamma}\left(1+r+a-b+\sqrt{(1+r+a-b)^2 - 4a(1+r)}\right) \tag{4.11}$$

$$\lambda_2 = \frac{1}{2\gamma}\left(1+r+a-b-\sqrt{(1+r+a-b)^2 - 4a(1+r)}\right) \tag{4.12}$$

where λ_1 and λ_2 are the roots of

$$\lambda^2 - \left(\frac{1+r+a-b}{\gamma}\right)\lambda + \frac{a\,(1+r)}{\gamma^2} = 0. \tag{4.13}$$

We prove in the Appendix:

Proposition 3 *The eigenvalues of M λ_1 and λ_2 are the roots of (4.13) and are given by (4.11) and (4.12). They are complex if and only if $a > 0$ and*

$$\left(\sqrt{1+r} - \sqrt{a}\right)^2 < b < \left(\sqrt{1+r} + \sqrt{a}\right)^2. \tag{4.14}$$

If λ_1 and λ_2 are complex they are complex conjugates. The roots λ_1 and λ_2 lie in the interior of the unit circle if and only if $a < \frac{\gamma^2}{1+r}$ and

$$(1+r-\gamma)\left(1-\frac{a}{\gamma}\right) < b < (1+r+\gamma)\left(1+\frac{a}{\gamma}\right)$$

which implies that $-\frac{\gamma^2}{1+r} < a < \frac{\gamma^2}{1+r}$. If $a > \frac{\gamma^2}{1+r}$ and

$$(1+r-\gamma)\left(1-\frac{a}{\gamma}\right) < b < (1+r+\gamma)\left(1+\frac{a}{\gamma}\right)$$

λ_1 and λ_2 both lie outside the unit circle. If

$$b < \min\left[(1+r-\gamma)\left(1-\frac{a}{\gamma}\right), (1+r+\gamma)\left(1+\frac{a}{\gamma}\right)\right]$$

both roots are real, $\lambda_1 > 1$ and $-1 < \lambda_2 < 1$. If

$$b > \max\left[(1+r-\gamma)\left(1-\frac{a}{\gamma}\right), (1+r+\gamma)\left(1+\frac{a}{\gamma}\right)\right]$$

both roots are real, $\lambda_2 < -1$ and $-1 < \lambda_2 < 1$. If

$$(1+r+\gamma)\left(1+\frac{a}{\gamma}\right) < b < (1+r-\gamma)\left(1-\frac{a}{\gamma}\right)$$

then $a < -\frac{\gamma^2}{1+r}$, both roots are real and $\lambda_2 < \lambda_1 < -1$.

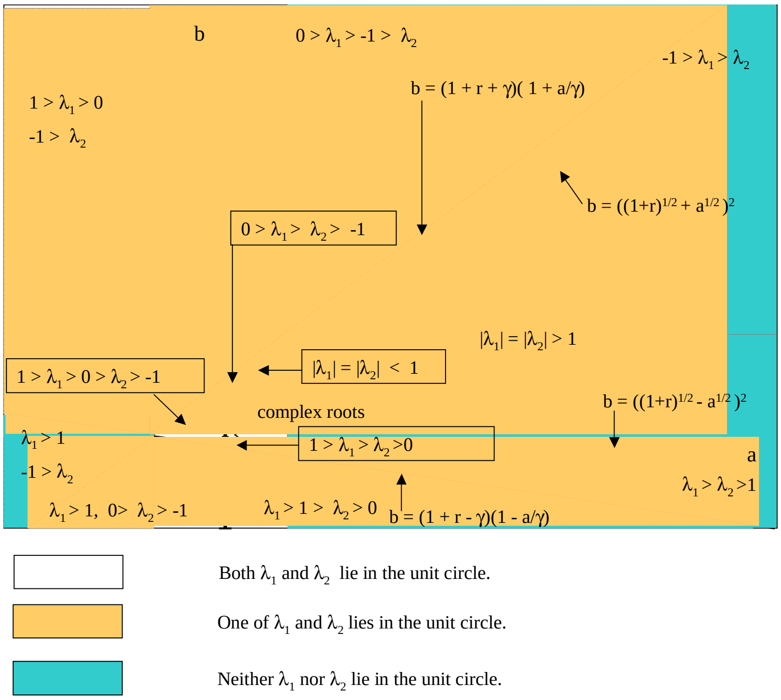

Figure 3: Stability condition with $\gamma = 1$ and $r = 0.5$.

The proof of Proposition 3 in the Appendix gives a more detailed description of how λ_1 and λ_2 relate to a and b, which is illustrated in Figure 3. This Figure is drawn under the completely unrealistic assumption that $\gamma = 1$ and $r = 0.5$, (recall that $\gamma - 1$ is the trend rate of growth of earnings and r is the interest rate). These values are chosen simply because it makes it possible to see in the Figure all the various regions in which λ_1 and λ_2 are positive and negative, real and complex, and inside and outside the unit circle. Figure 4 shows part of a similar plot to Figure 3 for the more realistic values of $\gamma = 1.05$ and $r = 0.07$. Note the very different scale on the vertical, b axis.

The stylised facts on dividend smoothing suggest that a is likely to be close to 1, and b is likely to be positive but small. As Figure 3 shows, this implies that λ_1 and λ_2 are close to 1, they are positive if they are real, but could be complex, and could lie either inside or outside the unit circle. A model in which dividends depend upon lagged equity prices but not dividends, with coefficients $a \approx 0$ and $b \approx 1 + r - \gamma$ is also close to the boundary of the stable set. There are no points in the stable set for which $b = 0$, dividends must be affected by the balance sheet. Recalling that we define earnings as being earnings before interest, the balance sheet can affect dividends through earnings after

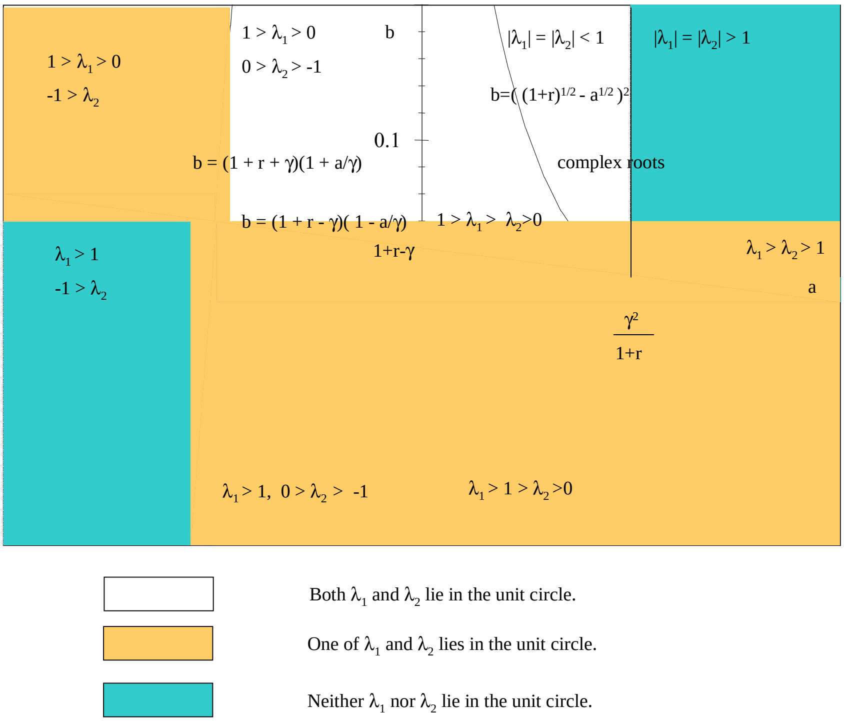

Figure 4: Stability conditions with $\gamma = 1.05$ and $r = 0.7$.

interest. If all three roots λ_1, λ_2 and ϕ lie in the unit circle detrended prices and dividends are stationary, if $\lambda_1$and λ_2 but not ϕ lie in the unit circle any two of prices, dividends and earnings are cointergrated. If, as seems plausible the price and dividend processes involve roots close to unity, their variances will be very large or infinite. We will return to this point when we have explicit expressions for the price and dividend processes.

5 The Paths of Prices and Dividends

We prove in the Appendix:

Proposition 4 (Prices and Dividends) *If dividends and prices are given by the equations*

$$\begin{bmatrix} \gamma^{-t} d_t \\ \gamma^{-t} p_t \\ y_t \end{bmatrix} = \begin{bmatrix} h \\ -h \\ 0 \end{bmatrix} + M \begin{bmatrix} \gamma^{-(t-1)} d_{t-1} \\ \gamma^{-(t-1)} p_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} f_1 \varepsilon_{1t} + f_2 \varepsilon_{2t} + f_3 \varepsilon_{3t} \\ (1-f_1)\varepsilon_{1t} + (1-f_2)\varepsilon_{2t} - f_3 \varepsilon_{3t} \\ \varepsilon_{2t} \end{bmatrix} \tag{5.1}$$

where

$$M = \frac{1}{\gamma}\left[\begin{array}{ccc} a & b & \gamma c \\ -a & 1+r-b & -\gamma c \\ 0 & 0 & \gamma\phi \end{array}\right].$$

then

$$\begin{aligned} \frac{d_t}{\gamma^t} - \mu_d &= \sum_{h=0}^{t-1}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1-k\lambda_i\right)\lambda_i^h \Delta_{ij}\varepsilon_{jt-h} + m_{dt} \\ &= \sum_{h=0}^{t-1}\Lambda_{dh}\Delta\varepsilon_{t-h} + m_{dt} \end{aligned} \tag{5.2}$$

$$\begin{aligned} \frac{p_t}{\gamma^t} - \mu_p &= \sum_{h=0}^{t-1}\sum_{i=1}^{3}\sum_{j=1}^{3} k\lambda_i\lambda_i^h \Delta_{ij}\varepsilon_{jt-h} + m_{pt} \\ &= \sum_{h=0}^{t-1}\Lambda_{ph}\Delta\varepsilon_{t-h} + m_{pt} \end{aligned} \tag{5.3}$$

$$y_t = \sum_{h=0}^{t-1}\lambda_3^h\varepsilon_{2t-h} + \lambda_3^t y_0$$

where $\lambda_3 = \phi$

$$\mu_d = \frac{(1+r-\gamma)\,h\gamma}{(1+r-\gamma)(\gamma-a)-b\gamma} \tag{5.4}$$

$$\mu_p = \frac{h\gamma^2}{(1+r-\gamma)(\gamma-a)-b\gamma} \tag{5.5}$$

$$\Delta = \frac{1}{k(\lambda_1-\lambda_2)}\left[\begin{array}{ccc} 1-k\lambda_2-f_1 & 1-k\lambda_2-f_2+\frac{c}{(\lambda_3-\lambda_1)} & -f_3 \\ -1+k\lambda_1+f_1 & -1+k\lambda_1+f_2-\frac{c}{(\lambda_3-\lambda_2)} & f_3 \\ 0 & \frac{-(\lambda_1-\lambda_2)c}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & 0 \end{array}\right] \tag{5.6}$$

$$\Lambda_{dh} = \Big(\left(1-k\lambda_1\right)\lambda_1^h, \left(1-k\lambda_2\right)\lambda_2^h, \left(1-k\lambda_3\right)\lambda_3^h\Big) \tag{5.7}$$

$$\Lambda_{ph} = \Big(k\lambda_1\lambda_1^h, k\lambda_2\lambda_2^h, k\lambda_3\lambda_3^h\Big) \tag{5.8}$$

$$\varepsilon_{t-h} = \begin{bmatrix} \varepsilon_{1t-h} \\ \varepsilon_{2t-h} \\ \varepsilon_{3t-h} \end{bmatrix} \tag{5.9}$$

and

$$m_{dt} = (1 - k\lambda_1)\, \lambda_1^t q_1 + (1 - k\lambda_2)\, \lambda_2^t q_2 - \frac{(1 - k\lambda_3)\, c\lambda_3^t y_0}{k\,(\lambda_3 - \lambda_1)\,(\lambda_3 - \lambda_2)} \tag{5.10}$$

$$m_{pt} = k\lambda_1 \lambda_1^t q_1 + k\lambda_2 \lambda_2^n q_2 - \frac{k\lambda_3 c\lambda_3^t y_0}{k\,(\lambda_3 - \lambda_1)\,(\lambda_3 - \lambda_2)} \tag{5.11}$$

where

$$q_1 = -k\lambda_2\,(d_0 - \mu_d) + (1 - k\lambda_2)\,(p_0 - \mu_p) + \frac{1}{(\lambda_3 - \lambda_1)} c y_0 \tag{5.12}$$

$$q_2 = k\lambda_1\,(d_0 - \mu_d) - (1 - k\lambda_1)\,(p_0 - \mu_p) - \frac{1}{(\lambda_3 - \lambda_2)} c y_0. \tag{5.13}$$

Note from (5.6) that $\Delta_{11} + \Delta_{21} + \Delta_{31} = \Delta_{12} + \Delta_{22} + \Delta_{32} = 1$, whilst $\Delta_{13} + \Delta_{23} = 0$ and $\Delta_{33} = 0$. Thus $\frac{d_t}{\gamma^t} - \mu_d$ and $\frac{p_t}{\gamma^t} - \mu_p$ can be interpreted as weighted averages of AR(1) processes with innovations $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ with weights determined by f_1, f_2 and c plus a multiple f_3 of the difference between two AR(1) processes with innovations $\{\varepsilon_{3t}\}$. However looking at (5.6) there is no guarantee that the weights are non-negative. Indeed they are complex if λ_1 and λ_2 are complex, and have zeros in the denominator if any of the roots coincide. Lemma 1 in the appendix gives alternative expressions for dividends and prices in which all the terms are finite and real valued, even if the roots are complex or coincident.

Proposition 4, and in particular (5.2) and (5.3) imply that a necessary and sufficient condition for the detrended dividend and price processes to be stationary in the sense that $E\left(\gamma^{-t} d_t | d_0, p_0\right)$, $\operatorname{var}\left(\gamma^{-t} d_t | d_0, p_0\right)$, $E\left(\gamma^{-t} p_t | d_0, p_0\right)$, and $\operatorname{var}\left(\gamma^{-t} p_t | d_0, p_0\right)$ all tend to finite limits is that m_{dt}, m_{pt} and $\sum_{h=0}^{t-1} \left|\Delta_{ij} \lambda_i^h\right|^2$ for $i = 1,2,3,\ \ j = 1,2,3$ all tend to finite limits as t tends to infinity. There is of course no reason why companies should keep econometricians happy by choosing a dividend policy which results in prices and dividends with bounded second moments, but they do need to consider their debt and equity holders. Recall from (2.6) that $p_t + B_t = \frac{E_t V_{t+1}}{1+r}$, where p_t is the value of equity, B_t is the face value of debt and V_t the expected present value of earnings. From Assumption 6 $E_t V_{t+1} = \gamma^t \left[\alpha \frac{1+r}{1+r-\gamma} + \phi y_t\right]$. Assume that y_t is stationary, with finite variance so $\gamma^{-t} E_t V_{t+1}$ is also stationary with finite variance, then if $\gamma^{-t} p_t$ is non-stationary with an infinite variance $\gamma^{-t} B_t$ must also be non-stationary with an infinite variance. But it seems highly unlikely that companies will be able to issue debt on terms which allow the face value of debt B_t to be much larger than the total value of the firm E_tV_{t+1}, so dividend policies which result in non-stationary detrended price processes are likely to become unsustainable, as indeed are those which although stationary have very large, but finite, variances.

Proposition 3 established conditions under which λ_1 and λ_2 both lie in the unit circle. If in addition $-1 < \lambda_3 = \phi < 1$ the solution is stable, and the conditional variances of $\{d_t, p_t, y_t\}$ given $\{d_0, p_0, y_0\}$ exist, whatever the starting point. However it is also possible to get stability when one or two of the roots lies outside the unit circle, provided all the weight is put on stable roots, and the starting point is appropriate. For example, consider the case when λ_1 and λ_2 are real, set $f_1 = f_2 = 1 - k\lambda_1$, and $f_3 = c = 0$. The sum of dividend and price at date 0 is determined by the previous history and shocks at date 0, so $d_0 - \mu_d + p_0 - \mu_p = s$ for some number s. It is possible to set dividends at date 0 so $d_0 - \mu_d = (1 - k\lambda_1)s$ and $p_0 - \mu_p = k\lambda_1 s$, which implies that $q_1 = k(\lambda_1 - \lambda_2)s$, $q_2 = 0$, and from Proposition 4

$$\Delta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$\frac{d_t}{\gamma^t} - \mu_d = \sum_{h=0}^{t-1} (1 - k\lambda_1)\, \lambda_1^h(\varepsilon_{1t-h} + \varepsilon_{2t-h}) + (1 - k\lambda_1)\, \lambda_1^t k(\lambda_1 - \lambda_2)s \quad (5.14)$$

$$\frac{p_t}{\gamma^t} - \mu_p = \sum_{h=0}^{t-1} k\lambda_1\lambda_1^h(\varepsilon_{1t-h} + \varepsilon_{2t-h}) + k\lambda_1\lambda_1^t k(\lambda_1 - \lambda_2)s. \quad (5.15)$$

This implies that, for $t = 0, 1, 2.....$,

$$k\lambda_1 \left(\frac{d_t}{\gamma^t} - \mu_d\right) = (1 - k\lambda_1) \left(\frac{p_t}{\gamma^t} - \mu_p\right)$$

i.e. prices and dividends are perfectly correlated AR(1) processes with root λ_1.

Similarly setting $f_1 = f_2 = 1 - k\lambda_2$, $f_3 = c = 0$, $d_0 - \mu_d = (1 - k\lambda_2)s$

$$\Delta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\frac{d_t}{\gamma^t} - \mu_d = \sum_{h=0}^{t-1} (1 - k\lambda_2)\, \lambda_2^h(\varepsilon_{1t-h} + \varepsilon_{2t-h}) + (1 - k\lambda_2)\, \lambda_2^t k(\lambda_1 - \lambda_2)s \quad (5.16)$$

$$\frac{p_t}{\gamma^t} - \mu_p = \sum_{h=0}^{t-1} k\lambda_2 \lambda_2^h (\varepsilon_{1t-h} + \varepsilon_{2t-h}) + k\lambda_2 \lambda_2^t k(\lambda_1 - \lambda_2) s \tag{5.17}$$

Again prices and dividends are perfectly correlated AR(1) processes with root λ_2 and

$$k\lambda_2 \left(\frac{d_t}{\gamma^t} - \mu_d \right) = (1 - k\lambda_2) \left(\frac{p_t}{\gamma^t} - \mu_p \right)$$

Thus it is possible to get a stable AR(1) process with finite variances in the knife edge case when one but not both of λ_1 and λ_2 lie outside the unit circle. The stability is however delicate, with the wrong initial value of dividends, dividends and prices are unstable. The possibility of perfect correlation is interesting, because Proposition 2 then implies that the variances of prices and dividends lie on the boundary of the feasible set. We concentrate on this case when we consider variance bounds in Section 7. One special case is noteworthy. If $a = 0$ so lagged dividends do not enter the dividend equation then $\lambda = 0$ is a root of the characteristic equation (4.13). Suppose $\lambda_1 = 0$, and consider (5.15) and (5.17) which imply that $\frac{d_t}{\gamma^t} - \mu_d = \varepsilon_{1t} + \varepsilon_{2t}$ and $\frac{p_t}{\gamma^t} - \mu_p = 0$. Thus it is possible to construct a dividend policy where the entire shock to current and expected future earnings goes into dividends which have no autocorrelation, and prices are constant. This is completely unrealistic, but does demonstrate the point that smoothing dividends by introducing autocorrelation in dividend policy, implicitly involves making equity prices more variable than they need be.

A more standard assumption is that $b = c = 0$ in which case

$$\frac{d_t}{\gamma^t} - \mu_d = \frac{a}{\gamma} \left(\frac{d_{t-1}}{\gamma^{t-1}} - \mu_d \right) + f_1 \varepsilon_{1t} + f_2 \varepsilon_{2t}$$

detrended dividends follow an AR(1) process. Equations (4.11) and (4.12) imply that $\lambda_1 = a/\gamma$ and $\lambda_2 = (1+r)/\gamma \equiv 1/k$. We assumed that $|k| < 1$ in order to make the expected present value of future earnings finite, so λ_2 is outside the unit circle. Equations (5.2), (5.3) and (5.6) imply that there is no weight on the unstable root if $f_1 = f_2 = 1 - k\lambda_1 = 1 - a/(1+r)$. If this condition is not satisfied the price process is unstable. This contrasts with the usual argument that if dividends follow an AR(1) process prices must also do so. The difference is that the usual argument assumes that equity is priced on the basis that dividends will continue to follow the same process in the future. This argument ignores the fact that net dividends are ultimately constrained by the firm's ability to borrow. In this paper we are in a Modigliani-Miller world, where the value of equity is the total value of the firm minus its debt, regardless of dividend policy, because dividends are ultimately constrained by capital structure.

In this model earnings before interest are given by equations (4.1) and (4.2), i.e.

$$\frac{x_t}{\gamma^t} = \alpha + \frac{(1+r-\phi\gamma)}{(1+r)} y_t + \varepsilon_{1t}$$

where $y_t = \phi y_{t-1} + \varepsilon_{2t}$. From Proposition 1 borrowing $B_t = \frac{E_t V_{t+1}}{1+r} - p_t$ where $E_t V_{t+1}$ is the expected present value at t of earnings from date $t+1$, onwards which is a linear function of y_t. This implies that transient shocks to earnings before interest can have persistent effects on earnings after interest, prices and dividends through their effect on borrowing and thus on interest payments. Lintner (1956) and the subsequent literature argues that dividends respond mainly to permanent shocks in earnings; we argue that the transience or persistence of earnings shocks is determined by dividend policy.

6 Variances

We are looking for the limits as t tends to infinity of expressions for the conditional variances of detrended prices and dividends $\operatorname{var}(\gamma^{-t} d_t | d_0, p_0)$ and $\operatorname{var}(\gamma^{-t} p_t | d_0, p_0)$. Proposition 4 implies that if λ_1, λ_2 and $\phi = \lambda_3$ all lie in the unit circle these limits exist. We derive general expressions for the variances in the Appendix. These expressions are very complicated unless $c = 0$, where c is the coefficient on the earnings state variable y_{t-1} in the dividend equation. The next Proposition provides formulae if $c = 0$. Note that setting $c = 0$ allows for an effect of earnings on dividends, but requires that the effect works entirely through the impact of shocks ε_{1t} and ε_{2t} to earnings. The proof is in the Appendix.

Proposition 5 (Variances) *Assume that*

$$-\frac{\gamma^2}{1+r} < a < \frac{\gamma^2}{1+r}$$

and

$$(1+r-\gamma)\left(1-\frac{a}{\gamma}\right) < b < (1+r+\gamma)\left(1+\frac{a}{\gamma}\right)$$

so λ_1 *and* λ_2 *lie in the interior of the unit circle. Let*

$$\pi = \lambda_1\lambda_2 \equiv \frac{a}{k\gamma} \equiv \frac{a(1+r)}{\gamma^2} \tag{6.1}$$

and

$$\tau = \frac{\lambda_1 + \lambda_2}{1+\lambda_1\lambda_2} \equiv \frac{\gamma(1+r+a-b)}{\gamma^2 + a(1+r)} \tag{6.2}$$

which implies that

$$-1<\pi<1 \text{ and } -1<\tau<1 \tag{6.3}$$

Then if $c=0$, $\operatorname{var}\varepsilon_{1t}=\sigma_1^2$, $\operatorname{var}\varepsilon_{2t}=\sigma_2^2$ *and* $\operatorname{var}\varepsilon_{3t}=\sigma_3^2$, *for* $t=0,1,2...$

$$\begin{aligned}&\lim_{t\to\infty}\operatorname{var}\left(\gamma^{-t}d_t|d_0,p_0\right)\\ =\;&\frac{\left[1-k\tau\left(1+\pi\right)+k^2\pi\right]^2}{\left[\left(\tau-k\right)^2+1-\tau^2\right]\left(1-\pi^2\right)}\left(\sigma_1^2+\sigma_2^2\right)+\\ &\frac{\left[k^2(1-\tau^2)+\left(1-k\tau\right)^2\right]}{k^2\left(1-\pi^2\right)\left(1-\tau^2\right)}\left(\left(f_1-f_0^d\right)^2\sigma_1^2+\left(f_2-f_0^d\right)^2\sigma_2^2+f_3^2\sigma_3^2\right)\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\lim_{t\to\infty}\operatorname{var}\left(\gamma^{-t}p_t|d_0,p_0\right)\;=\;&\frac{k^2\pi^2}{\left(1-\pi^2\right)}\left(\sigma_1^2+\sigma_2^2\right)\\ &+\frac{\left(f_1-f_0^p\right)^2\sigma_1^2+\left(f_2-f_0^p\right)^2\sigma_2^2+f_3^2\sigma_3^2}{\left(1-\pi^2\right)\left(1-\tau^2\right)}\end{aligned} \tag{6.5}$$

where

$$f_0^d=\frac{\left(1-k\tau\right)\left[1-k\tau\left(1+\pi\right)+k^2\pi\right]}{\left[\left(\tau-k\right)^2+1-\tau^2\right]}>0 \tag{6.6}$$

and

$$f_0^p=1-k\pi\tau. \tag{6.7}$$

The terms f_0^p *and* f_0^d *satisfy*

$$f_0^p-f_0^d=\frac{k^2\left(1-\pi\right)\left(1-\tau^2\right)}{\left[\left(\tau-k\right)^2+1-\tau^2\right]}>0 \tag{6.8}$$

and

$$1>f_0^p>f_0^d>0.$$

When $f_1=f_2=f_0^p$ *and* $f_3=0$

$$\lim_{t\to\infty}\operatorname{var}\left(\gamma^{-t}p_t|d_0,p_0\right)=\frac{k^2\pi^2\left(\sigma_1^2+\sigma_2^2\right)}{\left(1-\pi^2\right)} \tag{6.9}$$

and

$$\begin{aligned}\lim_{t\to\infty}\operatorname{var}\left(\gamma^{-t}d_t|d_0,p_0\right)\;=\;&\frac{\left[1-k\tau\left(1+\pi\right)+k^2\pi\right]^2\left(\sigma_1^2+\sigma_2^2\right)}{\left[\left(\tau-k\right)^2+1-\tau^2\right]\left(1-\pi^2\right)}\\ &+\frac{k^2\left(1-\pi\right)\left(1-\tau^2\right)\left(\sigma_1^2+\sigma_2^2\right)}{\left[\left(\tau-k\right)^2+1-\tau^2\right]}.\end{aligned} \tag{6.10}$$

When $f_1 = f_2 = f_0^d$ and $f_3 = 0$

$$\lim_{t\to\infty} \text{var}\left(\gamma^{-t} p_t | d_0, p_0\right) = \frac{k^2\pi^2\left(\sigma_1^2 + \sigma_2^2\right)}{\left(1-\pi^2\right)} + \frac{k^4\left(1-\pi\right)\left(1-\tau^2\right)\left(\sigma_1^2+\sigma_2^2\right)}{\left[\left(\tau-k\right)^2 + 1 - \tau^2\right]\left(1+\pi\right)} \tag{6.11}$$

$$\lim_{t\to\infty} \text{var}\left(\gamma^{-t} d_t | d_0, p_0\right) = \frac{\left[1 - k\tau\left(1+\pi\right) + k^2\pi\right]^2\left(\sigma_1^2+\sigma_2^2\right)}{\left[\left(\tau-k\right)^2 + 1 - \tau^2\right]\left(1-\pi^2\right)}. \tag{6.12}$$

Recall that if $c = 0$ the dividend equation is

$$\left(\frac{d_t}{\gamma^t}\right) = h + \left(\frac{a}{\gamma}\right)\left(\frac{d_{t-1}}{\gamma^{t-1}}\right) + \left(\frac{b}{\gamma}\right)\left(\frac{p_{t-1}}{\gamma^{t-1}}\right) + f_1\varepsilon_{1t} + f_2\varepsilon_{2t} + f_3\varepsilon_{3t}.$$

This proposition makes explicit the dependence of the variances of prices and dividends on the coefficients of lagged dividends and prices a and b which have their effect through π and τ defined in (6.1) and (6.2), and the coefficient f_1, f_2 which determines the impact of earnings shocks and f_3 which determines the response to a white noise process orthogonal to random shocks. Equations (6.4) and (6.5) imply that non-zero values of f_3 increase the variance of both dividends and prices. The total shock to the expected present value of current and future earnings is $\varepsilon_{1t} + \varepsilon_{2t}$ where ε_{2t} is the shock to the earnings state variable y_t which endures for many periods, ε_{1t} is the shock to earnings which has no effect in later periods. For given a and b and thus π and τ the variances are quadratic function of f_1 and f_2 which are zero if dividend policy does not respond at all to current news about earnings, and are one if the entire shock to the expected present value of current and future earnings is immediately incorporated into dividends.

Proposition 5 also implies that there is indeed a trade off, different values of f_1 and f_2 minimise the variances of dividends and prices. Consider the problem of choosing f_1 and f_2 to minimise the variance of prices whilst requiring that the variance of dividends is below a fixed level. This is a very well behaved problems, the first order conditions imply that at the optimum $f_1 = f_2$, so any dividend policy which solves such a problem responds only to the shock $\varepsilon_{1t} + \varepsilon_{2t}$ in the expected present value of current and future dividends. This is compatible with Lintner's (1956) observation that shocks to current earnings which are expected to endure have a bigger impact on earnings than temporary shocks, as the enduring shocks also affect expected future earnings.

We can also get some insight into the effects of the coefficients a of dividends and b of prices in the dividend equation. In the thoroughly unrealistic case that $a = 0$, (6.1) implies that $\pi = 0$ and from (6.7) $f_0^p = 1$, so if $f_1 = 1 = f_0^p$ from (6.9) and (6.10) $\lim_{t\to\infty} \text{var}\left(\gamma^{-t} p_t | d_0, p_0\right) = 0$ and $\lim_{t\to\infty} \text{var}\left(\gamma^{-t} d_t | d_0, p_0\right) = \left(\sigma_1^2 + \sigma_2^2\right)$, detrended prices are constant, and all the shocks to the expected present value of earnings are incorporated into dividends. When $a = \pi = 0$

there is considerable scope for trading off the variances of dividends and prices against each other by choice of $f_1$and f_2.

Rather more realistically a may be large enough to make π close to 1. If either π or τ tends to 1 the scope for trading off variances disappears. Equation (6.8) implies that there is very little difference between the values of f_1 which minimise the variances of prices and dividends, and as f_1 moves between these values there is little effect on variances. This is a situation where the interval $\left(f_0^d, f_0^p\right)$ is small, outside this interval changes in the value of f_1 make the variances of prices and dividends move in the same direction. If π tends to 1 both variances tend to infinity whatever the value of f_1 and f_2. This is because $\pi = \lambda_1\lambda_2$ so if π tends to 1 both λ_1 and λ_2 tend to the boundary of the unit circle, and it is impossible to avoid giving some weight to close to unit roots in the solutions given in Proposition 4. The scope for trading off variances also disappears as τ tends to 1, in which case (6.7) and (6.8) imply that f_0^p and f_0^d both tend to $1-k\pi$, but the variances remain bounded. Looking at (6.2) shows why this happens; τ tends to 1 if one or both of λ_1 and λ_2 tend to 1, for example if λ_1 tends to 1 but $|\lambda_2| < 1$ then π tends to λ_2 and f_0^p and f_0^d both tend to $1-k\lambda_2$. Setting $f_1 = f_2 = 1 - k\lambda_2$ is precisely the dividend policy which gives no weight to the unstable root λ_1 and results in perfectly correlated prices and dividends.

Stylised facts about dividend smoothing suggest that $a \approx 1 + r \approx \gamma$ and $b \approx 0$ in which case $\pi \approx 1$ and $\tau \approx 1$, a situation where there is little trade off between the variances of prices and dividends, and both variances are very large due to the presence of $1-\pi^2$ terms in the denominators of (6.11) - (6.12). If this is so, an explanation is needed as to why firms set dividends in this way.

The results on variances in the Appendix, make it possible in principle to trace out the dependence of the variances of prices and dividends on all the parameters of the dividend equation, but the algebra is nasty, and not very illuminating. Further we already know from Proposition 2 what the boundaries of the feasible set of prices and dividends look like, and that they are generated by perfectly correlated price and dividend processes. The results of Proposition 3 tell us that choosing a and b so that

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \equiv \lambda^2 - \left(\frac{1+r+a-b}{\gamma}\right)\lambda + \frac{a(1+r)}{\gamma^2} = 0$$

so λ is equal to λ_1 or λ_2, and setting $f_1 = f_2 = 1 - k\lambda$ and $f_3 = c = 0$, gives prices and dividends perfectly correlated AR(1) processes, for which

$$\lim_{t\to\infty} \operatorname{var}\left(\gamma^{-t} d_t | d_0, p_0\right) = \frac{(1-k\lambda)^2}{1-\lambda^2}\left(\sigma_1^2 + \sigma_2^2\right).$$

$$\lim_{t\to\infty} \operatorname{var}\left(\gamma^{-t} p_t | d_0, p_0\right) = \frac{k^2\lambda^2}{1-\lambda^2}\left(\sigma_1^2 + \sigma_2^2\right).$$

In particular this can be achieved by letting $b = 0$ so the roots are $\frac{1+r}{\gamma}$ and $\frac{a}{\gamma}$ and letting $\lambda = \frac{a}{\gamma}$. Evidently $\lambda = 0$ implies $\lim_{t\to\infty} \operatorname{var}(\gamma^{-t}p_t|d_0, p_0) = 0$ at which point $\lim_{t\to\infty} \operatorname{var}(\gamma^{-t}d_t|d_0, p_0) = \sigma_1^2 + \sigma_2^2$, which again gives us the dividend policy which makes prices grow at a deterministic rate γ and puts all earnings shocks immediately into dividends. As

$$\frac{(1-k\lambda)^2}{1-\lambda^2} = 1 - k^2 + \frac{(\lambda - k)^2}{1-\lambda^2}$$

$\lambda = k$ minimises the variance of detrended dividends and when $\lambda = k$, $\lim_{t\to\infty} \operatorname{var}(\gamma^{-t}d_t|d_0, p_0) = \left(1-k^2\right)\left(\sigma_1^2 + \sigma_2^2\right)$ and $\lim_{t\to\infty} \operatorname{var}(\gamma^{-t}p_t|d_0, p_0) = \frac{k^4}{1-k^2}\left(\sigma_1^2 + \sigma_2^2\right)$, so perfectly correlated price and dividend processes give considerable scope for trading off the variances of prices and dividends against each other. The variance of detrended prices increases and the variance of detrended dividends decreases as λ increases from 0 to k, increasing λ beyond k increases both the variance of dividends and prices. However

$$\frac{d_t}{d_{t-1}} = \lambda\gamma + \frac{\mu_d(1-\lambda)\gamma^t}{d_{t-1}} + \frac{(1-k\lambda)}{d_{t-1}}\varepsilon_t$$

so increasing λ beyond k makes dividends smoother, in the sense that they are more predictable, and their growth rate is closer to a constant. In the limit as λ tends to 1 detrended dividends and prices become a random walk with infinite variance.

7 Variance Bounds

One of the most striking results in empirical finance is Shiller's (1981) demonstration that the sample variance of perfect foresight share prices is dramatically smaller than the sample variance of equity prices, whereas asset pricing theory implies that the variance of perfect foresight share prices is greater than the variance of share prices. The subsequent literature (Flavin (1983), Kleidon (1986) and Marsh and Merton (1986) points to reasons why the sample variances may give misleading estimates of the variances. The apparatus of this paper makes it very easy to see why this can happen.

The perfect foresight price p_{tT}^* is defined as the price which would prevail at date t if investors had perfect foresight up to date T

$$p_{tT}^* = \sum_{j=1}^{T-t} \frac{d_{t+j}}{(1+r)^j} + \frac{p_T}{(1+r)^{T-t}}.$$

(Note this is not the same as Shiller's p_t^* which uses the sample average of prices, rather than the terminal price p_T. It is however what standard theory implies). From Proposition 2

$$d_{t+j} = (1+r)p_{t+j-1} + e_{t+j} - p_{t+j}$$

where

$$e_{t+j} = E_{t+j}V_{t+j} - E_{t+j-1}V_{t+j}$$

and

$$V_{t+j} = E_{t+j}\left[\sum_{i=0}^{\infty}\frac{x_{t+j+i}}{(1+r)^i}\right]$$

so

$$\begin{aligned} p^*_{tT} &= \sum_{j=1}^{T-t}\frac{(1+r)p_{t+j-1}+e_{t+j}-p_{t+j}}{(1+r)^j} + \frac{p_T}{(1+r)^{T-t}} \\ &= p_t + \sum_{j=1}^{T-t}\frac{e_{t+j}}{(1+r)^j}. \end{aligned} \quad (7.1)$$

Thus the perfect foresight price p^*_{tT} is the actual price p_t plus the present discounted value of innovations in the earnings process. Clearly $\operatorname{var} p^*_{tT} > \operatorname{var} p_t$. However we shall show that the situation can be very different for sample variances.

We consider cases where prices and dividends are perfectly correlated AR(1) processes, i.e. when either equations (5.14), (5.15), or (5.16) and (5.17) of Section 5 apply. By choosing the parameters of dividend policy a and b so that $\lambda^2 - \left(\frac{1+r+a-b}{\gamma}\right)\lambda + \frac{a(1+r)}{\gamma^2} = 0$, any value of λ can be chosen. Set $f_1 = f_2 = 1-k\lambda$, and $f_3 = c = 0$. The sum of dividend and price at date 0 is determined by the previous history and shocks at date 0, so $d_0 - \mu_d + p_0 - \mu_p = s$ for some number s. Set dividends at date 0 so $d_0 - \mu_d = (1-k\lambda)s$ and $p_0 - \mu_p = k\lambda s$. Then from (5.15), or (5.17)

$$\gamma^{-t}p_t - \mu_p = \lambda\left(\gamma^{-(t-1)}p_{t-1} - \mu_p\right) + k\lambda(\varepsilon_{1t} + \varepsilon_{2t})$$

For notational convenience let $\hat{e}_t = \gamma^{-t}e_t = \varepsilon_{1t} + \varepsilon_{2t}$. Then

$$\gamma^{-t}p_t - \mu_p = \lambda^t\left(p_0 - \mu_p\right) + k\lambda\hat{e}_t + k\lambda^2\hat{e}_{t-1} + k\lambda^3\hat{e}_{t-2} + ... + k\lambda^t\hat{e}_1. \quad (7.2)$$

Assume that $\hat{e}_t$ is stationary with variance $\hat{\sigma}^2_e$ and that $\operatorname{var} p_0 = \frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}^2_e$, which implies that

$$\operatorname{var}\gamma^{-t}p_t = \frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}^2_e \text{ for } t > 0.$$

As $e_{t+j} = \gamma^{t+j}\hat{e}_{t+j}$ and $k = \frac{\gamma}{1+r}$, equation (7.1) implies that

$$\begin{aligned} \gamma^{-t}p^*_{tT} - \mu_p &= \lambda^t\left(p_0 - \mu_p\right) + k\lambda\hat{e}_t + k\lambda^2\hat{e}_{t-1} + k\lambda^3\hat{e}_{t-2} + ... + k\lambda^t\hat{e}_1 \\ &\quad + k\hat{e}_{t+1} ... + k^{j-t}\hat{e}_j + ... + k^{T-t}\hat{e}_T. \end{aligned} \quad (7.3)$$

If $-1 < \lambda < 1$ and $-1 < k < 1$

$$\lim_{T\to\infty} \operatorname{var} \gamma^{-t} p^*_{tT} = \frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}^2_e + \frac{k^2}{1-k^2}\hat{\sigma}^2_e \text{ for } t > 0..$$

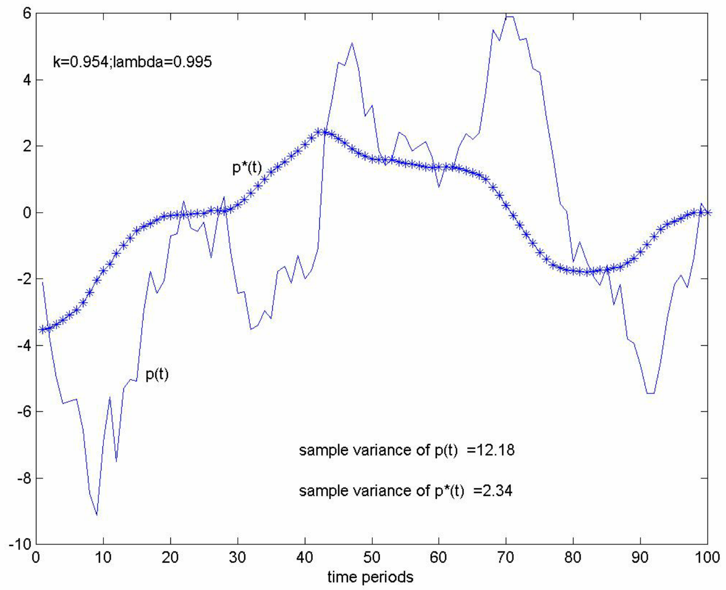

Figure 5 - Variance bound violation: Prices and perfect foresight prices for perfectly correlated prices and dividends.

We have assumed throughout this paper that $k = \frac{\gamma}{1+r}$ lies in $(0, 1)$. Recall that $\gamma - 1$ is the growth rate of expected earnings, so if $k \geq 1$ the expected present value of earnings is infinite, and the model does not make sense. If $\lambda \geq 1$ the price and dividend processes grow faster than expected earnings, and as we have argued this becomes economically implausible. However both k and λ may be close to 1. If $\lambda = 1$, $\gamma^{-t}p_t$ is a random walk, with infinite variance,

$$\gamma^{-t}p_t = p_0 + \sum_{j=1}^{t} \hat{e}_j \qquad \text{for all } t$$

and the sample variance of p_t is necessarily biased downwards as an estimator of $\operatorname{var} \gamma^{-t}p_t$.

If $\lambda = k = 1$ the variance of $\gamma^{-t}p^*_{tT}$ is also infinite. However equation (7.3) then implies that

$$\gamma^{-t}p^*_{tT} = p_0 + \sum_{j=1}^{T} \hat{e}_j \qquad \text{for all } t$$

so the sample variance of detrended perfect foresight prices is zero. This suggests that when λ and k are less than but close to 1 the sample variance of detrended perfect foresight prices may be very much more biased downwards as an estimator of $\operatorname{var}\gamma^{-t}p_{tT}^{*}$ than the sample variance of detrended prices is biased downwards as an estimator of $\operatorname{var} p\gamma_t^{-t}$.

Simulation results confirm this intuition. Figure 5 plots a realization of prices and perfect foresight prices for perfectly correlated prices and dividends in correspondence to $k = 0.954$ (as in Shiller, 1981) and $\lambda = 0.995$. The variance bound is violated and the plot clearly resembles the well known diagrams for real U.S. data used by Shiller (1981) to claim the failure of the efficient market hypothesis. The extreme case of $k = \lambda = 0.999$ is plotted in Figure 6 where the sample variance of detrended perfect foresight prices is (almost) zero.

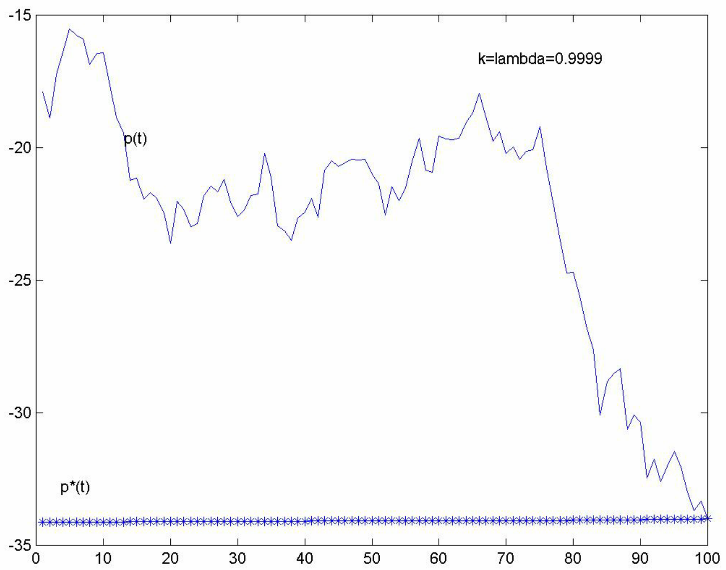

Figure 6 - Dramatic variance bound violation: Prices and perfect foresight prices for perfectly correlated prices and dividends.

The next Proposition rigorously formalize the argument (proof in the Appendix).

Proposition 6 (Variance Bounds) *Suppose that detrended prices $\gamma^{-t}p_t$ follows the AR(1) process*

$$\gamma^{-t}p_t - \mu_p = \lambda^t\left(p_0 - \mu_p\right) + k\lambda\hat{e}_t + k\lambda^2\hat{e}_{t-1} + k\lambda^3\hat{e}_{t-2} + ... + k\lambda^t\hat{e}_1.$$

$\operatorname{var}\hat{e}_t = \hat{\sigma}_e^2$ *for all* t *and* $\operatorname{var} p_0 = \frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2$. *The detrended perfect foresight price*

p^*_{tT} *is given by*

$$\begin{aligned}\gamma^{-t}p^*_{tT}-\mu_p &= \lambda^t\left(p_0-\mu_p\right)+k\lambda\hat{e}_t+k\lambda^2\hat{e}_{t-1}+k\lambda^3\hat{e}_{t-2}+......+k\lambda^t\hat{e}_1\\ &\quad +k\hat{e}_{t+1}...+k^{j-t}\hat{e}_j+....+k^{T-t}\hat{e}_T.\end{aligned}$$

Let $V_T(\lambda,k)$ *the sample variance of detrended prices* $\left(\gamma^{-1}p_1,......\gamma^{-T}p_T\right)$ *and* $V^*_T(\lambda,k)$ *the sample variance of detrended perfect foresight prices* $\left(\gamma^{-1}p^*_{1T},..\gamma^{-T}p^*_{TT}\right)$.

- *If* $-1<\lambda<1$ *and* $-1<k<1$

$$\lim_{T\to\infty}EV_T(\lambda,k)=\operatorname{var}\gamma^{-t}p_t=\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}^2_e$$

  and

$$\lim_{T\to\infty}EV^*_T(\lambda,k)=\lim_{T\to\infty}\operatorname{var}\gamma^{-t}p^*_{tT}=\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}^2_e+\frac{k^2}{1-k^2}\hat{\sigma}^2_e.$$

- *For fixed finite* $T>1$

$$\lim_{\lambda\to1^-,k\to1^-}EV_T(\lambda,k)=\hat{\sigma}^2_e\frac{\left(T^2-1\right)}{6T}>\lim_{\lambda\to1^-,k\to1^-}EV^*_T(\lambda,k)=0.$$

This first part of the Proposition implies that $V_T(\lambda,k)$ and $V^*_T(\lambda,k)$ are asymptotically unbiased as estimators of the limits of the variances of p_t and p^*_{tT} as the sample size T tends to infinity, so in the limit the expectations of the sample variances satisfy the variance bounds inequalities. Violation of the variance bounds is therefore a small sample problem, although 100 years of annual data may be a small sample as it is shown in Figure 7 which plots $EV_T(\lambda,k)$ and $EV^*_T(\lambda,k)$ as function of the sample size T in correspondence to $k=0.954$ (as in Shiller, 1981) and $\lambda=0.98$.

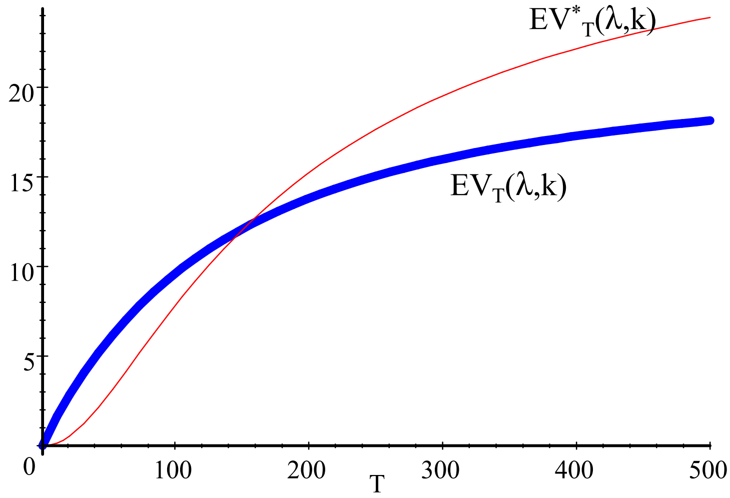

Figure 7 - The effect of sample size T on expected sample variances of actual prices $(EV_T(\lambda, k))$ and perfect foresight prices $(EV_T^*(\lambda, k))$ for $\lambda = 0.98$ and $k = 0.954$.

The second part of the Proposition shows that for any fixed finite sample size the expected sample variances will dramatically violate the variance bounds inequalities if λ and k are close enough to 1, that is when dividends are smoothed to the point where detrended prices are close to a random walk and the growth rate of earnings is close to the interest rate. This is clearly shown in Figure 8 which plots $EV_T(\lambda, k)$ and $EV_T^*(\lambda, k)$ as function of the parameter λ in correspondence of $T = 100$ and $k = 0.995$.

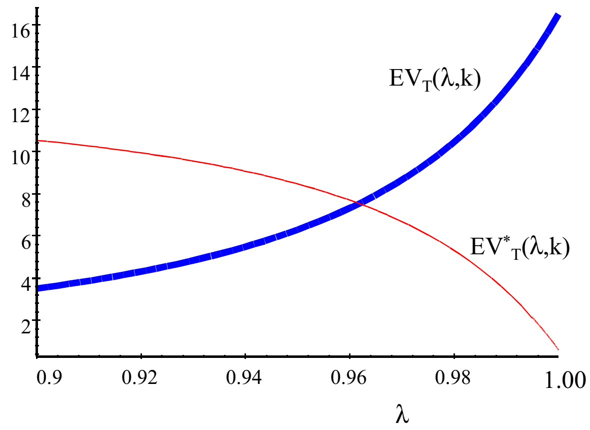

Figure 8 - Dividend smoothing; the effect of λ on the expected sample variances of actual prices ($EV_T(\lambda, k)$) and perfect foresight prices ($EV_T^*(\lambda, k)$) for $k = 0.995$ and sample size $T = 100$.

8 Conclusions

We started with a problem. What does standard financial theory imply about the relationship between the variance of dividends and the variance of equity prices? More precisely, is there scope for a trade off between these variances? Inevitably, being academics, the answer we have arrived at is "sometimes". There is in principle considerable scope for trade off, achieved by varying the parameters of the dividend equation (4.4). Perhaps the most obviously important of these parameters for the variances of prices and dividends are the coefficients determining the initial response of dividends to persistent and transient shocks to earnings (i.e. f_1 and f_2 in (4.4)). However Proposition 5 implies that there is in fact little scope for trade off through f_1 and f_2 if the coefficients a of lagged dividends and b of lagged equity prices are close to the boundary of the stable set defined in Proposition 3. The dividend processes which result in variances on the boundary of the feasible set defined in Proposition 2 and allow for substantial trade offs between the variances of dividends and prices are simple to describe. They are AR(1) processes, and result in prices and dividends which are perfectly correlated. However, and contrary to standard assumptions, an arbitrary AR(1) dividend process does not generally result in perfectly correlated AR(1) prices. For this to happen the initial conditions have to be very closely specified.

We argued in Section 5 that dividend processes which are unstable and make the variance of the total value of equity infinite are implausible because of their implications for the balance sheet. However the stability conditions of Proposition 3 show that the equation system is at best knife edge stable if the dividend equation does not include a balance sheet variable in the form of the lagged value of equity, debt or interest payments. Making a system knife-edge stable requires that the initial dividend be set at a level which depends on lagged equity prices. Thus the balance sheet matters for dividend policy in the sense in which it is used in this paper, setting the net dividend before interest, that is dividends minus share issues, or dividends plus share repurchases. In retrospect this is not surprising. However by assuming a linear dividend policy we may have misrepresented the way in which the balance sheet effect on dividends works. One possibility is a non-linear dividend policy which makes gross dividends follow a smoothed policy much of the time, with occasional substantial changes in net dividends reflecting the balance sheet, either in the form of share issues or repurchases, or big changes in gross dividends. The resulting stochastic process would be difficult to model analytically, but perhaps captures the stylised facts of dividend policy better than a linear model.

The effect of the balance sheet on net dividends may be difficult to pick up econometrically if dividends and equity prices are highly correlated, and if a is close to its maximal stable value of $\frac{\gamma^2}{1+r}$ and b is close to the lower boundary of the stable set, in which case $b \approx \frac{(1+r-\gamma)^2}{1+r}$ which is small if the rate of growth $\gamma - 1$ is close to the interest rate r. If net dividends are smoothed, the parameters a and b lie close to the boundary of the stable set and detrended dividends and equity prices have a root which is close to unity. As we showed in Section 7 this can give rise to data which satisfies variance bounds but makes the sample variances violate variance bounds inequalities by an arbitrarily large amount.

We have already pointed out that this paper is not about dividend policy as conventionally defined, which refers to the gross dividend. Dividend smoothing is about smoothing the dividend per share, which does not smooth the gross dividend if there are share sales or repurchases. In a world with frequent share repurchases it remains true that value of a share is the present discounted value of dividends per share, but a firm which regularly repurchases a fraction of its outstanding equity can make both dividends and earnings per share grow, whilst the total amount of earnings and dividends remains constant. Two firms with exactly the same underlying earnings could display radically different rates of growth of dividends per share, with the low growth firm paying out all or most of its net dividend as gross dividends, and the high growth firm paying out most of its net dividends in the form of share repurchases, giving shareholders a capital gain. This is simply another way of saying that in a Modigliani-Miller world gross dividend policy is irrelevant to the value of equity. Any explanation of the significance of the choice between share repurchases and dividends has to move beyond the Modigliani-Miller world of this paper.

Appendix

Proof of Proposition 3

We are considering the roots λ_1 and λ_2 of

$$\lambda^2 - \left(\frac{1+r+a-b}{\gamma}\right)\lambda + \frac{a\,(1+r)}{\gamma^2} = 0. \tag{A1}$$

The roots are complex if

$$\frac{4a\,(1+r)}{\gamma^2} > \left(\frac{1+r+a-b}{\gamma}\right)^2$$

which is equivalent to

$$\left(\sqrt{1+r} - \sqrt{a}\right)^2 < b < \left(\sqrt{1+r} + \sqrt{a}\right)^2.$$

The roots are

$$\lambda_1 = \frac{1}{2\gamma}\left(1+r+a-b+\sqrt{(1+r+a-b)^2 - 4a(1+r)}\right) \tag{A2}$$

$$\lambda_2 = \frac{1}{2\gamma}\left(1+r+a-b-\sqrt{(1+r+a-b)^2 - 4a(1+r)}\right). \tag{A3}$$

Considering

$$\lambda^2 - \left(\frac{1+r+a-b}{\gamma}\right)\lambda + \frac{a\,(1+r)}{\gamma^2} \equiv (\lambda_1 - \lambda)\,(\lambda_2 - \lambda)$$

as a function of λ and sketching curves demonstrates that

$$(\lambda_1 + 1)\,(\lambda_2 + 1) > 0 \text{ and } (\lambda_1 - 1)\,(\lambda_2 - 1) > 0$$

if and only if one of the four following mutually exclusive possibilities holds.

- λ_1 and λ_2 are complex in which case $|\lambda_1|^2 = |\lambda_2|^2 = \lambda_1\lambda_2 > 0$ so if $|\lambda_1\lambda_2| > 1$ both λ_1 and λ_2 lie outside the unit circle and if $|\lambda_1\lambda_2| < 1$ both λ_1 and λ_2 lie inside the unit circle.
- λ_1 and λ_2 are real and $\lambda_2 \leq \lambda_1 < -1$ in which case $|\lambda_1\lambda_2| > 1$.
- λ_1 and λ_2 are real and $-1 < \lambda_2 \leq \lambda_1 < 1$ in which case $|\lambda_1\lambda_2| < 1$.
- λ_1 and λ_2 are real and $1 < \lambda_2 \leq \lambda_1$ in which case $|\lambda_1\lambda_2| > 1$.

Thus the conditions that $(\lambda_1 + 1)(\lambda_2 + 1) > 0$, $(\lambda_1 - 1)(\lambda_2 - 1) > 0$ and $|\lambda_1 \lambda_2| < 1$ imply that both λ_1 and λ_2 lie in the unit circle. Conversely if λ_1 and λ_2 both lie in the unit circle then $|\lambda_1 \lambda_2| < 1$, and either λ_1 and λ_2 are complex in which case $(\lambda_1 - \lambda)(\lambda_2 - \lambda) > 0$ for all real λ and in particular for $\lambda = 1$ and $\lambda = -1$, or they are real in which case $-1 < \lambda_2 \leq \lambda_1 < 1$ so $(\lambda_1 + 1)(\lambda_2 + 1) > 0$ and $(\lambda_1 - 1)(\lambda_2 - 1) > 0$. Thus λ_1 and λ_2 both lie in the unit circle if and only if $(\lambda_1 + 1)(\lambda_2 + 1) > 0$, $(\lambda_1 - 1)(\lambda_2 - 1) > 0$ and $|\lambda_1 \lambda_2| < 1$.

More curve sketching implies that $(\lambda_1 + 1)(\lambda_2 + 1) < 0$ and $(\lambda_1 - 1)(\lambda_2 - 1) < 0$ if and only if λ_1 and λ_2 are real and $\lambda_2 < -1 < 1 < \lambda_1$. Similarly $(\lambda_1 + 1)(\lambda_2 + 1) > 0$ and $(\lambda_1 - 1)(\lambda_2 - 1) < 0$ if and only if λ_1 and λ_2 are real and $-1 < \lambda_2 < 1 < \lambda_1$, whilst $(\lambda_1 + 1)(\lambda_2 + 1) < 0$ and $(\lambda_1 - 1)(\lambda_2 - 1) > 0$ if and only if λ_1 and λ_2 are real and $\lambda_2 < -1 < \lambda_1 < 1$.

The conditions in terms of λ_1 and λ_2 can be translated into the conditions given in the theorem by noting firstly that $\lambda_1 \lambda_2 = \frac{a(1+r)}{\gamma^2}$ so $|\lambda_1 \lambda_2| < 1$ if and only if $\left|\frac{a(1+r)}{\gamma^2}\right| < 1$, and $\lambda_1 + \lambda_2 = \frac{1+r+a-b}{\gamma}$so

$$(\lambda_1 + 1)(\lambda_2 + 1) = 1 + \left(\frac{1 + r + a - b}{\gamma}\right) + \frac{a(1 + r)}{\gamma^2} > 0$$

if and only if

$$(1 + r + \gamma)\left(1 + \frac{a}{\gamma}\right) > b$$

and

$$(\lambda_1 - 1)(\lambda_2 - 1) = 1 - \left(\frac{1 + r + a - b}{\gamma}\right) + \frac{a(1 + r)}{\gamma^2} > 0$$

if and only if

$$(1 + r - \gamma)\left(1 - \frac{a}{\gamma}\right) < b.$$

Proof of Proposition 4

The matrix difference equation

$$\begin{bmatrix} \gamma^{-t} d_t \\ \gamma^{-t} p_t \\ y_t \end{bmatrix} = \begin{bmatrix} h \\ -h \\ 0 \end{bmatrix} + M \begin{bmatrix} \gamma^{-(t-1)} d_{t-1} \\ \gamma^{-(t-1)} p_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} f_1 \varepsilon_{1t} + f_2 \varepsilon_{2t} + f_3 \varepsilon_{3t} \\ (1 - f_1)\varepsilon_{1t} + (1 - f_2)\varepsilon_{2t} - f_3 \varepsilon_{3t} \\ \varepsilon_{2t} \end{bmatrix} \tag{A4}$$

where

$$M = \frac{1}{\gamma} \begin{bmatrix} a & b & \gamma c \\ -a & 1 + r - b & -\gamma c \\ 0 & 0 & \gamma \phi \end{bmatrix}. \tag{A5}$$

can be written as

$$z_t = Mz_{t-1} + u_t \tag{A6}$$

where

$$z_t = \begin{bmatrix} \gamma^{-t}d_t - \mu_d \\ \gamma^{-t}p_t - \mu_p \\ y_t \end{bmatrix} \tag{A7}$$

$$\mu_d = \frac{(1+r-\gamma)\, h\gamma}{(1+r-\gamma)(\gamma-a) - b\gamma} \tag{A8}$$

$$\mu_p = \frac{h\gamma^2}{(1+r-\gamma)(\gamma-a) - b\gamma} \tag{A9}$$

$$u_t = \begin{bmatrix} f_1 & f_2 & f_3 \\ 1-f_1 & 1-f_2 & -f_3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix} \tag{A10}$$

Iterating (A6) yields

$$z_t = \sum_{h=0}^{n-1} M^h u_{t-h} + M^n z_{t-n}. \tag{A11}$$

The matrix M defined in (A5) can be written as

$$M = LDL^{-1} \tag{A12}$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{A13}$$

is a diagonal matrix, λ_1, λ_2 and λ_3 are the eigenvalues of M, and L is a matrix of eigenvectors of M. From (A5) the characteristic equation of M is

$$\left[\left(\frac{a}{\gamma} - \lambda\right)\left(\frac{1+r-b}{\gamma} - \lambda\right) + \frac{ab}{\gamma^2}\right][\lambda_3 - \lambda] = 0$$

whose roots are λ_1, λ_2 and λ_3 where λ_1 and λ_2 are by given (A2) and (A3), and $\lambda_3 = \phi$. Let

$$k = \frac{\gamma}{1+r}.$$

A matrix of eigenvectors of M is

$$L = \begin{bmatrix} 1-k\lambda_1 & 1-k\lambda_2 & 1-k\lambda_3 \\ k\lambda_1 & k\lambda_2 & k\lambda_3 \\ 0 & 0 & -kc^{-1}(\lambda_3-\lambda_1)(\lambda_3-\lambda_2) \end{bmatrix} \tag{A14}$$

with inverse

$$L^{-1} = \frac{1}{k(\lambda_1-\lambda_2)} \begin{bmatrix} -k\lambda_2 & 1-k\lambda_2 & \frac{1}{(\lambda_3-\lambda_1)}c \\ k\lambda_1 & -1+k\lambda_1 & -\frac{1}{(\lambda_3-\lambda_2)}c \\ 0 & 0 & -\frac{(\lambda_1-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}c \end{bmatrix}. \tag{A15}$$

Equation (A12) implies that $M^h = LD^hL^{-1}$ so from (A11)

$$z_t = \sum_{h=0}^{t-1} LD^hL^{-1}u_{t-h} + LD^tL^{-1}z_0. \tag{A16}$$

From (A10) and (A15)

$$L^{-1}u_{t-h} = \Delta\varepsilon_{t-h} \tag{A17}$$

where

$$\Delta = \frac{1}{k(\lambda_1-\lambda_2)} \begin{bmatrix} 1-k\lambda_2-f_1 & 1-k\lambda_2-f_2+\frac{c}{(\lambda_3-\lambda_1)} & -f_3 \\ -1+k\lambda_1+f_1 & -1+k\lambda_1+f_2-\frac{c}{(\lambda_3-\lambda_2)} & f_3 \\ 0 & \frac{-(\lambda_1-\lambda_2)c}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} & 0 \end{bmatrix} \tag{A18}$$

and

$$\varepsilon_{t-h} = \begin{bmatrix} \varepsilon_{1t-h} \\ \varepsilon_{2t-h} \\ \varepsilon_{3t-h} \end{bmatrix} \tag{A19}$$

whilst from (A7) and (A15)

$$L^{-1}z_0 = \begin{bmatrix} q_1 \\ q_2 \\ -\frac{1}{k(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}cy_0 \end{bmatrix}$$

where

$$q_1 = \frac{1}{k(\lambda_1-\lambda_2)}\left(-k\lambda_2(d_0-\mu_d) + (1-k\lambda_2)(p_0-\mu_p) + \frac{1}{(\lambda_3-\lambda_1)}cy_0\right) \tag{A20}$$

$$q_2 = \frac{1}{k(\lambda_1-\lambda_2)}\left(k\lambda_1(d_0-\mu_d) - (1-k\lambda_1)(p_0-\mu_p) - \frac{1}{(\lambda_3-\lambda_2)}cy_0\right). \tag{A21}$$

Hence from (A13), (A14), (A16) and (A17)

$$\begin{aligned}\frac{d_t}{\gamma^t} - \mu_d &= \sum_{h=0}^{t-1}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(1-k\lambda_i\right)\lambda_i^h\Delta_{ij}\varepsilon_{jt-h} + m_{dt} \\ &= \sum_{h=0}^{t-1}\Lambda_{dh}\Delta\varepsilon_{t-h} + m_{dt}\end{aligned} \tag{A22}$$

$$\begin{aligned}\frac{p_t}{\gamma^t} - \mu_d &= \sum_{h=0}^{t-1}\sum_{i=1}^{3}\sum_{j=1}^{3}k\lambda_i\lambda_i^h\Delta_{ij}\varepsilon_{jt-h} + m_{pt} \\ &= \sum_{h=0}^{t-1}\Lambda_{ph}\Delta\varepsilon_{t-h} + m_{pt}\end{aligned} \tag{A23}$$

$$y_t = \sum_{h=0}^{t-1}\lambda_3^h\varepsilon_{2t-h} + \lambda_3^t y_0$$

where Δ and ε_{t-h} are given by (A18) and (A19),

$$\Lambda_{dh} = \Big(\left(1-k\lambda_1\right)\lambda_1^h, \left(1-k\lambda_2\right)\lambda_2^h, \left(1-k\lambda_3\right)\lambda_3^h\Big) \tag{A24}$$

$$\Lambda_{ph} = \Big(k\lambda_1\lambda_1^h, k\lambda_2\lambda_2^h, k\lambda_3\lambda_3^h\Big) \tag{A25}$$

$$m_{dt} = \left(1-k\lambda_1\right)\lambda_1^t q_1 + \left(1-k\lambda_2\right)\lambda_2^t q_2 - \frac{\left(1-k\lambda_3\right)c\lambda_3^t y_0}{k\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)} \tag{A26}$$

$$m_{pt} = k\lambda_1\lambda_1^t q_1 + k\lambda_2\lambda_2^t q_2 - \frac{k\lambda_3 c\lambda_3^t y_0}{k\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)} \tag{A27}$$

and Δ and ε_{t-h} are given by (A18) and (A19).

Proof of Proposition 5

The result is proved from using Lemmas 1, 2 and 3. Lemma 1 gives expressions for d_t and p_t which are equivalent to (5.2) and (5.3), but unlike the expressions in these equations involve only real valued variables even if λ_1 and λ_2 are complex, are finite and well defined for all values of λ_1, λ_2 and λ_3 even if two or more of the roots coincide, and disentangle the effects of λ_1, λ_2 and λ_3 from those of f_1, f_2,f_3 and c.

Lemma 1

$$\frac{d_t}{\gamma^t} - \mu_d = \sum_{h=0}^{t-1} \psi_{dh} A_d F \varepsilon_{t-h} + m_{dt} \tag{A28}$$

$$\frac{p_t}{\gamma^t} - \mu_p = \sum_{h=0}^{t-1} \psi_{ph} A_p F \varepsilon_{t-h} + m_{dt} \tag{A29}$$

where

$$\psi_{dh} = \left(\lambda_1^h + \lambda_2^h, \frac{\lambda_1^h - \lambda_2^h}{\lambda_1 - \lambda_2}, \frac{1}{k(\lambda_1 - \lambda_2)} \left[-(1 - k\lambda_1) \frac{\left(\lambda_1^h - \lambda_3^h\right)}{(\lambda_1 - \lambda_3)} + (1 - k\lambda_2) \frac{\left(\lambda_2^h - \lambda_3^h\right)}{(\lambda_2 - \lambda_3)} \right] \right)$$

$$\psi_{ph} = \left(\lambda_1^h + \lambda_2^h, \frac{\lambda_1^h - \lambda_2^h}{\lambda_1 - \lambda_2}, \frac{1}{k(\lambda_1 - \lambda_2)} \left[-k\lambda_1 \frac{\left(\lambda_1^h - \lambda_3^h\right)}{(\lambda_1 - \lambda_3)} + k\lambda_2 \frac{\left(\lambda_2^h - \lambda_3^h\right)}{(\lambda_2 - \lambda_3)} \right] \right)$$

$$A_d = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{k}(1 - k\lambda_1)(1 - k\lambda_2) & -\frac{1}{k} + \frac{1}{2}(\lambda_1 + \lambda_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A30}$$

$$A_p = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2}(\lambda_1 + \lambda_2) - k\lambda_1\lambda_2 & -\frac{1}{2}(\lambda_1 + \lambda_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A31}$$

$$F = \begin{bmatrix} 1 & 1 & 0 \\ f_1 & f_2 & f_3 \\ 0 & c & 0 \end{bmatrix}. \tag{A32}$$

The vectors ψ_{dh}, ψ_{ph} and matrices A_p, A_d and F are real valued and finite.

Proof. The Lemma is proved using (5.2) and (5.3) and showing that $\Lambda_{dh}\Delta = \psi_{dh} A_d F$ and $\Lambda_{ph}\Delta = \psi_{ph} A_p F$. Note firstly that $\Delta = HF$ where

$$H = \frac{1}{k(\lambda_1 - \lambda_2)} \begin{bmatrix} 1 - k\lambda_2 & -1 & \frac{1}{\lambda_3 - \lambda_1} \\ -1 + k\lambda_1 & 1 & -\frac{1}{\lambda_3 - \lambda_2} \\ 0 & 0 & -\frac{(\lambda_1 - \lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \end{bmatrix} \tag{A33}$$

so the Lemma can be proved by showing that $\Lambda_{dh}H = \psi_{dh}A_d$ and $\Lambda_{ph}H = \psi_{ph}A_p$. From (5.7) and (A33)

$$\Lambda_{dh}H = \left(\frac{1}{k}\left(1-k\lambda_1\right)\left(1-k\lambda_2\right)\frac{\left(\lambda_1^h-\lambda_2^h\right)}{\left(\lambda_1-\lambda_2\right)}, \frac{-\left(1-k\lambda_1\right)\lambda_1^h+\left(1-k\lambda_2\right)\lambda_2^h}{k\left(\lambda_1-\lambda_2\right)}, \alpha_d \right)$$

where

$$\begin{aligned} \alpha_d &= \frac{1}{k\left(\lambda_1-\lambda_2\right)}\left[\frac{\left(1-k\lambda_1\right)\lambda_1^h}{\lambda_3-\lambda_1} - \frac{\left(1-k\lambda_2\right)\lambda_2^h}{\lambda_3-\lambda_2} - \frac{\left(\lambda_1-\lambda_2\right)\left(1-k\lambda_3\right)\lambda_3^h}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)}\right] \\ &\equiv \frac{1}{k\left(\lambda_1-\lambda_2\right)}\left[-\left(1-k\lambda_1\right)\frac{\left(\lambda_1^h-\lambda_3^h\right)}{\left(\lambda_1-\lambda_3\right)} + \left(1-k\lambda_2\right)\frac{\left(\lambda_2^h-\lambda_3^h\right)}{\left(\lambda_2-\lambda_3\right)}\right] \equiv \psi_{3dh} \end{aligned}$$

so

$$\begin{aligned} \Lambda_{dh}H &= \left(\frac{1}{k}\left(1-k\lambda_1\right)\left(1-k\lambda_2\right)\psi_{2dh}, -\frac{1}{2}\psi_{1dh} - \left(1-\frac{1}{2}\left(1-k\left(\lambda_1+\lambda_2\right)\right)\right)\psi_{2dh}, \psi_{3dh}\right) \\ &= \psi_{dh}A_d. \end{aligned}$$

Also

$$\begin{aligned} \Lambda_{ph}H &= \frac{1}{k\left(\lambda_1-\lambda_2\right)}\left(k\lambda_1\lambda_1^h, k\lambda_2\lambda_2^h, k\lambda_3\lambda_3^h\right)\left[\begin{array}{ccc} 1-k\lambda_2 & -1 & \frac{1}{\lambda_3-\lambda_1} \\ -1+k\lambda_1 & 1 & -\frac{1}{\lambda_3-\lambda_2} \\ 0 & 0 & -\frac{\left(\lambda_1-\lambda_2\right)}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)} \end{array}\right] \\ &= \left(\frac{k\lambda_1\left(1-k\lambda_2\right)\lambda_1^h - k\lambda_2\left(1-k\lambda_1\right)\lambda_2^h}{k\left(\lambda_1-\lambda_2\right)}, \frac{-k\lambda_1\lambda_1^h+k\lambda_2\lambda_2^h}{k\left(\lambda_1-\lambda_2\right)}, \alpha_p\right) \end{aligned}$$

where

$$\begin{aligned} \alpha_p &= \frac{1}{k\left(\lambda_1-\lambda_2\right)}\left[\frac{k\lambda_1\lambda_1^h}{\lambda_3-\lambda_1} - \frac{k\lambda_2\lambda_2^h}{\lambda_3-\lambda_2} - \frac{\left(\lambda_1-\lambda_2\right)k\lambda_3\lambda_3^h}{\left(\lambda_3-\lambda_1\right)\left(\lambda_3-\lambda_2\right)}\right] \\ &= \frac{1}{k\left(\lambda_1-\lambda_2\right)}\left[-k\lambda_1\frac{\left(\lambda_1^h-\lambda_3^h\right)}{\left(\lambda_1-\lambda_3\right)} + k\lambda_2\frac{\left(\lambda_2^h-\lambda_3^h\right)}{\left(\lambda_2-\lambda_3\right)}\right] = \psi_{3ph} \end{aligned}$$

so

$$\begin{aligned} \Lambda_{ph}H &= \left(\frac{1}{2}\psi_{1ph} + \left(-k\lambda_1\lambda_2 + \frac{1}{2}\left(\lambda_1+\lambda_2\right)\right)\psi_{2ph}, -\frac{1}{2}\psi_{1ph} - \frac{1}{2}\left(\lambda_1+\lambda_2\right)\psi_{2ph}, \psi_{3ph}\right) \\ &= \psi_{ph}A_p. \end{aligned}$$

To see that the vectors ψ_{dh}, ψ_{ph} and matrices A_p, A_d and F are real valued even if λ_1 and λ_2 are complex, note that if complex λ_1 and λ_2 are complex conjugates. To see that they are finite even when roots coincide note from the definition of a derivative that terms such as $\frac{\left(\lambda_1^h-\lambda_3^h\right)}{\left(\lambda_1-\lambda_3\right)}$ tend to $h\lambda_1^{h-1}$. ■

Lemma 2 *Assume that* $-1 < \lambda_3 < 1$,

$$-\frac{\gamma^2}{1+r} < a < \frac{\gamma^2}{1+r}$$

and

$$(1+r-\gamma)\left(1-\frac{a}{\gamma}\right) < b < (1+r+\gamma)\left(1+\frac{a}{\gamma}\right)$$

so λ_1, λ_2 *and* λ_3 *all lie in the interior of the unit circle. Then if* $\operatorname{var}\varepsilon_{it-h} = \sigma_i^2$ *for* $i = 1, 2.3$ *and* $h = 0, 1, 2....$

$$\begin{aligned}\operatorname{var}\gamma^{-t}d_t &= \begin{bmatrix} 1 & f_1 & 0 \end{bmatrix} A_d'\Omega_d A_d \begin{bmatrix} 1 \\ f_1 \\ 0 \end{bmatrix} \sigma_1^2 + \begin{bmatrix} 1 & f_2 & c \end{bmatrix} A_d'\Omega_d A_d \begin{bmatrix} 1 \\ f_2 \\ c \end{bmatrix} \sigma_2^2 \\ &\quad + \begin{bmatrix} 0 & f_3 & 0 \end{bmatrix} A_d'\Omega_d A_d \begin{bmatrix} 0 \\ f_3 \\ 0 \end{bmatrix} \sigma_3^2\end{aligned}$$

$$\begin{aligned}\operatorname{var}\gamma^{-t}p_t &= \begin{bmatrix} 1 & f_1 & 0 \end{bmatrix} A_p'\Omega_p A_p \begin{bmatrix} 1 \\ f_1 \\ 0 \end{bmatrix} \sigma_1^2 + \begin{bmatrix} 1 & f_2 & c \end{bmatrix} A_p'\Omega_p A_p \begin{bmatrix} 1 \\ f_2 \\ c \end{bmatrix} \sigma_2^2 \\ &\quad + \begin{bmatrix} 0 & f_3 & 0 \end{bmatrix} A_p'\Omega_p A_p \begin{bmatrix} 0 \\ f_3 \\ 0 \end{bmatrix} \sigma_3^2\end{aligned}$$

where A_d *and* A_p *are given by (A30) and (A31) and*

$$\Omega_d = \sum_{h=0}^{\infty} \psi_{dh}'\psi_{dh}$$

$$\Omega_p = \sum_{h=0}^{\infty} \psi_{ph}'\psi_{ph}$$

Proof. From Lemma 1 if λ_1, λ_2 and λ_3 lie in the unit circle

$$\operatorname{var}\gamma^{-t}d_t = \sum_{h=0}^{\infty} \psi_{dh} A_d F S^2 F' A_d' \psi_{dh}'$$

where

$$S = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}.$$

But

$$
\begin{aligned}
\psi_{dh} A_d F S^2 F' A_d' \psi_{dh}' &= \operatorname{trace} \psi_{dh} A_d F S^2 F' A_d' \psi_{dh}' \\
&= \operatorname{trace} S F' A_d' \psi_{dh}' \psi_{dh} A_d F S \\
&= \begin{bmatrix} 1 & f_1 & 0 \end{bmatrix} A_d' \psi_{dh}' \psi_{dh} A_d \begin{bmatrix} 1 \\ f_1 \\ 0 \end{bmatrix} \sigma_1^2 \\
&\quad + \begin{bmatrix} 1 & f_2 & c \end{bmatrix} A_d' \psi_{dh}' \psi_{dh} A_d \begin{bmatrix} 1 \\ f_2 \\ c \end{bmatrix} \sigma_2^2 \\
&\quad + \begin{bmatrix} 0 & f_3 & 0 \end{bmatrix} A_d' \psi_{dh}' \psi_{dh} A_d \begin{bmatrix} 0 \\ f_3 \\ 0 \end{bmatrix} \sigma_3^2
\end{aligned}
$$

since S is a diagonal matrix and F is given by (A32). Noting that h appears only in $\psi_{dh}' \psi_{dh}$ gives the result for dividends.

Similarly

$$
\operatorname{var} \gamma^{-t} p_t = \sum_{h=0}^{\infty} \psi_{ph} A_p F S^2 F' A_p' \psi_{ph}'
$$

and

$$
\begin{aligned}
\psi_{ph} A_p F S^2 F' A_p' \psi_{ph}' &= \operatorname{trace} \psi_{ph} A_p F S^2 F' A_p' \psi_{ph}' \\
&= \operatorname{trace} S F' A_p' \psi_{ph}' \psi_{ph} A_p F S \\
&= \begin{bmatrix} 1 & f_1 & 0 \end{bmatrix} A_p' \psi_{ph}' \psi_{ph} A_p \begin{bmatrix} 1 \\ f_1 \\ 0 \end{bmatrix} \sigma_1^2 \\
&\quad + \begin{bmatrix} 1 & f_2 & c \end{bmatrix} A_p' \psi_{ph}' \psi_{ph} A_p \begin{bmatrix} 1 \\ f_2 \\ c \end{bmatrix} \sigma_2^2 \\
&\quad + \begin{bmatrix} 0 & f_3 & 0 \end{bmatrix} A_p' \psi_{ph}' \psi_{ph} A_p \begin{bmatrix} 0 \\ f_3 \\ 0 \end{bmatrix} \sigma_3^2
\end{aligned}
$$

which as h appears only in $\psi_{ph}' \psi_{ph}$ gives the result for prices. ■

Lemma 3 *If*

$$
\pi = \lambda_1 \lambda_2 = \frac{a\,(1+r)}{\gamma^2}
$$

and

$$
\tau = \frac{\lambda_1 + \lambda_2}{1 + \lambda_1 \lambda_2} = \frac{\gamma\,(1 + r + a - b)}{\gamma^2 + a\,(1+r)}
$$

$$\Omega_{11}^d = \Omega_{11}^p = \frac{4(1-\tau^2)+\tau^2(1-\pi)^2}{(1-\tau^2)(1-\pi^2)}$$

$$\Omega_{22}^d = \Omega_{22}^p = \frac{1}{(1-\tau^2)(1-\pi^2)}$$

$$\Omega_{12}^d = \Omega_{21}^d = \Omega_{12}^p = \Omega_{21}^p = \frac{\tau(1-\pi)}{(1-\tau^2)(1-\pi^2)}.$$

Proof. We have

$$\begin{aligned}
\Omega_{11}^d &= \Omega_{11}^p = \sum_{h=0}^{\infty}\psi_{1hd}^2 = \sum_{h=0}^{\infty}\psi_{1hp}^2 = \sum_{h=0}^{\infty}\left(\lambda_1^h+\lambda_2^h\right)^2 = \\
&= \frac{1}{1-\lambda_1^2}+\frac{1}{1-\lambda_2^2}+\frac{2}{1-\lambda_1\lambda_2} \\
&= \frac{4(1+\lambda_1\lambda_2)-(\lambda_1+\lambda_2)^2(3-\lambda_1\lambda_2)}{\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right) &= (1-\lambda_1-\lambda_2+\lambda_1\lambda_2)(1+\lambda_1+\lambda_2+\lambda_1\lambda_2) \\
&= (1+\pi)(1-\tau)(1+\pi)(1+\tau) = \left(1-\tau^2\right)(1+\pi)^2
\end{aligned}$$

so

$$\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right) = \left(1-\tau^2\right)(1+\pi)^2 \tag{A34}$$

$$1-\lambda_1\lambda_2 = 1-\pi \tag{A35}$$

$$\begin{aligned}
4(1+\lambda_1\lambda_2)-(\lambda_1+\lambda_2)^2(3-\lambda_1\lambda_2) &= 4(1+\pi)-\tau^2(1+\pi)^2(3-\pi) \\
&= 4(1-\tau^2)+\tau^2(1-\pi)^2
\end{aligned}$$

so

$$\Omega_{11}^d = \Omega_{11}^p = \frac{4(1-\tau^2)+\tau^2(1-\pi)^2}{(1-\tau^2)(1-\pi^2)}. \tag{A36}$$

Similarly

$$\begin{aligned}
\Omega_{12}^d &= \Omega_{21}^d = \Omega_{12}^p = \Omega_{21}^p = \sum_{h=0}^{\infty}\psi_{1hd}\psi_{2hd} = \sum_{h=0}^{\infty}\psi_{1hp}\psi_{2hp} \\
&= \sum_{h=0}^{\infty}\left(\lambda_1^h+\lambda_2^h\right)\left(\frac{\lambda_1^h-\lambda_2^h}{\lambda_1-\lambda_2}\right) \\
&= \frac{1}{\lambda_1-\lambda_2}\left[\frac{1}{1-\lambda_1^2}-\frac{1}{1-\lambda_2^2}\right] \\
&= \frac{\lambda_1+\lambda_2}{\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)} = \frac{\tau(1-\pi)}{(1-\tau^2)(1-\pi^2)}
\end{aligned}$$

and

$$
\begin{aligned}
\Omega_{22}^{d} &= \Omega_{22}^{p}=\sum_{h=0}^{\infty}\psi_{2hd}^{2}=\sum_{h=0}^{\infty}\psi_{2hp}^{2}=\sum_{h=0}^{\infty}\left(\frac{\lambda_1^h-\lambda_2^h}{\lambda_1-\lambda_2}\right)^2 \\
&= \frac{1}{\left(\lambda_1-\lambda_2\right)^2}\left[\frac{1}{1-\lambda_1^2}+\frac{1}{1-\lambda_2^2}-\frac{2}{1-\lambda_1\lambda_2}\right] \\
&= \frac{1+\lambda_1\lambda_2}{\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)\left(1-\lambda_1\lambda_2\right)} \\
&= \frac{1}{\left(1-\tau^2\right)\left(1-\pi^2\right)}.
\end{aligned}
$$

■

The next Lemma gives formulae which taken with Lemmas 2 and 3 give general expressions for the variances of prices and dividends.

Lemma 4

$$
\Omega_{13}^{d}=\Omega_{31}^{d}=\frac{k\left\{\tau\left(1-\pi\right)-\lambda_3\pi\left[2-\tau^2\left(1+\pi\right)\right]\right\}-\tau^2\left(1+\pi\right)+2\pi+\pi\tau\lambda_3\left(1-\pi\right)}{k(1-\tau^2)\left(1-\pi^2\right)\left[1-\tau\left(1+\pi\right)\lambda_3+\pi\lambda_3^2\right]}
$$

$$
\Omega_{23}^{d}=\Omega_{32}^{d}=\frac{k\left(1-\lambda_3\tau\pi\right)+\lambda_3\pi-\tau}{k(1-\tau^2)\left(1-\pi^2\right)\left[1-\tau\left(1+\pi\right)\lambda_3+\pi\lambda_3^2\right]}
$$

$$
\Omega_{33}^{d}=\Omega_{33}\left(1\right)=\frac{\left(1-\lambda_3^2\pi\right)\left[1+k^2-2k\tau\right]+\lambda_3\left(1-\pi\right)\left[\left(1+k^2\right)\tau-2k\right]}{k^2\left(1-\lambda_3^2\right)\left(1-\tau^2\right)\left(1-\pi^2\right)\left[1-\tau\left(1+\pi\right)\lambda_3+\pi\lambda_3^2\right]}
$$

$$
\Omega_{13}^{p}=\Omega_{31}^{p}=\frac{\lambda_3\pi\left[2-\tau^2\left(1+\pi\right)\right]-\tau\left(1-\pi\right)}{\left(1-\tau^2\right)\left(1-\pi^2\right)\left[1-\tau\left(1+\pi\right)\lambda_3+\pi\lambda_3^2\right]}
$$

$$
\Omega_{23}^{p}=\Omega_{32}^{p}=\frac{-1+\lambda_3\tau\pi}{\left(1-\tau^2\right)\left(1-\pi^2\right)\left[1-\tau\left(1+\pi\right)\lambda_3+\pi\lambda_3^2\right]}
$$

and

$$
\Omega_{33}^{p}=\Omega_{33}\left(0\right)=\frac{1-\lambda_3^2\pi+\lambda_3\left(1-\pi\right)\tau}{\left(1-\lambda_3^2\right)\left(1-\tau^2\right)\left(1-\pi^2\right)\left[1-\tau\left(1+\pi\right)\lambda_3+\pi\lambda_3^2\right]}.
$$

Proof. Let

$$\psi_{3h}(\alpha)=\frac{1}{k(\lambda_1-\lambda_2)}\left[-(\alpha-k\lambda_1)\frac{\left(\lambda_1^h-\lambda_3^h\right)}{(\lambda_1-\lambda_3)}+(\alpha-k\lambda_2)\frac{\left(\lambda_2^h-\lambda_3^h\right)}{(\lambda_2-\lambda_3)}\right] \tag{A37}$$

and observe that when $\alpha=1$, $\psi_{3h}(\alpha)=\psi_{3hd}$ and when $\alpha=0$, $-\psi_{3h}(\alpha)=\psi_{3hp}$. Let $\psi_{ih}=\psi_{ihp}\neq\psi_{ih}d$ for $i=1,2$ so

$$\Omega_{13}(\alpha)=\sum_{h=0}^{\infty}\psi_{1h}\psi_{3h}(\alpha)$$

and if $\alpha=1$, $\Omega_{13}(\alpha)=\Omega_{13}^d=\Omega_{31}^d$ whilst if $\alpha=0$, $\Omega_{13}(\alpha)=-\Omega_{13}^p=-\Omega_{31}^p$. Now

$$\begin{aligned}\sum_{h=0}^{\infty}\lambda_i^h\psi_{3h}(\alpha) &= \frac{1}{k(\lambda_1-\lambda_2)}\left[-\frac{(\alpha-k\lambda_1)}{(\lambda_1-\lambda_3)}\left(\frac{1}{1-\lambda_1\lambda_i}-\frac{1}{1-\lambda_3\lambda_i}\right)\right]\\ &\quad+\frac{1}{k(\lambda_1-\lambda_2)}\left[\frac{(\alpha-k\lambda_2)}{(\lambda_2-\lambda_3)}\left(\frac{1}{1-\lambda_2\lambda_i}-\frac{1}{1-\lambda_3\lambda_i}\right)\right]\\ &= \frac{\lambda_i(k-\alpha\lambda_i)}{k(1-\lambda_i\lambda_1)(1-\lambda_i\lambda_2)(1-\lambda_i\lambda_3)}\end{aligned}$$

so

$$\begin{aligned}\Omega_{13}(\alpha) &= \sum_{h=0}^{\infty}\left(\lambda_1^h+\lambda_2^h\right)\psi_{3h}(\alpha)\\ &= \frac{\lambda_1(k-\alpha\lambda_1)}{k(1-\lambda_1^2)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)}+\frac{\lambda_2(k-\alpha\lambda_2)}{k(1-\lambda_1\lambda_2)\left(1-\lambda_2^2\right)(1-\lambda_2\lambda_3)}\\ &= \frac{\lambda_1(k-\alpha\lambda_1)\left(1-\lambda_2^2\right)(1-\lambda_2\lambda_3)+\lambda_2(k-\alpha\lambda_2)\left(1-\lambda_1^2\right)(1-\lambda_1\lambda_3)}{k(1-\lambda_1^2)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_2\lambda_3)}\\ &= \frac{(\lambda_1+\lambda_2)(1-\lambda_1\lambda_2)-\lambda_3\lambda_1\lambda_2\left(2-\lambda_1^2-\lambda_2^2\right)}{\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_2\lambda_3)}\\ &\quad-\frac{\alpha\left[\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2-\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2)(1-\lambda_1\lambda_2)\right]}{k(1-\lambda_1^2)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_2\lambda_3)}.\end{aligned}$$

Furthermore

$$\lambda_1^2+\lambda_2^2=\tau^2(1+\pi)^2-2\pi \tag{A38}$$

$$(1-\lambda_1\lambda_3)(1-\lambda_2\lambda_3)=1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2 \tag{A39}$$

and so using (A34) and (A35)

$$\begin{aligned}&\Omega_{13}(\alpha)\\ =\;&\frac{k\left\{\tau(1-\pi)-\lambda_3\pi\left[2-\tau^2(1+\pi)\right]\right\}+\alpha\left\{-\tau^2(1+\pi)+2\pi+\pi\tau\lambda_3(1-\pi)\right\}}{k(1-\tau^2)(1-\pi^2)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]}\end{aligned}$$

implying that

$$\begin{aligned}
\Omega_{13}^d &= \Omega_{31}^d = \Omega_{13}(1) \\
&= \frac{k\left\{\tau(1-\pi) - \lambda_3\pi\left[2-\tau^2(1+\pi)\right]\right\}}{k(1-\tau^2)(1-\pi^2)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]} \\
&\quad - \frac{\tau^2(1+\pi) - 2\pi - \pi\tau\lambda_3(1-\pi)}{k(1-\tau^2)(1-\pi^2)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]} \qquad \text{(A40)}
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{13}^p &= \Omega_{31}^p = -\Omega_{13}(0) \\
&= \frac{\lambda_3\pi\left[2-\tau^2(1+\pi)\right] - \tau(1-\pi)}{(1-\tau^2)(1-\pi^2)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]}. \qquad \text{(A41)}
\end{aligned}$$

Similarly

$$\begin{aligned}
\Omega_{23}(\alpha) &= \sum_{h=0}^{\infty} \frac{\left(\lambda_1^h - \lambda_2^h\right)}{(\lambda_1-\lambda_2)}\psi_{3h}(\alpha) \\
&= \frac{\lambda_1(k-\alpha\lambda_1)}{k(\lambda_1-\lambda_2)(1-\lambda_1^2)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)} \\
&\quad - \frac{\lambda_2(k-\alpha\lambda_2)}{k(\lambda_1-\lambda_2)(1-\lambda_1\lambda_2)\left(1-\lambda_2^2\right)(1-\lambda_2\lambda_3)} \\
&= \frac{\lambda_1(k-\alpha\lambda_1)\left(1-\lambda_2^2\right)(1-\lambda_2\lambda_3) - \lambda_2(k-\alpha\lambda_2)\left(1-\lambda_1^2\right)(1-\lambda_1\lambda_3)}{k(\lambda_1-\lambda_2)\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_2\lambda_3)} \\
&= \frac{1+\lambda_1\lambda_2 - \lambda_3\lambda_1\lambda_2(\lambda_1+\lambda_2)}{\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_2\lambda_3)} \\
&\quad - \frac{\alpha\left[\lambda_1+\lambda_2-\lambda_3\lambda_1\lambda_2(1+\lambda_1\lambda_2)\right]}{k(1-\lambda_1^2)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_1\lambda_3)(1-\lambda_2\lambda_3)}.
\end{aligned}$$

Thus using (A35) and (A39)

$$\begin{aligned}
\Omega_{23}(\alpha) &= \sum_{h=0}^{\infty} \frac{\left(\lambda_1^h - \lambda_2^h\right)}{(\lambda_1-\lambda_2)}\psi_{3h}(\alpha) \\
&= \frac{k(1-\lambda_3\tau\pi) + \alpha(\lambda_3\pi - \tau)}{k(1-\tau^2)(1-\pi^2)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]}
\end{aligned}$$

so

$$\Omega_{23}^d = \Omega_{32}^d = \Omega_{23}(1) = \frac{k(1-\lambda_3\tau\pi) + \lambda_3\pi - \tau}{k(1-\tau^2)(1-\pi^2)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]} \qquad \text{(A42)}$$

and

$$\Omega_{23}^p = \Omega_{32}^p = -\Omega_{23}(0) = \frac{-1+\lambda_3\tau\pi}{(1-\tau^2)(1-\pi^2)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]}.$$

As

$$
\begin{aligned}
&\sum_{h=0}^{\infty} \frac{\left(\lambda_i^h - \lambda_3^h\right)}{(\lambda_i - \lambda_3)} \frac{\left(\lambda_k^h - \lambda_3^h\right)}{(\lambda_k - \lambda_3)} \\
&= \frac{1}{(\lambda_i - \lambda_3)(\lambda_k - \lambda_3)} \left[\frac{1}{1 - \lambda_i \lambda_k} - \frac{1}{1 - \lambda_3 \lambda_i} - \frac{1}{1 - \lambda_3 \lambda_k} + \frac{1}{1 - \lambda_3^2} \right] \\
&= \frac{\left(1 - \lambda_i \lambda_k \lambda_3^2\right)}{\left(1 - \lambda_3^2\right)(1 - \lambda_i \lambda_k)(1 - \lambda_3 \lambda_i)(1 - \lambda_3 \lambda_k)}
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{33}(\alpha) &= \\
&= \frac{(\alpha - k\lambda_1)}{k^2 (\lambda_1 - \lambda_2)^2 \left(1 - \lambda_3^2\right)(1 - \lambda_3 \lambda_1)} \left[\frac{(\alpha - k\lambda_1)(1 + \lambda_3 \lambda_1)}{\left(1 - \lambda_1^2\right)} - \frac{(\alpha - k\lambda_2)\left(1 - \lambda_3^2 \lambda_1 \lambda_2\right)}{(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_2)} \right] \\
&\quad + \frac{(\alpha - k\lambda_2)}{k^2 (\lambda_1 - \lambda_2)^2 \left(1 - \lambda_3^2\right)(1 - \lambda_3 \lambda_2)} \left[\frac{(\alpha - k\lambda_2)(1 + \lambda_3 \lambda_2)}{\left(1 - \lambda_2^2\right)} - \frac{(\alpha - k\lambda_1)\left(1 - \lambda_3^2 \lambda_1 \lambda_2\right)}{(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_1)} \right]
\end{aligned}
$$

but

$$
\begin{aligned}
&\frac{(\alpha - k\lambda_1)(1 + \lambda_3 \lambda_1)}{\left(1 - \lambda_1^2\right)} - \frac{(\alpha - k\lambda_2)\left(1 - \lambda_3^2 \lambda_1 \lambda_2\right)}{(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_2)} \\
&= \frac{(\alpha - k\lambda_1)(1 + \lambda_3 \lambda_1)(1 - \lambda_3 \lambda_2)(1 - \lambda_1 \lambda_2) - (\alpha - k\lambda_2)\left(1 - \lambda_3^2 \lambda_1 \lambda_2\right)\left(1 - \lambda_1^2\right)}{\left(1 - \lambda_1^2\right)(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_2)} \\
&= \frac{\left(1 - \lambda_3^2 \lambda_1 \lambda_2\right)\left[(\alpha - k\lambda_1)(1 - \lambda_1 \lambda_2) - (\alpha - k\lambda_2)\left(1 - \lambda_1^2\right)\right]}{\left(1 - \lambda_1^2\right)(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_2)} \\
&\quad + \frac{\lambda_3 (\lambda_1 - \lambda_2)(\alpha - k\lambda_1)(1 - \lambda_1 \lambda_2)}{\left(1 - \lambda_1^2\right)(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_2)} \\
&= (\lambda_1 - \lambda_2) \frac{\left(1 - \lambda_3^2 \lambda_1 \lambda_2\right)(\alpha \lambda_1 - k) + \lambda_3 (\alpha - k\lambda_1)(1 - \lambda_1 \lambda_2)}{\left(1 - \lambda_1^2\right)(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_2)}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\frac{(\alpha - k\lambda_2)(1 + \lambda_3 \lambda_2)}{\left(1 - \lambda_2^2\right)} - \frac{(\alpha - k\lambda_1)\left(1 - \lambda_3^2 \lambda_1 \lambda_2\right)}{(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_1)} \\
&= -(\lambda_1 - \lambda_2) \frac{\left(1 - \lambda_3^2 \lambda_1 \lambda_2\right)(\alpha \lambda_2 - k) + \lambda_3 (\alpha - k\lambda_2)(1 - \lambda_1 \lambda_2)}{\left(1 - \lambda_2^2\right)(1 - \lambda_1 \lambda_2)(1 - \lambda_3 \lambda_1)}
\end{aligned}
$$

so

$$\begin{aligned}
&\Omega_{33}(\alpha) \\
= &\frac{(\alpha-k\lambda_1)\left[\left(1-\lambda_3^2\lambda_1\lambda_2\right)(\alpha\lambda_1-k)+\lambda_3(\alpha-k\lambda_1)(1-\lambda_1\lambda_2)\right]}{k^2(\lambda_1-\lambda_2)\left(1-\lambda_3^2\right)\left(1-\lambda_1^2\right)(1-\lambda_1\lambda_2)(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)} \\
&-\frac{(\alpha-k\lambda_2)\left[\left(1-\lambda_3^2\lambda_1\lambda_2\right)(\alpha\lambda_2-k)+\lambda_3(\alpha-k\lambda_2)(1-\lambda_1\lambda_2)\right]}{k^2(\lambda_1-\lambda_2)\left(1-\lambda_3^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)} \\
= &\frac{\left(1-\lambda_3^2\lambda_1\lambda_2\right)\left[(\alpha-k\lambda_1)(\alpha\lambda_1-k)\left(1-\lambda_2^2\right)-(\alpha-k\lambda_2)(\alpha\lambda_2-k)\left(1-\lambda_1^2\right)\right]}{k^2(\lambda_1-\lambda_2)\left(1-\lambda_3^2\right)\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)} \\
&+\frac{\lambda_3(1-\lambda_1\lambda_2)\left[(\alpha-k\lambda_1)^2\left(1-\lambda_2^2\right)-(\alpha-k\lambda_2)^2\left(1-\lambda_1^2\right)\right]}{k^2(\lambda_1-\lambda_2)\left(1-\lambda_3^2\right)\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)} \\
= &\frac{\left(1-\lambda_3^2\lambda_1\lambda_2\right)\left[\left(\alpha^2+k^2\right)(1+\lambda_1\lambda_2)-2\alpha k(\lambda_1+\lambda_2)\right]}{k^2\left(1-\lambda_3^2\right)\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)} \\
&+\frac{\lambda_3(1-\lambda_1\lambda_2)\left[\left(\alpha^2+k^2\right)(\lambda_1+\lambda_2)-2\alpha k(1+\lambda_1\lambda_2)\right]}{k^2\left(1-\lambda_3^2\right)\left(1-\lambda_1^2\right)\left(1-\lambda_2^2\right)(1-\lambda_1\lambda_2)(1-\lambda_3\lambda_1)(1-\lambda_3\lambda_2)}.
\end{aligned}$$

Using (A34), (A35) and (A39)

$$\Omega_{33}(\alpha)=\frac{\left(1-\lambda_3^2\pi\right)\left[\alpha^2+k^2-2\alpha k\tau\right]+\lambda_3(1-\pi)\left[\left(\alpha^2+k^2\right)\tau-2\alpha k\right]}{k^2\left(1-\lambda_3^2\right)\left(1-\tau^2\right)\left(1-\pi^2\right)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]}$$

so

$$\Omega_{33}^d=\Omega_{33}(1)=\frac{\left(1-\lambda_3^2\pi\right)\left[1+k^2-2k\tau\right]+\lambda_3(1-\pi)\left[\left(1+k^2\right)\tau-2k\right]}{k^2\left(1-\lambda_3^2\right)\left(1-\tau^2\right)\left(1-\pi^2\right)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]} \tag{A43}$$

and

$$\Omega_{33}^p=\Omega_{33}(0)=\frac{1-\lambda_3^2\pi+\lambda_3(1-\pi)\tau}{\left(1-\lambda_3^2\right)\left(1-\tau^2\right)\left(1-\pi^2\right)\left[1-\tau(1+\pi)\lambda_3+\pi\lambda_3^2\right]} \tag{A44}$$

Proof of Proposition 5

If $c=0$

$$\begin{pmatrix} 1 & f & c \end{pmatrix} A^d\Omega^d A^{d\prime}\begin{pmatrix} 1 & f & c \end{pmatrix}'=\begin{pmatrix} 1 & f \end{pmatrix} B^d\Omega B^{d\prime}\begin{pmatrix} 1 & f \end{pmatrix}'$$

and

$$\begin{pmatrix} 1 & f & c \end{pmatrix} A^p\Omega^p A^{p\prime}\begin{pmatrix} 1 & f & c \end{pmatrix}'=\begin{pmatrix} 1 & f \end{pmatrix} B^p\Omega B^{p\prime}\begin{pmatrix} 1 & f \end{pmatrix}'$$

where

$$B^d = \begin{bmatrix} 0 & \frac{1}{k}\left[1 - k\tau(1+\pi) + k^2\pi\right] \\ \frac{1}{2} & -\frac{1}{k}\left(1 - \frac{1}{2}k\tau(1+\pi)\right) \end{bmatrix} \tag{A45}$$

$$B^p = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\tau(1+\pi) - k\pi \\ -\frac{1}{2} & -\frac{1}{2}\tau(1+\pi) \end{bmatrix} \tag{A46}$$

and

$$\begin{aligned} \Omega &= \frac{1}{(1-\pi^2)(1-\tau^2)} \begin{bmatrix} 4\left(1-\tau^2\right) + \tau^2(1-\pi)^2 & \tau(1-\pi) \\ \tau(1-\pi) & 1 \end{bmatrix} \\ &= \frac{4}{(1-\pi^2)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{(1-\pi^2)(1-\tau^2)} \begin{bmatrix} \tau(1-\pi) \\ 1 \end{bmatrix} \begin{bmatrix} \tau(1-\pi) & 1 \end{bmatrix}. \end{aligned}$$

From (A45)

$$B^d \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B^{d\prime} = \frac{1}{4} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^d \begin{bmatrix} \tau(1-\pi) \\ 1 \end{bmatrix} = \frac{1}{k} \begin{bmatrix} 1 - k\tau(1+\pi) + k^2\pi \\ -1 + k\tau \end{bmatrix}$$

so

$$\begin{aligned} \begin{pmatrix} 1 & f \end{pmatrix} B^d \Omega B^{d\prime} \begin{pmatrix} 1 & f \end{pmatrix}' &= \frac{f^2}{(1-\pi^2)} + \frac{\left[1 - k\tau(1+\pi) + k^2\pi - f(1-k\tau)\right]^2}{k^2(1-\pi^2)(1-\tau^2)} \\ &= \frac{\left[1 - k\tau(1+\pi) + k^2\pi\right]^2}{\left[(\tau-k)^2 + 1 - \tau^2\right](1-\pi^2)} \\ &\quad + \frac{\left[k^2(1-\tau^2) + (1-k\tau)^2\right]}{k^2(1-\pi^2)(1-\tau^2)} \left(f - f_0^d\right)^2 \end{aligned}$$

where

$$f_0^d = \frac{(1-k\tau)\left[1 - k\tau(1+\pi) + k^2\pi\right]}{\left[(\tau-k)^2 + 1 - \tau^2\right]} > 0 \tag{A47}$$

since both k and τ lie in $(-1, 1)$, and $1 - k\tau(1+\pi) + k^2\pi$ is linear in π and positive at both $\pi = 1$ and $\pi = -1$. Similarly (A46) implies that

$$B^p \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B^{p\prime} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

and

$$B^p \left[\begin{array}{c} \tau(1-\pi) \\ 1 \end{array} \right] = \left[\begin{array}{c} \tau - k\pi \\ -\tau \end{array} \right]$$

so

$$\begin{aligned} \left(\begin{array}{cc} 1 & f \end{array} \right) B^p \Omega B^{p\prime} \left(\begin{array}{cc} 1 & f \end{array} \right)' &= \frac{(1-f)^2}{(1-\pi^2)} + \frac{(\tau - k\pi - f\tau)^2}{(1-\pi^2)(1-\tau^2)} \\ &= \frac{k^2\pi^2}{(1-\pi^2)} + \frac{(f - f_0^p)^2}{(1-\pi^2)(1-\tau^2)} \quad \text{(A48)} \end{aligned}$$

where

$$f_0^p = 1 - k\pi\tau.$$

From (A47)

$$\begin{aligned} f_0^p - f_0^d &= 1 - k\pi\tau - \frac{(1-k\tau)\left[1 - k\tau(1+\pi) + k^2\pi\right]}{\left[(\tau-k)^2 + 1 - \tau^2\right]} \\ &= \frac{(1-k\pi\tau)\left[(\tau-k)^2 + 1 - \tau^2\right] - (1-k\tau)\left[1 - k\tau(1+\pi) + k^2\pi\right]}{\left[(\tau-k)^2 + 1 - \tau^2\right]} \\ &= \frac{k^2(1-\pi)\left(1-\tau^2\right)}{\left[(\tau-k)^2 + 1 - \tau^2\right]} > 0. \end{aligned}$$

Thus

$$1 > f_0^p > f_0^d > 0$$

and when $f = f_0^p$

$$\left(\begin{array}{cc} 1 & f \end{array} \right) B^p \Omega B^{p\prime} \left(\begin{array}{cc} 1 & f \end{array} \right)' = \frac{k^2\pi^2}{(1-\pi^2)}$$

$$\begin{aligned} & \left(\begin{array}{cc} 1 & f \end{array} \right) B^d \Omega B^{d\prime} \left(\begin{array}{cc} 1 & f \end{array} \right)' \\ = \quad & \frac{\left[1 - k\tau(1+\pi) + k^2\pi\right]^2}{\left[(\tau-k)^2 + 1 - \tau^2\right](1-\pi^2)} + \frac{k^2(1-\pi)\left(1-\tau^2\right)}{\left[(\tau-k)^2 + 1 - \tau^2\right]} \end{aligned}$$

and when $f = f_0^d$

$$\left(\begin{array}{cc} 1 & f \end{array} \right) B^p \Omega B^{p\prime} \left(\begin{array}{cc} 1 & f \end{array} \right)' = \frac{k^2\pi^2}{(1-\pi^2)} + \frac{k^4(1-\pi)\left(1-\tau^2\right)}{\left[(\tau-k)^2 + 1 - \tau^2\right](1+\pi)}$$

$$\left(\begin{array}{cc} 1 & f \end{array}\right) B^d \Omega B^{d\prime} \left(\begin{array}{cc} 1 & f \end{array}\right)' = \frac{\left[1 - k\tau\left(1+\pi\right) + k^2\pi\right]^2}{\left[\left(\tau - k\right)^2 + 1 - \tau^2\right]\left(1-\pi^2\right)}.$$

■

Proof of Proposition 6

The first step is the proof of the following Lemma.

Lemma 5 *Under the condition of Proposition 6 the expected sample variance of detrended prices* $\left(\gamma^{-1}p_1, ...\gamma^{-T}p_T\right)$ *is*

$$\begin{aligned} EV_T(\lambda, k) &= S_T\left(\lambda\right)\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2 + \frac{k^2}{T}\sum_{h=1}^{T}\left(T-h+1\right)S_{T-h+1}\left(\lambda\right)\hat{\sigma}_e^2 \\ &+ \frac{1}{T^2}\sum_{h=1}^{T}\left(T-h+1\right)\left(h-1\right)\left[m_{T-h+1}\left(\lambda\right)\right]^2\hat{\sigma}_e^2 \qquad (A49) \end{aligned}$$

and the expected sample variance of detrended perfect foresight prices $\left(\gamma^{-1}p_{1T}^*, ...\gamma^{-T}p_{TT}^*\right)$ *is*

$$\begin{aligned} EV_T^*(\lambda, k) &= S_T\left(\lambda\right)\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2 + \frac{k^2}{T}\sum_{h=1}^{T}\left(T-h+1\right)S_{T-h+1}\left(\lambda\right)\hat{\sigma}_e^2 \\ &+ \frac{1}{T^2}\sum_{h=1}^{T}\left(T-h+1\right)\left(h-1\right)\left[m_{h-1}\left(k\right) - m_{T-h+1}\left(\lambda\right)\right]^2\hat{\sigma}_e^2 \\ &+ \frac{1}{T}\sum_{h=1}^{T}\left(h-1\right)S_{h-1}\left(k\right)\hat{\sigma}_e^2. \qquad (A50) \end{aligned}$$

where

$$\begin{aligned} m_i(\theta) &= \frac{1}{i}\sum_{h=1}^{i}\theta^h \\ S_i\left(\theta\right) &= \frac{1}{i}\sum_{h=1}^{i}\left(\theta^h - m_i(\theta)\right)^2 = \frac{1}{i}\sum_{h=1}^{i}\theta^{2h} - m_i(\theta)^2. \end{aligned}$$

Proof. Let

$$\hat{p}_t(\alpha) = \lambda^t\left(p_0 - \mu_p\right) + k\lambda^t\hat{e}_1 + k\lambda^{t-1}\hat{e}_2 + ... + k\lambda\hat{e}_t + \alpha(k\hat{e}_{t+1}... + k^{h-t}\hat{e}_h + ... + k^{T-t}\hat{e}_T)$$

which implies that

$$\gamma^{-t}p_t = \mu_p + \hat{p}_t(0)$$

$$\gamma^{-t}p^*_{tT} = \mu_p + \hat{p}_t(1)$$

so $V_T(\lambda, k)$, the sample variance of detrended prices $\left(\gamma^{-1}p_1, \gamma^{-2}p_2, ...\gamma^{-t}p_t...\gamma^{-T}p_T\right)$, is the same as the sample variance of $(\hat{p}_1(\alpha), \hat{p}_2(\alpha), ...\hat{p}_t(\alpha)...\hat{p}_T(\alpha))$ with $\alpha = 0$. Furthermore, $V^*_T(\lambda, k)$, the sample variance of detrended perfect foresight prices $\left(\gamma^{-1}p^*_{1T}, \gamma^{-2}p^*_{2T}, ...\gamma^{-t}p^*_{tT}...\gamma^{-T}p^*_{TT}\right)$ is the same as the sample variance of $(\hat{p}_1(\alpha), \hat{p}_2(\alpha), ...\hat{p}_t(\alpha)...\hat{p}_T(\alpha))$ with $\alpha = 1$. It is convenient to write

$$\hat{p}_t(\alpha) = A_{t0}\left(p_0 - \mu_p\right) + \sum_{h=1}^{T} A_{th}\hat{e}_h$$

where $A_{t0} = \lambda^t$ and for $1 \leq h \leq T$, $A_{th} = k\lambda^{t-h+1}$ if $h \leq t$ and $A_{th} = \alpha k^{h-t}$ if $h > t$. Thus, the matrix A_{th} for $1 \leq h \leq T$ and $1 \leq t \leq T$ is

$$A = \begin{bmatrix} k\lambda & \alpha k & . & \alpha k^{h-1} & . & \alpha k^{T-2} & \alpha k^{T-1} \\ k\lambda^2 & k\lambda & . & \alpha k^{h-2} & . & \alpha k^{T-3} & \alpha k^{T-2} \\ . & . & . & . & . & . & . \\ k\lambda^t & k\lambda^{t-1} & . & k\lambda & . & .\alpha k^{T-t-1} & \alpha k^{T-t} \\ . & . & . & . & . & . & . \\ k\lambda^{T-1} & k\lambda^{T-2} & . & k\lambda^{T-h} & . & k\lambda & \alpha k \\ k\lambda^T & k\lambda^{T-1} & . & k\lambda^{T-h+1} & . & k\lambda^2 & k\lambda \end{bmatrix}.$$

Let $\hat{V}_T(\lambda, k, \alpha)$ be the sample variance of $(\hat{p}_1(\alpha), ...\hat{p}_T(\alpha))$ so $\hat{V}_T(\lambda, k, 0) = V_T(\lambda, k)$ is the sample variance of $\left(\gamma^{-1}p_1, ...\gamma^{-T}p_T\right)$ and $\hat{V}_T(\lambda, k, 1) = V^*_T(\lambda, k)$ is the sample variance of $\left(\gamma^{-1}p^*_1, ...\gamma^{-T}p^*_T\right)$. Now

$$\begin{aligned} E\hat{V}_T(\lambda, k, \alpha) &= E\left(\frac{1}{T}\sum_{t=1}^{T}\left(\hat{p}_t(\alpha) - \left(\frac{1}{T}\sum_{t=1}^{T}\hat{p}_t(\alpha)\right)\right)^2\right) \\ &= E\left(\frac{1}{T}\sum_{t=1}^{T}\left(\hat{p}_t(\alpha)\right)^2\right) - E\left(\frac{1}{T}\sum_{t=1}^{T}\hat{p}_t(\alpha)\right)^2 \end{aligned}$$

but

$$\begin{aligned} E\left(\frac{1}{T}\sum_{t=1}^{T}\left(\hat{p}_t(\alpha)\right)^2\right) &= \left(\frac{1}{T}\sum_{t=1}^{T}A^2_{t0}\right)\operatorname{var}p_0 + \frac{1}{T}\sum_{t=1}^{T}\left(\left(\sum_{h=1}^{T}A^2_{th}\right)\hat{\sigma}^2_e\right) \\ &= \left(\frac{1}{T}\sum_{t=1}^{T}A^2_{t0}\right)\operatorname{var}p_0 + \frac{1}{T}\sum_{h=1}^{T}\left(\sum_{t=1}^{T}A^2_{th}\right)\hat{\sigma}^2_e \end{aligned}$$

and

$$\begin{aligned} E\left(\frac{1}{T}\sum_{t=1}^{T}\hat{p}_t(\alpha)\right)^2 &= \left(\frac{1}{T}\sum_{t=1}^{T}A_{t0}\right)^2\operatorname{var}p_0 + \frac{1}{T^2}\left(\sum_{t=1}^{T}E\left(\sum_{h=1}^{T}A_{th}\hat{e}_h\right)\right)^2 \\ &= \left(\frac{1}{T}\sum_{t=1}^{T}A_{t0}\right)^2\operatorname{var}p_0 + \frac{1}{T^2}\sum_{h=1}^{T}\left(\sum_{t=1}^{T}A_{th}\right)^2\hat{\sigma}^2_e \end{aligned}$$

so as $\operatorname{var} p_0 = \frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2$

$$\begin{aligned} EV_T(\lambda,k,\alpha) &= \left[\left(\frac{1}{T}\sum_{t=1}^{T}A_{t0}^2\right)-\left(\frac{1}{T}\sum_{t=1}^{T}A_{t0}\right)^2\right]\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2 \\ &\quad +\sum_{h=1}^{T}\left[\frac{1}{T}\left(\sum_{t=1}^{T}A_{th}^2\right)-\frac{1}{T^2}\left(\sum_{t=1}^{T}A_{th}\right)^2\right]\hat{\sigma}_e^2 \\ &= V_{A0}\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2+\sum_{h=1}^{T}V_{Ah}\hat{\sigma}_e^2 \qquad \text{(A51)} \end{aligned}$$

where

$$V_{Ah}=\frac{1}{T}\left(\sum_{t=1}^{T}A_{th}^2\right)-\frac{1}{T^2}\left(\sum_{t=1}^{T}A_{th}\right)^2=\frac{1}{T}\sum_{t=1}^{T}\left(A_{th}-\frac{1}{T}\sum_{t=1}^{T}A_{th}\right)$$

is the sample variance of $(A_{1h}, A_{2h}, ...A_{Th})'$, the hth column of the matrix A. Furthermore, it is straightforward to confirm that

$$\begin{aligned} m_i(\theta) &= \frac{1}{i}\sum_{h=1}^{i}\theta^h \qquad \text{(A52)} \\ S_i(\theta) &= \frac{1}{i}\sum_{h=1}^{i}\left(\theta^h-m_i(\theta)\right)^2=\frac{1}{i}\sum_{h=1}^{i}\theta^{2h}-m_i(\theta)^2 \qquad \text{(A53)} \end{aligned}$$

$$V_{A0}=S_T(\lambda) \qquad \text{(A54)}$$

and for $h\geq 1$, $(A_{1h}, A_{2h}, ...A_{Th}) = \left(\alpha k^{h-1}, \alpha k^{h-2}, ...\alpha k, k\lambda, k\lambda^2, ...k\lambda^{T-h+1}\right)$ and has sample variance

$$\begin{aligned} V_{Ah} &= \frac{h-1}{T}\alpha^2 S_{h-1}(k)+\frac{T-h+1}{T}k^2 S_{T-h+1}(\lambda) \\ &\quad +\frac{(T-h+1)(h-1)}{T^2}\left[\alpha m_{h-1}(k)-m_{T-h+1}(\lambda)\right]^2 \qquad \text{(A55)} \end{aligned}$$

Using (A54) and (A55) with $\alpha = 1$ gives (A50). Using (A54), (A55) with $\alpha = 0$ and the condition that $\operatorname{var} p_0 = \frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2$, gives (A49). ■

We are now ready to prove the Proposition. As $S_T(\lambda)$ is a continuous functions of λ, $S_T(1) = 0$ and $1-\lambda^2 = 0$ when $\lambda = 1$

$$\lim_{\lambda\to 1}\frac{S_T(\lambda)}{1-\lambda^2}=\lim_{\lambda\to 1}\frac{\frac{\partial S_T(\lambda)}{\partial\lambda}}{\frac{\partial(1-\lambda^2)}{\partial\lambda}}.$$

But as

$$S_T(\lambda) = \frac{1}{T}\sum_{h=1}^{T}\lambda^{2h} - \left(\frac{1}{T}\sum_{h=1}^{T}\lambda^{h}\right)$$

$\frac{\partial S_T(\lambda)}{\partial\lambda} = 0$ when $\lambda = 1$ whereas $\frac{\partial(1-\lambda^2)}{\partial\lambda} = -2$ when $\lambda = 1$, and so $\lim_{\lambda\to 1}\frac{S_T(\lambda)}{1-\lambda^2} = 0$. Thus from (A54)

$$\lim_{\lambda\to 1^-, k\to 1^-} V_{A0}\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2 = \lim_{\lambda\to 1^-, k\to 1^-}\frac{k^2\lambda^2 S_T(\lambda)}{1-\lambda^2}\hat{\sigma}_e^2 = 0. \tag{A56}$$

The expressions for $m_i(\theta)$ and $S_i(\theta)$ in (A52) and (A53) imply that they are continuous in θ, $m_i(1) = 1$ and $S_i(1) = 0$. Thus the limit of the expected sample variance of p_t is

$$\begin{aligned}
\lim_{\lambda\to 1^-, k\to 1^-} EV_T(\lambda, k) &= \frac{\hat{\sigma}_e^2}{T^2}\sum_{h=1}^{T}(T-h+1)(h-1) \\
&= \frac{\hat{\sigma}_e^2}{T^2}\left(\frac{T^2(T-1)}{2} - \frac{T(T-1)(2T-1)}{6}\right) \\
&= \hat{\sigma}_e^2\frac{\left(T^2-1\right)}{6T}
\end{aligned}$$

since

$$\sum_{h=1}^{T}(h-1)^2 = \frac{1}{6}T(T-1)(2T-1).$$

Finally, the limit of the expected sample variance of p_{tT}^* is

$$\lim_{\lambda\to 1^-, k\to 1^-} EV_T^*(\lambda, k) = 0.$$

and the first part of the Proposition is proved. We now prove the second part of the Proposition by deriving explicit expressions for the expected sample variances. As $A_{t0} = \lambda^t$

$$\left[\left(\frac{1}{T}\sum_{t=1}^{T}A_{t0}^2\right) - \left(\frac{1}{T}\sum_{t=1}^{T}A_{t0}\right)^2\right] = \frac{1}{T}\frac{\lambda^2(1-\lambda^{2T})}{1-\lambda^2} - \frac{1}{T^2}\left(\frac{\lambda(1-\lambda^T)}{1-\lambda}\right)^2. \tag{A57}$$

As for $1 \leq h \leq T$, $A_{th} = k\lambda^{t-h+1}$ if $h \leq t$ and $A_{th} = \alpha k^{h-t}$ if $h > t$

$$
\begin{aligned}
& \frac{1}{T}\left(\sum_{t=1}^{T} A_{th}^{2}\right) - \frac{1}{T^{2}}\left(\sum_{t=1}^{T} A_{th}\right)^{2} \\
= & \frac{1}{T}\left(\alpha^{2}\sum_{i=1}^{h-1} k^{2i} + k^{2}\sum_{i=1}^{T-h+1}\lambda^{2i}\right) - \frac{1}{T^{2}}\left(\alpha\sum_{i=1}^{h-1} k^{i} + k\sum_{i=1}^{T-h+1}\lambda^{i}\right)^{2} \\
= & \frac{1}{T}\left(\alpha^{2}k^{2}\frac{1-k^{2(h-1)}}{1-k^{2}} + k^{2}\lambda^{2}\frac{1-\lambda^{2(T-h+1)}}{1-\lambda^{2}}\right) - \frac{1}{T^{2}}\left(\alpha k\frac{1-k^{(h-1)}}{1-k} + k\lambda\frac{1-\lambda^{(T-h+1)}}{1-\lambda}\right)^{2} \\
= & \frac{1}{T}\left(\frac{\alpha^{2}k^{2}}{1-k^{2}} + \frac{k^{2}\lambda^{2}}{1-\lambda^{2}}\right) - \frac{1}{T^{2}}\left(\frac{\alpha k}{1-k} + \frac{k\lambda}{1-\lambda}\right)^{2} - \frac{1}{T}\left(\alpha^{2}k^{2}\frac{k^{2(h-1)}}{1-k^{2}} + k^{2}\lambda^{2}\frac{\lambda^{2(T-h+1)}}{1-\lambda^{2}}\right) \\
& -\frac{1}{T^{2}}\left(\frac{\alpha k k^{(h-1)}}{1-k} + \frac{k\lambda\lambda^{(T-h+1)}}{1-\lambda}\right)^{2} + \frac{2}{T^{2}}\left(\frac{\alpha k}{1-k} + \frac{k\lambda}{1-\lambda}\right)\left(\frac{\alpha k k^{(h-1)}}{1-k} + \frac{k\lambda\lambda^{(T-h+1)}}{1-\lambda}\right) \\
= & \frac{1}{T}\left(\frac{\alpha^{2}k^{2}}{1-k^{2}} + \frac{k^{2}\lambda^{2}}{1-\lambda^{2}}\right) - \frac{1}{T^{2}}\left(\frac{\alpha k}{1-k} + \frac{k\lambda}{1-\lambda}\right)^{2} - \frac{1}{T}\alpha^{2}k^{2h}\left(\frac{1}{1-k^{2}} + \frac{1}{T\left(1-k\right)^{2}}\right) \\
& -\frac{1}{T}k^{2}\lambda^{2}\lambda^{2(T-h+1)}\left(\frac{1}{\left(1-\lambda^{2}\right)} + \frac{1}{T\left(1-\lambda\right)^{2}}\right) - 2\frac{\alpha k^{2}\lambda k^{h-1}\lambda^{T-h+1}}{T^{2}\left(1-k\right)\left(1-\lambda\right)} \\
& +\frac{2}{T^{2}}\left(\frac{\alpha k}{1-k} + \frac{k\lambda}{1-\lambda}\right)\left(\frac{\alpha k^{h}}{1-k} + \frac{k\lambda^{(T-h+2)}}{1-\lambda}\right).
\end{aligned}
$$

hence

$$
\begin{aligned}
& \sum_{h=1}^{T}\left[\frac{1}{T}\left(\sum_{t=1}^{T} A_{th}^{2}\right) - \frac{1}{T^{2}}\left(\sum_{t=1}^{T} A_{th}\right)^{2}\right] \\
= & \frac{\alpha^{2}k^{2}}{1-k^{2}} + \frac{k^{2}\lambda^{2}}{1-\lambda^{2}} - \frac{1}{T}\left(\frac{\alpha k}{1-k} + \frac{k\lambda}{1-\lambda}\right)^{2} - \frac{1}{T}\alpha^{2}\left(\frac{k^{2}\left(1-k^{2T}\right)}{1-k^{2}}\right)\left(\frac{1}{1-k^{2}} + \frac{1}{T\left(1-k\right)^{2}}\right) \\
& -\frac{1}{T}k^{2}\frac{\lambda^{4}\left(1-\lambda^{2T}\right)}{1-\lambda^{2}}\left(\frac{1}{\left(1-\lambda^{2}\right)} + \frac{1}{T\left(1-\lambda\right)^{2}}\right) - 2\frac{\alpha k^{2}\lambda^{2}\left(\lambda^{T}-k^{T}\right)}{T^{2}\left(1-k\right)\left(1-\lambda\right)\left(\lambda-k\right)} \\
& +\frac{2}{T^{2}}\left(\frac{\alpha k}{1-k} + \frac{k\lambda}{1-\lambda}\right)\left(\frac{\alpha k\left(1-k^{T}\right)}{\left(1-k\right)^{2}} + \frac{k\lambda^{2}\left(1-\lambda^{T}\right)}{\left(1-\lambda\right)^{2}}\right). \qquad \text{(A58)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{EV_T\left(\lambda,k,\alpha\right)}{\hat{\sigma}_e^2} \\
= \quad & \left(\frac{1}{T}\frac{\lambda^2(1-\lambda^{2T})}{1-\lambda^2}-\frac{1}{T^2}\left(\frac{\lambda(1-\lambda^T)}{1-\lambda}\right)^2\right)\frac{k^2\lambda^2}{1-\lambda^2} \\
&+\frac{\alpha^2k^2}{1-k^2}+\frac{k^2\lambda^2}{1-\lambda^2}-\frac{1}{T}\left(\frac{\alpha k}{1-k}+\frac{k\lambda}{1-\lambda}\right)^2-\frac{1}{T}\alpha^2\left(\frac{k^2\left(1-k^{2T}\right)}{1-k^2}\right)\left(\frac{1}{1-k^2}+\frac{1}{T\left(1-k\right)^2}\right) \\
&-\frac{1}{T}k^2\frac{\lambda^4\left(1-\lambda^{2T}\right)}{1-\lambda^2}\left(\frac{1}{\left(1-\lambda^2\right)}+\frac{1}{T\left(1-\lambda\right)^2}\right)-2\frac{\alpha k^2\lambda^2\left(\lambda^T-k^T\right)}{T^2\left(1-k\right)\left(1-\lambda\right)\left(\lambda-k\right)} \\
&+\frac{2}{T^2}\left(\frac{\alpha k}{1-k}+\frac{k\lambda}{1-\lambda}\right)\left(\frac{\alpha k\left(1-k^T\right)}{\left(1-k\right)^2}+\frac{k\lambda^2\left(1-\lambda^T\right)}{\left(1-\lambda\right)^2}\right).
\end{aligned}
$$

The expected sample variance of p_t is $EV_T(\lambda,k)=EV_T\left(\lambda,k,0\right)$ so

$$
\begin{aligned}
&EV_T\left(\lambda,k\right) \\
= \quad & \left(\frac{1}{T}\frac{\lambda^2(1-\lambda^{2T})}{1-\lambda^2}-\frac{1}{T^2}\left(\frac{\lambda(1-\lambda^T)}{1-\lambda}\right)^2\right)\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2 \\
&+\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2-\frac{1}{T}\left(\frac{k\lambda}{1-\lambda}\right)^2\hat{\sigma}_e^2 \\
&-\frac{1}{T}k^2\frac{\lambda^4\left(1-\lambda^{2T}\right)}{1-\lambda^2}\left(\frac{1}{\left(1-\lambda^2\right)}+\frac{1}{T\left(1-\lambda\right)^2}\right)\hat{\sigma}_e^2 \\
&+\frac{2}{T^2}\frac{k^2\lambda^3\left(1-\lambda^T\right)}{\left(1-\lambda\right)^3}\hat{\sigma}_e^2. \qquad \text{(A59)}
\end{aligned}
$$

and if $-1<\lambda<1$ and $-1<k<1$

$$\lim_{T\to\infty}EV_T\left(\lambda,k\right)=\frac{k^2\lambda^2}{1-\lambda^2}\hat{\sigma}_e^2.$$

The expected sample variance of $p_t^*$is $EV_T^*(\lambda, k) = EV_T(\lambda, k, 1)$ so

$$
\begin{aligned}
& EV_T^*(\lambda, k) \\
= \quad & \left(\frac{1}{T} \frac{\lambda^2(1-\lambda^{2T})}{1-\lambda^2} - \frac{1}{T^2} \left(\frac{\lambda(1-\lambda^T)}{1-\lambda} \right)^2 \right) \frac{k^2\lambda^2}{1-\lambda^2} \hat{\sigma}_e^2 \\
& + \frac{k^2}{1-k^2} \hat{\sigma}_e^2 + \frac{k^2\lambda^2}{1-\lambda^2} \hat{\sigma}_e^2 - \frac{1}{T} \left(\frac{k}{1-k} + \frac{k\lambda}{1-\lambda} \right)^2 \hat{\sigma}_e^2 \\
& - \frac{1}{T} \left(\frac{k^2\left(1-k^{2T}\right)}{1-k^2} \right) \left(\frac{1}{1-k^2} + \frac{1}{T\left(1-k\right)^2} \right) \hat{\sigma}_e^2 \\
& - \frac{1}{T} k^2 \frac{\lambda^4\left(1-\lambda^{2T}\right)}{1-\lambda^2} \left(\frac{1}{\left(1-\lambda^2\right)} + \frac{1}{T\left(1-\lambda\right)^2} \right) \hat{\sigma}_e^2 - 2 \frac{k^2\lambda^2\left(\lambda^T - k^T\right)}{T^2\left(1-k\right)\left(1-\lambda\right)\left(\lambda-k\right)} \hat{\sigma}_e^2 \\
& + \frac{2}{T^2} \left(\frac{k}{1-k} + \frac{k\lambda}{1-\lambda} \right) \left(\frac{k\left(1-k^T\right)}{\left(1-k\right)^2} + \frac{k\lambda^2\left(1-\lambda^T\right)}{\left(1-\lambda\right)^2} \right) \hat{\sigma}_e^2 \qquad \text{(A60)}
\end{aligned}
$$

and if $-1 < \lambda < 1$ and $-1 < k < 1$

$$\lim_{T\to\infty} EV_T^*(\lambda, k) = \frac{k^2\lambda^2}{1-\lambda^2} \hat{\sigma}_e^2 + \frac{k^2\lambda^2}{1-\lambda^2} \hat{\sigma}_e^2.$$

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