B for Bifurcations

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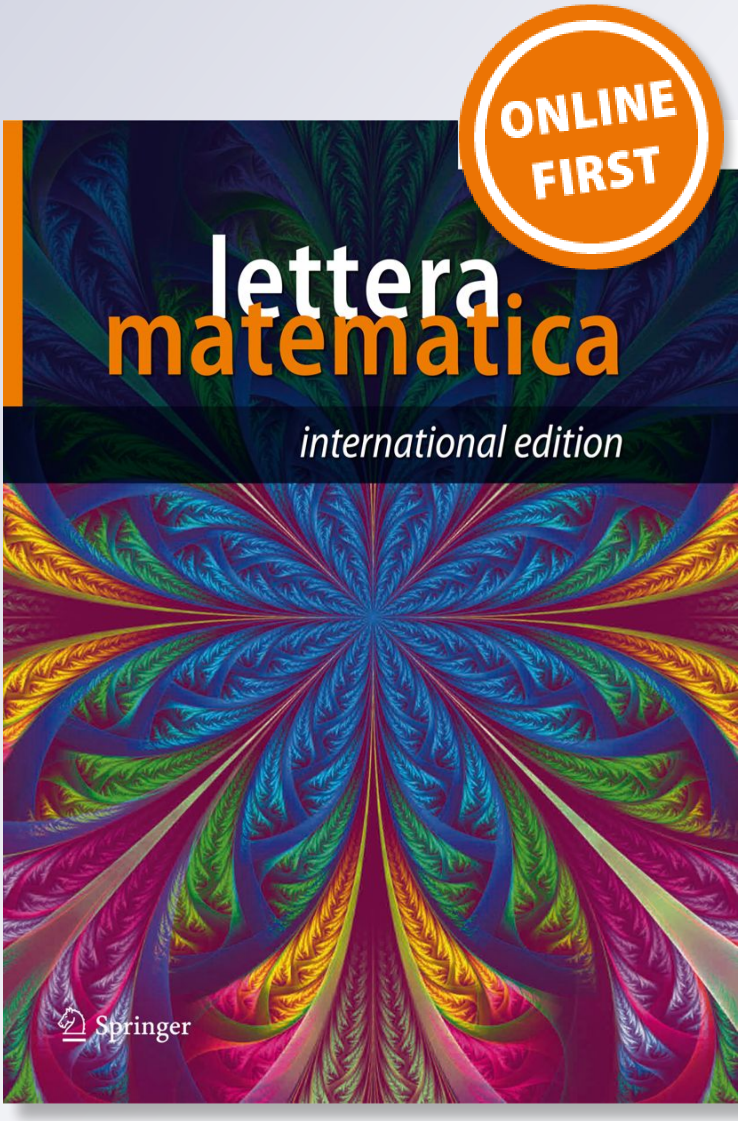

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B for Bifurcations

Singularities and catastrophes

Gian Italo Bischi[1]

Abstract The paper gives an elementary introduction to the concept of bifurcation in the context of dynamical systems, providing a historical background as well as examples of applications in real life.

Keywords Dynamical systems · Bifurcations · Singularity theory

A faint tendril of alarm moved through Jason Dill. …

A decided bifurcation of society seems in the making. …

After a pause, Vulcan 3 added: I sense a rapidly approaching crisis. … A new orientation appears to be on the verge of verbalization. …

Somewhere, lost in the fog of random and meaningless sound, were faint traces of words. …

...progressive bifurcation... … ...social elements according to new patterns…

(Philip K. Dick, Vulcan's Hammer, 1960)

In the language of mathematics, the term *bifurcation* is commonly used to denote a qualitative change of a mathematical object described in general by an equation or a system of equations (sets of points, curves or surfaces) as a function of one or more coefficients (or parameters) on which the properties of the object depend. For instance, the equation $y^2 = x\,(x^2 + bx + c)$ represents a family of curves on the Cartesian plane (x,y) whose shape and position depend on the parameters b and c (Fig. 1). By gradually varying the parameter values, these can be used as "knobs" to adjust the position and the shape of the curve: sometimes small variations cause small, quantitative changes, that is, small deformations or translations, as can be seen passing from the situation of Fig. 1a to that of Fig. 1b; in other cases, even minor variations cause the transition between two qualitatively different situations, such as the transition from a curve consisting of a single component to one consisting of two components, which occurs here when the variation of the parameters causes a change of sign of the discriminant $b^2 - 4c$. In that case we say that a bifurcation occurs and the locus of points $b^2 = 4c$, represented by a parabola in the parameter space, is called the bifurcation curve.

At the bifurcation condition a situation such as that illustrated in Fig. 1c occurs, characterized by the presence of a cusp point, which represents a case of structural instability, that is, one such that the slightest perturbation of one or more parameters, transverse with respect to the bifurcation set, leads to qualitatively different situations. In other words, the bifurcation condition separates two equivalence classes, which in the case of the example in Fig. 1 are represented respectively by curves consisting of a single component and curves consisting of two components. Problems of this type had already been studied in the eighteenth century, by Euler among others, and have since given rise to *singularity theory*, which shows that the global properties of a curve or a surface can be deduced from the knowledge of certain points, the so-called singular points, consisting of special folds or angular points like the cusp of Fig. 1. This theory has undergone considerable development during the twentieth century, thanks to the work in the 1930s of Marston Morse, in the 1940s by Hasser Whitney, and in the 1960s by René Thom and John Mather, leading to the

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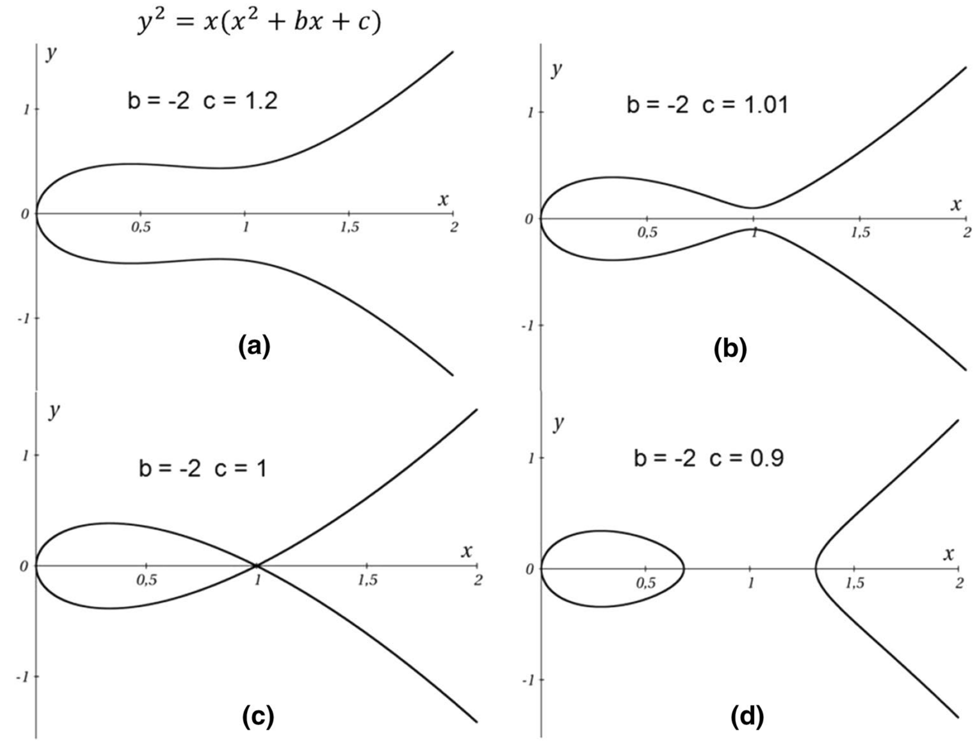


Fig. 1 Family of curves depending on parameters b and c

conclusion that the generality of the possible bifurcations can be reduced to a limited number of elementary cases (see e.g. [1, 5]), called *elementary catastrophes* (a term proposed by Christopher Zeeman commenting on the work of René Thom). This is the origin of the term "catastrophe theory", which has led to not a few misunderstandings on the scope and goals of this area of mathematics.

However, the most systematic and comprehensive study of bifurcations was carried out within the qualitative (or topological) analysis of differential equations proposed by Poincaré at the beginning of the twentieth century, which paved the way for the modern theory of nonlinear dynamical systems, within which the fundamental concept of *structural stability* was introduced thanks to the pioneering work of Russian scholars, especially of the school of Aleksandr Andronov and Lev Pontryagin in the 1950s, up to the more recent work by Vladimir Arnold.

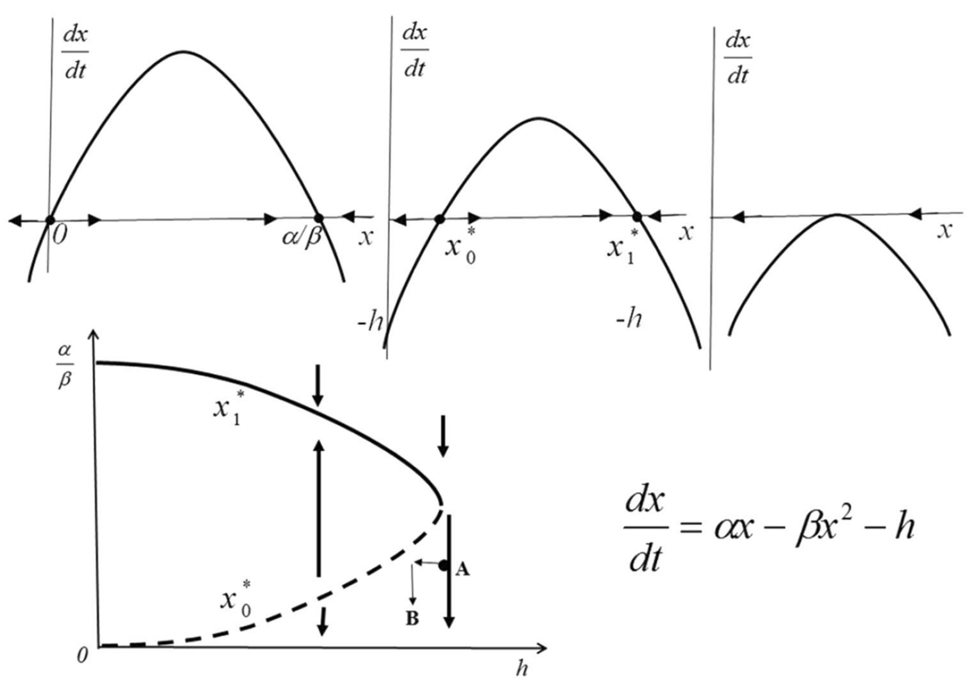
A continuous-time dynamical system of dimension n is represented by n state variables which are functions of time $x_i(t), i = 1, \dots, n$, whose temporal evolution is represented by a system of n first-order differential equations of the form $\frac{dx_i}{dt} = f_i(x_1, \dots, x_n; \mu)$, where on the left hand there are the time derivatives of the variables, that is, their rates of change, and on the right hand there is a vector field defined in the set of admissible values of the variables, also called phase space. The letter μ denotes a set of one or more parameters appearing in the definition of the vector field. Each point of the phase space represents a state of the system, while the vector associated with that point denotes the direction of the evolution of the dynamical system in that state. Given an initial condition $x_i(0), i = 1, \dots, n$, the vector field uniquely defines the trajectory through that initial condition and tangent at each point to the vectors. The overall representation of the trajectories in the phase space is called the *phase diagram* of the dynamical system. This set of curves is the mathematical object whose changes we study when the parameters vary, with its possible qualitative changes, that is, the bifurcations.

A point at which all the components of the vector on the right hand are zero is an equilibrium point, because if the initial condition is taken there the state of the system will remain in it. An equilibrium point is also considered to be a particular invariant trajectory. An equilibrium is said to be stable if the initial conditions taken in a neighbourhood of it generate trajectories that do not depart from it. There are other special invariant trajectories, such as closed curves (periodic trajectories), which may be stable or unstable, and, for dynamical systems with dimensions greater than two, we can also have invariant curves twisted like skeins, called strange, or chaotic, attractors. One of the main results in the qualitative theory of dynamical systems consists in having classified the types of invariant sets and having acquired the ability of characterising the entire phase diagram by just knowing these particular singular sets. When the parameters vary, the vector field changes, and consequently the set of curves that constitute the phase diagram is deformed. In general, a small change in a parameter will result in a small quantitative deformation of the phase diagram (that is, it produces a qualitatively equivalent phase diagram) and then it is said that the system is structurally stable. However, there are situations in which arbitrarily small variations of one parameter lead to qualitative changes in the phase diagram, such as the creation or disappearance of equilibrium points or other invariant orbits (e.g., curves covered by periodic trajectories or other types of attractors) and/or changes in stability. In these cases it is said that the system is structurally unstable and that it is located close to a bifurcation that separates two different classes of equivalent systems.

As a first example, consider the following one-dimensional dynamical system that describes the growth of a population subject to a constant quota removal (for instance, a fish population subject to commercial fishing): $\frac{dx}{dt} = \alpha x - \beta x^2 - h$, where $x(t)$ represents the population at time t, the parameter α the population growth rate at low density, β the mortality rate due to overcrowding and h the constant instantaneous removal through fishing. If $h=0$, we have a typical logistic growth, with two equilibria: $x=0$ (extinction equilibrium) and $x=\alpha/\beta$ (carrying capacity), the first unstable and the second stable (see [2]). This can be seen from the one-dimensional phase diagram represented by the arrows on the horizontal axis of Fig. 2, where a right-pointing arrow represents a population growth (positive derivative) and a left-pointing arrow a decrease (negative derivative). If the removal parameter h, taken as bifurcation parameter, is allowed to grow, we observe that the first effect is only a quantitative one: the stable equilibrium settles at a lower value (a predictable effect in the presence of removal), while the unstable equilibrium increases, causing a narrowing of the basin of attraction of the stable equilibrium, so making the system more vulnerable to possible unforeseen decreases in population or shifts to the left of the initial condition. When the parameter h increases further, the two equilibria get even closer, until at $h = \frac{\alpha^2}{4\beta}$ the two equilibria overlap. This is a condition of bifurcation (called *fold*), a typical structurally unstable situation: if h is further increased, even very little, there are no more equilibria and the only possible dynamics is a decrease toward extinction. This sequence of dynamical scenarios can be summarised by means of a bifurcation diagram, giving on the horizontal axis the parameter h and on the vertical axis the corresponding invariant sets (in this case the two equilibria, distinguishing the stable from the unstable one). As shown in Fig. 2, once the bifurcation value is surpassed, returning the parameter h to a value slightly lower than that of bifurcation is not sufficient to bring the system back to its original stable equilibrium, because we are now out of its basin of attraction (that is, the phase point is under the unstable equilibrium, which is a survival threshold for the species). This is a typical situation of irreversibility, or hysteresis, which may partly justify the term catastrophe used in the 1970s to identify similar bifurcations.

Another interesting type of bifurcation is the so-called *pitchfork*, characterised by the transition from a single equilibrium point to three distinct equilibria due to the

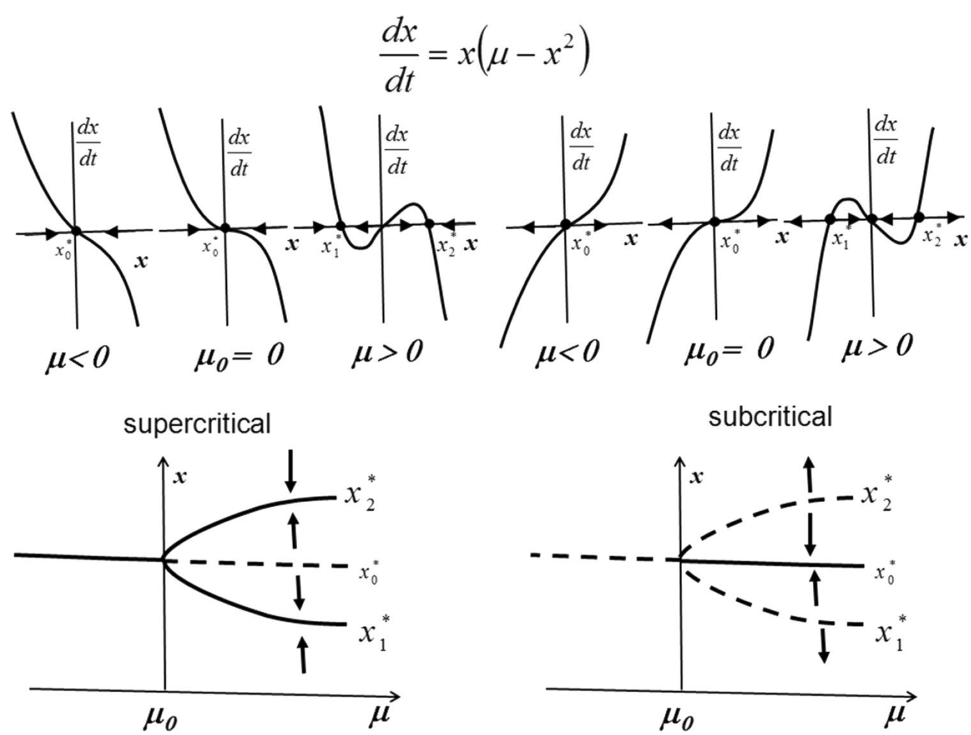
Fig. 2 A simple one-dimensional dynamical system



change in a parameter, with a simultaneous change of stability of the initial equilibrium. Clearly, if the parameter changes in the opposite direction we observe the collapse of the three equilibria into a unique central one. A typical example is given by $\frac{dx}{dt} = x(\mu - x^2)$ which, when the parameter μ changes from negative to positive values, presents a pitchfork bifurcation (called supercritical; see the left part of Fig. 3) in correspondence with the value of $\mu = 0$. Such a bifurcation leads to a significant qualitative change in the long-term dynamics: from the global (that is, from any initial condition) convergence towards the unique stable equilibrium, we reach a situation of *bistability* (with two attractors, one high and one low) with a watershed (the central, now unstable, equilibrium) that separates the two basins of attraction. In this case the initial condition becomes crucial for the ultimate fate of the evolution of the system. By changing the sign of the right hand of the differential equation, we obtain a subcritical pitchfork bifurcation (right part of Fig. 3). In this case, when the parameter μ increases, the unique (unstable) equilibrium becomes stable at the bifurcation and simultaneously two unstable equilibria that mark its basin of attraction appear. In this case it is interesting to examine the effect of the bifurcation when μ decreases: the basin of attraction bounded by the two unstable equilibria becomes narrower and narrower until it collapses on the central equilibrium, which at the bifurcation becomes unstable; after that, the trajectories starting from initial conditions near to it recede indefinitely, an undoubtedly catastrophic effect.

Increasing the dimension of the dynamical system, that is, the number of dynamical variables, we find other interesting bifurcations. With two dynamical variables $x(t)$ and $y(t)$ controlled by vector fields of the type $\frac{dx}{dt} = f(x, y); \frac{dy}{dt} = g(x, y)$, we can have the *Andronov-Hopf bifurcation*, which occurs when an equilibrium point towards which there is a convergence through damped oscillations (a convergent spiral or stable focus) becomes unstable (a divergent spiral or unstable focus). If the vector field is nonlinear in the dynamical variables, then this change of stability is generally associated with the creation of an invariant closed orbit, along which the system moves indefinitely with a periodic motion. Such a closed orbit can be stable and surround the unstable equilibrium (supercritical case, see Fig. 4) or be unstable around the stable equilibrium, delimiting the basin of attraction (subcritical case). Thus, the supercritical case provides a mechanism for the creation of stable, or self-sustaining, oscillations. This is a very important case in practical applications, since it describes stable cyclical trends that repeat periodically in time, such as phenomena observed in many systems in the fields of physics (for instance, convection in fluids), ecology (the predator–prey systems) and economics (the endogenous economic cycle of capitalist economies). In the subcritical case we have instead a gradual narrowing of the basin of attraction of

Fig. 3 Pitchfork bifurcation



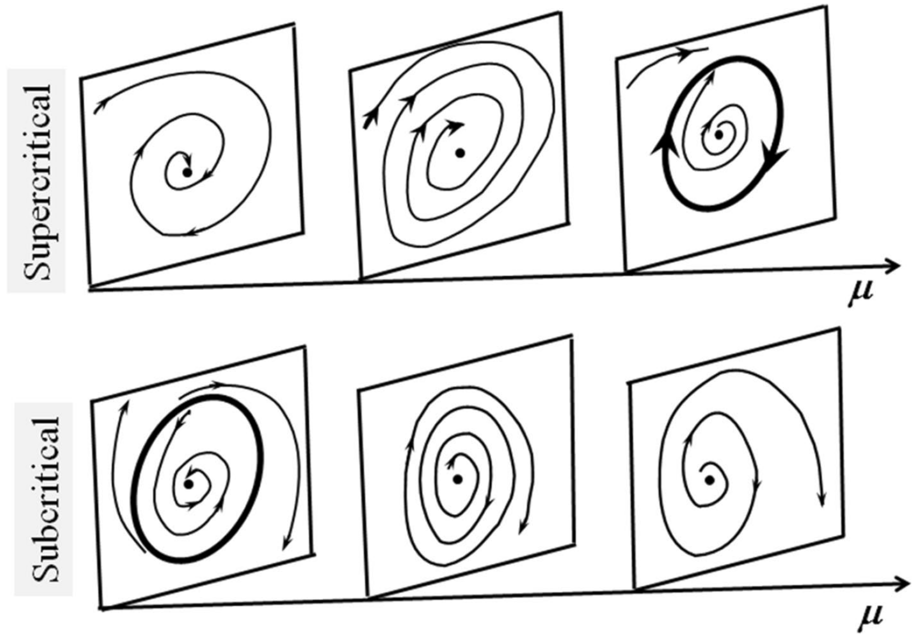


Fig. 4 Andronov-Hopf bifurcation

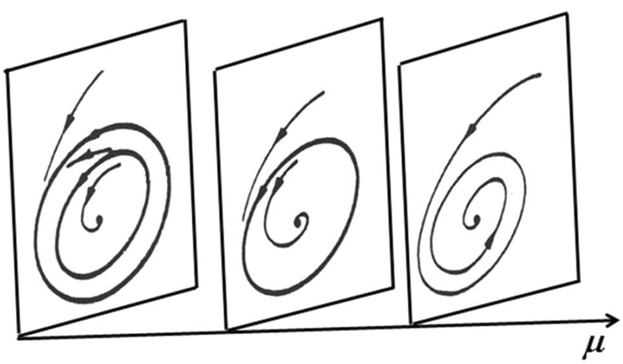


Fig. 5 Collision between stable and unstable closed orbits

the equilibrium, until the unstable curve delimiting it collapses on the point making it unstable and causing a catastrophic transition towards another, distant attractor.

There are other types of bifurcation, although their classification includes a fairly limited number of cases. These can often be described in terms of collisions between invariant sets that lead to situations of structural instability: collisions between equilibrium points, as we have seen in the case of the folds and the pitchforks; between equilibrium points and invariant closed orbits, as we have seen in the Andronov-Hopf bifurcation; and between stable and unstable closed orbits (Fig. 5) with their consequent disappearance (which can be considered as a sudden creation of a pair of orbits, a stable and an unstable one, reversing the direction of the parameter variation). We refer the interested reader to [3] or [4] for a more complete description.

As we have seen in this brief description of some typical bifurcations, the adjective "catastrophic" is sometimes used to describe certain particularly evident transitions that can follow a situation of structural instability, justifying the term adopted by Zeeman when he applied some results from the theory of singularities to the study of dynamical systems. Today we prefer to simply call them bifurcations or structurally unstable situations, but the powerful combination of the singularity theory and the qualitative theory of dynamical systems continues to provide important results in the study of the general properties and practical applications of dynamical systems.

Translated from the Italian by Daniele A. Gewurz.

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