Interdisciplinary Workshop
FROM WAVES TO DIFFUSION AND BEYOND
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Dynamic Modelling and Complexity in Economics

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I attended the course in Mathematical Physics held by Prof. Francesco Mainardi in the academic year 1982/83, when I was a third year student in the Physics degree program at the University of Bologna. The program of the course was focused on Fluid Dynamics and Waves, and during the course I was very attracted by nonlinearity. This was an important step in the development of my scientific interests, because I moved to study the effects of nonlinearity in several fields, from fluid dynamics to biology and, more recently, in economic modelling.

My chaotic path among so different applications of mathematical modelling crossed several times the path followed by Francesco: in fact, when I was working about problems in biomathematics I realized that he was developing advanced researches in the close fields of biomechanics and nerve conduction; and when, more recently, I started to deal with problems in dynamic economics and finance I crossed again over his researches in the new field known as Econophysics. So, I am very glad to give my modest contribution to the meeting in honour of the 60th birthday of Francesco, because it gives me the opportunity to outline some points of contact and differences between the dynamic modelling approaches in Physics and in Economics.

Many problems of Mathematical Physics are formulated by the methods and the formalism of the qualitative theory of *nonlinear dynamical systems*. From the three body problem in astronomy to the mathematical description of turbulence in fluid mechanics, mathematical physics was the main motivation for the study of nonlinear dynamical systems. However, this formalism recently (and rapidly) spread through several other field, such as biology, economics, sociology etc., where new and challenging problems have been proposed. For example, the models in economics and sociology presents a peculiarity with respect to the ones of physics and biology : they are based on decisions of humans, hence they must involve some assumptions on the degree of rationality involved in the decision processes. Moreover, the decisions taken at a given time are often influenced by agents' expectations about future scenarios, and this leads to some strong modifications about the classical view of dynamical systems. In fact, the classical paradigm "The actual state of a system evidently arises from the state at a previous time..." in economics and sociology is often modified into "The actual state of a system is influenced by agents' expectation about its future state...". Economists often assume full rationality of economic agents, and in a deterministic systems this implies perfect foresight, i.e. agents are assumed to be able to predict the future states of the system. Equilibria that are obtained under this strong assumption are called rational expectations equilibria (REE). However, full rationality is often considered a too strong assumption, so dynamic models have been proposed where agents are only boundedly rational, in the sense that they learn to make forecasts by trial and error (or adaptive) methods. Several kinds of learning mechanism have been proposed, by which economic agents make forecasts based on observations of past data, and many authors investigated when a system endowed with a learning mechanism converges to a REE, i.e. to the same equilibria as in the presence of the fully rational agents. In many dynamic economic models multiple REE emerge, and economists are generally interested in rational expectations equilibria which are stable under some learning mechanism. However, several attractors may coexist, and in this case the study of their basins of attraction becomes crucial in order to understand the long-run dynamics of the system, and interesting cases arise when these basins have complicated structures. This becomes especially important when locally stable non rational equilibria may be generated by the learning mechanism itself, and these equilibria coexist with stable rational equilibriua. This is the case analysed in Barucci, Bischi and Gardini (1999) where the presence of basins of attraction with extremely complex topological structures has been shown.

Another peculiar problem related to modelling decisions processes in economics and sociology concerns the presence of strategic interaction, i.e. the effect (or payoff) arising from a given decisions also depends on the decisions of other agents. This leads to the mathematical methods of game theory. Also in this case, players' expectations about future decisions of other players influence the actual decisions. Fully rational players, able to correctly forecast the decisions of their competitors, are assumed to choose strategies according to a Nash equilibrium. However, boundedly rational players often have to guess the decisions of other players by using some learning methods. Such games are often repeated over time, so that players can adapt their strategies on the basis of the outcome of previous decisions. So, in the recent literature, dynamic situations are considered where players interact with each other repeatedly over time, and convergence to a Nash equilibrium of a dynamic game played by boundedly rational agents reinforces its meaning as a real word outcome. Also in this case, the presence of multiple Nash equilibria leads to the problem of which one will be obtained as the long-run outcome of some evolutionary process. However, several coexisting Nash equilibria may be stable, and this leads to a situation of strategic uncertainty, because in this case the selected equilibrium may be path-dependent. This, again, leads to the study of the basins of attraction, a typical problem which is faced in the framework of the global analysis of nonlinear dynamical systems.

Both the problems outlined above are often modelled in the form of a discrete dynamical system, defined by the iteration of a map defined in an n-dimensional phase space. Such dynamic models may have time evolutions that exhibit bounded dynamics which may be periodic, quasi-periodic or chaotic. In such cases, a delimitation of a bounded region of the strategy space where the system dynamics are ultimately trapped, despite of the complexity of the long-run time patterns, may be an useful information for practical applications. Moreover, in the case of several attractors, the dynamic process becomes path-dependent, i.e. which kind of long run dynamics is chosen depends on the starting condition of the game. This naturally leads to the delimitation of the basins of attraction and their changes as the parameters of the model vary. These two problems lead to two different routes to complexity, one related to the complexity of the attracting sets which characterize the long run time evolution of the dynamic process, the other one related to the complexity of the boundaries which separate the basins when several coexisting attractors are present.

The study of both these questions require an analysis of the global dynamical properties of the dynamical system. Indeed, discrete time dynamic models in economics are often represented by the iteration of a noninvertible (or "many-to-one") map, i.e. a point transformation which maps distinct points into the same point. Loosely speaking, this can be expressed by saying that the map "folds and pleats" the state space. In this case, the global dynamical properties can be usefully characterized by the method of critical sets (see e.g. the survey in Agliari et al., 2002) as the repeated application of a noninvertible map repeatedly folds the state space along the critical sets and their images, and this allows one to define a bounded region where asymptotic dynamics are trapped. Instead, the repeated application of the inverses "repeatedly unfold" the state space, so that a neighborhood of an attractor may have preimages far from it. This may give rise to complicated topological structures of the basins, which may be formed by the union of non connected portions.

These methods are useful when applied to repeated games characterized by the presence of identical players, so that the dynamical systems that describes the time evolution of the choices of the players have symmetry properties related to the fact that they must remain the same as two or more identical players are swapped. This leads to the presence of invariant submanifolds of lower dimensionality than the phase space, and problems of chaos synchronization, intermittency and riddled basins arise. These problems are at the centre of a flourishing literature in mathematical physics and engineering applications. The application of the method of the critical curves to problems of chaos synchronization arising from symmetric games gave us the opportunity to contribute to this kind of literature (see Bischi and Gardini, 1998, 2000).

References

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