7. Compartmental analysis of economic systems with heterogeneous agents: an introduction

Gian Italo Bischi

1 INTRODUCTION

Economic and social systems are usually made up of a great number of units - economic agents, such as consumers or producers, goods, countries ... - and aggregated (or lumped) variables representing homogeneous groups of such units must often be introduced in order to model the time evolution of these systems. An extreme assumption is that all the units constituting the system are identical, that is, the whole system is perfectly homogeneous. In this case the behaviour of the system can be summarised by that of a *representative unit*, whose time evolution represents that of the whole system. At the opposite extreme is the 'microscopic' modelling of all the distinct interacting units. Compartmental modelling is placed at an intermediate level, between complete aggregation and complete disaggregation, being based on the subdivision of the system into a finite number of component parts, each formed by a sufficiently great number of homogeneous units, called compartments. Each compartment is characterised by the value of a measurable quantity, and different compartments interact by exchanging units.

Compartmental models have been traditionally used in physiology and pharmacokinetics to describe the distribution of a substance among different tissues of an organism. In this case a compartment represents the amount of the substance inside a certain tissue and the flows are due to diffusion processes. In physics the compartments may represent the energy content of the different parts of a system and the flows are due to heat or work exchange between compartments. In population dynamics the compartments may represent the number of individuals living in different regions and the flows are due to migration currents; a country can be divided into rural and urban compartments, a city into residential, industrial and services compartments. In these examples the compartments are used to describe the distribution of a measurable quantity among distinct regions in space and the flows are represented by transfer rates between the regions. In other cases the compartments represent qualitatively different quantities in the same space, and the inter-compartment flows represent transformation rates instead of transfer rates. This is the case of many models in chemistry where the compartments are quantities of different chemical substances contained in the same vessel, and the flows are due to the transformation rates of reactants into products.

Compartmental models, whose development began in the 1940s in physiology, have been extensively used in chemistry (see e.g. Nicolis and Prigogine, 1977, Ladde, 1976a, 1976b), medicine (Anderson, 1983, Jacquez, 1972), epidemiology (Murray, 1989), ecology (Matis et al., 1979), pharmacokinetics (Anderson, 1983, Rescigno and Segre, 1966, Solimano et al., 1990). Many of the models and methods developed in these fields can be usefully applied, by analogy, in the description of economical and social systems. In many social and economical models the population is divided into compartments (or classes) in relation to social or economical behaviour, or geographical location. For example a population of consumers can be subdivided into age classes (young and old, in the simplest case) or classes differing by different income; a population of producers can be ‘compartmentalised’ by defining size classes (small and large firms, for example). By decreasing the number of compartments a more aggregated model is obtained, whereas an increase in the number of compartments leads to a higher disaggregation into heterogeneous subsystems, However each compartment should contain a sufficiently great number of individual units, so that the exchanges between the compartments can be modelled as continuous flows that can be represented by the use of differential calculus. In marketing models the diffusion of goods among different regions or different social classes can be described by compartmental models similar to those used in ecology. The description of the propagation of an information, an innovation or a new product among a certain population can be developed on the basis of a subdivision into compartments related to economical criteria (income for example), or geographical criteria, or by personal activity (see e.g. Bass, 1969, Rogers and Shoemaker, 1971) in analogy with the models used to simulate the spread of a disease. Other compartmental models, used in enterprises for manpower planning, have compartments which correspond to different grades, and the flows represent promotions, arrivals and departures (Vajda, 1978). In models for decision making in collective systems the compartments are interpreted as different options available for individuals, and flow rates represent probabilities of individual choice (De Palma and Lefèvre, 1987, Leonardi, 1987).

In this chapter a general overview of the method of construction of the compartmental models and of their main properties is given. The most important results are described and many references are suggested where their rigorous proofs can be found. The main purpose of this overview is to guide the reader interested in the application of compartmental analysis to economic and social modelling among the references, many of which are out of the economic and social literature.

2 DEFINITIONS AND NOTATION

In order to obtain a deterministic mathematical description of a compartmental system the compartments are usually denoted by a set of indices:

$$I = \{1, 2, ..., n\} \tag{7.1}$$

and are graphically represented by boxes linked by arrows indicating flows between compartments, as in figure 7.l.

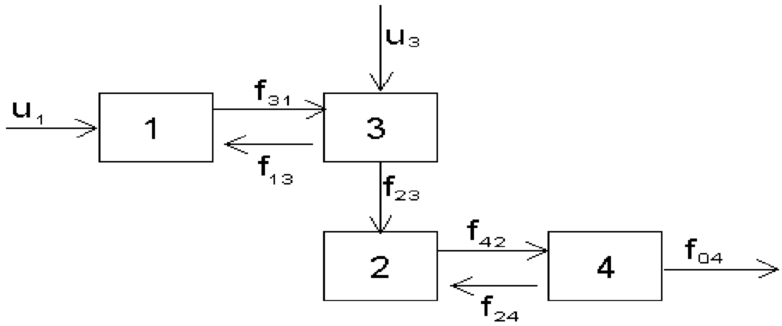


Figure 7.1

It can be noticed that in figure 7.1, as usual in compartmental analysis, f_{ij} denotes the flow rate from compartment i to compartment j (unfortunately some authors, especially in connection with stochastic modelling, use the opposite subscript convention). The index 0 denotes the outer environment, or external world, that is, everything which is not included in the compartments. If $x(t)$, i=l,2,...,n, denotes the quantity which characterises the ith compartment, the vector $\mathbf{x}(t) = (x_l(t),...,x_n(t))$ represents the state of the dynamical system at time t. The time evolution of the compartmental system is described by a set of n ordinary first-order differential equations obtained through the application, to each compartment, of the simple balance equation:

$$\frac{dx_i}{dt} = (\text{rate of inflow - rate of outflow})$$

With the symbols adopted in figure 7.1 the most general form of compartmental equation is:

$$\dot{x}_i = u_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} f_{ij} - \sum_{\substack{j=0 \\ j \neq i}}^{n} f_{ij} \text{ , } i=1,...,n \tag{7.2}$$

where a dot over a character represents the time derivative, $u_i = u_i(t)$ represents the input into the ith compartment from the outer environment and the flow rates $f_{ij} = f_{ij}(t, x)$ are in general functions of time and state variables. If we define the fractional transfer coefficients as the ratios of the flow from jth to ith compartment to the state variable in the donor compartment j:

$$a_{ij} = \frac{f_{ij}}{x_j} \tag{7.3}$$

the equations (7.2) can be written in matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{b} \tag{7.4}$$

where $\mathbf{A}=\{a_{ij}\}$ is an $n \bullet n$ matrix whose off-diagonal entries are the fractional transfer coefficients (7.3) and the diagonal entries are defined as the negative sum of all the fractional transfer coefficients of the outflows:

$$a_{ii} = -\sum_{\substack{j=0 \\ j \neq i}}^{n} a_{ij} \tag{7.5}$$

In many cases the parameters describing the flow rates, contained in the matrix **A**, are not a priori known. The estimation of such parameters through a planned input-output experiment is one of the main purposes of compartmental analysis. The procedure followed in the estimation of such parameters is typical of inverse problems: a compartmental model, in the form (7.4), is obtained on the basis of system knowledge and practical considerations; then an experiment is performed with inputs in some accessible compartments and data collected from observable compartments; finally the collection of gathered data is compared, by fitting techniques, with the solutions of the model. By this procedure a system of equations is obtained to estimate the unknown parameters. Since it is seldom possible to sample all compartments it may not be possible to obtain a unique determination of the unknown parameters. The possibility of uniquely computing the parameters by a given input-output experiment is known as the identification problem. To this problem is devoted a great part of the literature on compartmental analysis (see e.g. Anderson, 1983, Bellman, 1960, Seber and Wild., 1988, ch.8).

The biggest body of theory and applications of compartmental analysis has been concerned with linear models, where the matrix **A** and the vector **b** in (7.4) are constant. Such models are obtained if the flow rates f_{ij} are directly proportional to the quantity in the donor compartment (first order processes) or independent of the state variables (zero order processes). A typical case of first order flows is that of transfer between compartments due to passive diffusion, where the flow rate is proportional to concentration gradients. To give an example, suppose that the compartments of figure 7.2 represent two regions and let $c_i=x_i/S_i$, $c_j=x_j/S_j$ denote the population densities in the regions of surfaces S_i and S_j respectively. If migration currents are only due to passive diffusion the balance equations become

$$\dot{x}_i = k(c_j - c_i), \qquad \dot{x}_j = -k(c_j - c_i)$$

or, with the state variables,

$$\begin{aligned} \dot{x}_i &= -\frac{k}{S_i}x_i + \frac{k}{S_j}x_j \\ \dot{x}_j &= \frac{k}{S_i}x_i - \frac{k}{S_j}x_j \end{aligned} \qquad (7.6)$$

and the submatrix relative to these compartments is given in figure 7.2.

$$\begin{bmatrix} -\dfrac{k}{S_i} & \dfrac{k}{S_j} \\ \dfrac{k}{S_i} & -\dfrac{k}{S_j} \end{bmatrix}$$

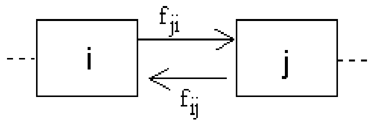


Figure 7.2

First order transfer processes are frequently met in compartmental models: the traffic flowing out of a city is proportional to the concentration of cars inside the city, the know-how transfer is proportional to the amount of information contained in the donor compartment and so on. A nonlinear diffusion model can be obtained from the linear model (7.6) if we suppose that the transport of people between the two regions is made by transport units, so that a saturation effect must be considered: the migration rate becomes a zero order process if the saturation level of carriers is exceeded, whereas the flow continues to be a first order process if the carriers have not reached the saturation point. Such a diffusion process, called active diffusion in biological literature, is characterised by state dependent transfer rates.

In the remaining part of this section we are concerned with linear time-invariant (that is, autonomous) compartmental models of the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{7.7}$$

where **A** is a constant *n•n* matrix and **b** is a constant vector. From the definitions (7.3) and (7.5) the matrix **A** has nonnegative off-diagonal elements:

$$a_{ij} \geq 0, i \neq j \tag{7.8}$$

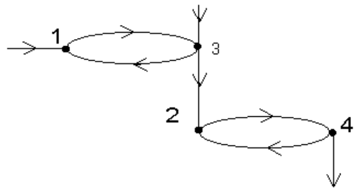
non positive diagonal elements:

$$a_{ii} \leq 0, i = 1,\ldots,n \tag{7.9}$$

and nonpositive column sum:

$$\sum_{i=1}^{n} a_{ij} \leq 0, j = 1,\ldots,n \tag{7.10}$$

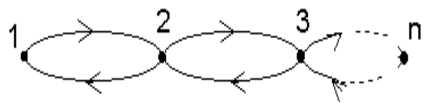
Every matrix whose elements satisfy (7.8), (7.9) and (7.10) is called a *compartmental matrix*. Many properties of a linear compartmental model depend on the disposition of zero and nonzero elements in the compartmental matrix. A useful representation of the structure of a compartmental system is the *connectivity diagram* which is a directed graph where the nodes represent compartments and the directed edges connecting certain nodes represent the flows. For example, the connectivity diagram and the corresponding matrix structure relative to the compartmental system of figure 7.l are given in figure 7.3.



$$\begin{bmatrix} x & 0 & x & 0 \\ 0 & x & x & x \\ x & 0 & x & 0 \\ 0 & x & 0 & x \end{bmatrix}$$

Figure 7.3

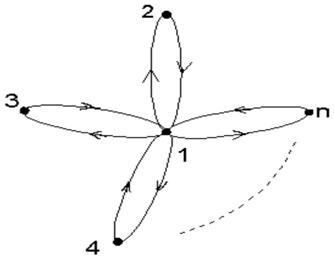
We describe now some typical connectivity diagrams that are frequently met in compartmental models. In figure 7.4 the connectivity diagram of a *catenary system* is represented, where only adjacent compartments communicate. The corresponding compartmental matrix has a tridiagonal structure.



$$\begin{bmatrix} x & x & 0 & 0 & \ldots & 0 \\ x & x & x & 0 & \ldots & 0 \\ 0 & x & x & x & \ldots & 0 \\ 0 & 0 & x & x & x & 0 \\ \vdots & & & \ldots & & \vdots \\ 0 & 0 & \ldots & 0 & x & x \end{bmatrix}$$

Figure 7.4

In figure 7.5 there is a *mamillary system*, where flows only take place between a central (or ‘mother’) compartment and each individual ‘daughter’ compartment. If the central compartment is denoted by the index 1 the corresponding compartmental matrix has nonzero entries only on the first row and first column and, of course, the main diagonal.



$$\begin{bmatrix} x & x & x & \dots & x \\ x & x & & & \\ x & & x & & \\ \vdots & & & \ddots & \\ x & & & & x \end{bmatrix}$$

Figure 7.5

Now let us suppose that for a system of n compartments an ordering of the compartments exists such that there is a subset $T = \{p, p+1,...,n\}$, $1 < p \leq n$, of the indices set I such that there are no flows from the compartments in T to those in I-T, nor to the environment, that is, a_{ij} = 0 for each $j \in T$ and i = 0, ... , p-1. Such a set I is called a *trap* and the corresponding compartmental matrix has the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{D} & \mathbf{C} \end{bmatrix} \qquad (7.l1)$$

where **B** and **C** are square submatrices and **0** is a (p-1)•(n- p+1) matrix of zeros. For example, the compartmental system of figure 7.3 has a trap formed by the nodes 2 and 4. If the order of the compartments is changed through permutation of nodes 2,3 the trap T={3,4} is obtained and the compartmental matrix assumes the structure (7.11). A system that has no exit flows to the outer environment (that is, a_{0j}=0 for each j) is itself a trap. Compartmental matrices which can be brought into the partitioned form (7.ll) through a renumbering of nodes are called *reducible matrices*; otherwise the matrix is said to be *irreducible*. Following a common terminology in graph theory (see e.g. Bertocchi et al., 1992), if starting from any node any other node in the connectivity diagram can be reached by some path, the system is said to be *strongly connected*. Of course a strongly connected compartmental system has an irreducible matrix. As stressed above, linear compartmental systems are frequently met when the modelled systems are characterised by flows due to passive diffusion or other first order processes. However the great importance of linear compartmental equations is due to the mathematical description of tracer experiments. These experiments are performed on systems which are in a steady-state. In such a situation measurements of $x_i(t)$ give no information about the dynamical behaviour of the system and nothing can be said about the flow rates. In a tracer experiment a small amount of labelled 'units' is added to one or more compartments. A *tracer* is a 'material' which behaves like the original material flowing between compartments, but it can always be distinguished and easily detected by an observer. In a compartmental system in which a substance flows, a tracer may be a fluorescent dye or a radioactive isotope with flow rates equal to those of the original substance. In a model of population migrations a tracer may be a group of labelled individuals, in a marketing model a stock of labelled goods which can be detected in different compartments at regular time intervals. In order to obtain information about the dynamics of the system the time evolution of the tracer distribution is followed through successive measurements of concentrations of the labelled material inside the observable compartments. A perfect tracer has the following properties: (a) the system should be unable to distinguish between the original material and its tracer; (b) the tracer should be added in small amounts so that it does not disturb the steady-state; (c) the tracer should not be added at its equilibrium values, so that its concentrations inside the compartments are nonconstant functions of time; (d) the tracer should not interfere with the dynamical behaviour of the original material. Under these hypotheses the dynamics of the tracer's concentrations in the compartments is approximately described by linear compartmental equations, even if the original equations are nonlinear (see e.g. Seber and Wild, 1988, Anderson, 1983, Jacquez, 1972). Let e_i represent the amount of labelled material in the ith compartment at time t, measured in the same units as x_i. If $a_{ij}(x)$ denotes the fractional transfer coefficient from compartment j to i, the total flow in a time interval dt is $a_{ij}(\bar{\mathbf{x}}+\mathbf{e})(\bar{x}_j+e_j)$ of which the fraction $e_j/\bar{x}_j$ is labelled. If, on the basis of assumption (b), we consider $\bar{\mathbf{x}}+\mathbf{e}\cong\bar{\mathbf{x}}$, the tracer equations become (see Godfrey, 1983, Jacquez, 1972)

$$\dot{\mathbf{e}} = A\mathbf{e} \tag{7.12}$$

In other words, in considering the ideal tracer dynamics we can ignore the original material, provided that it is in a steady-state, and the linear equations (7.12) give a basis for estimating the steady-state fractional transfer coefficients. A more complete derivation of the linear approximation of the tracer equations, based on a Taylor expansion around the steady-state, can be found in Anderson (1983) or Seber and Wild (1988). A typical tracer experiment consists in selecting an input and deciding what compartments should be sampled. Let $\mathbf{u} = (u_1,..., u_q)$ be an input vector and $\mathbf{B}$ an $(n\bullet q)$ matrix whose entry b_{ik} is positive if a fraction of the input u_k enters compartment $\boldsymbol{i}$. Let $\mathbf{y}=(y_1,...,y_p)$ be the output vector of the p measures, linked to the state vector by the output connection matrix **C** whose entry c_{ij} is positive if x_j influences the component y_i of the output vector. The tracer experiment can be described by the following system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad\qquad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{7.13}$$

For example, if in a three-compartment model an amount u_1 of tracer is injected into the compartments 1 and 2 at t=0 and then its concentrations in compartments 1 and 3 are measured at successive discrete times to obtain the output functions $y_1(t)$ and $y_2(t)$ we have:

$$\mathbf{u} = [u_1 d(t)]; \quad \mathbf{B} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}; \quad \mathbf{y} = [y_1 y_2]; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

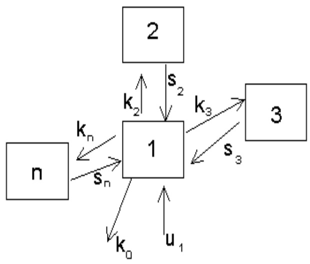
where d is the Dirac delta. The structural identification problem is solved if a choice of **B** and **C** exists such that the parameters in the compartmental matrix **A** can be uniquely determined from experimental data. If the answer is affirmative then the parameter estimation problem can be carried out by regression methods. Necessary and sufficient conditions for structural identifiability can be given in terms of connectivity of the compartmental system (see e.g. Anderson, 1983, Cobelli and Thomaseth, 1985, Godfrey, 1983).

In the remainder of this section we give examples which will also be used to illustrate the topics of the following sections.

Example 7.1. Compartmental models have been used to simulate the evolution of a population with age structure, that is, a population formed by groups of different ages. The population is split into n age groups (compartments) and the flows represent growth rates. This may be useful in social and economic modelling when the individuals of different ages have different social or economical behaviour. Suppose we want to study the problem of management of a living resource, for example the trees of a plantation. These are planted when they are very small (age class 1) and then they grow. If the population of trees is divided into n age classes, let $x_i(t)$ be the number of individuals in the ith class at time t, f_i the fraction of them which survives and moves into the next age class at a rate r_i, d_i the fraction of them which dies. The nonzero elements of the compartmental matrix $\mathbf{A}=\{a_{ij}\}$ are along the main diagonal, $a_{ii} =(d_i+f_ir_i)$ and under this diagonal $a_{i+1,i} = f_ir_i$. An input/output vector $\mathbf{b} = (b_1,...,b_n)$ can be used to simulate plantation (b_i>0) or uptaking (b_i< 0) of trees in each age class.

Models with age classes are discussed in Svirezhev and Logofet (1983) and Getz and Haight (1989).

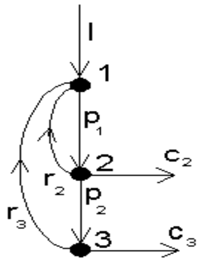
Example 7.2. Consider an economy consisting of m regions and a central stock of capital. Let $x_1(t)$ be the capital in the central agency at time t and let $x_i(t)$, $i = 2, \dots, n$, with $n = m+1$, be the capital of the ith region at the same time. Suppose that the central compartment can get profits from the outer world at a rate $u_1(t)$, receives the savings $s_i x_i$ from each region (s_i is the propensity to save for the ith region), distributes capital among the regions at rates k_i, $i = 2,\dots, n$, where the vector $\mathbf{k} = (k_2,\dots,k_n)$ represents a distribution criterion which may depend on state variables or other decision policies, and distributes capital at a rate k_0 towards the outer world, that is, everything that is not included in the model. The system is modelled as a mamillary compartmental system (figure 7.6).



$$\begin{bmatrix} -k_0-k_2\dots-k_n & s_2 & s_3 & \cdots & s_n \\ k_2 & -s_2 & 0 & & 0 \\ \vdots & & & & \\ k_n & 0 & & & -s_n \end{bmatrix}$$

Figure 7.6

Example 7.3. Consider an industrial system which produces goods starting from a raw material (say iron). Let $x_1(t)$ be the amount of raw material stored at time t, suppose that it is supplied at a rate u and is used by a ‘primary industry’ to produce simple objects and components for other factories (like screws, bolts). Let $x_2(t)$ be the amount of iron contained in the objects of this primary production, and suppose that a fraction of it will be used for a ‘secondary production’ of more sophisticated objects, another fraction is directly bought by consumers and then recycled as new raw material after its usage while the remaining fraction is lost after usage, so that it leaves the production system (for example buried in dumps). A third level $x_3(t)$ can be considered, and so on. At the ith stage let $c_i x_i$ be the fraction of iron which is lost, $p_i x_i$ be the fraction used for a higher level production (if any), $r_i x_i$ the fraction which will be recycled after usage. The connectivity diagram, the compartmental matrix and the input vector for a system with three compartments are shown in figure 7.7.



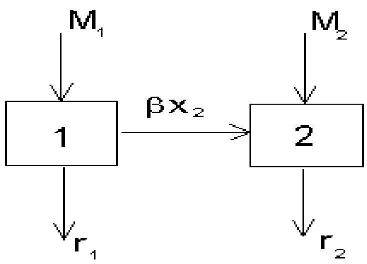
$$\mathbf{A}=\begin{bmatrix}-p_1 & r_2 & r_3\\ p_1 & -c_2-p_2-r_2 & 0\\ 0 & p_2 & -c_3-r_3\end{bmatrix}\mathbf{b}=\begin{bmatrix}I\\0\\0\end{bmatrix}$$

Figure 7.7

Example 7.4. Consider an information which can diffuse among a population by being talked about. In order to model the spread of such an information the population is divided into two compartments: x_1 is the number of susceptibles, that is, those who have not received the information but are able to take it, x_2 is the number of informed people, that is, those who are currently informed and capable of transmitting the information by direct communication. It is reasonable to assume that f_{12}, the rate at which susceptible individuals become informed, is proportional to the product of the two population densities (which represents the probability of encounters between individuals of the two compartments). The balance equations are:

$$\dot{x}_1 = -bx_1x_2 - r_1x_1 + M_1$$
$$\dot{x}_2 = bx_1x_2 - r_2x_2 + M_2$$

where r_1 and r_2 represent the rates at which individuals are removed from the population, M_1 and M_2 are input rates, from the outer world, of susceptible and informed people respectively (figure 7.8).



$$\mathbf{A}=\begin{bmatrix}-bx_2-r_1 & 0\\ bx_2 & -r_2\end{bmatrix}\mathbf{b}=\begin{bmatrix}M_1\\M_2\end{bmatrix}$$

Figure 7.8

Since the fractional transfer coefficient a_{21} is a function of a state variable (of the acceptor compartment in this case) this is a nonlinear model. Models like this have been extensively used in epidemiology (see e.g. Olinick, 1978).

Example 7.5. In this example a model is proposed to simulate the evolution of an economic system with scarcity of raw materials and, as a consequence, shortage of commodities in the market. Consider a resource (for example a raw material) which is an essential input for the production of m different goods. Let $x_1(t)$ be the amount of available resource at time t and $x_i(t)$, $i = 2, ..., n$, with $n = m+1$, the amount of the ith good whose production requires the given resource as input. The resource is essential in the sense that when it is available in small amounts the production of goods is limited, that is, proportional to x_1 (like in a first order transfer), whereas when the resource is very abundant the production rate is independent of x_1 (zero order transfer). The connectivity diagram is given in figure 7.9 and the corresponding balance equations can be written as:

$$\dot{x}_1 = u - \sum_{i=2}^{n} p_i \frac{x_1}{A_i + x_1}$$

$$\dot{x}_i = \frac{p_i x_1}{A_i + x_1} - \frac{d_i x_i}{B_i + x_i} \quad i = 2,...,n$$

where u is the input rate of resource, p_i is the maximum production rate of the ith good, A_i is the production half-saturation constant for the ith good, that is, the amount of resource necessary to obtain half of the maximum production, d_i is the maximum demand for the ith good and B_i is the half-saturation constant for the ith demand, that is, the amount of available good necessary to satisfy half of the ith good demand in the simulated market. In this model the compartment 0, that is, the outer world, represents the consumers, so the vector $(d_2,..,d_n)$ gives the demand of the consumers for the commodity bundle $(x_2, ..., x_n)$. This is a nonlinear compartmental model since the fractional transfer coefficients are functions of the state variables.

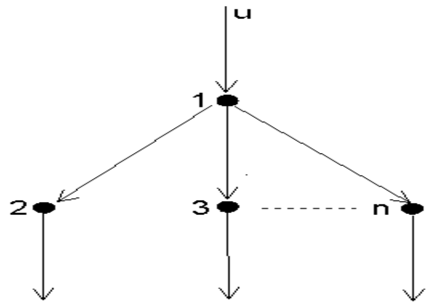


Figure 7.9

Example 7.6. In Leonardi (1987), a compartmental model is proposed to simulate the dynamics of collective choices in an urban system, where a population is made up of individuals having the possibility of choosing the district in which they prefer to live. Let $P_j(t)$ be the population living at time t in the district labelled j (j = l,..., n) and let r_j be the fractional coefficient of flow from zone j to zone i (that is, the number of individuals moving from j to i in a small time interval dt is $r_{ij}P_j dt$). The balance equations have the usual form:

$$\dot{P}_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} r_{ij} P_j - P_i \sum_{\substack{j=1 \\ j \neq i}}^{n} r_{ji}$$

If the system is closed, the total number of individuals in the system, say T, is fixed, and $\sum \dot{P}_i = 0$. In Leonardi (1987), each compartment is supposed to have a finite capacity Q_j , $j = l, ..., n$ so that not more than C_j individuals can be in compartment j at the same time. The model features are described in the particular functional form adopted for the coefficients r_{ij} . In Leonardi (1987), the following parameters are defined in order to model the dynamics of population migration in an urban system: l = intensity parameter scaling the speed of the moving process; f = origin-destination-specific factor summarising exogenous bilateral effects (distance, information, moving cost and so on) which might reduce or increase the likelihood of moving between compartments j and i; $V_j(t)$ = attractiveness factor, measuring the benefits (or disbenefits) of living in the compartment j (it is a utility function depending on dwelling prices, accessibility to shops or other facilities and certain microeconomic assumptions). Combining these factors in a multiplicative way one gets: $r_{ij}(t) = l f_{ij}(V_i(t)-V_j(t))(Q_j-P_j(t))$, where the last term is the limited capacity factor and can be looked at as a logistic term inhibiting the mobility because of scarce capacity. The presence of this last term makes the model nonlinear and the dependence on time of the utility functions *V* makes the model nonautonomous.

3 SUMMARY OF THE MOST RELEVANT PROPERTIES OF LINEAR COMPARTMENTAL MODELS

Consider the linear model (7.7) where **A** is a compartmental matrix. From (7.8) every compartmental matrix is a Metzler matrix, whose properties are well known in mathematical economics literature (Takayama, 1985, Newman, 1959). The following theorem, which can be found in standard textbooks on linear dynamical systems (see e.g. Bellman, 1960), is very important for systems whose state variables are required to be nonnegative:

Theorem 7.1. In a linear system of the form (7.7) the solutions $x(t)$ starting with $x(t_0) \geq 0$ are nonnegative for $t \geq t_0$ if and only if $\mathbf{b} \geq 0$ and **A** is a Metzler matrix.

Many of the spectral properties of a compartmental matrix are based on (7.10) which implies:

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \tag{7.14}$$

From the Gerschgorin Circle Theorem (see e.g. Bertocchi et al., 1992, p.213) with (7.14) the following theorem can be easily proved (see e.g. Anderson, 1983):

Theorem 7.2. The real part of any eigenvalue of a compartmental matrix is nonpositive. Moreover the matrix has no purely imaginary eigenvalues.

This implies that oscillations (if any) in the solutions of a compartmental model are necessarily damped. From the definition (7.5) of the diagonal elements of the matrix **A** it follows that:

$$\sum_{i=1}^{n} a_{ij} = -a_{0j}, \quad j = 1,...,n \tag{7.15}$$

that is, the jth column sum is equal to the negative of the fractional outflow rate from compartment j toward the outer world. In particular if $a_{0j}=0$ for each j, that is, in closed systems, then the compartmental matrix **A** is singular since all its columns have zero sum, and this implies that there is at least a zero eigenvalue. On the other hand if $a_{0j} \neq 0$ for each j, that is, every compartment of the system has an exit toward the outer world, then zero is not an eigenvalue **A**, hence **A** is nonsingular and stable, that is, all its eigenvalues have negative real parts. This can be seen as a corollary of theorem 7.2, or it can be deduced from the fact that if $a_{0j} \neq 0$ for each j then the (7.1) are satisfied with strict inequalities, that is, **A** is a column diagonally dominant matrix in the sense of the classical definition of Hadamard ((Takayama, 1985, p.381). This is a well known result in the theory of Metzler matrices, as stated in the following theorem (see Takayama, 1985, for a proof):

Theorem 7.3. Let **A** be an $n{\bullet}n$ Metzler matrix. Then the following conditions are mutually equivalent:

a) **A** has diagonal dominance, that is, positive numbers $d_1, d_2, ..., d_n$ exist such that

$$d_j \left| a_{jj} \right| > \sum_{i=1}^{n} d_i \left| a_{ij} \right|, \; j = 1,...,n$$

b) The real parts of all the eigenvalues of **A** are negative.
c) There exists an $\bar{\mathbf{x}} \geq \mathbf{0}$ such that $\mathbf{A}\bar{\mathbf{x}} < \mathbf{0}$.
d) For any $\mathbf{c} \leq \mathbf{0}$ there exists an $\bar{\mathbf{x}} \geq \mathbf{0}$ such that $\mathbf{A}\bar{\mathbf{x}} = \mathbf{c}$.
e) **A** is nonsingular and $\mathbf{A}^{-1} \leq 0$, that is, its entries are nonpositive for each i,j and strictly negative for some i,j.
f) The matrix **A** is Hicksian, that is, its successive principal minors alternate in sign.

A steady state (or equilibrium point) of the dynamical system (7.7) is a state vector **x*** which satisfies the linear algebraic system:

$$\mathbf{A}\mathbf{x}^* + \mathbf{b} = 0. \tag{7.16}$$

Hence if **A** is nonsingular there is a unique equilibrium $\mathbf{x}^*=-\mathbf{A}^{-1}\mathbf{b}$. Moreover, from the propositions (c) and (e) of Theorem 7.3, in a compartmental system with nonnegative input vector **b** a unique nonnegative equilibrium always exists. The invertibility condition for a compartmental matrix can also be given in terms of the connectivity diagram. In fact the strong connectivity, equivalent to the irreducibility of the compartmental matrix, is linked to invertibility by the following theorem (see Anderson, 1983, Takayama, 1985):

Theorem 7.4. If the matrix **A** is irreducible then the following condition is equivalent to any of the six conditions of theorem 7.3:

g) **A** is nonsingular and $\mathbf{A}^{-1} < 0$, that is, all its elements are negative (negative matrix).

Moreover for a compartmental matrix the following theorem can be proved (see Anderson, 1983):

Theorem 7.5. A compartmental matrix is invertible if and only if the corresponding compartmental system contains no traps, that is, every compartment has a path towards the outer environment.

The stability of an equilibrium point **x*** of (7.7) is equivalent to the stability of the solution **x**=0 of the homogeneous system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. This is asymptotically stable if and only if all the eigenvalues of **A** lie in the left half of the complex plane. Thus theorems 7.3, 7.4 and 7.5 state that there is a direct correspondence between existence and stability of a positive equilibrium point of a linear compartmental system, and these properties can easily be proved by an inspection of the connectivity diagram. In order to examine the transient behaviour of a stable compartmental system when its state is displaced from the equilibrium point, let us define the dominant eigenvalue $\bar{l}$ of **A** as the eigenvalue with greatest real part, that is, the one which is closest to the imaginary axis. This eigenvalue represents the slowest transient since the solution term exp($\bar{l}$ t) affects the trajectory for the longest time. The relaxation time (or characteristic return time) is defined as the time required to reduce the distance of the phase point $x(t)$ from the steady state **x*** by the factor 1/e:

$$T_r = -\frac{1}{\mathrm{Re}(\bar{l})} \tag{7.17}$$

Some interesting results about the dominant eigenvalue can be obtained by the application of the Perron-Frobenius theorem to the positive matrix **M** := **A**+c**I**, where **A** is the compartmental matrix, **I** the identity matrix and $c \geq \max|a_{ii}|$ (see e.g. Luenberger, 1979, p.205, Hearon, 1963, p.44):

Theorem 7.6. For any compartmental matrix **A** the dominant eigenvalue is real and the corresponding eigenvector is nonnegative.

Moreover, for strongly connected compartmental systems the following theorem holds (see Anderson, 1983).

Theorem 7.7. If the compartmental matrix is irreducible then:
(i) the dominant eigenvalue $\bar{l}$ is simple;
(ii) the corresponding eigenvector is positive, that is all its components are positive;
(iii) if l_i is any other eigenvalue of **A** then Re(l_i)< $\bar{l}$;
(iv) $\bar{l}$ increases when any outflow fractional transfer rate a_{0j} decreases.

The last thesis of this theorem states that if in a strongly connected system a flow rate toward the outer environment is weakened, then the relaxation time increases, that is, the return to the stable equilibrium is slower. Bounds on $\bar{l}$ can be obtained, without its effective calculation, in terms of row or column sums of the compartmental matrix. Let R and r be the largest and smallest row sum respectively:

$$R = \max_i \sum_{j=1}^{n} a_{ij} \qquad r = \min_i \sum_{j=1}^{n} a_{ij} \tag{7.18}$$

Theorem 7.8. If in a compartmental matrix **A** the numbers R and r are defined as in (7.5) then

(i) $r \leq \bar{l} \leq \min(0, R)$
(ii) $|\bar{l}| \leq \min_i |a_{ii}|$
(iii) $\min_j (a_{0j}) \leq |\bar{l}| \leq \max_j (a_{0j})$

Let **A** = $\{a_{ij}\}$ be a compartmental matrix with connectivity diagram Z. A circuit in G is an ordered set of k distinct nodes $\{j_1, j_2, \ldots, j_k\}$ such that the digraph of Z contains the directed edges $\overrightarrow{j_1 j_2}, \overrightarrow{j_2 j_3}, \ldots, \overrightarrow{j_k j_1}$. The equivalent condition in terms of matrix elements is that the product $a_{j_1 j_k} \cdots a_{j_1 j_2} \neq 0$. The number k of distinct nodes forming the circuit is said to be the length of the circuit. The following theorem is proved in Kellog and Stephens (1978).

Theorem 7.9. If the longest circuit in the connectivity diagram of a compartmental model is of length 2, then each eigenvalue of the corresponding compartmental matrix **A** is real.

The hypotheses of this theorem are satisfied by catenary and mamillary systems.

Example 7.7. Consider the model of example 7.2 with both saving rates and capital distribution rates independent of the state variables. Under these assumptions the model is linear with a mamillary compartmental matrix and input vector **b** = $(u_1, 0, \ldots, 0)$. Only the first column has a nonzero sum: $a_{01} = k_0$, $a_{0j} = 0$, $j = 2, \ldots, n$. From theorem 7.5 the matrix **A** is nonsingular since every compartment has a path toward the outer environment through compartment l. Thus a unique equilibrium exists: $\mathbf{x}^* = -\mathbf{A}^{-1}\mathbf{b}$. The matrix **A** is irreducible since the corresponding connectivity diagram is strongly connected (see figure 7.6). Thus from theorem 7.4 $-\mathbf{A}^{-1}$ is a positive matrix and since **b** is a nonnegative vector the equilibrium point **x*** has positive components. In this case the steady state solution can be easily computed:

$$x_1^* = \frac{u_1}{k_0}, \; x_i^* = \frac{k_i}{s_i k_0} u_1 \, ; i = 2, \ldots, n$$

In order to study the transient behaviour of the system when the starting condition is out of the equilibrium, an analysis of the eigenvalues of the matrix **A** is necessary. Since **A** is a mamillary matrix all its eigenvalues are real and, since it is nonsingular, from theorem 7.3 all the eigenvalues are negative. This implies that the equilibrium **x*** is globally asymptotically stable and the convergence is nonoscillatory. Moreover from theorem 7.8 we obtain the following bounds for the dominant eigenvalue:

$$\left|\bar{l}\right| \leq k_0 \;\text{ and }\; \left|\bar{l}\right| \leq \min_i \{k_0 + \ldots + k_n, s_2, \ldots, s_n\}$$

that is to say: $\max\{-k_0, s_2,\ldots,s_n\} \le \bar{l} < 0$ which in turn gives an estimate of the characteristic return time according to (7.4). It can be noted that $T_r \to +\infty$ if k_0 or any of the s_i becomes zero. In fact in both these cases the matrix **A** becomes singular and the equilibrium **x*** no longer exists. Finally, from theorem 7.7, we can deduce that the characteristic return time is a decreasing function of k_0.

4 RELATIONS BETWEEN DETERMINISTIC AND STOCHASTIC COMPARTMENTAL MODELS

Several authors have noted that a linear time-invariant compartmental model admits a stochastic interpretation if the compartments are considered as the states of a Markov process and the flows as probabilistic transitions of particles from one compartment to another, under the assumption that the particles of the ‘flowing material’ move independently of each other (see e.g. Eisenfeld, 1979, and references therein). Such models are particularly appropriate in situations where the number of particles (or individuals) becomes small. When such a number grows the stochastic model tends in the limit towards the deterministic linear model described by equation (7.4) (Seber and Wild, 1988; De Palma and Lefèvre, 1987). However even in the case of a large number of individuals (as is often the case in models for economy or sociology) the stochastic interpretation can provide useful interpretations of analytic results obtained in compartmental analysis. Let us first recall the following definitions which can be found in any standard textbook on probability or stochastic processes.

Definition 7.1. A vector **u** = $(u_1,\ldots,u_n)$ is called a *probability vector* if:

$$u_i \ge 0,\ i = 1,\ldots,n \qquad \text{and} \sum_{i=1}^{n} u_i = 1 \tag{7.19}$$

Definition 7.2. A square matrix **P** = $\{p_{ij}\}$ is called a *stochastic matrix* if each of its rows is a probability vector.

Definition 7.3. Given a system which can exist in n different states, a sequence of states (events) is said to form a Markov chain if the state at any step depends at most upon the previous state, and for each pair of states (a_i,a_j) there is a given probability p_{ij} that a_j occurs immediately after a_i occurs. The matrix $\mathbf{P}=\{p_{ij}\}$ is called a *transition matrix*.

Note that in these definitions the subscript convention is different from that used in compartmental analysis. In order to have a stochastic description consistent with the compartmental one we shall interpret the entry $p_{ij}(h)$ of a transition matrix to represent the probability of going from compartment j to compartment i in time h and we shall define a column stochastic matrix as a matrix whose columns are probability vectors. Let us consider first a closed compartmental system. In this case $\sum \dot{x}_i = 0$ and $\sum x_i(t) = \sum x_i(0)$. If $X_T = \sum x_i^0$ is the total amount of material in the compartmental system at t=0, the initial load distribution can be normalised as

$$x_i(0) \leftarrow \frac{x_i^0}{X_T} \tag{7.20}$$

so that the state vector $x(t)$ is, at each time, a probability vector. This allows us to consider the evolution of the state of a closed compartmental system from a stochastic point of view. The following results are given in three papers of Eisenfeld (1979, 1981, 1980).

Theorem 7.10. Let $\mathbf{A} = \{a_{ij}\}$ be an $n\bullet n$ closed compartmental matrix, that is, all its column sums are zero. Let $d = \max\left\{ \left|a_{ij}\right| \right\}$ and $0 \le h \le 1/d$. Then $\mathbf{Q}(h)=\mathbf{I}+h\mathbf{A}$ ($\mathbf{I}$ identity matrix) is a column stochastic matrix. Conversely if $\mathbf{P}$ is a column stochastic matrix, then for $h > 0$, $\mathbf{A}(h)=1/h(\mathbf{P}-\mathbf{I})$ is a closed compartmental matrix.

Suppose now that $\mathbf{A}$ is an open compartmental matrix, that is, at least one j exists for which a_{0j}>0. From $\mathbf{A}$ we can construct the matrix $\overline{\mathbf{A}}$, called the closure of $\mathbf{A}$, by attaching an $(n+1)\bullet l$ column vector of zeros on the left-hand side of $\mathbf{A}$ and the $l\bullet n$ row vector $(a_{0l},\ldots, a_{0n})$ at the top of $\mathbf{A}$:

$$\overline{\mathbf{A}} = \begin{bmatrix} 0 & a_{01} & \cdots & a_{0n} \\ 0 & & & \\ 0 & & \mathbf{A} & \\ 0 & & & \end{bmatrix}$$

This is equivalent to saying that a new compartment representing the outer world has been added with index 0. In this case the column stochastic matrix

$$\overline{\mathbf{Q}} = \mathbf{I} + h\overline{\mathbf{A}} \tag{7.21}$$

defined, as before, for sufficiently small values of h, is the transition matrix of an (n+1)-state Markov process:

$$\mathbf{x}(t+h) = \overline{\mathbf{Q}}(h)\mathbf{x}(t) \tag{7.22}$$

which, for a fixed time increment, gives the sequence $\mathbf{x}(nh)$, n=0,1,… which is a Markov chain. Such a Markov chain is absorbing since the outer environment is an absorbing state (the (0,0) entry of the corresponding transition matrix is equal to 1). If the compartmental matrix **A** is invertible the corresponding compartmental system has no traps and this implies that it is possible to pass from any compartment to the outside compartment. In this case the matrix:

$$\mathbf{M}(h) = [\mathbf{I} - \mathbf{Q}(h)]^{-1} = -\frac{1}{h}\mathbf{A}^{-1} \tag{7.23}$$

can be defined. It is known as the fundamental matrix associated with $\mathbf{Q}(h)$ and its generic element $m_{ij}(h)$, which is nonnegative (see theorem 7.3), is interpreted as the expected number of times the Markov process is in compartment i having started at t = 0 from compartment j, before the process enters the absorbing state. Since h is the time step between two transitions we have that hm_{ij} is the average time that a unit of flowing material spends in compartment i having been loaded in compartment j at the initial time. This is the essence of the following theorem:

Theorem 7.11. Let **A** be the compartmental matrix associated with an open system with no traps (so that $\mathbf{A}^{-1}$ exists). Then the entry t_{ij} of the matrix $\mathbf{T}=-\mathbf{A}^{-1}$ represents the mean time the process is in compartment i, having started in compartment j, before exiting the system.

If **A** is not invertible the compartmental system has at least one trap and this is equivalent to saying that the mean residence time for a compartment in the trap is infinite. The second stochastic matrix defined in the papers of Eisenfeld is:

$$\mathbf{P}(t) = \exp(t\mathbf{A}),\ t \geq 0 \tag{7.24}$$

whose generic element p_{ij} is shown to represent the probability that the process is in compartment ï at time t if it started in compartment j at t = 0. If **A** is an open matrix the column sum of **P** is not necessarily l, but **P** has the properties of a partial stochastic matrix, that is

$$p_{ij} \geq 0 \text{ and} \sum_{j=1}^{n} p_{ij} \leq 1 \tag{7.25}$$

Even if **A** is a closed compartmental matrix the two stochastic matrices **P**(h)=exp(h**A**) and **Q**(h)=**I**+h**A** are different. In fact the directed graph of the matrix **P**(h) exhibits more connections than the graph of **A**, whereas **Q**(h) has the same connectivity diagram as **A**. This is due to the fact that if a compartment i is reachable from compartment j along some path of arbitrary length (which may be of several steps connecting many compartments) then $p_{ij}(h)$>0 whereas a_{ij} and $q_{ij}(h)$ are positive only if the connectivity diagram of **A** has a one-step path from j to i. On the other hand from the expansion

$$\mathbf{P}(h) = \exp(h\mathbf{A}) = I + h\mathbf{A} + \frac{h^2\mathbf{A}^2}{2!} + \ldots \tag{7.26}$$

it follows that, for sufficiently small h, **P**(h)=**I**+h**A**. In other words the stochastic matrix **Q**(h) is the transition matrix for small time steps, whereas the stochastic matrix **P**(h) gives the long term behaviour of the Markov process. Another stochastic parameter which can be defined from the analysis of the compartmental matrix is the *mean system residence time*, defined as the sum of the elements of a column of $-\mathbf{A}^{-1} = \{t_{ij}\}$

$$T_m = \sum_{i=1}^{n} t_{im} \tag{7.27}$$

This represents the mean residence time of a particle in the entire system given that it was in compartment m at the initial time t=0. The parameter

$$r_{ij} = \frac{a_{ij}}{|a_{jj}|} \tag{7.28}$$

can be interpreted as the probability that a particle in compartment j will enter compartment i when a transition occurs. In fact $|a_{jj}|x_j(t)$ is the total outflow from compartment j and $a_{ij}x_j(t)$ is that portion of the outflow which enters compartment i.

Example 7.8. Consider the model of example 7.3 with n compartments (one raw material and (n-1) production levels) and with the following assumptions: (i) all the coefficients p_i, r_i, c_i, i=l,…, n, are real positive constants; (ii) no losses of the raw material exist which are not considered in the parameters c_i, that is at each stage $p_i+r_i+c_i=1$.

The connectivity diagram is strongly connected (see figure 7.7) and, if at least one of the outflows $a_{0j}=c_j$, j=2,…, n, is positive, the system has no traps and the compartmental matrix is nonsingular. Under this condition a unique equilibrium **x*** exists and is globally asymptotically stable (as usual in linear compartmental models equilibrium existence implies stability and vice versa, as stated in theorem 7.3). The equilibrium components are:

$$x_1^* = \frac{I}{c_2 + p_2 c_3 + p_2 p_3 c_4 + ... + p_2 \cdot\cdot p_{n-1} c_n}, \quad x_n^* = \left(\prod_{i=2}^{n} p_i\right) x_1^*$$

Even if the recycling parameters do not appear in the equilibrium components they influence the other parameters because of assumption (ii). If one r_i is increased the corresponding p_i and c_i decrease since $p_i + c_i = 1 - r_i$, so all the equilibrium components increase. From assumption (ii) the diagonal elements of the compartmental matrix are equal to l. From (7.28) the off-diagonal entries of the compartmental matrix can be interpreted as transfer probabilities: r_i is the probability that a particle of iron in the ith compartment will be recycled, c_i the probability it will be eliminated out of the production system. We consider now the case n=3 as in figure 7.7. The inverse of the compartmental matrix can be easily computed and the matrix of the mean residence times is obtained according to theorem 7.11:

$$\mathbf{T} = -\mathbf{A}^{-1} = \frac{1}{1 - r_3 p_2 - r_2}\begin{bmatrix} 1 & r_2 + r_3 p_2 & r_3 \\ 1 & 1 & r_3 \\ p_2 & p_2 & 1 - r_2 \end{bmatrix}$$

If only raw material enters the system, that is, the only input is in compartment l, then only the first column of **T** is meaningful. The mean time a particle of iron spends in the production system after having entered as a raw material is, according to (7.27)

$$\frac{p_2 + 2}{1 - r_3 p_2 - r_2} > 0$$

since $r_3 p_2 + r_2 < p_2 + r_2 < 1 - c_2$.

The mean time a particle of iron spends in the objects of primary production (compartment 2) before exiting the system is

$$t_{21} = \frac{1}{1 - r_3 p_2 - r_2}$$

and in the goods of second-level production (compartment 3) $t_{31} = p_2 t_{21}$. The meaning of the other entries of matrix **T** is that of mean residence times of particles of iron which were introduced into the system as goods, that is, directly in compartments 2 or 3, rather than as raw material. For example a particle of iron introduced into objects of compartment 3 will spend a mean time

$$t_{31} = \frac{r_3}{1 - r_3 p_2 - r_2}$$

in compartment 1 as a raw material after recycling.

5 NONLINEAR COMPARTMENTAL MODELS

A compartmental model is nonlinear if some fractional transfer coefficients (7.3) are functions of the state variables. A general autonomous dynamic system $\dot{\mathbf{x}} = f(\mathbf{x})$, with $f : \Re^n \to \Re^n$, can be considered as a compartmental model if it can be factored in the form:

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{b} \tag{7.29}$$

where $\mathbf{A}$(x)={$a_{ij}(\mathbf{x})$} is an n•n compartmental matrix, that is

$$a_{ij}(\mathbf{x}) \geq 0 \text{ for } i \neq j;\ a_{ii}(\mathbf{x}) \leq 0;\ \sum_{i=1}^{n} a_{ij}(\mathbf{x}) \leq 0 \quad j = 1,\ldots,n \tag{7.30}$$

The properties of this general class of compartmental models are examined in Anderson and Roller (1991) and its stability properties are also considered in Ladde (1976b). A less general class of nonlinear compartmental models is that of donor-controlled equations, where the flow rates are functions of the state of the donor compartments only:

$$a_{ij}(\mathbf{x}) = k_{ij} f_j(x_j) \tag{7.31}$$

The properties of such systems have been examined in Maeda and Kodama (1978). In the following we summarise some of the results given in the papers quoted above, chiefly in Anderson and Roller (1991), in order to apply them to the nonlinear models presented in the examples of section 2. Let $\Omega = \{x \in \Re^n : \mathbf{A}(\mathbf{x}) \text{ is compartmental}\}$.

If $\Re^n \subseteq \Omega$ and $\mathbf{b} \geq \mathbf{0}$ then every solution of (7.29) starting with $\mathbf{x}(t_0) \geq 0$ remains nonnegative for $t>t_0$. In fact if $x_i = 0$ and $x_j \geq 0$ for $j \neq i$ from (7.29) with (7.30) we have $\dot{\mathbf{x}} \geq 0$; hence $x_i=0$ is a repulsive, or invariant, hyperplane for any trajectory in the nonnegative cone of $\Re^n$.

Some questions arise concerning existence and uniqueness of steady states. In fact the components of an equilibrium point must be found by solving the system of nonlinear algebraic equations:

$$\mathbf{A}(\mathbf{x})\mathbf{x} = -\mathbf{b} \tag{7.32}$$

whose solution, in general, may be a difficult task, so conditions on existence, uniqueness and bounds of equilibrium points are needed. Let

$\Omega_{-1} = \{x \in \Omega : A(\mathbf{x}) \text{ is nonsingular}\}$

that is, the set of **x** such that every compartment has a path towards the outer world, and let $g : \Omega_{-1} \to \Re^n$ be the function:

$$g(\mathbf{x}) = -\mathbf{A}^{-1}(\mathbf{x})\mathbf{b} \tag{7.33}$$

so that any equilibrium point in Ω_{-1} is a fixed point of g, that is, **x***=*g*(**x***). The following theorem gives a range for *g*(**x**) and hence a bound for the steady states if they exist:

*Theorem 7.12.*Suppose $\|\mathbf{A}(\mathbf{x})\| \le L$ and $\|\mathbf{A}^{-1}(\mathbf{x})\| \le \mathrm{U}$. Then

$$\frac{\|\mathbf{b}\|}{L} \le \|g(\mathbf{x})\| \le U\|\mathbf{b}\| \qquad (7.34)$$

Let $S = \sum_{i=1}^{n} x_i$ and $b = \sum_{i=1}^{n} b_i$. Then, by summing equations (7.29),

$$\frac{dS}{dt} = -\sum_{j=1}^{n} a_{0j} x_j + b \qquad (7.35)$$

At a steady state *dS/dt* = 0, so the following theorem holds:

Theorem 7.13. If an equilibrium solution of (7.29) exists then the total input *b* must be in the range of the function $\sum_{i=1}^{n} x_i a_{0i}(\mathbf{x})$.

A lower bound for **x*** can be obtained if the excretion functions are bounded above, that is, $a_{0i}(\mathbf{x}) \le u_i$. In this case we have:

$$\|g(\mathbf{x})\|_2 \ge \frac{b}{\|\mathbf{u}\|_2} \qquad (7.36)$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Furthermore, if $M = \max_i \{u_i\}$ then:

$$\|\mathbf{x}(t)\|_1 \ge \|\mathbf{x}(0)\|_1 e^{-Mt} + \frac{b}{M}\left(1 - e^{-Mt}\right) \qquad (7.37)$$

where $\|\mathbf{x}\|_1 = \sum_i |x_i|$. If the excretion functions are bounded away from zero an existence theorem can be proved (see Anderson and Roller, 1991):

Theorem 7.14. If $\mathbf{b} \geq 0$ and for each i the excretion rate function $a_{0i}(\mathbf{x})$ is bounded away from zero as $x_i \to +\infty$, then there exists an equilibrium point $\mathbf{x}^* \geq 0$ of (7.29).

Moreover if $a_{0i}(\mathbf{x}) \geq d_i > 0$, then

$$x_i^* \leq \frac{b}{d_i} \tag{7.38}$$

and if $m = \min_i \{d_i\}$ we have:

$$\|\mathbf{x}(t)\|_1 \leq \|\mathbf{x}(0)\|_1 e^{-mt} + \frac{b}{m}\left(1 - e^{-mt}\right) \tag{7.39}$$

Furthermore, if the excretion functions are strictly positive the following inequality holds:

$$\frac{1}{\max_j \{a_{0j}(\mathbf{x})\}} \leq \left\|\mathbf{A}^{-1}(\mathbf{x})\right\|_1 \leq \frac{1}{\min_j \{a_{0j}(\mathbf{x})\}} \tag{7.40}$$

where $\|\mathbf{A}\|_1 = \max_j \sum_i |a_{ij}|$. The last result can be useful for the following existence theorem:

Theorem 7.15. Let $\mathbf{b} \geq 0$, $\Omega = \Omega_{-1} = \Re_+^n$ and suppose that a constant U exists such that $\left\|\mathbf{A}^{-1}(\mathbf{x})\right\| \leq U$ for some matrix norm, for each $\mathbf{x}$ in $S = \{\mathbf{x} \in \Omega : \|\mathbf{x}\| \leq U\|\mathbf{b}\|\}$. Then a solution **x*** exists for equations (7.32).

The proofs of these results are given in Anderson and Roller (1991). The proof of theorem 7.14 is based on a fixed point result since under the hypotheses of the theorem the function $g(\mathbf{x})$ maps S into itself. This result provides an alternative to the computation of **x*** from equations (7.32). In fact when the function g acts as a contractor on a subset of Ω_{-1} the recursive sequence $x_{p+1}=g(x_p)$ converges to a fixed point of g, that is, to an equilibrium point of (7.29). Conditions for the uniqueness of the equilibrium point are also given in Anderson and Roller (1991).

Stability results for nonlinear compartmental models are much more complicated than those for linear compartmental models. In the latter case systems with no traps have a unique equilibrium point which is globally asymptotically stable. In the case of nonlinear models local stability results do not necessarily imply global stability. Moreover many equilibrium points may exist. A first approach to the question of stability is through linearisation of the system around an equilibrium point. For this the computation of the Jacobian matrix of the function $f(\mathbf{x})=\mathbf{A}(\mathbf{x})\mathbf{x}+\mathbf{b}$ at the equilibrium point is needed. If the eigenvalues of the Jacobian have negative real parts the equilibrium is locally asymptotically stable. If the Jacobian is a compartmental matrix the spectral results given in section 3 can be used, but such an assumption is not necessarily true. Some results on these questions are given in Maeda and Kodama (1978). Global stability results can be obtained through Lyapunov's second method (see e.g. La Salle and Lefschetz, 1961; Brock and Malliaris, 1989). Some interesting results about the use of Lyapunov functions in nonlinear compartmental models are given in Ladde (1976b).

Example 7.9. Consider the model of information diffusion given in example 7.4. Some entries of the matrix **A** of figure 7.8 are functions of the state variables, hence the model is nonlinear. The matrix **A** is compartmental for each $x_2>0$, and since the input vector **b** is nonnegative the positive cone $\Re^n_+$ is positively invariant. From the connectivity diagram we can deduce that **A** is nonsingular provided that $r_2>0$, because in this case both compartments have a path towards the outer world, whereas if $r_2=0$ compartment 2 is a trap. Thus if $r_2>0$ $\Omega_{-1}=\Omega=\Re^n_+$. The inverse of the compartmental matrix is given by

$$-\mathbf{A}^{-1}(\mathbf{x})=\begin{bmatrix} 1/(bx_2+r_1) & 0 \\ bx_2/r_2(bx_2+r_1) & 1/r_2 \end{bmatrix} \le \begin{bmatrix} 1/r_1 & 0 \\ 1/r_2 & 1/r_2 \end{bmatrix}$$

Let $\mathbf{T}(\mathbf{x}):=-\mathbf{A}^{-1}(\mathbf{x})=\{t_{ij}\}$. In linear compartmental models the entries of **T** represent mean residence times, but now the $\mathbf{t_{ij}}$ are not constant. However a constant upper bound can be obtained for the mean residence times. The inequality $t_{21}<1/r_2$ states that the average time that an individual, not informed at the initial time, remains without being informed is less than

$1/r_0$. Of course r_{12}=0 since an individual who enters the system in compartment 2 will never reach compartment 1.

In this model the excretion rates are constant: $a_{01}=r_1$ and $a_{02}=r_2$; hence if $r_i>0$, i=1,2, the excretion coefficients are bounded away from zero. Thus theorem 7.3 guarantees the existence of at least one equilibrium point. The uniqueness of the equilibrium point can be easily proved in this case because it is the intersection of the two nullclines $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ which have equations

$$x_2 = \frac{M_1 - r_1 x_1}{b x_1} \text{ and } x_2 = \frac{M_2}{r_2 - b x_1}$$

respectively. From (7.38) we have

$$x_i^* \le \frac{M_1 + M_2}{r_i}$$

and if $m=\min\{r_1,r_2\}$, from (7.39)

$$x_1(t) + x_2(t) \le \|\mathbf{x}(0)\|_1 e^{-mt} + \frac{b}{m}\left(1 - e^{-mt}\right)$$

The unique positive equilibrium $\mathbf{x}^* = \left(x_1^*, x_2^*\right)$ is the solution of the equations:

$$\begin{aligned} b x_1 x_2 + r_1 x_1 &= M_1 \\ b x_1 x_2 - r_2 x_2 &= -M_2 \end{aligned}$$

The global stability of the equilibrium can be proved by the following Lyapunov function (see e.g. Goh, 1980):

$$V(x_1, x_2) = \left(x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) + \left(x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} \right)$$

The derivative of V along the trajectories of the dynamical system is:

$$\dot{V} = \frac{x_1 - x_1^*}{x_1} \dot{x}_1 + \frac{x_2 - x_2^*}{x_2} \dot{x}_2 = -\sum_{i=1}^{2} \frac{M_i}{x_i x_i^*} \left(x_i - x_i^*\right)^2 \le 0$$

and $\dot{V} = 0$ only if **x**=**x***. Then the global stability of **x*** follows from the Lyapunov second method.

Example 7.10. Consider the model for goods production with scarcity of a raw material, described in example 7.5. The system of differential equations given in example 7.5 can be written in the form (7.29) with:

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} -\sum_{i=2}^{n} \frac{p_i}{A_i + x_i} & 0 & \cdots & 0 \\ \frac{p_2}{A_2 + x_i} & -\frac{d_2}{B_2 + x_2} & \cdots & 0 \\ \vdots & & \cdots & \vdots \\ \frac{p_n}{A_n + x_n} & 0 & \cdots & -\frac{d_n}{B_n + x_n} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} u \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The matrix **A** is a lower triangular compartmental matrix for each feasible value of **x**, that is, for each $\mathbf{x} \in \Re_{+}^{n}$. The column sums, that is, the excretion functions, are:

$$a_{01} = 0,\ a_{0j} = \frac{d_j}{B_j + x_j}\ j = 2,\dots,n$$

From theorem 7.13 a necessary condition for the existence of an equilibrium is that the input *u* must be in the range of the function:

$$f(\mathbf{x}) = \sum_{i=1}^{2} x_i a_{0i}(\mathbf{x}) = \sum_{i=1}^{2} \frac{d_i x_i}{B_i + x_i}$$

Hence it must be $0 \le u < d_1 + d_2$.

The excretion functions are bounded above, $a_{0i} \le d_i / B_i$, hence from (7.36) a lower bound for the equilibrium components can be obtained:

$$\sum_{i=1}^{n} x_i^{*2} \ge \frac{u^2}{d_1^2 + d_2^2}$$

and from (7.37), if $M = \max_i \left\{ \frac{d_i}{B_i} \right\}$, we obtain

$$\sum_{i=1}^{n}|x_i| \geq \|\mathbf{x}(0)\|_1 e^{-Mt} + \frac{u}{M}\left(1-e^{-Mt}\right)$$

In this model a unique positive equilibrium exists whose components x_i^*, i=1,..., n, can be found from the nonlinear algebric equations (7.32). Let

$$F(x_1) = \sum_{i=1}^{2} \frac{p_i x_1}{A_i + x_1}$$

This is a continuous and increasing function, hence invertible, and $x_1^* = F^{-1}(u)$, provided that the input rate u is in the range of F^{-1}, that is, $0 \leq u \leq p_1 + p_2$. Now let

$$U_i^* = \frac{p_i x_1^*}{A_i + x_1^*}$$

Then the other equilibrium components are:

$$x_i^* = \frac{B_i U_i^*}{d_i + U_i^*}$$

The positive equilibrium exists provided that $0 \leq u \leq p_1 + p_2$ and $U_i^* < d_i$.

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