NONINVERTIBLE MAPS AND COMPLEX BASIN BOUNDARIES IN DYNAMIC ECONOMIC MODELS WITH COEXISTING ATTRACTORS

Gian-Italo Bischi[*]
Istituto di Scienze Economiche, University of Urbino, Italy
Laura Gardini[†]
Istituto di Scienze Economiche, University of Urbino, Italy and
Istituto di Matematica "E. Levi", University of Parma, Italy
Michael Kopel[‡]
Department of Managerial Economics and Industrial Organization,
University of Technology, Theresianumgasse 27, 1040 Vienna, Austria

Abstract

In this paper we study the boundaries which separate the basins of attraction of coexisting attractors for discrete dynamical systems represented by the iteration of noninvertible maps. This problem is often met in dynamic models which describe economic and social systems, where the presence of several attractors raises questions related to equilibrium selection. For noninvertible maps the basins may have rather complex topological structures, since they may be non connected or multiply connected sets. The analysis, as well as the study of the global bifurcations which change their qualitative properties, is carried out by the method of critical sets. As an example, a Cournot duopoly game is proposed, where the profit-maximizing production choices of the two players are taken over time according to non-monotonic Best Reply functions. The players are supposed to show some kind of inertia, as their choices are described by a partial adjustment to the Best Response. This duopoly game has multiple stable Nash equilibria, which gives rise to strategic uncertainty among players, and its time evolution is obtained by the iteration of a noninvertible map of the plane. The structure of the basins of attraction of the emerging Nash equilibria is analyzed, and by employing the method of critical curves some global bifurcations are evidenced that change the topological structure of the basins from simple to complex.

1 Introduction

In the last two decades the literature on dynamic modeling of economic and social systems was mainly concerned with the study of the attracting sets and the bifurcations which lead

[*]E-mail address: bischi@econ.uniurb.it

[†]E-mail address: gardini@econ.uniurb.it

[‡]E-mail address: kopel@ebwnov.tuwien.ac.at

to more and more complex asymptotic dynamics. When several coexisting attractors are present – a situation often met in dynamic models of economic and social systems – another route to complexity is related to more and more complex boundaries which separate the basins of attraction. Generally, these two different routes to complexity are not correlated, in the sense that simple attractors may have complex basins and complex attractors may have simple basin boundaries.

The study of the complexity related to the structure of the basins has been rather neglected in the economic literature, because it requires an analysis of the global properties of the dynamical systems (i.e. not based on the linear approximation) and the global bifurcations that change the qualitative structure of the basins are generally detected through geometric and numerical methods. This task may be particularly challenging in the case of discrete dynamical systems represented by the iteration of noninvertible maps. Indeed, in this case the basins may have complicated topological structures, since they may be multiply-connected sets or non-connected, often formed by the union of infinitely many disjoint portions (see e.g. Mira et al. [26, 28], Abraham et al. [1]).

Recent results on basin bifurcations in noninvertible maps, mainly based on the method of critical sets (see Gumowski and Mira [21], Mira et al. [28]), allow now to obtain insights into the structure of the basins and into the creation of complex basin boundaries. As some parameter is varied, such changes in the structure can be characterized by global bifurcations: they are the consequence of *contact bifurcations*, i.e. due to contacts between critical sets and invariant sets (such as as fixed points or cycles or their stable sets). For two-dimensional maps, such kinds of bifurcations can be very rarely studied by analytical methods, since the equations of such singularities are not known in general. Hence these global bifurcations are mainly studied by geometrical and numerical methods.

In economic modeling, discrete dynamical systems obtained by the iteration of noninvertible maps are often met, and many of them are characterized by the presence of coexisting attractors. For example, in models with learning, attracting sets which represent rational (or perfect foresight) intertemporal equilibria may coexist with attractors characterized by bounded rationality (see e.g. Barucci et al. [4]), and in models of strategic interaction with boundedly rational competitors different Nash equilibria may be observed as the long run outcome depending on the initial condition (see e.g. Agiza et al. [2] and Bischi and Kopel [10]). For other economic models where noninvertible maps emerge, see Bischi and Naimzada [6], Bischi et al. [8, 9].

The transition to a complicated structure of the basin boundaries usually causes a loss in the system's predictability, in the sense that small changes in the initial condition may give a completely different asymptotic evolution. In fact, if a point is far from the basin boundaries, a slight perturbation has no effect on the long run behavior. On the other hand, if a point is very close to a basin boundary (and many point are in the presence of complex basin boundaries) a small perturbation has a high probability to cause a crossing of the boundary and the long run evolution will be very different. The trajectory may then reach a completely different region of the phase space (it may even diverge, if infinity is an attractor). It is important to note that this kind of unpredictability of the long-run behavior as a consequence of small changes of the initial conditions is different from the sensitive dependence on initial conditions which characterizes a chaotic attractor: When we have a chaotic attractor, even if two initially very close chaotic trajectories fastly depart as time

increases (at an exponential rate), such trajectories are finally trapped inside the same compact invariant set (the chaotic attractor).

In this paper, the problem characterized above is illustrated via a discrete-time two-dimensional dynamical system which models the time evolution of a Cournot duopoly game. The Cournot oligopoly model is one of the most widely used concepts in the industrial organization literature. Since the work of Cournot [14] researchers have been interested in the stability properties of the resulting Cournot-Nash equilibria since - although it seems unlikely that firms immediately coordinate on such an equilibrium - general intuition suggests that they would learn to play according to a Nash equilibrium profiles over time (see Lucas [25]). For theoretical work on the stability properties of Cournot-Nash equilibria, see Theocharis [36], Okuguchi [29], and Szidarovszky and Yen [35]. In our example, the players are assumed to be boundedly rational. They exhibit some kind of inertia and adjust their production quantities only partially toward their profit-optimizing choices. The Best Replies of the players are given by unimodal functions, and this gives rise to multiplicity of equilibria. When several coexisting stable equilibria are present, we focus on the delimitation of their *basins of attraction*. We try to explain the origin of complex topological structures of the basins by analyzing the global properties of noninvertible maps via the method of critical curves.

The paper is organized as follows. In section 2 we recall some definitions and properties of noninvertible maps. In section 3 we give a description of the duopoly model and the adjustment dynamic which characterizes the dynamic game. We give conditions for the existence and local stability of the emerging Nash equilibria of the model. Furthermore, we give results concerning the delimitation of the basins of attraction and the global bifurcations causing qualitative changes in their structure as some parameter is allowed to vary. Throughout section 3 comparison is made between the cases of homogeneous and heterogeneous players. We end the paper with a discussion in section 4.

2 Discrete Dynamical Systems, Noninvertible Maps and Critical Curves

A map $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$p' = T(p) \qquad (1)$$

transforms a point $p \in \mathbb{R}^n$ into a unique point $p' \in \mathbb{R}^n$. The point p' is the *rank-1 image* of p, and a point p such that $T(p) = p'$ is a *rank-1 preimage* of p'. A *discrete-time dynamical system* is defined inductively by the difference equation

$$p_{t+1} = T(p_t)$$

with a given initial condition p_0. So, if $p \in \mathbb{R}^n$ represents the state of a system, T can be seen as a "unit time advancement operator"

$$T : p_t \to p_{t+1}.$$

Starting from an *initial condition* $p_0 \in \mathbb{R}^n$, the repeated application (*iteration*) of T gives the time evolution of the system, represented by the *trajectory*

$$\tau(p_0) = \left\{ p_t | p_t = T^t(p_0), t = 0, 1, 2, ... \right\},$$

where T^0 is the identity map and $T^n = T(T^{n-1})$.

We recall now some definitions. A set $A \subset \mathbb{R}^n$ is *trapping* if it is mapped into itself, $T(A) \subseteq A$, i.e. $\forall x \in A \; T(x) \in A$. A trapping set is *invariant* if it is mapped onto itself: $T(A) = A$, i.e. all the points of A are images of points of A. A closed invariant set A is an attractor if it is *asymptotically stable*, i.e. if a neighborhood U of A exists such that $T(U) \subseteq U$ and $T^n(x) \to A$ as $n \to +\infty$ for each $x \in U$.

The *Basin* of an attractor A is the set of all points that generate trajectories converging to A

$$\mathcal{B}(A) = \left\{ p \,|\, T^t(x) \to A \quad \text{as } t \to +\infty \right\}$$

Starting from the definition of stability, let $U(A)$ be a neighborhood of an attractor A whose points converge to A. Of course $U(A) \subseteq \mathcal{B}(A)$, but also the points of the phase space which are mapped inside U after a finite number of iterations belong to $\mathcal{B}(A)$. Hence, in general the *total basin of A* (or briefly the basin of A) is given by

$$\mathcal{B}(A) = \bigcup_{n=0}^{\infty} T^{-n}(U(A))$$

where $T^{-1}(x)$ represents the set of the rank-1 preimages of x (i.e. the points mapped into x by T), and $T^{-n}(x)$ represents the set of the rank-n preimages of x (i.e. the points mapped into x after n applications of T).

If $p_1 \neq p_2$ implies $T(p_1) \neq T(p_2)$ then T is an *invertible map*, because the inverse mapping that gives $p = T^{-1}(p')$ is uniquely defined; otherwise T is a *noninvertible map*. Noninvertible in this sense means "many-to-one", that is, distinct points $p_1 \neq p_2$ may have the same image, $T(p_1) = T(p_2) = p$. Hence, several rank-1 preimages may exist and the inverse relation $p = T^{-1}(p')$ may be multivalued. Geometrically, the action of a noninvertible map T can be expressed by saying that it "folds and pleats" the plane, so that the two distinct points p_1 and p_2 are mapped into the same point p. This is equivalently stated by saying that several inverses are defined in p, and these inverses "unfold" the plane.

For a noninvertible map (1) $\mathbb{R}^n$ can be subdivided into regions Z_k, $k \geq 0$, whose points have k distinct rank-1 preimages. Generally, as the point p' varies in $\mathbb{R}^n$, pairs of preimages appear or disappear as this point crosses the boundaries separating different regions. Hence, such boundaries are characterized by the presence of at least two coincident (merging) preimages. This leads to the definition of the critical sets, one of the distinguishing features of noninvertible maps. Following the notations of Gumowski and Mira [21] and Mira et al. [27, 28], the *critical set LC* (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages, located on a set (*set of merging preimages*) called LC_{-1}. LC is the two-dimensional generalization of the notion of critical value (when it is a local minimum or maximum value) of a one-dimensional map[1], LC_{-1} is the generalization of the notion of critical point (when it is a local extremum point). Portions of LC separate the regions Z_k characterized by a different number of rank-1 preimages.

The relation $LC = T(LC_{-1})$ holds, and the points of LC_{-1} in which the map is differentiable are necessarily points where the Jacobian determinant vanishes:

$$LC_{-1} \subseteq J_0 = \left\{ (x,y) \in \mathbb{R}^2 \,|\, \det DT(x,y) = 0 \right\}. \tag{2}$$

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[1] This terminology, and notation, originates from the notion of critical points as it is used in the classical works of Julia and Fatou.

In fact in any neighborhood of a point of LC_{-1} there are at least two distinct points which are mapped by T in the same point. Accordingly, the map is not locally invertible in points of LC_{-1}. More generally, for a continuous map T the set LC_{-1} is included in the set where $\det DT(x,y)$ changes sign, since T is locally an orientation preserving map near points (x,y) such that $\det DT(x,y) > 0$ and orientation reversing if $\det DT(x,y) < 0$. An intuitive visualization in $\mathbb{R}^2$ is given in Fig. 1, where the *folding properties* of two-dimensional noninvertible maps is illustrated.
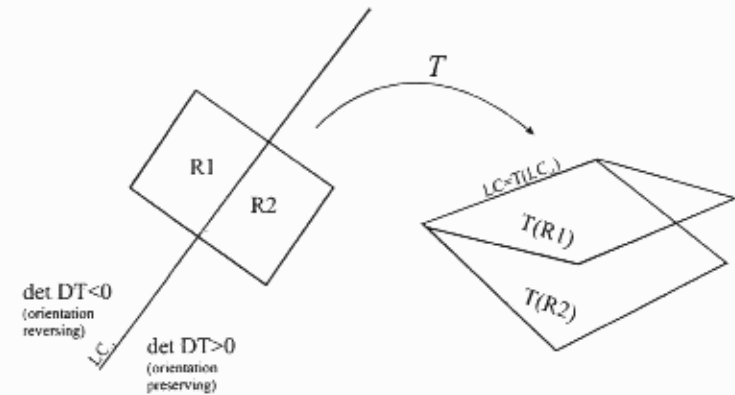

Figure 1: Folding action of the critical curves

In order to give a geometrical interpretation of the action of the multivalued inverse relation T^{-1}, it is useful to consider a region Z_k as the superposition of k sheets, each one associated with a different inverse. Such a representation is known as *Riemann foliation* of the plane (see e.g. Mira et al. [27, 28]). Different sheets are connected by folds joining two sheets, and the projections of such folds on the phase plane are arcs of LC. This is shown in the qualitative sketch of Fig. 2, where the case of a $Z_0 - Z_2$ noninvertible map is considered.
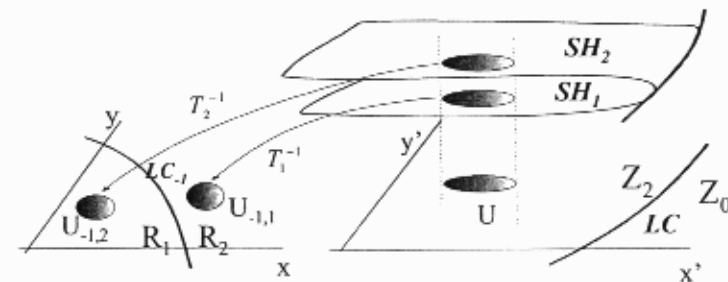

Figure 2: Riemann foliation to represent the unfolding action of the inverses in a $Z_0 - Z_2$ noninvertible map

The backward iteration of a noninvertible map *repeatedly unfolds* the phase space, and this implies that the basins may be non-connected, i.e. formed by several disjoint portions:

$$\mathcal{B}(A) = \bigcup_{n=0}^{\infty} T^{-n}(\mathcal{B}_0(A))$$

where $\mathcal{B}_0(A)$ is the *immediate basin*. The immediate basin is defined as the widest connected component of the basin which contains A.

The global bifurcations which give rise to complex topological structures of the basins like those formed by non connected sets, can be explained in terms of contacts of basins boundaries and critical sets. In fact, if a parameter variation causes a crossing between a basin boundary and a critical set which separates different regions Z_k so that a portion of a basin enters a region where an higher number of inverses is defined, then new components of the basin may suddenly appear at the contact. This is the basic mechanism which causes the creation of more and more complex structures of the basins, as we shall see in the example given in the following section.

3   A Nonlinear Duopoly Game with Multiple Nash Equilibria

As an example, we consider a Cournot duopoly game. This game describes a market where two firms 1 and 2 in each time period t produce the quantities $q_1(t)$ and $q_2(t)$ respectively. They determine their production quantities for the next period such that their profits are maximized,

$$q_i(t+1) = \arg\max \Pi_i\left(q_i, q_j^{(e)}(t+1)\right), i,j = 1,2, \ j \neq i \tag{3}$$

where $q_j^{(e)}(t+1)$ represents the expectation of producer i about the production decision of producer j. We assume that:

(i) the problem (3) has a unique solution, expressed as $q_i(t+1) = r_i(q_j^{(e)}(t+1))$;

(ii) the firms have naive (or Cournot) expectations $q_j^{(e)}(t+1) = q_j(t)$. That is, firm i expects that the quantity of firm j will remain the same as in the current period.
Under these assumptions the solution of (3) can be expressed as $q_i(t+1) = r_i(q_j(t))$, where r_i are called *reaction functions* or *Best Replies*. We further assume that the competitors do not immediately adjust to the optimal quantity they computed on the basis of the profit maximization problem, but that they exhibit some kind of inertia: they only adjust their previous production quantities in the direction of the Best Response:

$$q_1(t+1) = q_1(t) + \lambda_1 \left(r_1(q_2(t)) - q_1(t)\right)$$

$$q_2(t+1) = q_2(t) + \lambda_2 \left(r_2(q_1(t)) - q_2(t)\right) \tag{4}$$

The parameter $\lambda_i \in [0,1]$, $i = 1,2$, represent the *speeds of adjustment*. Such inertial adjustment dynamics can be due to a lack of confidence of firms in their own forecasts of rivals' behavior or due to a conservative attitude in the decision making of firms. Recent experimental evidence suggests that such partial adjustment to the Best Response or inertial selection dynamics describes the behavior of real decision makers quite well (see Rassenti et al. [34], Huck et al. [22]).

In order to get a complete description of the dynamic adaptive system we have to specify the reaction functions. In the original work of Cournot, as well as in many of the subsequent

papers, the reaction functions of the players are assumed to be decreasing functions that intersect in a unique point of the positive quadrant, which is the unique Nash equilibrium. In this case the equilibrium in the Cournot oligopoly model has very simple stability properties as the trajectories can either converge to the Nash equilibrium or diverge. Cournot oligopolies where the reaction functions of the players are non-monotonic, have been studied to a far lesser extent. It is important to notice that the non-monotonicity of reaction functions arises naturally due to nonlinearities or externalities in the cost functions (Furth [16], Puhakka and Wissink [30]), when competitors regard their products as strategic complements over a certain range of possible actions (Bulow et al. [11], Cooper and John [13]), or just simply because the market can be described by a constant elasticity demand function (Bulow et al. [12]). Note that in the case of non-monotonic reactions functions multiple Nash equilibria may occur. In economic dynamics non-monotonic reaction functions have been introduced with several economic justifications, and it has been demonstrated that the Cournot tâtonnement (or Best Response Dynamic) and inertial adjustment dynamics may lead to nonconvergence with complicated, but bounded behaviors; see Rand [33], Puu, [31, 32], Kopel [23]. These authors have been mainly concerned with the question of local stability of the Nash equilibria and the creation of complex attractors when convergence to an equilibrium failed.

In our example, we consider reaction functions in the form of logistic maps

$$\begin{aligned} r_1(q_2) &= \mu_1 q_2 (1-q_2) \\ r_2(q_1) &= \mu_2 q_1 (1-q_1) \end{aligned} \tag{5}$$

proposed by Kopel [23], where it is shown that the functions given in (5) can be derived as Best Responses. The parameters $\mu_i, i = 1,2$ are a measure of the intensity of the positive externality the actions of one player exert on the payoff of the other player.

By inserting (5) into (4), the time evolution of the game is obtained by the iteration of the two-dimensional map $T : (q_1, q_2) \rightarrow (q_1', q_2')$ defined by

$$T : \begin{cases} q_1' = (1-\lambda_1)q_1 + \lambda_1 \mu_1 q_2 (1-q_2) \\ \\ q_2' = (1-\lambda_2)q_2 + \lambda_2 \mu_2 q_1 (1-q_1) \end{cases} \tag{6}$$

Of course, only non-negative trajectories are suitable to represent feasible time evolutions of the duopoly system.

The map (6) contains four parameters: $\mu_i > 0$, $i = 1,2$, and $\lambda_i \in [0,1]$, $i = 1,2$. If $\mu_i \in [0,4]$, $i = 1,2$, as assumed in Kopel [23], then the region $\mathcal{R} = \{[0,1] \times [0,1]\}$ is trapping for each value of λ_i, i.e. any trajectory which starts inside $\mathcal{R}$ remains inside $\mathcal{R}$ for each $t \geq 0$. However, feasible trajectories of (6) can be obtained even if $\mu_1 > 4$ or $\mu_2 > 4$ provided that the adjustment speeds λ_1 and λ_2 are sufficiently small.

3.1   Nash Equilibria and their Local Stability

The fixed points of the discrete dynamical system (4) are the solutions of the system

$$\begin{aligned} q_1 &= r_1(q_2) \\ q_2 &= r_2(q_1) \end{aligned} \tag{7}$$

Hence, they are independent of the parameters λ_1 and λ_2, and are located at the intersections of the two reaction curves. If r_1 and r_2 are the reaction functions (5), it is easy to show that the fixed points with positive coordinates coincide with the Nash equilibria of the duopoly game (see Kopel [23]). So, in what follows we will often use the terms fixed points and Nash equilibrium interchangeably.

The algebraic system (7) has two or four real solutions: this can be seen by a graphical representation of the two reaction curves or analytically, by the study of the fourth degree algebraic system.

Besides the trivial fixed point $O = (0,0)$, another fixed point $S = (q_1^s, q_2^s)$ exists which merges with O if $\mu_1 \mu_2 = 1$ and has positive coordinates, $q_i^s > 0$, $i = 1,2$, provided that $\mu_1 \mu_2 > 1$. Two further positive fixed points exist for sufficiently high values of μ_1 and μ_2.

In order to reduce the number of parameters in our model, we will assume that

$$\mu_1 = \mu_2 = \mu. \tag{8}$$

This means that the positive externality which the actions of each player exerts on the payoff of the other player is equal. Under this symmetry assumption, the fixed points of (6) can be expressed by simple analytical expressions. In fact, in this case the fixed point S belongs to the diagonal Δ, or *line of equal quantities*

$$\Delta = \{(q,q), q \in \mathbb{R}\} \tag{9}$$

and is given by

$$S = \left(1 - \frac{1}{\mu}, 1 - \frac{1}{\mu}\right)$$

For $\mu > 1$, S represents a symmetric Nash equilibrium, since it is characterized by identical quantities produced by the two firms. Two further Nash equilibria, given by

$$E_1 = \left(\frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{2\mu}, \frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{2\mu}\right) \tag{10}$$

and

$$E_2 = \left(\frac{\mu+1-\sqrt{(\mu+1)(\mu-3)}}{2\mu}, \frac{\mu+1+\sqrt{(\mu+1)(\mu-3)}}{2\mu}\right), \tag{11}$$

are created at $\mu = 3$, and for $\mu > 3$ they are located in symmetric positions with respect to the diagonal Δ. Each of them represents a nonsymmetric Nash equilibrium, being characterized by different quantities of the two players. However, E_1 and E_2 represent two (locally) optimal situations which are symmetric in the following sense: in E_1 firm 1 produces more than firm 2 (i.e. firm 1 dominates the market) in exactly the same way as firm 2 produces more than firm 1 in E_2 (where firm 2 dominates the market).

In the presence of multiple Nash equilibria the problem of equilibrium selection arises (see Van Huyck and Battalio [39]), and this naturally leads to the question of stability (see Van Huyck et al. [37, 38]). We recall that a fixed point q is *locally asymptotically stable* if for every neighborhood U of p there exists a neighborhood V of p such that $T^t(V) \subset U$ for each $t \geq 0$, and each initial condition in V generates a trajectory converging to p. The

local stability of a Nash equilibrium of our duopoly game means in other words: provided that the initially chosen quantities of the duopolists are not too far from the equilibrium point, even if the players are boundedly rational and behave adaptively, then the repeated applications of the adjustment dynamic leads to a situation where no player can gain by unilateral deviation from the prescribed equilibrium quantity. It is worth noticing that no Nash equilibrium can be globally attracting because diverging trajectories can always be obtained by starting from sufficiently high quantities q_1 and q_2 (see Appendix A). In other words, an attracting set at infinite distance always exists, and we call *basin of infinity* $\mathcal{B}(\infty)$ the open set of points that generate diverging trajectories,

$$\mathcal{B}(\infty) = \{(q_1,q_2) \mid \|T^t(q_1,q_2)\| \to \infty \text{ as } t \to +\infty\} . \tag{12}$$

The analysis of the local stability of a fixed point is obtained through the localization of the eigenvalues of the Jacobian matrix in the complex plane, where the Jacobian

$$DT(q_1,q_2) = \begin{bmatrix} 1-\lambda_1 & \lambda_1\mu_1(1-2q_2) \\ \lambda_2\mu_2(1-2q_1) & 1-\lambda_2 \end{bmatrix} \tag{13}$$

computed at the corresponding fixed point has to be considered. We limit our analysis to the case (8). Under this assumption, from the analysis of the eigenvalues given in Appendix B, we can derive a complete characterization of the regions of existence and stability of the Nash equilibria in the three-dimensional parameters space $(\mu, \lambda_1, \lambda_2)$. However, we shall first consider the duopoly model under the assumptions (8) and

$$\lambda_1 = \lambda_2 = \lambda . \tag{14}$$

as a benchmark case, in order to highlight the effects of *heterogeneous behavior* of the two firms, i.e., when the speeds of adjustment differ, $\lambda_1 \neq \lambda_2$. A study of the map (6) with $\lambda_1 = \lambda_2 = \lambda$ in the particular case $\mu_1 = \mu_2 = 4$ is given in Gardini et al. [17].

With the assumption (14) the map (6) has the following symmetry property: it remains the same after an exchange of the state variables q_1 and q_2. This means that the map T commutes with the operator $P : (q_1,q_2) \to (q_2,q_1)$, which represents a reflection with respect to the diagonal Δ, $T(P(q_1,q_2)) = P(T(q_1,q_2))$, and consequently the diagonal Δ is a trapping submanifold for the map T, i.e., $T(\Delta) \subseteq \Delta$. For the economic model this means that if two firms start with equal quantities $q_1(0) = q_2(0)$ and behave identically, then their choices will be the same for each future time period. The trajectories, embedded into the one-dimensional submanifold Δ, are governed by the restriction of the two-dimensional map T to Δ, given by

$$q' = f(q) = (1 + \lambda(\mu-1))q - \lambda\mu q^2 \tag{15}$$

which may be seen as the model of a "representative player" (see Bischi et al. [5] or Kopel et al. [24]). The map (15) is conjugated to the logistic map

$$z' = az(1-z) \tag{16}$$

through the linear transformation $q = \frac{\lambda(\mu-1)+1}{\lambda\mu}z$, and the parameters are related by $\mu = 1 + (a-1)/\lambda$. The standard analysis of the local stability of the four fixed points in
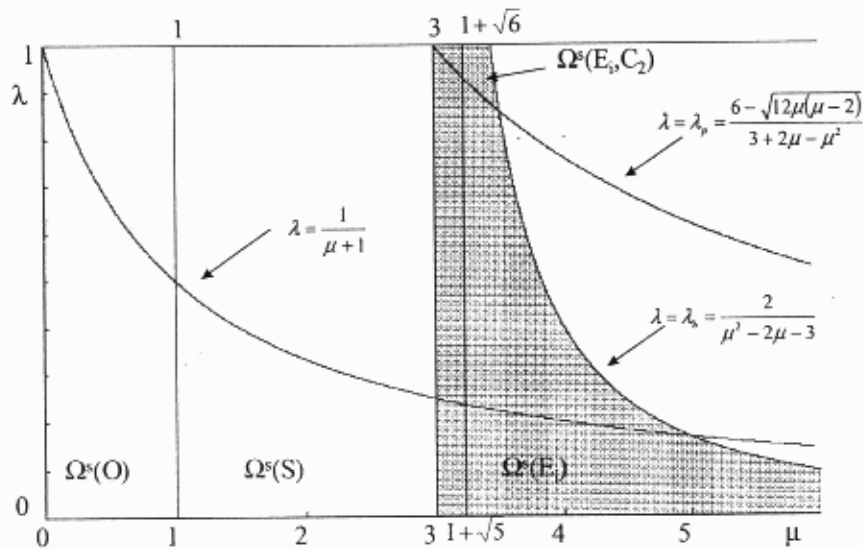
Figure 3: Stability regions in the two-dimensional space Ω_2 of the parameters μ and λ for the case of homogeneous behavior.

the case of homogeneous behavior can be summarized by the following proposition, which defines the stability regions for each Nash equilibrium in the parameters space $\Omega = \left\{ (\mu,\lambda) \in \mathbb{R}^3 | \mu > 0, 0 \leq \lambda \leq 1 \right\}$ (see also Fig. 3)

Proposition 1 (homogeneous behavior). *Let (8) and (14) hold. Then*

(i) The fixed point $O = (0,0)$ exists for each $(\mu,\lambda) \in \Omega$ and it is a stable node for $0 < \mu < 1$, for $1 < \mu < 2/\lambda - 1$ it is a saddle point, with unstable set along Δ and local stable set which crosses through O perpendicular to Δ, and for $\mu > 2/\lambda - 1$ it is an unstable node;

(ii) the equilibrium $S = \{1 - 1/\mu, 1 - 1/\mu\}$ exists for each $(\mu,\lambda) \in \Omega$ and it is a stable node $(\mu,\lambda) \in \Omega^s(S)$, where

$$\Omega^s(S) = \{(\mu,\lambda) \in \Omega |\ 1 < \mu < 3\}; \tag{17}$$

for $3 < \mu < 1 + 2/\lambda$ it is a saddle point, with local stable set along Δ and unstable set which crosses through S perpendicular to Δ, and for $\mu > 1 + 2/\lambda$ it is an unstable node;

(iii) The fixed points E_i, $i = 1,2$, given in (10) and (11) are created at $\mu = 3$ through a pitchfork bifurcation of S and are both locally asymptotically stable for $(\mu,\lambda) \in \Omega^s(E_i)$, with

$$\Omega^s(E_i) = \left\{ (\mu,\lambda) \in \Omega |\ \mu > 3, 0 < \lambda < \lambda_h(\mu) = \frac{2}{\mu^2 - 2\mu - 3} \right\}; \tag{18}$$

they are stable nodes for $3 < \mu < 1 + \sqrt{5}$, stable foci for $1 + \sqrt{5} < \mu < 1 + \sqrt{4 + \frac{2}{\lambda}}$ and at $\mu = 1 + \sqrt{4 + \frac{2}{\lambda}}$ they become unstable foci through a Neimark-Hopf bifurcation.

(iv) in the subset of $\Omega^s(E_i)$ given by

$$\Omega^s(E_i,C_2) = \left\{ (\mu,\lambda) \in \Omega^s(E_i) |\ \lambda > \lambda_p(\mu) = \frac{6 - \sqrt{12\mu(\mu-2)}}{3 + 2\mu - \mu^2} \right\} \tag{19}$$

the two stable equilibria E_i, $i = 1,2$, coexist with the stable cycle of period two

$$C_2 = \{(p_1,p_1),(p_2,p_2)\} \in \Delta \tag{20}$$

with $p_1 = \frac{\lambda(\mu-1)+2-\sqrt{\lambda^2(\mu-1)^2-4}}{2\lambda\mu}$ and $p_2 = \frac{\lambda(\mu-1)+2+\sqrt{\lambda^2(\mu-1)^2-4}}{2\lambda\mu}$

The proof immediately follows from the analysis of the eigenvalues (see Appendix B). In our analysis of the global dynamic properties we shall be mainly concerned with the set of parameters in region $\Omega^s(E_i)$, which gives coexistence of stable Nash equilibria.

We now turn to the case of heterogeneous behavior, i.e. we relax assumption (14). Although the same Nash equilibria are obtained in this case since the fixed points of (6) do not depend on the speeds of adjustment, the eigenvalues of the Jacobian matrix (13) depend on both of the parameters λ_1 and λ_2. Furthermore, the diagonal Δ is no longer trapping. The following proposition, whose proof immediately follows from the analysis of the eigenvalues (see Appendix B), defines the stability regions for each Nash equilibrium in the three-dimensional parameters space

$\Omega_3 = \left\{ (\mu,\lambda_1,\lambda_2) \in \mathbb{R}^3 | \mu > 0, 0 \leq \lambda_1 \leq 1, 0 \leq \lambda_2 \leq 1 \right\}$.

Proposition 2. *Let (8) hold. Then*

(i) for $0 < \mu < 1$ the fixed point $O = (0,0)$ is a stable node, for $1 < \mu < \sqrt{1 + \frac{4-2(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}}$ it is a saddle point, for $\mu > \sqrt{1 + \frac{4-2(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}}$ it is an unstable node;

(ii) the steady state $S = (1 - 1/\mu, 1 - 1/\mu)$ is a stable node for $(\mu,\lambda_1,\lambda_2) \in \Omega_3^s(S)$, where

$$\Omega_3^s(S) = \{(\mu,\lambda_1,\lambda_2) \in \Omega_3 | 1 < \mu < 3\}, \tag{21}$$

for $3 < \mu < 2 + \sqrt{1 + \frac{4-2(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}}$ it is a saddle point, for $\mu > 2 + \sqrt{1 + \frac{4-2(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}}$ it is an unstable node;

(iii) The fixed points E_i, $i = 1,2$, given in (10) and (11) respectively, are created at $\mu = 3$ through a pitchfork bifurcation of S, and are both stable for $(\mu,\lambda_1,\lambda_2) \in \Omega_3^s(E_i)$ where

$$\Omega_3^s(E_i) = \left\{ (\mu,\lambda_1,\lambda_2) \in \Omega_3 | 3 < \mu < 1 + \sqrt{4 + \frac{1}{\lambda_2} + \frac{1}{\lambda_1}} \right\}; \tag{22}$$

they are stable nodes for $3 < \mu < 1 + \sqrt{\frac{9}{2} + \frac{\lambda_1}{4\lambda_2} + \frac{\lambda_2}{4\lambda_1}}$, stable foci for $1 + \sqrt{\frac{9}{2} + \frac{\lambda_1}{4\lambda_2} + \frac{\lambda_2}{4\lambda_1}} < \mu < 1 + \sqrt{4 + \frac{1}{\lambda_2} + \frac{1}{\lambda_1}}$ and at $\mu = 1 + \sqrt{4 + \frac{1}{\lambda_2} + \frac{1}{\lambda_1}}$ they become unstable foci through a Neimark-Hopf bifurcation.

This proposition generalizes Proposition 1 in the sense that the bifurcation curves that constitute the boundaries of the stability regions defined in Proposition 1 are the intersections of the bifurcation surfaces given in Proposition 2 with the plane $\lambda_1 = \lambda_2$. The main difference between the two propositions lies in the fact that if we do not deal with homogeneous players, i.e. (14) is not assumed, then the local stable and unstable sets associated

with the fixed points O and S are no longer along the diagonal Δ and orthogonal to it, because a symmetry property no longer holds. This has important consequences for the structure of the basins, as we shall see in the following. From a comparison of the two propositions given above, it appears that the influence of heterogeneous behavior on the stability of the Nash equilibria is not too strong. However, in the case of coexisting stable Nash equilibria, an important question concerns the delimitation of their basins of attraction, and we will demonstrate that heterogeneity plays an important role in this context.

3.2   Critical Curves and Structure of the Basins

The delimitation of the basins of attraction of coexisting Nash equilibria requires a study of the global properties of the dynamical system, i.e., a study which is not based on the linear approximation of the map (6) around an equilibrium. In fact, if the initial quantities are not taken in a small neighborhood of a Nash equilibrium, the nonlinear terms of the map may have a strong influence on the time evolution of the repeated quantity choices. The process may then converge to a stable Nash equilibrium, to a more complex bounded attractor or diverge.

As argued in section 2, the properties of the inverses of the map (6) become important in order to understand the structure of the basins.

Given a point $q' = (q'_1, q'_2) \in \mathbb{R}^2$, its rank-1 preimages $T^{-1}(q')$ can be computed by solving the algebraic system (6) with respect to the quantities q_1 and q_2:

$$\begin{cases} (1-\lambda_1)q_1 + \lambda_1\mu_2 q_2(1-q_2) = q'_1 \\ (1-\lambda_2)q_2 + \lambda_2\mu_1 q_1(1-q_1) = q'_2 \end{cases} \tag{23}$$

This is a fourth degree algebraic system, which may have four, two or no real solutions, so the map (6) is a noninvertible map of $Z_0 - Z_2 - Z_4$ type.

Being the map T continuously differentiable, LC_{-1} coincides with the set of points on which $\det DT = 0$, which gives

$$\left(q_1 - \frac{1}{2}\right)\left(q_2 - \frac{1}{2}\right) - \frac{(1-\lambda_1)(1-\lambda_2)}{4\lambda_1\lambda_2\mu_1\mu_2} \tag{24}$$

This equation represents an equilateral hyperbola, whose two branches are denoted by $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$ in Fig. 4a. It follows that also $LC = T(LC_{-1})$ is the union of two branches, say $LC^{(a)} = T(LC_{-1}^{(a)})$ and $LC^{(b)} = T(LC_{-1}^{(b)})$. The branch $LC^{(b)}$ separates the region Z_0, whose points have no preimages, from the region Z_2, whose points have two distinct rank-1 preimages. The other branch $LC^{(a)}$ separates the region Z_2 from Z_4, whose points have four distinct preimages (see Fig. 4b). Notice that a point of $LC^{(a)}$ has two coincident rank-1 preimages which are located at a point of $LC_{-1}^{(a)}$. Furthermore, a point of $LC^{(b)}$ has two coincident rank-1 preimages which are located at a point of $LC_{-1}^{(b)}$ plus two further distinct rank-1 preimages, called *extra preimages* (see e.g. Mira et al. [28]).

In the case of homogeneous behavior, i.e., when the assumptions (8) and (14) hold, the diagonal Δ is a trapping set and the point C_{-1} at which $LC_{-1}^{(a)}$ intersects Δ corresponds to
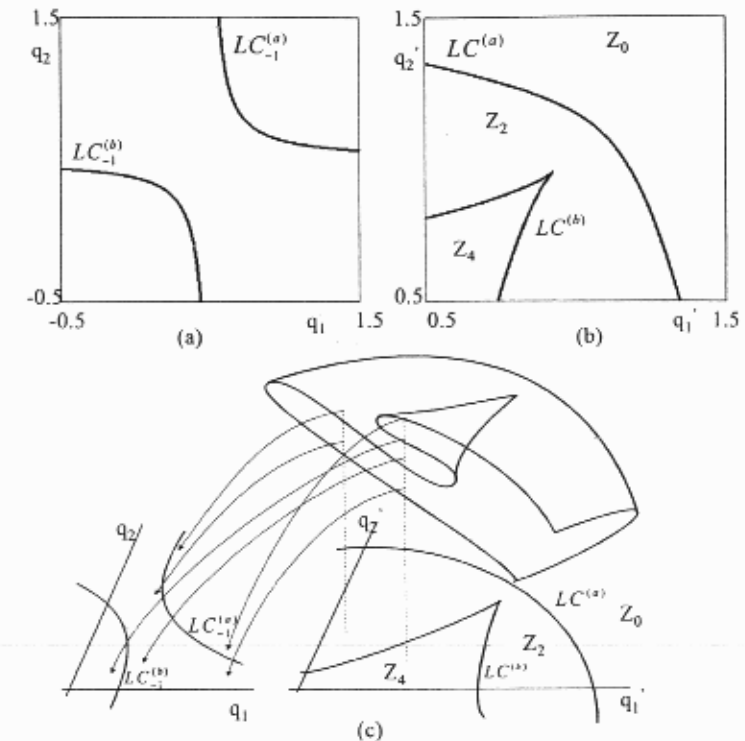
Figure 4: (a) Critical curves of rank-0, LC_{-1}. (b)Critical curves of rank-1, $LC = T(LC_{-1})$. (c) Riemann foliation of the plane for the $Z_4 - Z_2 - Z_0$ map T.

the unique critical point of rank-0 (local maximum point) of the restriction (15) of T to Δ. Formally,

$$LC_{-1}^{(a)} \cap \Delta = C_{-1} = (c_{-1}, c_{-1}) \quad \text{with} \quad c_{-1} = \frac{\lambda(\mu-1)+1}{2\lambda\mu}$$

In fact, in any point of LC_{-1} at least one eigenvalue of DT vanishes, and in C_{-1} the eigenvalue $z_\|$ with eigendirection along Δ vanishes (see Appendix B). Of course

$$LC^{(a)} \cap \Delta = C = (c,c) \quad \text{with} \quad c = f(c_{-1}) = \frac{(\lambda(\mu-1)+1)^2}{4\lambda\mu}$$

that is, the point C at which $LC^{(a)}$ intersects Δ corresponds to the unique critical point of rank-1 (maximum value) of the restriction (15) of T to Δ.

The other intersection of LC_{-1} with Δ, given by

$$LC_{-1}^{(b)} \cap \Delta = K_{-1} = (k_{-1}, k_{-1}) \quad \text{with} \quad k_{-1} = \frac{\lambda(\mu-1)-1}{2\lambda\mu}$$

does not correspond to a critical point of the restriction (15), because in K_{-1} the eigenvalue $z_\perp$, with eigendirection perpendicular to Δ, vanishes (see Appendix B). As also the tangent

to $LC_{-1}^{(b)}$ in K_{-1} is perpendicular to Δ, the curve $LC^{(b)} = T(LC_{-1}^{(b)})$ has a *cusp point* (see e.g. Arnold et al. [3]) given by

$$K = LC^{(b)} \cap \Delta = (k,k) \quad \text{with} \quad k = f(k_{-1}) = \frac{(\lambda(\mu+1)-1)(\lambda\mu+3(1-\lambda))}{4\lambda\mu} \qquad (25)$$

The *Riemann* foliation associated with the map (6) is qualitatively represented in Fig. 4c. It can be noticed that the cusp point of LC is characterized by three merging preimages at the junction of two folds.

3.2.1 Basin of Infinity

The first task in order to gain some insight into the robustness of our model is the determination of the set of initial quantities the players can choose, such that bounded trajectories are obtained. In other words, it is first of all necessary to determine the boundaries $\partial\mathcal{B}(\infty)$ of the basin $\mathcal{B}(\infty)$. The points belonging to $\mathcal{B}(\infty)$ generate diverging trajectories, and the points in the complementary set

$$\mathcal{B} = \mathbb{R}^2 \setminus \mathcal{B}(\infty)$$

form the basin of bounded trajectories. Points of $\partial\mathcal{B}(\infty)$ are mapped into points of $\partial\mathcal{B}(\infty)$, and the interior points of $\mathcal{B}$ may converge to a Nash equilibrium or to some other bounded attracting set. The boundary $\partial\mathcal{B} = \partial\mathcal{B}(\infty)$ behaves as a repelling set for the points near it, since it acts like a watershed for the trajectories of the map T. Points belonging to $\partial\mathcal{B}$ are mapped into $\partial\mathcal{B}$ both under forward and backward iteration of T, that is, the boundary is invariant both with regard to T and T^{-1}. More exactly (see Mira et al. [26], Mira et al. [28])

$$T(\partial\mathcal{B}) \subseteq \partial\mathcal{B}, T^{-1}(\partial\mathcal{B}) = \partial\mathcal{B}.$$

This implies that if an unstable fixed point or cycle belongs to $\partial\mathcal{B}$, then $\partial\mathcal{B}$ must also contain all of its preimages of any rank. Moreover, if a saddle-point, or a saddle-cycle, belongs to $\partial\mathcal{B}$, then $\partial\mathcal{B}$ must also contain the whole stable set (see Gumowski and Mira [21], Mira et al. [28]).

These properties allow us to obtain a delimitation of $\partial\mathcal{B}$ and to understand the influence of the parameters on the extension and the structure of the basins. Let us first consider the case of homogeneous behavior (14). According to Proposition 1, for $\mu > 1$ and $0 < \lambda < 2/(\mu+1)$ the fixed point O is a saddle, with unstable set $W^u(O)$ reaching S along the diagonal Δ and local stable set $W_{loc}^s(O)$ crossing through O perpendicular to Δ. The stable set $W_{loc}^s(O) \subseteq \partial\mathcal{B}$. In fact, if we consider a neighborhood of O, $W_{loc}^s(O)$ is a separatrix between the trajectories which converge to a bounded attractor (generated by the points above $W_{loc}^s(O)$) and those which diverge to $-\infty$ (generated by the points below $W_{loc}^s(O)$). The whole boundary $\partial\mathcal{B}$ is given by the whole stable set $W^s(O)$, obtained by taking the preimages of any rank of $W_{loc}^s(O)$

$$\partial\mathcal{B} = W^s(O) = \bigcup_{k \geq 0} T^{-k}(W_{loc}^s(O))$$

In the case of symmetric players (14), the preimages of O can be analytically computed, and their coordinates allow us to obtain a rough estimate of the extension of $\mathcal{B}$.

If $0 < \lambda < 1/(\mu+1)$ then $O \in Z_2$ (see (25)), i.e. it has two rank-1 preimages both belonging to Δ. One is O itself (being O a fixed point) the other one is

$$O_{-1}^{(1)} = \left(\frac{1+\lambda(\mu-1)}{\lambda\mu}, \frac{1+\lambda(\mu-1)}{\lambda\mu} \right). \qquad (26)$$

Notice that, since $O_{-1}^{(1)} \in \Delta$, its coordinates can be easily computed by using the restriction (15).

If $\lambda > 1/(\mu+1)$ then $O \in Z_4$, i.e. it has four rank-1 preimages. Two of them, O and $O_{-1}^{(1)}$, belong to Δ, and the other two, say $O_{-1}^{(2)}$ and $O_{-1}^{(3)}$, are located in symmetric positions with respect to Δ and belong to the line Δ_{-1}, of equation

$$q_1 + q_2 = 1 + \frac{1}{\mu}\left(1 - \frac{1}{\lambda}\right). \qquad (27)$$

In general, the preimages of any point of Δ are located on Δ or on Δ_{-1}, as it can be seen by setting $q_1' = q_2'$ in (23) and adding or subtracting the two symmetric equations. In particular, with $q_1' = q_2' = 0$ we get the solution

$$O_{-1}^{(2)} = \left(\frac{\lambda(\mu+1)-1+\sqrt{\lambda^2\mu^2+2\lambda\mu(1-\lambda)-3(\lambda^2+1)+6\lambda}}{2\lambda\mu}, \frac{\lambda(\mu+1)-1-\sqrt{\lambda^2\mu^2+2\lambda\mu(1-\lambda)-3(\lambda^2+1)+6\lambda}}{2\lambda\mu} \right) \qquad (28)$$

and the symmetric solution $O_{-1}^{(3)}$ is obtained from $O_{-1}^{(2)}$ by swapping the two coordinates. If $0 < \lambda < 1/(\mu+1)$, the stable set $W^s(O)$ consists of two smooth arcs connecting O and $O_{-1}^{(1)}$, symmetric with respect to Δ. If $\lambda > 1/(\mu+1)$, then $W^s(O)$ has a similar shape, with the symmetric arcs connecting O and $O_{-1}^{(1)}$ which pass through the points $O_{-1}^{(2)}$ and $O_{-1}^{(3)}$. The latter situation is shown in Fig. 5a, obtained for $\mu = 2.8$ and $\lambda = 0.5$, and the former situation is shown in Fig. 6, obtained for $\mu = 3.5$ and $\lambda = 0.22$. In these figures the dark grey region represents the basin of diverging trajectories $\mathcal{B}(\infty)$.

The knowledge of the coordinates of O and $O_{-1}^{(1)}$ allows us to get an estimate of the "size" of $\mathcal{B}$ and the influence of the parameters λ and μ on it. In the case of homogeneous behavior (14) the length of the segment $OO_{-1}^{(1)}$, given by

$$l(OO_{-1}^{(1)}) = \sqrt{2}\frac{1+\lambda(\mu-1)}{\lambda\mu} \qquad (29)$$

is a decreasing function of both parameters λ and μ, and it goes to infinity as $\lambda \to 0^+$, i.e. the basin of bounded trajectories tends to include the whole diagonal in such a limiting case. It is also interesting to note that in the other limiting case, $\lambda \to 1^-$, we get $O_{-1}^{(1)} \to (1,1)$, $O_{-1}^{(2)} \to (1,0)$, $O_{-1}^{(3)} \to (0,1)$. Hence in the case of instantaneous adjustment ($\lambda = 1$), the basin $\mathcal{B}$ becomes the square $(0,1) \times (0,1)$. This result also holds for $\mu_1 \neq \mu_2$, as proved in Bischi et al. [7].

Many of the arguments given above continue to hold in the case of heterogeneous behavior, characterized by different speeds of adjustment $\lambda_1 \neq \lambda_2$. However, a simple analytical expression of the preimages of O cannot be obtained, since in this case they are given

by the solution of the fourth degree nonsymmetric algebraic system (23). The diagonal Δ is no longer invariant, and even if the fixed points remain the same, the basins are no longer symmetric with respect to Δ. Numerical explorations show that if player 1 exhibits more inertia than player 2, i.e. $\lambda_1 < \lambda_2$, then the extent of the basin $\mathcal{B}$ of bounded trajectories is smaller in the q_1 direction and larger in the q_2 direction. In other words, the system is more robust with respect to perturbations of the quantity of the player who adjusts faster to the Best Response (see Fig. 5b).

It is also important to notice that even after the bifurcation occurring at $\lambda(\mu+1) = 2$, when O is transformed from a saddle point into an unstable node with the simultaneous creation of a saddle cycle C_2 of period 2, the boundary $\partial \mathcal{B}$ remains practically the same. In fact, in this case $\partial \mathcal{B} = \overline{W^s(C_2)}$, which continues to include O and its preimages of any rank.
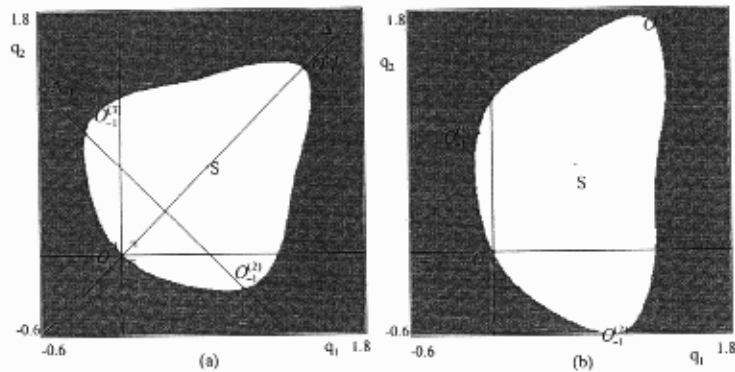

Figure 5: The white regions represent the set $\mathcal{B}$ whose points generate bounded trajectories and the dark-grey regions represent the basin of infinity $\mathcal{B}(\infty)$. (a) Case of homogeneous behavior with $\mu = 2.8$ and $\lambda = 0.5$. (b) Case of heterogeneous behavior with $\mu = 2.8$, $\lambda_1 = 0.25$ and $\lambda_2 = 0.75$.

To end this subsection we remark that in the figures we have also considered regions of the plane which are out of the feasible region $\mathbb{R}^2_+$. Of course, from an economic point of view such regions are not meaningful. However, in order to obtain a more complete understanding of the whole basins structure from a mathematical point of view, we must not exclude these regions from our analysis. In fact, as the basins are obtained by backward iteration of the map (6), it is natural that basins of positive attractors also include negative points, whose (non feasible) trajectories enter the positive orthant $\mathbb{R}^2_+$ after a finite number of iterations. It is plain that only the feasible trajectories, i.e. those which start in $\mathbb{R}^2_+$ and remain inside $\mathbb{R}^2_+$ forever, are interesting for the economic model. However, in the following we will continue to consider regions outside of $\mathbb{R}^2_+$ since this helps us to understand the properties and the qualitative changes of the basins.

3.2.2   Basins of Coexisting Bounded Attractors and their Global Bifurcations

As far as $1 < \mu < 3$, the fixed point $S \in \Delta$ is the only stable Nash equilibrium, and the set $\mathcal{B}$ coincides with the basin $\mathcal{B}(S)$. In this case the robustness of S with respect to exogenous perturbations can be deduced from the knowledge of the boundary which separates $\mathcal{B}(S)$ from $\mathcal{B}(\infty)$. For $\mu > 3$, according to the stability analysis given in Propositions 1 and 2, S is no longer stable, and two coexisting stable Nash equilibria, or more complex attracting sets, are present.

We now consider values of the parameter μ just after the pitchfork bifurcation such that the two Nash equilibria E_1 and E_2 are both stable. As mentioned before, in such a situation the issue of equilibrium selection arises. Note that since both equilibria are stable, the property of "stability" can not be used as a proper equilibrium refinement (see Cox and Walker [15]). Hence, a delimitation of the basins of the two equilibria becomes crucial in order to understand the long run evolution of the adjustment dynamic. In the case where only these two stable Nash equilibria coexist, the set $\mathcal{B}$ is shared by the two basins $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$. The boundary of each of these two basins is formed by an "outer portion", which separates them from $\mathcal{B}(\infty)$, and an "inner portion", separating $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$, which contains the symmetric Nash equilibrium S (which is a saddle point just after the pitchfork bifurcation) as well as its whole stable set $W^s(S)$ (see Fig. 6). In fact, just after the pitchfork bifurcation, occurring at $\mu = 3$, at which the two stable fixed points E_1 and E_2 are created, the symmetric Nash equilibrium $S \in \Delta$ is a saddle, provided that $0 < \lambda < \frac{2}{\mu-1}$, and the two branches of unstable set $W^u(S)$ departing from it reach the two stable nodes E_1 and E_2. Hence the local stable set $W^s_{loc}(S)$ belongs to the boundary that separates the two basins, as well as its preimages of any rank:

$$W^s(S) = \bigcup_{k \geq 0} T^{-k}\left(W^s_{loc}(S)\right) = \partial \mathcal{B}(E_1) \cap \partial \mathcal{B}(E_2) \qquad (30)$$

In order to study the shape of $W^s(S)$ and the global bifurcations which change its qualitative structure, in the following we consider the symmetric case (14) of homogeneous players and the case of heterogeneous players separately.

In the homogeneous case, because of the symmetry property of the map (6), the local stable set of S belongs to the invariant diagonal Δ. Indeed, as far as

$$\lambda(\mu+1) < 1 \qquad (31)$$

the whole stable set belongs to Δ and is given by

$$W^s(S) = OO^{(1)}_{-1}$$

where $O^{(1)}_{-1}$ is given in (26) and $OO^{(1)}_{-1}$ is the segment joining O with $O^{(1)}_{-1}$. In fact, if (31) holds, the cusp point K of the critical curve $LC^{(b)}$ has negative coordinates and, consequently, the whole segment $OO^{(1)}_{-1}$ belongs to the region Z_2. This implies that the two preimages of any point of $OO^{(1)}_{-1}$ belong to Δ, and can be computed by the restriction (15). This proves that the segment $OO^{(1)}_{-1}$ is backward invariant, i.e., $T\left(OO^{(1)}_{-1}\right) = OO^{(1)}_{-1}$.

Our arguments imply that if (31) holds, the structure of the basins $\mathcal{B}(E_i)$, $i = 1, 2$, is very simple: $\mathcal{B}(E_1)$ is the portion of $\mathcal{B}$ below the diagonal Δ and $\mathcal{B}(E_2)$ is the portion of $\mathcal{B}$ above it. This situation is shown in Fig. 6, where the numerically computed basins $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ are represented by light grey and intermediate grey respectively. The line Δ is the only boundary between the two basins, hence any bounded trajectory starting with $q_1(0) > q_2(0)$ converges to the Nash equilibrium E_1 and any bounded trajectory starting with $q_1(0) < q_2(0)$ converges to the Nash equilibrium E_2. In economic terms this means that an initial difference in the quantities uniquely determines which of the equilibria is selected in the long run. If player 1 offers a larger quantity than player 2, then E_1 is selected, and vice versa. In general, *if (31) holds, we can state that if* $q_1(0) > q_2(0)$ $(q_1(0) < q_2(0))$ *then* $q_1(t) > q_2(t)$ $(q_1(t) < q_2(t))$ *for any t*. In other words, the following monotonicity property of the adjustment dynamic holds: any initial order of the quantities of the two players is maintained during the whole time evolution of the duopoly game.
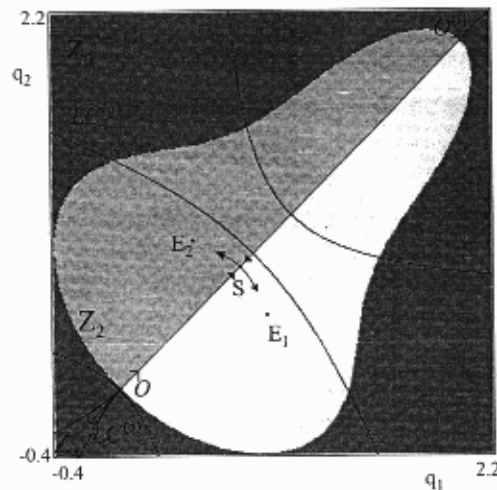

Figure 6: Case of homogeneous behavior with parameters $\mu = 3.5$ and $\lambda = 0.22$. The dark-grey region represents the basin $\mathcal{B}(\infty)$ of diverging trajectories, the light-grey region represents the basin $\mathcal{B}(E_1)$ of the Nash equilibrium E_1, the intermediate-grey region represents the basin $\mathcal{B}(E_2)$ of the Nash equilibrium E_2.

Both of the basins $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ are simply connected sets if (31) holds. On the other hand, their structure becomes a lot more complex for $\lambda(\mu + 1) > 1$. This is shown in Fig. 7a, which is obtained with $\mu = 3.5$ as in Fig. 6, but $\lambda = 0.4 > 1/(\mu + 1) = 0.\overline{2}$. In order to understand the bifurcation occurring at $\lambda(\mu + 1) = 1$, we consider the critical curves of the map (6). In fact, at $\lambda(\mu + 1) = 1$ a contact between $LC^{(b)}$ and the fixed point O occurs, due to the merging between O and the cusp point K given in (25). For $\lambda(\mu + 1) > 1$ the portion KO of the segment $OO^{(1)}_{-1}$ belongs to the region Z_4 where four inverses of T exist. This implies that besides the two rank-1 preimages on Δ the points of KO have two further preimages located on the segment $O^{(2)}_{-1}O^{(3)}_{-1}$ of the line Δ_{-1} of equation

(27). Since $OO^{(1)}_{-1} = W^s_{loc}(S) \subset \partial \mathcal{B}(E_1) \cap \partial \mathcal{B}(E_2)$, also all the preimages of this segment belong to the boundary which separates $\mathcal{B}(E_1)$ from $\mathcal{B}(E_2)$. Furthermore, also the segment $O^{(2)}_{-1}O^{(3)}_{-1}$ has rank-1 preimages, because portions of it are included in the regions Z_2 and Z_4. These are preimages of rank-2 of $OO^{(1)}_{-1}$ and, consequently, belong to $W^s(S)$ according to (30). This repeated procedure, based on the iteration of the multivalued inverse of T, leads to the construction of the stable set $W^s(S)$ which is formed by the union of infinitely many arcs which accumulate on the boundary $\partial \mathcal{B}$. In fact the invariant set $\partial \mathcal{B}$, being a repelling set for the forward iteration of T, attracts the invariant manifolds under the iteration of the inverses of T.

The results given above can be summarized as follows:

Proposition 3. *If* $\mu_1 = \mu_2 = \mu$, $\lambda_1 = \lambda_2 = \lambda$ *and* $(\mu, \lambda) \in \Omega_2(E_i)$, *the bounded trajectories of (6) converge to one of the stable Nash equilibria E_1 or E_2, given by (10) and (11) respectively, and the common boundary $\partial \mathcal{B}(E_1) \cap \partial \mathcal{B}(E_2)$ which separates the basin $\mathcal{B}(E_1)$ from the basin $\mathcal{B}(E_2)$ is given by the stable set $W^s(S)$ of the saddle point S. If $\lambda(\mu + 1) < 1$ then $W^s(S) = OO^{(1)}_{-1}$, where $O = (0,0)$ and $O^{(1)}_{-1}$ is given by (26), and the two basins are simply connected sets; if $\lambda(\mu + 1) > 1$ then the two basins are non connected sets, formed by infinitely many simply connected components.*

We would like to emphasize that the bifurcation occurring at $\lambda(\mu + 1) = 1$ is a global bifurcation, i.e. it cannot be revealed by a study of the linear approximation of the dynamical system. The occurrence of such a bifurcation has been characterized by a contact between the stable set of S and a critical curve. This kind of bifurcation has been called *contact* (or *non classical*) bifurcation in Mira et al. [28].

The occurrence of the bifurcation which transforms the basins from simply connected to non connected causes a loss of predictability about the long-run evolution of our Cournot game starting from given initial quantities of the two players. In fact, in contrast to what happens in the case of simply connected basins, when the basins are no longer simply connected, the adjustment dynamic starting with $q_1(0) > q_2(0)$ may lead to convergence to either of the Nash equilibria. Furthermore, if the initial quantities are sufficiently far away from a Nash equilibrium – for example near the boundary $\partial \mathcal{B}$ of $\mathcal{B}$ – then the presence of the infinitely many components of both basins causes a sort of unpredictability with respect to these initial conditions. Even a very small perturbation of the starting point of the Cournot game may lead to a crossing of the boundary which separates the two basins and, consequently, may result in the convergence to a different Nash equilibrium. The complex structure of the basin boundaries in the region close to the boundary of $\mathcal{B}$ becomes quite obvious in the enlargement shown in Fig. 7b.

It is important to notice that even after the flip bifurcation occurring at $\lambda(\mu - 1) = 2$ the boundary that separates $\mathcal{B}(E_1)$ from $\mathcal{B}(E_2)$ remains practically the same. This bifurcation transforms S from a saddle point into an unstable node and a saddle cycle of period 2 is created with stable set along Δ. The boundary of the basins is then given by the closure of the stable set of the 2-cycle, which continues to include S and its preimages of any rank.

We now turn to the case of heterogeneous behavior, in which the assumption (14) is relaxed. In this case, the common boundary $\partial \mathcal{B}(E_1) \cap \partial \mathcal{B}(E_2)$ is still formed by the whole stable set $W^s(S)$, but if $\lambda_1 \neq \lambda_2$ the local stable set $W^s_{loc}(S)$ is not along the diagonal Δ, because T is no longer symmetric and, consequently, Δ is no longer invariant. However, by
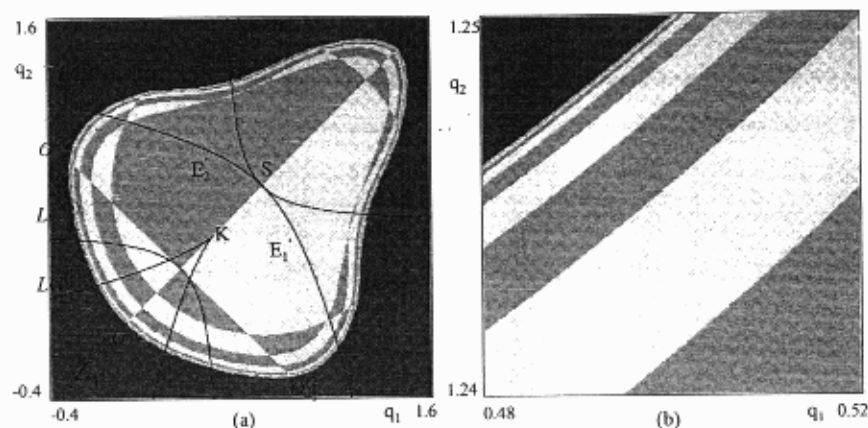
Figure 7: Case of homogeneous behavior with parameters $\mu = 3.5$ and $\lambda = 0.4$. (a) The fixed point $O = (0,0)$ is a saddle, the fixed point $S = (0.71, 0.71)$ is a saddle, the fixed points E_1 and E_2 are stable foci.(b) Enlargement of a portion of fig.(a)

numerical and graphical investigations, guided by the knowledge of the critical curves, we can analyze the structure of the basins of the two coexisting stable Nash equilibria and we can characterize the bifurcations that cause their qualitative changes.

Proposition 2 shows that, as in the case of homogeneous beliefs, there exists a rather large set of parameter values for which two stable equilibria exist. Moreover, it is easy to realize that slight differences between the two adjustment coefficients do not introduce significant changes in the local stability properties, i.e. in the modulus of the eigenvalues. However, as will be demonstrated below, even small heterogeneities between the players may cause remarkable effects with regard to the structure of the basins. Indeed, many of the arguments given in the previous section for the study of the boundaries of the basins and their global bifurcations continue to hold in the case of heterogeneous beliefs. The main difference is that the diagonal Δ is no longer invariant and, even if the fixed points remain the same, the basins are no longer symmetric with respect to Δ.

First, it is still true that for $\mu > 1$ the preimages of the unstable fixed point O belong to the boundary that separates the basin of infinity from the set of points which generate bounded trajectories, but a simple analytical expression of the preimages of O cannot be obtained. However, they can be easily computed by standard numerical routines, as solutions of a fourth degree algrebraic equation.

Second, it is still true that for increasing values of μ or λ_i the point O enters the region Z_4, but the exact values of the parameters at which this occurs cannot be computed analytically.

Third, the boundary which separates the basins $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ is still formed by the whole stable set $W^s(S)$, but in the case $\lambda_1 \neq \lambda_2$ the local stable set $W^s_{loc}(S)$ is not along the diagonal Δ. The contact between $W^s(S)$ and $LC^{(b)}$, which causes the transition from simple to complex basins, does not occur at the fixed point O (since now $O \notin W^s(S)$) and no longer involves the cusp point of $LC^{(b)}$. Again, the parameter values at which such

contact bifurcations occur cannot be computed analytically. However, the bifurcation is always caused by a contact between LC and a basin boundary.

In what follows, we will demonstrate, however, that the occurrence of these bifurcations can be detected by computer-assisted proofs, based on the knowledge of the properties of the critical curves and their graphical representation (see e.g. Mira et al. [28] for many examples). This "modus operandi" is typical in the study of the global bifurcations of nonlinear two-dimensional maps.

In order to understand how complex basin structures are obtained, we start from a situation in which $W^s(S)$ has a simple shape, like the one shown in Fig. 8a. This figure has been obtained for the same value of the parameter $\mu = 3.5$ as the previous figures, but different speeds of adjustment for the two players, $\lambda_1 = 0.6$, $\lambda_2 = 0.8$. The introduction of an asymmetry in the adaptive behavior of the players has a negligible effect on the local stability properties, since the eigenvalues of the two fixed points are exactly the same and are very close to the ones obtained in the homogeneous case with the same value for μ and with $\lambda = (\lambda_1 + \lambda_2)/2$. On the other hand, it causes an evident asymmetry of the basins of attraction. As shown in Fig. 8a, when $\lambda_2 > \lambda_1$ the extension of $\mathcal{B}(E_2)$ is in general greater than the extension of $\mathcal{B}(E_1)$, and the complementary situation is obtained if λ_1 and λ_2 are swapped. Numerical simulations show that in general *the Nash equilibrium E_i dominates the other Nash equilibrium E_j in terms of the extension of the basin if $\lambda_i > \lambda_j$.* Furthermore, even in situations characterized by a simple structure of the basins' boundaries, like the one shown in Fig. 8a where both of the basins are connected sets, the statement that the initial order of the quantities is maintained along the whole trajectory is no longer true. In fact, in the case of different speeds of adjustment, say $\lambda_i > \lambda_j$, the typical occurrence is that the smaller basin $\mathcal{B}(E_j)$ is surrounded by points of $\mathcal{B}(E_i)$. Hence, the adjustment dynamic in our Cournot game may lead to convergence to E_i in the long run, even if players start with quantities which are closer to E_j. Thus, the initial order of the quantities would be reversed.

In the simple situation shown in Fig. 8a, the smaller basin $\mathcal{B}(E_1)$ is a simply connected set. The basin $\mathcal{B}(E_2)$ is a multiply connected set, due to the presence of a big "hole" (or "island", following Mira et al. [26]) nested inside it, whose points belong to $\mathcal{B}(E_1)$. Furthermore, $W^s(S)$, i.e. the boundary which separates the two basins, is entirely included inside the regions Z_2 and Z_0. However, the fact that in Fig. 8a a portion of $W^s(S)$ is close to LC suggests the occurrence of a global bifurcation. If the parameters are changed, so that a contact between $W^s(S)$ and LC occurs, this contact will mark a bifurcation which causes qualitative changes in the structure of the basins. If a portion of $\mathcal{B}(E_1)$ enters Z_4 after a contact with $LC^{(b)}$, new rank-1 preimages of that portion will appear near $LC^{(b)}_{-1}$, and such preimages must belong to $\mathcal{B}(E_1)$. Indeed, this is the situation shown in Fig. 8b, obtained after a small change of λ_1. The portion of $\mathcal{B}(E_1)$ inside Z_4 is denoted by H_0. It has two rank-1 preimages, denoted by $H^{(1)}_{-1}$ and $H^{(2)}_{-1}$, which are located at opposite sides with respect to $LC^{(b)}_{-1}$ and merge on it (in fact, by definition, the rank-1 preimages of the arc of $LC^{(b)}$ which bound H_0 must merge along $LC^{(b)}_{-1}$). The set $H_{-1} = H^{(1)}_{-1} \cup H^{(2)}_{-1}$ constitute a non connected portion of $\mathcal{B}(E_1)$. Moreover, since H_{-1} belongs to the region Z_4, it has four rank-1 preimages, denoted by $H^{(j)}_{-2}$, $j = 1,...,4$ in Fig. 8b, which constitute other four "islands" of $\mathcal{B}(E_1)$. Points of these "islands" are mapped into H_0 in two iterations of the map T. Indeed, infinitely many higher rank preimages of H_0 exist, thus giving infinitely

many smaller and smaller disjoint "islands" of $\mathcal{B}(E_1)$. Hence, at the contact between $W^s(S)$ and LC the basin $\mathcal{B}(E_1)$ is transformed from a simply connected into a non connected set, and LC the basin $\mathcal{B}(E_1)$ is transformed from a simply connected into a non connected set, constituted by infinitely many disjoint components. The larger connected component of $\mathcal{B}(E_1)$ which contains E_1 is called *immediate basin* $\mathcal{B}_0(E_1)$, and the whole basin is given by the union of the infinitely many preimages of $\mathcal{B}_0(E_1)$: $\mathcal{B}(E_1) = \bigcup_{k \geq 0} T^{-k}(\mathcal{B}_0(E_1))$.
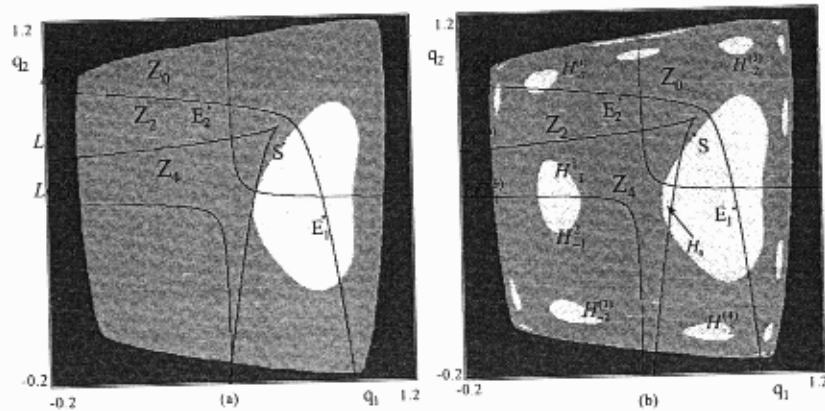

Figure 8: Case of heterogeneous behavior. (a) $\mu = 3.5$, $\lambda_1 = 0.6$, $\lambda_2 = 0.8$. (b) $\mu = 3.5$, $\lambda_1 = 0.65$, $\lambda_2 = 0.8$.

To sum up, also in the case of heterogeneous behavior, changes in the speeds of adjustment may cause global bifurcations, related to a contact between basin boundaries and critical curves. Such bifurcations change the qualitative structure of the basins, and give rise to a higher degree of uncertainty with respect to the possibility of forecasting the effects of small changes in the initial quantities on the long-run evolution of the duopoly game. However, note that due to the heterogeneous behavior of the two competing firms, the symmetry property of the dynamical system which allowed us to obtain a simple analytical expression of the global bifurcation value no longer holds. Hence, the occurrence of contact bifurcations can only be revealed numerically. This happens frequently in nonlinear dynamical systems of dimension greater than one, where the study global bifurcations is generally obtained through an interplay between theoretical and numerical methods.

3.2.3 More Complex Attractors

When the equilibria E_1 and E_2 lose stability through the Neimark-Hopf bifurcation occurring at $\mu = 1 + \sqrt{4 + \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}}$, two stable closed invariant curves are created around them, on which the trajectory exhibit a quasi-periodic motion. In this case, even if a Cournot tâtonnement starts very close to a Nash equilibrium, the trajectories move around it, without converging to any equilibrium point. However, the results on the delimitation of the basins, as well as the analysis of the contact bifurcations that change their structure, also hold in the presence of the more complex attractors which appear around the Nash equilibria

E_1 and E_2 when they lose their stability through the Neimark-Hopf bifurcation.

As the parameters move far from the bifurcation surface, in the region where no stable Nash equilibria exist, the amplitude of the two stable closed orbits increases and then more and more complex attractors appear, such as annular chaotic attractors (see Kopel [23], Gardini et al. [17]).

These changes in the shape and the properties of the attractors are independent of the changes in the boundaries of the basins and their bifurcations. In fact, complex attractors may exist with complex basins (see e.g. Gardini et al. [17]), but we may have simple attractors, such as stable fixed points, with complex structures of the basins (as demonstrated in the previous section), or complex attractors whose basins have boundaries with a simple structure. For example, in the case of homogeneous behaviors, we have simply connected and symmetric basins for $\mu > 3$ and $\lambda < 1/(\mu + 1)$, and the Nash equilibria E_i, $i = 1, 2$, lose their stability at $\lambda = 2/\left((\mu - 1)^2 - 4\right)$. Hence, for $\mu > 5$ and $2/\left((\mu - 1)^2 - 4\right) < \lambda < 1/(\mu + 1)$ we have coexisting attracting sets[2], say A_1 and A_2, which are more complex than fixed points, whose basins $\mathcal{B}(A_1)$ and $\mathcal{B}(A_2)$ are simply connected sets separated by the segment $OO_{-1}^{(1)}$ of the diagonal Δ, as stated in Proposition 3. This is the situation shown in Fig. 9, obtained with $\mu = 5.2$ and $\lambda = 0.16$.
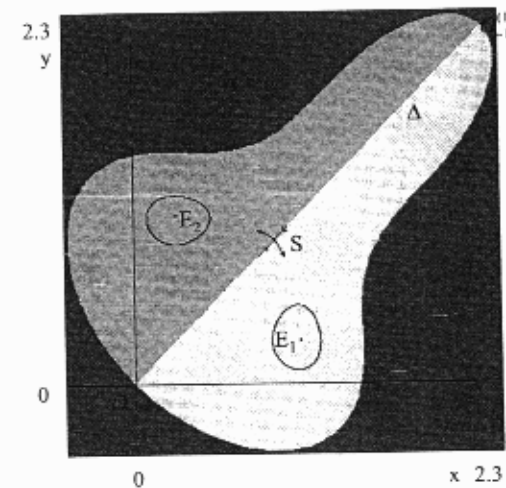

Figure 9: Case of homogeneous behavior with parameters $\mu = 5.2$ and $\lambda = 0.16$. The Nash equilibria E_1 and E_2 are unstable foci, and two quasi-periodic attractors exist around them, whose basins are simply connected sets represented by yellow and pale-blue regions respectively.

The creation and the qualitative changes of the attractors in the symmetric case as the parameters μ and λ are increased, have been extensively studied in the literature (see e.g. Kopel [23], and Gardini et al. [17] for the particular case $\mu = 4$).

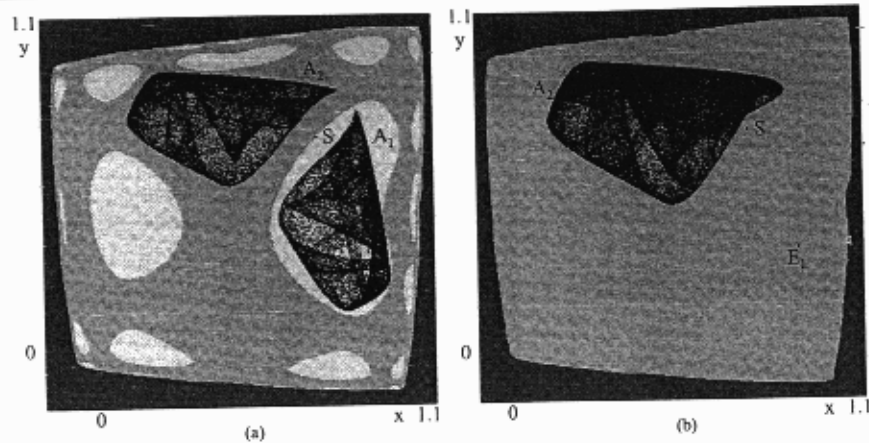[2]Notice that $2/\left((\mu - 1)^2 - 4\right) < 1/(\mu + 1)$ for $\mu > 5$

Figure 10: Case of heterogeneous behavior with $\lambda_1 = 0.7$ and $\lambda_2 = 0.8$. (a) $\mu = 3.9$. Two chaotic attractors A_1 and A_2 exist around the Nash equilibria E_1 and E_2 respectively. (b) $\mu = 3.95$. A unique chaotic attractor A_2 exist, to which all the bounded trajectories converge, which includes the Nash equilibrium E_2.

Here we are mainly interested in the effects of heterogeneous behavior on the properties of the attracting sets and, in particular, in situations where the asymmetry in the basins caused by the heterogeneity induces asymmetries in the structure of the attracting sets.

Consider, for example, the situation shown in Fig. 10a, obtained with $\mu_1 = \mu_2 = 3.9$, $\lambda_1 = 0.7$ and $\lambda_2 = 0.8$. In this case, two chaotic attractors A_1 and A_2 exist around the two Nash equilibria E_1 and E_2 respectively. According to Proposition 2, E_1 and E_2 are unstable foci with the same eigenvalues. Also the two chaotic attractors A_1 and A_2 appear to be very similar, despite of the difference between the two speeds of adjustment. In other words, even if the two firms are heterogeneous in our model, two trajectories τ_1 and τ_2 starting close to the Nash equilibria E_1 and E_2 respectively show similar evolutions in the sense that they depart from the respective Nash equilibria at the same rate (due to the identical eigenvalues). They then move erratically, covering the almost symmetric chaotic areas A_1 and A_2. However, Fig. 10a reveals an evident asymmetry in the structure of the basins. Indeed, $\mathcal{B}(A_1)$ is non connected and the boundary of the immediate basin $\mathcal{B}_0(A_1)$ is rather close to the boundary of the chaotic area A_1. Starting from this situation, a small change of some parameter will cause the occurrence of a contact between the chaotic attractor A_1 and the boundary of its immediate basin. Such a contact bifurcation causes the destruction of the chaotic attractor (Gumowski & Mira [20, 21]). Such a bifurcation is called *final bifurcation* in Mira et al. [28] and Abraham et al. [1] or *boundary crisis* in Grebogi et al. [18]. In our case, the contact between A_1 and $\partial \mathcal{B}_0(A_1)$ occurs, for example, by increasing μ. As shown in Fig. 10b, obtained with the same values of λ_1 and λ_2 as in Fig. 10a and a slightly increased value of μ, namely $\mu = 3.95$, after this contact bifurcation, the chaotic attractor A_1 disappears. More exactly, it becomes a chaotic repellor and the generic bounded trajectory converges to the attractor A_2, located around E_2, which remains the only bounded attracting set. This leads to a remarkable asymmetry in the structure of the

attractors, since the trajectories starting close to the Nash equilibrium E_2 remain around E_2, even if they exhibit an erratic motion, whereas the trajectories which start close to the other Nash equilibrium E_1 reach, in the long run, the chaotic attractor A_2 around E_2. Due to the presence of the "ghost" represented by the chaotic repellor, these trajectories converge to A_2 only after a relatively long chaotic transient in the region which was occupied by A_1.

A symmetric reasoning applies if we consider the case with $\lambda_1 > \lambda_2$. In this case the final bifurcation transforms the chaotic attractor A_2 into a chaotic repellor, whereas the attractor A_1 survives and attracts the generic bounded trajectory.

4   Conclusions

In this paper we have considered the structure of the basins of attraction in noninvertible maps and some global bifurcations which mark the route to more complex topological structures of the basins.

As an example, we have analyzed a Cournot Duopoly game in which the reaction functions are non-monotonic and it has been assumed that players do not adjust their quantity choices instantaneously to the Best Response, but exhibit some kind of inertial behavior. The assumption of non-monotonic reaction functions arises quite naturally in economic contexts and give rise to multiple coexisting Nash equilibria. In such cases an equilibrium selection problem arises and it becomes crucial to obtain some information on the stability extent of either of the equilibria. This information then enables us to assess the robustness of the Nash equilibria with respect to exogenous perturbations. Stability arguments are often used in economics to select among multiple equilibria, but this approach fails when multiple equilibrium assignments coexist. The selected equilibrium is then path-dependent in the sense that "the equilibrium predicted to emerge depends on the historical accident of the initial condition, rather than on deductive concepts of efficiency." (Van Huyck et al. [38], p. 484). We somehow made more precise what it means that eventually selected equilibria are "path-dependent", and how this selection process depends on "historical accidents".

Our study of the basins covers an aspect which is rather neglected in the economic dynamics literature, where the questions which traditionally are addressed are related to the local stability properties of equilibria and the creation of complex attractors when the steady states loose stability. In contrast to the existing work, we have shifted the emphasis to the study of the *global* stability properties of the Nash equilibria and global bifurcations that cause qualitative changes of the basins of attraction.

In fact, a study limited to the local stability analysis based on the linear approximation of the system, may be quite unsatisfactory. This is due to the fact that an analysis of local stability properties only determines the attractivity of a Nash equilibrium for games starting in some region around the equilibrium. However, such a region may be so small that every practical meaning of the (mathematical) concept of stability is lost. In these cases the study of the *stability extent*, that is the determination of the boundaries of the basin of attraction, can give a clear idea of the robustness of an equilibrium with respect to exogenous perturbations, since it permits one to understand if a given shock of finite amplitude can be recovered by the endogenous dynamics of the system, or if it will cause an irreversible departure from the Nash equilibrium. This aspect has been often left out, as an analysis of

the global properties of the dynamical system, i.e. an analysis not based on a linear approximation, is required. Instead, this aspect may be important for understanding the long-run behavior of any adjustment dynamic, since the convergence to one of several coexisting Nash equilibria crucially depends on the initially chosen quantities of the players.

We have shown that despite the local stability of coexisting Nash equilibria, the situation might become quite complicated as more and more complex basins with complicated topological structures (such as non connected sets formed by many disjoint portions) emerge as the adjustment speed (i.e. the extent of inertia of the players) changes. Such a route to complexity leads to some kind of unpredictability with respect to slight changes of initial conditions, because in the presence of complex basin boundaries a small change in the initial conditions may give a completely different long run evolution of the system if the change causes a crossing of some basin boundary. The route to more and more complex basin boundaries, as some parameter is varied, is characterized by global bifurcations, also called *contact bifurcations*. Such bifurcations can be rarely studied by analytical methods, since the analytical equations of such singularities are not known in general. Hence such an analysis is mainly performed by geometrical and numerical methods, based on theoretical results.

On the basis of the properties of noninvertible maps, we have explained why the basins of attraction may have very complex structures. We have shown that the bifurcations can be studied by using critical curves, a powerful tool for the investigation of the global behavior of noninvertible two-dimensional discrete dynamical systems. For the Cournot game with an inertial adjustment, the main qualitative changes of the structure of the basins of the equilibria can be explained through analytical and geometrical arguments, which are based on the knowledge of the critical curves and of the stable sets of the saddle fixed points or cycles.

In this paper we have only considered the case $\mu_1 = \mu_2$, where the parameters $\mu_i, i = 1, 2$ measure the extent of the positive externality the actions of one player exert on the payoff of the other player. That is, we somehow assumed that the game-theoretic situation we analyze is symmetric. The only source of heterogeneity of the players arose from a difference in the inertia of the players, i.e. different adjustment speeds. An asymmetric situation, in which $\mu_1 \neq \mu_2$, under the Best Response dynamic, $\lambda_1 = \lambda_2 = 1$ (instantaneous adjustment), has been analyzed in Bischi et al. [7], where it is shown that such a dynamic game is characterized by the coexistence of many periodic or chaotic attractors, with rather intermingled basins.

Appendix A. The attractivity of infinity

In order to prove that an attracting set of the map (6) exists at infinite distance, i.e. the iteration of (6) may generate diverging sequences, we show that

$$V(t) = \|(q_1(t), q_2(t))\| = q_1(t)^2 + q_2(t)^2$$

is diverging if it is computed along the trajectories which start from $(q_1(0), q_2(0))$ with $\|(q_1(0), q_2(0))\|$ sufficiently large.

From (6) it follows that for any set of parameters $\mu_i > 0$, $\lambda_i > 0$, $i = 1, 2$, we have $q'_1 < -M$

and $q'_2 < -M$ for a finite $M > 0$, provided that

$$q_1 > \frac{1}{2} \left[1 + \sqrt{1 + \frac{4}{\lambda_2 \mu_2} (1 - \lambda_2) q_2 + M} \right] \text{ and } q_2 > \frac{1}{2} \left[1 + \sqrt{1 + \frac{4}{\lambda_1 \mu_1} (1 - \lambda_1) q_1 + M} \right]$$

Furthermore, from (6) it can be immediately seen that if $q_1 < 0$ and $q_2 < 0$ then also $q'_1 < 0$ and $q'_2 < 0$. Hence, the negative cone

$$N = \{(q_1, q_2) | q_1 < 0 \text{ and } q_2 < 0\}$$

is a trapping set for the map (6).

By using the expression of the map (6) we obtain that along the trajectories of T

$$V(t+1) - V(t) = q_1'^2 + q_2'^2 - q_1^2 - q_2^2 =$$
$$\lambda_2^2 \mu_2^2 q_1^4 + \lambda_1^2 \mu_1^2 q_2^4 - 2\left(\lambda_2^2 \mu_2^2 q_1^3 + \lambda_1^2 \mu_1^2 q_2^3\right) + \left(\lambda_1 (\lambda_1 - 2) + \lambda_2^2 \mu_2^2\right) q_1^2 + \left(\lambda_2 (\lambda_2 - 2) + \lambda_1^2 \mu_1^2\right) q_2^2$$
$$- 2\lambda_1 (1 - \lambda_1) \mu_1 q_1 q_2^2 - 2\lambda_2 (1 - \lambda_2) \mu_2 q_1^2 q_2 + 2\left(\lambda_1 (1 - \lambda_1) \mu_1 + \lambda_2 (1 - \lambda_2) \mu_2\right) q_1 q_2$$

This expression is always positive if $(q_1, q_2) \in N$ and (q_1, q_2) is outside of the ellipse of equation

$$\lambda_1^2 \mu_1^2 (q_2 - 1)^2 + \lambda_2^2 \mu_2^2 (q_1 - 1)^2 = 2.$$

Appendix B. Local stability analysis of the fixed points

Here we analyze the local stability of the fixed points of the map (6) with $\mu_1 = \mu_2 = \mu$. Such an analysis can be carried out by studying the localization of the eigenvalues of the Jacobian matrix in the complex plane, i.e. the solutions of the characteristic equation

$$P(z) = z^2 - Tr \cdot z + Det = 0 , \tag{32}$$

where Tr and Det are the trace and the determinant of the Jacobian matrix (13) computed at the fixed point. A sufficient condition for the stability of a fixed point is that both the eigenvalues are inside the unit circle in the complex plane.

B.1. Homogeneous case

Under the assumptions (8) and (14) the Jacobian matrix becomes

$$DT(x, y) = \begin{bmatrix} 1 - \lambda & \lambda\mu(1 - 2y) \\ \lambda\mu(1 - 2x) & 1 - \lambda \end{bmatrix} \tag{33}$$

In the points of the diagonal Δ, DT assumes the structure

$$DT(x, x) = \begin{bmatrix} A & B \\ B & A \end{bmatrix} \tag{34}$$

with $A = 1 - \lambda$ and $B = \lambda\mu(1 - 2x)$. Such a matrix has real eigenvalues, given by

$$z_{\parallel} = A + B \quad \text{with eigenvector } \mathbf{r}_{\parallel} = (1, 1) \text{ along } \Delta$$
$$z_{\perp} = A - B \quad \text{with eigenvector } \mathbf{r}_{\perp} = (1, -1) \text{ perpendicular to } \Delta.$$

It is easy to see that the product of matrices with the structure (34) has the same structure. Hence all the fixed points and the cycles embedded in the invariant diagonal Δ have real eigenvalues with eigenvectors along Δ and perpendicular to Δ respectively.

For the symmetric equilibrium $S = (1 - 1/\mu, 1 - 1/\mu)$

$$z_\parallel(S) = 1 + \lambda(1-\mu) \text{ and } z_\perp(S) = 1 + \lambda(\mu-3)$$

Being

$-1 < z_\parallel(S) < 1$ for $0 < \lambda(\mu-1) < 2$;
$-1 < z_\perp(S) < 1$ for $-2 < \lambda(\mu-3) < 0$

S is a stable node in the region (17).

At $\mu = 3$, $z_\perp(S) = 1$, S loses stability in the direction transverse to Δ through a supercritical pitchfork bifurcation (see the remark below) at which the equilibria E_1 and E_2 are created for $\mu > 3$ and are stable just after the bifurcation. After such bifurcation S becomes a saddle point with unstable set in the direction transverse to Δ and local stable set along the invariant diagonal Δ. At $\lambda(\mu-1) = 2$ a flip bifurcation along Δ occurs at which S becomes a repelling node and a saddle cycle C_2 of period 2, whose periodic points (20) can be easily computed from the restriction of T to Δ, is created along the diagonal Δ, with stable set along Δ and unstable set transverse to it. The eigenvalues of C_2 are the eigenvalues of the matrix $DT(p_1, p_1) \cdot DT(p_2, p_2)$, given by

$$z_\parallel(C_2) = (1 - \lambda + \lambda\mu(1-2p_1))(1 - \lambda + \lambda\mu(1-2p_2)) = 5 - \lambda^2(\mu-1)^2$$

and

$$z_\perp(C_2) = (1 - \lambda - \lambda\mu(1-2p_1))(1 - \lambda - \lambda\mu(1-2p_2)) = (3 + 2\mu - \mu^2)\lambda^2 - 12\lambda + 13$$

We have $-1 < z_\parallel(C_2) < 1$ for $\frac{2}{\mu-1} < \lambda < \frac{\sqrt{6}}{\mu-1}$, and $0 < z_\perp(C_2) < 1$ for $0 < \lambda < \frac{6-\sqrt{12\mu(\mu-2)}}{3+2\mu-\mu^2}$.

Notice that $\frac{\sqrt{6}}{\mu-1} > 1$ for $\mu < 1 + \sqrt{6}$. At $\lambda = \lambda_p(\mu) = \frac{6-\sqrt{12\mu(\mu-2)}}{3+2\mu-\mu^2}$ we have $z_\perp(C_2) = 1$ and at $\lambda = \lambda_p$ it holds that $\frac{\partial z_\perp(C_2)}{\partial \lambda} = -2\sqrt{12\mu(\mu-2)} < 0$.

So, C_2 becomes a stable node for $\lambda > \lambda_p$ and two saddle cycles of period 2 are created through a subcritical pitchfork bifurcation (see the remark below).

In the fixed points E_1 and E_2 the Jacobian matrix (33) is given by

$$DT(E_1) = \begin{bmatrix} A & B_1 \\ B_2 & A \end{bmatrix} \text{ and } DT(E_2) = \begin{bmatrix} A & B_2 \\ B_1 & A \end{bmatrix}$$

respectively, with $B_1 = -\lambda(1 - \sqrt{(\mu+1)(\mu-3)})$ and $B_2 = -\lambda(1 + \sqrt{(\mu+1)(\mu-3)})$. It is easy to see that E_1 and E_2 have the same characteristic equation because the two matrices $DT(E_i)$, $i = 1, 2$, have the same trace and determinant. Being $Tr^2 - 4Det = 4\lambda^2(4 + 2\mu - \mu^2)$ the eigenvalues are real for $\mu \leq 1 + \sqrt{5}$ and complex for $\mu > 1 + \sqrt{5}$. It is easy to verify that at $\lambda(\mu^2 - 2\mu - 3) = 2$ the eigenvalues exit the unit circle, so that the region of stability of both equilibria E_i, $i = 1, 2$, is (18). Furthermore, the two fixed points are transformed from stable to unstable foci through a supercritical Neimark-Hopf bifurcation at which two stable closed orbits are created around the two unstable Nash equilibria E_1 and E_2.

Remark. A rigorous proof of the supercritical or subcritical nature of a Neimark-Hopf, or Pitchfork, bifurcation requires a center manifold reduction and the evaluation of higher order derivatives, up to the third order (see e.g. Guckenheimer and Holmes [19]). This is rather tedious in

a two-dimensional map, and we claim numerical evidence in order to ascertain the nature of such bifurcations.

B.2. Non homogeneous case

Under assumption (8), the Jacobian matrix becomes

$$DT(x,y) = \begin{bmatrix} 1-\lambda_1 & \lambda_1\mu(1-2y) \\ \lambda_2\mu(1-2x) & 1-\lambda_2 \end{bmatrix} \tag{35}$$

The system of inequalities (see e.g. Gumowski and Mira 1980, p.159)

$$P(1) = 1 - Tr + Det > 0; \quad P(-1) = 1 + Tr + Det > 0; \quad 1 - Det > 0$$

gives necessary and sufficient conditions for the two eigenvalues to be inside the unit circle of the complex plane.

At $S = (1 - 1/\mu, 1 - 1/\mu)$ we have
$Tr^2 - 4Det = \lambda_1^2 + \lambda_2^2 + 14\lambda_1\lambda_2 + 4\lambda_1\lambda_2\mu(\mu-4) \geq (\lambda_1 - \lambda_2)^2 \geq 0$, being $\mu(\mu-4) \geq -4$.
So the eigenvalues are always real at the fixed point S, and the stability conditions reduce to
$P(1) = \lambda_1\lambda_2(-\mu^2 + 4\mu - 3) > 0$ for $1 < \mu < 3$;
$P(-1) = \lambda_1\lambda_2\mu^2 - 4\lambda_1\lambda_2\mu + 3\lambda_1\lambda_2 + 2(\lambda_1 + \lambda_2) - 4 > 0$ for $\mu < 2 + \sqrt{1 + 2\frac{2-(\lambda_1+\lambda_2)}{\lambda_1\lambda_2}}$.
Then, S is a stable node in the region (21).

At $\mu = 1$ a transcritical bifurcation occurs at which O and S exchange stability, and at $\mu = 3$ a pitchfork bifurcation of S occurs at which the fixed points E_1 and E_2 are created.
Since

$$DT(E_1) = \begin{bmatrix} 1-\lambda_1 & -\lambda_1\left(1 - \sqrt{(\mu+1)(\mu-3)}\right) \\ -\lambda_2\left(1 + \sqrt{(\mu+1)(\mu-3)}\right) & 1-\lambda_2 \end{bmatrix}$$

and

$$DT(E_2) = \begin{bmatrix} 1-\lambda_1 & -\lambda_1\left(1 + \sqrt{(\mu+1)(\mu-3)}\right) \\ -\lambda_2\left(1 - \sqrt{(\mu+1)(\mu-3)}\right) & 1-\lambda_2 \end{bmatrix}$$

it is easy to realize that E_1 and E_2 have the same characteristic equation. The fixed points E_i are transformed from stable nodes into stable foci when

$$Tr^2 - 4Det = -4\lambda_1\lambda_2\mu^2 + 8\lambda_1\lambda_2\mu + 14\lambda_1\lambda_2 + \lambda_1^2 + \lambda_2^2 = 0$$

i.e. at $\mu = 1 + \sqrt{\frac{9}{2} + \frac{\lambda_1}{4\lambda_2} + \frac{\lambda_2}{4\lambda_1}}$. In this case, since $P(1) = \lambda_1\lambda_2(\mu+1)(\mu-3) > 0$ for $\mu > 3$ and $P(-1) = 4 - 2(\lambda_1 + \lambda_2) + \lambda_1\lambda_2(\mu+1)(\mu-3) > 0$ for $\mu > 3$, the stability conditions for E_i, $i = 1, 2$, reduce to

$$Det - 1 = \lambda_1\lambda_2\mu^2 - 2\lambda_1\lambda_2\mu - 3\lambda_1\lambda_2 - \lambda_1 - \lambda_2 < 0$$

which is satisfied in the region $\Omega_3^s(E_1)$ of the parameters space Ω_3. The equation

$$\mu = 1 + \sqrt{4 + \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2}}$$

defines a bifurcation surface through which a Neimark-Hopf bifurcation occurs.

Acknowledgments

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PHASE-PORTRAIT IN CONVERSATION-PROCESSES

Giovanna Morgavi

Institute of Electronics and Engineering of Information and Telecommunication ,
National Research Council, via De Marini 6, 16149 Genova Italy

Valentina Florini

Centro di Promozione Culturale per la Prima infanzia, "Il Piccolo Principe",
Comune di Genova,, 16129 Genova, Italy

ABSTRACT

Our goal was to extract information on communicative process evolution avoiding simplification and classification. We analysed 50 motivational research interview made from students during their university course. The nature intrinsically interactive of the dialogue concretises, shapes and evolves within time dimension. A reciprocal adaptation, where each partner learns, step by step, to lead in the interlocutor's reference frame, without quitting its own, turns into a common system exceeding those of both fellow. Our approach founds on analogies between conversation processes and chaotic systems: even in conversation, time evolution shapes, defines and characterises the process. During the interaction, the turn alternation is fundamental, specially when the mutual definition of the relationship involves the acknowledgement of different roles. Through word counting, we estimated the " conversation process phase-portrait". This procedure allowed information extraction on the communication evolution: plots with anomalous paths indicate situations where the communication has been troubled from external references.

1. INTRODUCTION

The conversation is the most common interaction process in the daily life: we were born in essentially dialogic and linguistic universe and our whole relationship history is, also, a continuous learning of the art to cope with conversations in an effective way. Nevertheless, this process is unconscious: our senses are used to live what is manifest without any question. A reflection on an aspect usually regarded as obvious seems particularly interesting. What is self-evident escapes from inquiry; however its characteristic measurement, its components evaluation were the incentive and the aim of our work.

The theory of the communication process suggests some rules in the conduction of a psychological interview: it underlines the turn importance, the need of giving space to the interviewee, the weight of the usage of a uniform language. In practice, these rules are often