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PLANE MAPS WITH DENOMINATOR. PART II: NONINVERTIBLE MAPS WITH SIMPLE FOCAL POINTS

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This paper is the second part of an earlier work devoted to the properties specific to maps of the plane characterized by the presence of a vanishing denominator, which gives rise to the generation of new types of singularities, called *set of nondefinition, focal points* and *prefocal curves.* A *prefocal curve* is a set of points which are mapped (or "focalized") into a single point, called *focal point*, by the inverse map when it is invertible, or by at least one of the inverses when it is noninvertible. In the case of noninvertible maps, the previous text dealt with the simplest geometrical situation, which is nongeneric. To be more precise this situation occurs when several focal points are associated with a given prefocal curve. The present paper defines the generic case for which only one focal point is associated with a given prefocal curve. This is due to the fact that only one inverse of the map has the property of focalization, but with properties different from those of invertible maps. Then the noninvertible maps of the previous Part I appear as resulting from a bifurcation leading to the merging of two prefocal curves, without merging of two focal points.

Keywords: Noninvertible maps; vanishing denominator; focal points; prefocal sets.

1. Introduction

In a previous paper [Bischi *et al.*, 1999] (Part I henceforth), we studied some global dynamical properties of two-dimensional maps T, related to the presence of a denominator which vanishes in a one-dimensional subset of the plane. Differently from the present paper, which is limited to the case of noninvertible maps, in Part I we considered both invertible and noninvertible maps. We remind that the phase space of a noninvertible map is subdivided into open regions (or zones) Z_k, whose points have k distinct rank-1 preimages, obtained by the application of k distinct inverses of the map. A specific feature of noninvertible maps is the existence of the *critical set LC* (from "Lignes Critique", see [Gumowski & Mira, 1980; Mira *et al.*, 1996]) defined as the locus of points having at least two coincident rank-1 preimages, located on the set of merging preimages denoted by LC_{-1}, $LC = T(LC_{-1})$. Segments of LC are boundaries that separate different regions Z_k, but the converse is not generally true, that is, also in the two-dimensional case, as in the one-dimensional one, boundaries of regions Z_k which are not portions of LC may exist.

In Part I we have evidenced some new kinds of global bifurcations, due to the contacts of two singular sets of different nature, whose description requires the definition of new concepts which are specific of maps defined by functions having a vanishing denominator, like the *set of nondefinition*, the *focal points* and the *prefocal curves*. Roughly speaking, a *prefocal curve* is a set of points which are mapped (or "focalized", as we shall say for short) into a single point, called *focal point*, by the inverse function (if the map is invertible) or by at least one of the inverses (if the map is noninvertible). Such global bifurcations cause the creation of structures of the basins which are peculiar of maps with a vanishing denominator, called *lobes* and *crescents*, and have been explained in terms of contacts between basin boundaries and prefocal curves (see also [Mira, 1999; Bischi & Gardini, 1999; Bischi *et al.*, 2001a]). These structures have been recently observed in discrete dynamical systems of the plane arising in different contexts, see e.g. [Yee & Sweby, 1994; Billings & Curry, 1996; Bischi & Naimzada, 1997; Brock & Hommes, 1997; Billings *et al.*, 1997; Gardini *et al.*, 1999; Bischi *et al.*, 2001b].

These properties were studied by considering the image of an arc crossing through a *focal point*, and it has been shown that, in the generic case, given by a focal point which is simple (i.e. located at a transverse intersection of the curve of vanishing denominator and that of vanishing numerator) a one-to-one correspondence is obtained between the slopes of the arcs through a focal point and the points in which their images cross the corresponding prefocal set. This implies that the preimages of any curve crossing the prefocal set in two points includes a loop with a knot in the focal point, and this is the basic mechanism leading to the formation of *lobes* and *crescents*. In the case of noninvertible maps, the previous text was not sufficiently clear in the sense that it dealt with the simplest geometric situation which lead us to improperly indicate it as "generic" in the title. To be more precise, the situation described in Part I occurs when several focal points are associated with a given prefocal curve. This gives a simple geometric situation for the prefocal set, which may be considered as the result of a bifurcation characterized by the merging of two prefocal curves, without merging of focal points, as we shall see in the present paper. So, the noninvertible maps considered in Part I concern nongeneric situations in the family of maps having a vanishing denominator.

In the case of two-dimensional noninvertible maps, this paper will investigate more deeply the properties of two-dimensional iterated maps associated with the existence of *simple* focal points and related prefocal sets. The role played by the existence of two or more inverses at a point of a prefocal set is not obvious. In fact, from the definitions of focal points and prefocal sets it follows that at least one inverse must exist, and we shall call it the "*focalizing inverse*". But which one is it? And what is its geometric action? Consider the simplest case in which only two inverses exist. We shall see that, generally, the focalizing inverse is only one along the whole prefocal set, but not always the same. If two focalizing inverses exist along a prefocal set (like in the maps studied in Part I) then this situation may be the result of the merging of two prefocal sets. Furthermore, when two distinct focal points exist, the related prefocal sets may be either different sets (generic case considered now) or the same set (bifurcation case considered in Part I). This qualitative difference is associated with a change of structure in the Riemann foliation of the plane (for the definitions and properties related to the Riemann foliation of two-dimensional noninvertible maps we refer to [Mira *et al.*, 1996a; Mira *et al.*, 1996b]).

So the present Part II essentially deals with the generic case of noninvertible maps for which only one focal point is associated with a given prefocal curve, i.e. when only one inverse of the map has the property of focalization, with properties different from those of invertible maps. More particularly, we shall see that the properties of the critical curves of the map are closely related to the presence of focal points and prefocal sets. As we shall see in the present paper, the focal points generally belong to the closure of the critical curve denoted by LC_{-1}, and consequently the critical curve LC is tangent to the prefocal sets.

These properties of simple focal points and related prefocal sets are important in order to understand the dynamics of iterated maps which are not defined in the whole plane. For example, the appearance/disappearance of the particular structures called *lobes* and *crescents*, may be related to the critical curves of the map, as shown in Part I, but also to the simple crossing of invariant phase curves through prefocal sets, as we shall see in this work. Other new results given in this paper concern the geometric action of the preimages, and the foliation structure, in the region between two prefocal sets, and several kinds of new bifurcation mechanisms related to arcs crossing the prefocal sets giving rise to loops and arcs through the focal points.

The paper is organized as follows. In Sec. 2 we remind some definitions related to the maps having a vanishing denominator, distinguishing between invertible maps and noninvertible ones. In particular, in Sec. 2.3 we shall see some relations between the critical set LC_{-1} and other sets, such as the set in which the Jacobian determinant vanishes, and the set of points across which a change in sign of the Jacobian determinant may occur, as well as its relation with the focal points of a map. We show that in the generic case the focal points belongs to $\overline{LC_{-1}}$, the closure of LC_{-1}. In Sec. 3 we stress some properties of the simplest noninvertible maps (of $Z_0 - Z_2$ type) having simple focal points. First, in Sec. 3.2, we discuss the case of focal points that do not belong to $\overline{LC_{-1}}$. This may also be considered as a bifurcation case with respect to the situations described in Sec. 3.3, where maps with focal points belonging to $\overline{LC_{-1}}$ are considered. In the latter case, which is the one mainly considered in the present paper, we show the particular role of the two inverses, which is at the basis of the particular geometric behavior associated with the focal points, and the foliation associated with the prefocal sets, which is peculiar of such kind of maps. Section 4 deals with a new kind of bifurcation which gives rise to the formation of *crescents* between two focal points. The results of the previous sections are then evidenced through the numerical examples of Sec. 5.

This paper will be followed by another one, Part III (to appear), where we shall consider the effects of the merging of two or more focal points, which thus satisfy the condition of "*nonsimple*" focal points, and some of the related bifurcations.

2. Definitions and Basic Properties

2.1. *Data in common with invertible and noninvertible maps*

In order to simplify the exposition, we assume that only one of the two functions defining the map T has a denominator which can vanish. Let

$$T : \begin{cases} x' = F(x, y) \\ y' = G(x, y) = \dfrac{N(x, y)}{D(x, y)} \end{cases} \tag{1}$$

where x and y are real variables, $F(x, y)$, $N(x, y)$ and $D(x, y)$ are continuously differentiable functions defined in the whole plane $\mathbb{R}^2$ (and without common factors). Then the *set of nondefinition* of the map T (that is, the set of points where at least one denominator vanishes) is given by

$$\delta_s = \{(x, y) \in \mathbb{R}^2 | D(x, y) = 0\}. \tag{2}$$

As in Part I we assume that δ_s is a smooth curve, or made up of pieces of smooth curves of the plane. The two-dimensional recurrence obtained by the successive iterations of T is well defined provided that the initial condition belongs to the set E given by $E = \mathbb{R}^2 \setminus \bigcup_{k=0}^{\infty} T^{-k}(\delta_s)$, where $T^{-k}(\delta_s)$ denotes the set of the rank-k preimages of δ_s, i.e. the set of points which are mapped into δ_s after k applications of T ($T^0(\delta_s) \equiv \delta_s$). Indeed, in order to generate uninterrupted sequences by the iteration of the map T, the points of δ_s, as well as all their preimages of any rank, constitute a set of zero Lebesgue measure which must be excluded from the set of initial conditions so that $T : E \to E$.

In order to analyze the role of the set of nondefinition δ_s, we consider a bounded and smooth simple arc γ, parametrized as $\gamma(\tau)$, transverse to δ_s, such that $\gamma(0) = (x_0, y_0)$ and $\gamma \cap \delta_s = \{(x_0, y_0)\}$, and we study its image $T(\gamma)$. As $(x_0, y_0) \in \delta_s$ we have, according to the definition of δ_s, $D(x_0, y_0) = 0$, but in general $N(x_0, y_0) \neq 0$. Hence

$$\lim_{\tau \to 0\pm} T(\gamma(\tau)) = (F(x_0, y_0), \infty) \tag{3}$$

where ∞ means either $+\infty$ or $-\infty$. This means that the image $T(\gamma)$ is made up of two disjoint unbounded arcs asymptotic to the line of equation $x = F(x_0, y_0)$. A different situation may occur if the point $(x_0, y_0) \in \delta_s$ is such that not only the denominator but also the numerator of $G(x, y)$ vanishes in it, i.e. $D(x_0, y_0) = N(x_0, y_0) = 0$. In this case, the second component of the limit (3) takes the form 0/0. This implies that this limit may give rise to a finite value, so that the image $T(\gamma)$ is a bounded arc crossing the line $x = F(x_0, y_0)$ at the point $(F(x_0, y_0), y)$, where

$$y = \lim_{\tau \to 0} G(x(\tau), y(\tau)) \tag{4}$$

It is clear that the value y in (4) must depend on the arc γ. Furthermore it may have a finite value along some arcs and be infinite along other ones. This leads us to the following definition of the singular sets of *focal point* and *prefocal curve* (given in Part I):

Definition. A point $Q = (x_0, y_0)$ is a focal point of a two-dimensional map T if at least one component of T takes the form 0/0 in Q and there exist smooth simple arcs $\gamma(\tau)$, with $\gamma(0) = Q$, such that $\lim_{\tau\to 0} T(\gamma(\tau))$ is finite. The set of all such finite values, obtained by taking different arcs $\gamma(\tau)$ through Q, is the prefocal set δ_Q.

In this paper we shall only consider *simple focal points*, i.e. points which are simple roots of the algebraic system

$$N(x, y) = 0\,, \quad D(x, y) = 0$$

Thus a focal point $Q = (x_0, y_0)$ is simple if

$$\overline{N}_x\overline{D}_y - \overline{N}_y\overline{D}_x \neq 0 \tag{5}$$

where $\overline{N}_x = (\partial N/\partial x)(x_0, y_0)$ and analogously for the other partial derivatives. In the case of a simple focal point there exists a one-to-one correspondence between the point $(F(Q), y)$, in which $T(\gamma)$ crosses δ_Q, and the slope m of $\overline{\gamma}$ in Q:

$$m \to (F(Q),\, y(m)),\ \text{with}\ y(m) = \frac{\overline{N}_x + m\overline{N}_y}{\overline{D}_x + m\overline{D}_y} \tag{6}$$

and

$$(F(Q),\, y) \to m(y)\ \text{with}\ m(y) = \frac{\overline{D}_x y - \overline{N}_x}{\overline{N}_y - \overline{D}_y y}\,. \tag{7}$$

As shown in Part I, these relations can be obtained by using a method either based on a series expansion of the functions $N(x, y)$ and $D(x, y)$ in a neighborhood of $Q = (x_0, y_0)$, or by considering the Jacobian determinant of the inverse map T^{-1} (or one of the inverses if the map is noninvertible).

If Q is simple, the point $(F(Q), y)$ spans the whole line $x = F(Q)$ as m varies. In fact, if $\overline{D}_y \neq 0$ then the graph of $y(m)$ is an hyperbola, so that for any $m \in \mathbb{R}$, $m \neq -\overline{D}_x/\overline{D}_y$, $y(m) \in \{\mathbb{R}\backslash y^*\}$, with $y^* = \overline{N}_y/\overline{D}_y$. Moreover, if $m \to -\overline{D}_x/\overline{D}_y$ then $y(m) \to \pm\infty$ (i.e. for γ tangent to δ_s, $T(\gamma)$ is unbounded with δ_Q as asymptote) and as $m \to \pm\infty$ then $y(m) \to y^*$ (i.e. the image of an arc γ with vertical tangent in Q is mapped by T into an arc which crosses the line δ_Q in $(F(Q), y^*)$. Furthermore, if $\overline{D}_y = 0$ then $y(m)$ is a linear function of m so that its range is $\mathbb{R}$. So, we can state the following proposition:

Proposition. *Let T be the map defined in* (1). *If Q is a simple focal point of T then the related prefocal set δ_Q is given by the whole line $x = F(Q)$.*

In this paper we shall consider only simple focal points and the structure of the plane foliation related to them, whereas the case of focal points which are not simple, as may occur in bifurcations which have been called class two in Part I, will be considered elsewhere (in Part III).

From the definition of the prefocal curve, it follows that the Jacobian determinant of the inverse of T, $\det(DT^{-1})$, if T is invertible, or of at least one of the inverses, if T is noninvertible, must necessarily vanish in the points of δ_Q. Indeed, if the map T^{-1} is defined in δ_Q, then all the points of the line δ_Q are mapped by at least one inverse T^{-1} into the focal point Q. *This means that at least one inverse T^{-1} is not locally invertible in the points of δ_Q*, it being a many-to-one map, and this implies that its Jacobian cannot be different from zero in the points of δ_Q.

From the relations (6) and (7) it follows that different arcs γ_j, crossing through a focal point Q with different slopes m_j, are mapped by T into bounded arcs $T(\gamma_j)$ crossing δ_Q at different points $(F(Q), y(m_j))$, see the qualitative picture in Fig. 1.

Interesting properties are obtained if the inverse (or inverses) of T is (are) applied to a curve that crosses a prefocal set. This is the basic mechanism which leads to the creation of lobes, a feature which often characterizes the structure of the basins of maps with denominator (see e.g. [Bischi & Gardini, 1997; Bischi & Naimzada, 1997; Billings *et al.*, 1997;

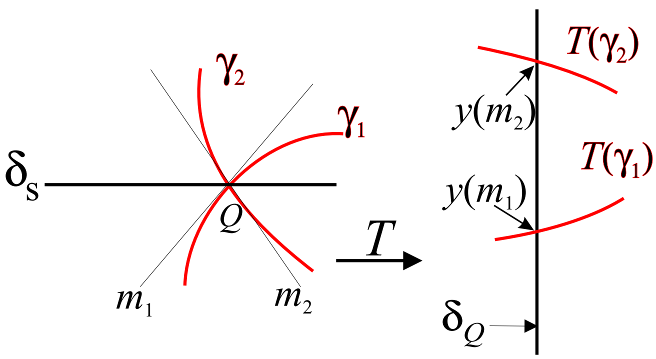


Fig. 1. Qualitative sketch to describe how different arcs crossing through a focal point Q, with different slopes, are mapped into bounded arcs crossing the corresponding prefocal curve δ_Q at different points. δ_s is the line of nondefinition (set where the denominator vanishes).

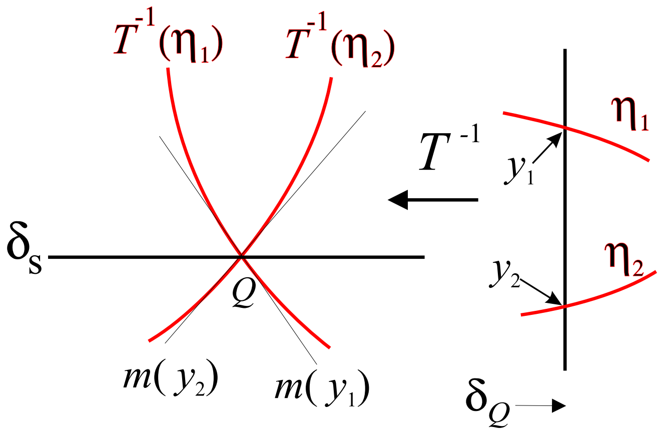
Brock & Hommes, 1997; Gardini *et al.*, 1999; Bischi *et al.*, 1999]). As we shall see, the properties of the inverses are different for invertible and noninvertible maps, so that we shall describe them separately. We first remind the properties of an invertible map, and then we shall consider those of a noninvertible one, which is the main object of this paper.

2.2. *Properties specific to invertible maps*

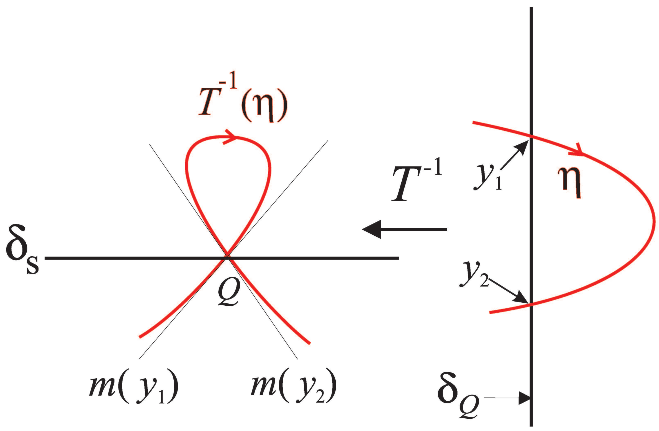
Let the map T be invertible, and let δ_Q be a prefocal curve whose corresponding focal point is Q. As noted above, the inverse map is not locally invertible in the points of the prefocal set δ_Q, thus we consider T as an invertible map in its domain of definition $(\mathbb{R}^2 \backslash \delta_s)$ and its range cannot include the prefocal set δ_Q. However, it is clear that each point sufficiently close to δ_Q has its rank-1 preimage in a neighborhood of the focal point Q. Moreover, if the inverse T^{-1} is defined and continuous along δ_Q (as generally occurs) then all the points of δ_Q are mapped by T^{-1} in the focal point Q. Roughly speaking we can say that the prefocal curve δ_Q is "*focalized*" by T^{-1} in the focal point Q, or, more concisely, that $T^{-1}(\delta_Q) = Q$. We stress again however that the map T is not defined in Q, thus T^{-1} cannot to be strictly considered as "an inverse of T in the points of δ_Q", even if T^{-1} is defined in δ_Q.

The relation (7) implies that the preimages of different arcs crossing the prefocal curve δ_Q in the same point $(F(Q), y)$ are given by arcs crossing the singular set through Q, all with the same slope $m(y)$ in Q. Indeed, if we consider different arcs η_j, crossing δ_Q in the same point $(F(Q), y)$ with different slopes, then these arcs are mapped by the inverse T^{-1} into different arcs $T^{-1}(\eta_j)$ through Q, all with the same tangent of slope $m(y)$, according to (7); they differ by the curvature at the point Q. Clearly, arcs crossing through δ_Q in different points y_i have preimages which cross the focal point Q in arcs with different slopes $m(y_i)$ according to (7), see the qualitative picture in Fig. 2(a). On the other hand, if we consider an arc crossing through δ_Q in two points y_1 and y_2 then its preimage is necessarily an arc with a knot in the focal point, with different tangents $m(y_i)$ according to (7), see the qualitative picture in Fig. 2(b). We can so state the following

Proposition 2.1. *Let T be an invertible map, Q a focal point of T with prefocal set δ_Q; then any arc crossing the prefocal set δ_Q in two points has a preimage which includes a loop with knot in Q.*



(a)



(b)

Fig. 2. (a) Arcs crossing through the prefocal curve δ_Q at different points have preimages which cross the focal point Q through arcs with different slopes. (b) The preimages of an arc crossing δ_Q at two points form an arc with a knot in the focal point.

As shown in Part I, such kind of loops with knots in Q are very important when studying, for example, the structure of basins and their boundaries. In fact, if the arc η belongs to a basin boundary, then the loops are responsible for the formations of the particular areas called "lobes", as we shall also see in the examples of the last section. Thus the contacts of a basin boundary with the prefocal set δ_Q may cause the occurrence of particular structures.

How these properties persist or change when T is not invertible is the object of Sec. 3, while in the next subsection we give some basic definitions and properties concerning the case of noninvertible maps.

2.3. *Properties specific to noninvertible maps*

We remind here some definitions and properties specific to noninvertible maps and also give some new insight about the relations between critical sets and focal points. In the case of a noninvertible map, the phase space is divided into open regions (or zones) Z_k, whose points have k distinct rank-1 preimages, obtained by the application of k distinct inverse maps (see e.g. [Gumowski & Mira, 1980; Mira *et al.*, 1996a]). In other words, if $(x, y) \in Z_k$ then k distinct points (x_j, y_j) are mapped into (x, y), i.e. $T(x_j, y_j) = (x, y)$, $j = 1, \ldots, k$, or equivalently we can say that k distinct inverse maps T_j^{-1} exist such that $T_j^{-1}(x, y) = (x_j, y_j)$, $j = 1, \ldots, k$. A specific feature of noninvertible maps is the existence of the *critical set LC* (from the French "Ligne Critique", see [Gumowski & Mira, 1980]), *defined as the locus of points having at least two coincident rank-1 preimages, which belong to the set of merging preimages, denoted by* LC_{-1}. In any neighborhood of a point of LC_{-1} there are at least two distinct points mapped by T into the same point, so that the map T is not locally invertible in the points of LC_{-1}, and this implies that for differentiable maps the set LC_{-1} is included in the set J_0 of points in which the Jacobian of T vanishes:

$$LC_{-1} \subseteq J_0 \text{ where } J_0 = \{(x, y) \in \mathbb{R}^2 | \det DT = 0\} \tag{8}$$

of course $LC = T(LC_{-1})$.

Segments of LC are boundaries that separate different regions Z_k, but the converse is not generally true, that is boundaries of regions Z_k, which are not portions of LC, may exist. This happens, for example, in polynomial maps having an inverse function with a vanishing denominator. In fact, as shown in Part I, this may imply the existence of a set which is mapped by T in one point, and such a set must belong to J_0 even if it is not critical, so that we have a strict inclusion: $LC_{-1} \subset J_0$.

Another distinguishing feature in many noninvertible maps is the existence of a set of points, which we shall denote by J_C, through which we have a change in the sign of the Jacobian of T, det DT. From the geometric action of the foliation of the Riemann plane we can also say that the critical set LC_{-1} must belong to J_C. In fact, a plane region U which intersects LC_{-1} is "folded" along LC into the side with more preimages, and the two folded images have opposite orientation. This implies that the map has the Jacobian with a different sign in the two portions of U separated by LC_{-1}, so that

$$LC_{-1} \subseteq J_C\,.$$

Previous results (for example see [Mira *et al.*, 1996a]) have shown that for a smooth map *without* a vanishing denominator we often have $LC_{-1} = J_C \subseteq J_0$, the strict inclusion occurring when there are sets mapped into a point (as shown in Part I of this paper), and when there are points X such that $J_0(X) = 0$ with the map T locally invertible at X (in the one-dimensional case such a point X is an inflection point). In the case of a smooth map *with* a vanishing denominator the situation is more complex. Indeed it can be seen (an example will be given in Sec. 5.3) that the set J_C may include also the singular set δ_S, although this can be considered as a bifurcation case, so that it occurs with the strict inclusion $LC_{-1} \subset J_C$.

So it is plain that $LC_{-1} \subseteq J_C \cap J_0$. However, from the geometric properties of T, we conjecture that the following relation holds:

$$LC_{-1} = J_C \cap J_0\,. \tag{9}$$

With continuously differentiable *maps without a vanishing denominator*, we may have $LC_{-1} = J_C \subset J_0$, although $LC_{-1} = J_C = J_0$ often occurs in the simplest situations. In smooth *maps with a vanishing denominator* $LC_{-1} = J_0 \subset J_C$ may occur. All these are particular cases of (9).

For the class of maps (1) considered in this paper, the Jacobian matrix is given by

$$DT(x, y) = \begin{bmatrix} F_x & F_y \\ \dfrac{N_x D - N D_x}{D^2} & \dfrac{N_y D - N D_y}{D^2} \end{bmatrix} \tag{10}$$

and we shall assume that F_y is not identically zero, i.e. we do not consider the particular case of a triangular map T (whose properties are often related to those of a one-dimensional map), because we are interested in true two-dimensional properties, and in particular we shall assume that $\overline{F}_y$ evaluated at a focal point Q is different from zero.

The Jacobian determinant is

$$\det DT(x, y) = \frac{A(x, y)}{D^2} \tag{11}$$

where

$$A(x, y) = [F_x(N_y D - N D_y) - F_y(N_x D - N D_x)] \tag{12}$$

DT is defined in $\mathbb{R}^2\backslash\delta_s$, and LC_{-1} is included in the set of points in which $A(x, y)$ vanishes,

$$J_0 = \{(x, y) \in \mathbb{R}^2 | A(x, y) = 0\} \cap \{\mathbb{R}^2\backslash\delta_s\}\,. \quad (13)$$

As $N = 0$ and $D = 0$ at a focal point, from $A(x, y) = 0$ in (12) it follows that *in general the focal points belong to the closure* $\overline{LC_{-1}}$ of LC_{-1} (not defined at these points). Moreover, we can easily compute the slope of the tangent to $\overline{LC_{-1}}$ in a simple focal point Q. In fact, the partial derivative with respect to y, computed at Q, of the expression $A(x, y)$ in (12) is $\overline{F}_y(\overline{N}_y\overline{D}_x - \overline{N}_x\overline{D}_y)$, and the partial derivative with respect to x, computed at Q, is $\overline{F}_x(\overline{N}_y\overline{D}_x - \overline{N}_x\overline{D}_y)$. Hence, the slope of the tangent to $\overline{LC_{-1}}$ at a *simple* focal point Q (for which (5) holds) is

$$\overline{m} = -\frac{\overline{A}_x}{\overline{A}_y} = -\frac{\overline{F}_x}{\overline{F}_y}\,. \quad (14)$$

We also note that when $F(x, y)$ is a linear or an affine function, then the partial derivatives are constant and the slope in (14) is independent on the focal point Q (i.e. it has the same value for each focal point), so that the tangents to $\overline{LC_{-1}}$ in all the focal points are parallel.

The fact that the closure of LC_{-1} intersects the singular set δ_S has some implications concerning the critical set $LC = T(LC_{-1})$. In fact, if a simple focal point $Q_i \in \overline{LC_{-1}}$, then the critical curve LC has a contact with the corresponding prefocal line δ_Q at a point V whose y coordinate can be easily computed by (6) with m given in (14):

$$y_V = y(\overline{m}) = \frac{\overline{N}_x\overline{F}_y - \overline{F}_x\overline{N}_y}{\overline{D}_x\overline{F}_y - \overline{F}_x\overline{D}_y} \quad (15)$$

From (12) and (8) it is also evident that $\overline{LC_{-1}}$ may intersect δ_s at a nonfocal point, i.e. a point (x_0, y_0) such that $D(x_0, y_0) = 0$ and $N(x_0, y_0) \neq 0$, provided that the equality

$$F_yD_x - F_xD_y = 0 \quad (16)$$

holds in (x_0, y_0) so that $A = N(F_yD_x - F_xD_y) = 0$ at the point (x_0, y_0). This suggests that, generally, $\overline{LC_{-1}}$ may intersect δ_s at *isolated* nonfocal points. However, it may occur, in particular cases, that (16) holds in all the points of δ_s. In this case, from (11) written in the form

$$\det DT = \frac{1}{D^2}[D(F_xN_y - F_yN_x) + N(F_yD_x - F_xD_y)]$$

we can see that when (16) holds for any point of δ_s and D changes its sign crossing the set of nondefinition, $(F_xN_y - F_yN_x)$ being finite, the Jacobian determinant also changes sign on the two sides of δ_s, i.e. $\delta_s \subset J_C$. So, even if the map T is not defined at the points of δ_s, in this case we are led to guess that δ_s may have properties similar to those of LC_{-1}. This is not so unusual if we consider that, as stated in Part I, δ_s may be seen as the two-dimensional analogue of a vertical asymptote. Indeed, such cases remind us what occurs in one-dimensional maps like $x' = 1/x^2$, which is a $Z_0 - Z_2$ map such that for each $x' > 0$ the two preimages $x = \pm 1/\sqrt{x'}$, are located at opposite sides with respect to the vertical asymptote of equation $x = 0$, so that the set of nondefinition, $x = 0$, plays the role of a critical point of rank-0 with critical point of rank-1 "at infinity". In Sec. 5.3 we shall see that a similar geometric behavior also occurs in noninvertible maps not defined everywhere.

3. Noninvertible Maps: Focal Points and Critical Curves Properties

3.1. *Description of two possible situations*

When T is a continuous noninvertible map with focal points, then one of the following two cases arises:

(i) Several focal points may be associated with a given prefocal curve δ_Q, each with its own one-to-one correspondence between slopes and points, like the one defined in (6) and (7). It is the situation considered in Part I, for which no focal point belongs to $\overline{LC_{-1}}$.

(ii) Only one focal point Q is associated with a given prefocal curve δ_Q, because only one inverse focalizes, but with properties different from those recalled in Sec. 2.1 for invertible maps. In this second case the focal point is related to the existence of several focalizing inverses and a particular structure of the foliation.

The analysis of the properties related to noninvertible maps of the simplest form (i.e. maps of type $Z_0 - Z_2$) is the object of this section. In the previous section we have seen, from the properties of a noninvertible map with a vanishing denominator, that generally a focal point Q belongs to the set $\overline{LC_{-1}} \cap \delta_S$, where $\overline{LC_{-1}}$ denotes the closure of

LC_{-1} (this fact results from $A(x, y) = 0$ in (12) with $N = D = 0$). Nevertheless in the particular cases in which δ_S belongs to J_C, it happens that a focal point Q may not belong to $\overline{LC_{-1}}$. The geometric behavior and the plane foliation are different in these two cases. This leads to different situations, according to the fact that the focal points belong or not to the set $\overline{LC_{-1}}$. Of course, it may occur that only some focal points belong to the set $\overline{LC_{-1}}$, so that a map T may have mixed properties, with respect to the cases described below.

3.2. *No focal point belongs to* $\overline{LC_{-1}}$

The following properties of this nongeneric case have been shown in Part I.

(a) For each prefocal curve δ_Q we have $LC \cap \delta_Q = \emptyset$ (ruling out the points at infinity).
(b) If all the inverses are continuous along a prefocal curve δ_Q, then the whole prefocal set δ_Q belongs to a unique region Z_k in which k inverse maps T_j^{-1}, $j = 1, \ldots, k$, are defined.

It is plain that for a prefocal δ_Q at least one inverse is defined that "focalizes" it into a focal point Q. However, other inverses may exist that "focalize" δ_Q into distinct focal points, all associated with the same prefocal curve δ_Q. These focal points are denoted as $Q_j = T_j^{-1}(\delta_Q)$, $j = 1, \ldots, n$, with $n \le k$. For each focal point Q_j the same results described in the previous sections can be obtained with T^{-1} replaced by T_j^{-1}, so that for each Q_j a one-to-one correspondence $m_j(y)$ in the form (6) is defined. Following arguments similar to those given above, it is easy to see that an arc η crossing δ_Q at a point $(F(Q), y)$, where $F(Q) = F(Q_j)$ for any j, is mapped by each T_j^{-1} into an arc $T_j^{-1}(\eta)$, through the corresponding Q_j with the slope $m_j(y)$. If different arcs are considered, crossing δ_Q at the same point, then these are mapped by each inverse T_j^{-1} into different arcs through Q_j, all with the same tangent.

Considering a nonfocal point $D = LC_{-1} \cap \delta_S$, then the critical set $LC = T(LC_{-1})$ includes an asymptote $x = F(D)$. Examples given in Part I (e.g. the maps (17) and (28) in that paper) show that this asymptote is a prefocal curve. It will appear below that the case in which no focal point belongs to $\overline{LC_{-1}}$, corresponds to a bifurcation of merging of two prefocal curves without merging of focal points, and it is also associated with the intersection between LC_{-1} and δ_S in a nonfocal point D. Then the fact that the asymptote $x = F(D)$ is a prefocal curve will be deduced from the limit process giving the points $LC \cap \delta_Q$ at infinity. So *this situation is nongeneric*, in spite of the (improperly chosen) title of Part I which might lead to think it as "generic".

3.3. *All the focal points belong to* $\overline{LC_{-1}}$

3.3.1. *Contact between the critical curve and the prefocal set*

When the focal points belong to $\overline{LC_{-1}}$, the geometric properties of the phase plane, and the bifurcations which may occur are more complex with respect to those described in Sec. 3.1. This is due to the fact that now $\overline{LC}$ has contact points at finite distance with the prefocal curves. Moreover, the property $Q_j = T_j^{-1}(\delta_Q)$, $j = 1, \ldots, n$, with $n \le k$, does not hold. Indeed, in the generic case a given prefocal curve δ_Q is not associated with several focal points Q_j as in the case described in Sec. 3.1. Only one of the inverses T_j^{-1} maps a noncritical point of a prefocal curve into its related focal point, so that we can write $Q = T_j^{-1}(F(Q), y)$, but the index j, defining the focalizing inverse function, depends on the point $(F(Q), y)$ considered on δ_Q. For this reason the previous situation of δ_Q (i.e. with focal points not belonging to $\overline{LC_{-1}}$) appears as nongeneric (and in the examples of Sec. 5 we shall see that it may result from the merging of two prefocal curves δ_{Q_1}, and δ_{Q_2} without merging of the corresponding focal points Q_1 and Q_2).

Let us start by analyzing the properties associated with the arcs of $\overline{LC_{-1}}$ which cross through a focal point Q. As we have seen in Sec. 2, if an arc crosses the singular set δ_s at a focal point, then its image by T must have a contact with the prefocal curve at a specific point. This also holds for an arc of critical curve LC_{-1}, and we can compute the contact point $V = (F(Q), y_V)$ between $\overline{LC}$ (image of $\overline{LC_{-1}}$) and δ_Q by using (15). However $\overline{LC}$ generally does not intersect transversely the prefocal curve δ_Q in V. This can be easily explained in the case in which the map $T : (x, y) \to (x', y')$ is of type Z_0–Z_2. In fact, the image of an arc of LC_{-1} belongs to the critical curve LC, which belongs to the boundary between the regions Z_0 and Z_2. If, by contradiction, we assume that an arc of $\overline{LC}$ intersects transversely δ_Q in V, then we have a part of δ_Q in Z_0, which is impossible, because by definition

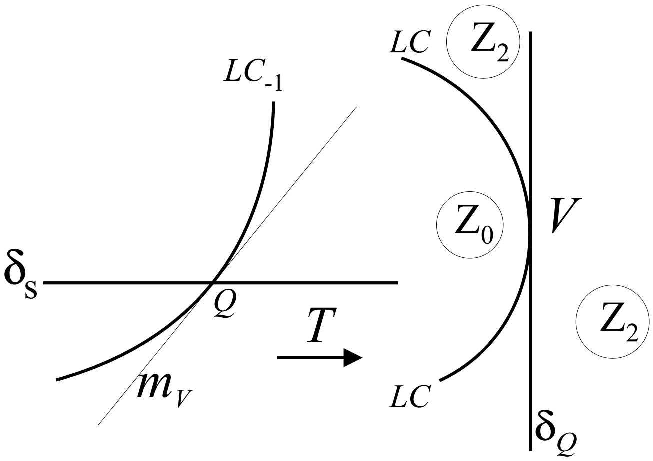


Fig. 3. Qualitative sketch to show that an arc of the critical curve closure $\overline{LC_{-1}}$ crossing the focal point Q (where LC is not defined) is mapped by T into an arc of $\overline{LC}$ which is tangent to the prefocal set δ_Q.

for each point of δ_Q associated with a simple focal point Q, at least one focalizing inverse must exist (see the qualitative picture in Fig. 3). We have so proved the following

Proposition 3.1. *Let T be a noninvertible map of type Z_0–Z_2, Q a simple focal point of T with prefocal set δ_Q. Then an arc of $\overline{LC_{-1}}$ crossing the focal point Q has slope $\overline{m} = -\overline{F}_x/\overline{F}_y$ in Q, and is mapped by T into an arc of $\overline{LC}$ which is tangent to the prefocal set δ_Q at the point $V = (F(Q), y_V)$ where y_V is given in (15).*

Being $\overline{LC}$ tangent to δ_Q in the point V, we have that the rank-1 preimages of any point $(F(Q), y) \in \delta_Q$, with y different from y_V, include two distinct points, one of which is necessarily focalized at the focal point Q and the other belongs to a curve which is tangent to $\overline{LC_{-1}}$ in Q. This comes from the fact that $\overline{LC} \cap \delta_Q = V$ and the slope of arcs issuing from Q associated with the point V of the prefocal set is $\overline{m}$ (see (14), (15), and (6) (7)), where $\overline{m}$ is the slope of the tangent to $\overline{LC_{-1}}$ in Q. Thus $T^{-1}(LC \cup \delta_Q)$ is made up of two arcs, LC_{-1} and an arc τ, which have a common tangent of slope $\overline{m}$ in Q.

Noticing that both the preimages of the tangency point V are "focalized" at the same focal point, we are led to consider the tangency point V (which belongs to $\overline{LC}$ but not to LC) as a particular point for the foliation structure of the two-dimensional noninvertible maps. In fact, V really behaves as a critical point (with two merging preimages) and crossing V we may have a change of role of the focalizing inverse on the prefocal set. We remark the difficulty in the definition of the critical sets for this class of maps. That is, in Sec. 2.3 we consider the critical points of T as those having two merging rank-1 preimages, then V ought to be a point of LC, but the merging preimages are of a focal point Q, where the map is not defined, thus this point is not a point of LC_{-1} and consequently also V is not a critical point.

3.3.2. *Preimages of an arc crossing through a prefocal set*

In order to clarify this particular geometric behavior, a qualitative illustration of the simplest case is given in Fig. 4. This case is related to a situation with two prefocal curves generated by a noninvertible map $T : (x, y) \to (x', y')$ of type $(Z_0$–$Z_2)$, i.e. the plane contains only regions whose points have no rank-1 preimages, and regions whose points have two rank-2 preimages. In other words, the inverse relation $T^{-1}(x', y')$ has two components in the region Z_2, denoted by T_1^{-1} and T_2^{-1}, and no real components in the region Z_0. The set of nondefinition δ_s is a simple straight line, and there are two prefocal lines, δ_{Q_i}, of equation $x = F(Q_i)$, associated with the focal points Q_i, $i = 1, 2$, respectively, and $V_i = \overline{LC} \cap \delta_{Q_i} = (F(Q), y(V_i))$ (where $y(V_i)$ is given in (15)) are the points of tangency between LC and the two prefocal curves. Let δ'_{Q_i} be the segment of δ_{Q_i} such that $y > y(V_i)$ (pecked line in Fig. 4), and δ''_{Q_i} the segment of δ_{Q_i} such that $y < y(V_i)$ (continuous line in Fig. 4). The "focalization" of the inverses occurs in the following way:

$$T_1^{-1}(\delta'_{Q_i}) = Q_i,\ T_2^{-1}(\delta''_{Q_i}) = Q_i \quad i = 1, 2 \qquad (17)$$

while the other preimages give

$$T_2^{-1}(\delta'_{Q_i}) \cup T_1^{-1}(\delta''_{Q_i}) = \pi_i\,, \quad i = 1, 2$$

each π_i being a set crossing through the focal points Q_i and tangent to LC_{-1} at the focal point, and in the qualitative picture in Fig. 4 (corresponding to an example which will be explicitly described in Sec. 5) π_i is exactly the line tangent to LC_{-1} in Q_i. It follows that the vertical strip between the two prefocal lines δ_{Q_i} is entirely included in the region Z_2, so that each point (x', y') in this strip has two distinct rank-1 preimages. However, differently from what occurs in two-dimensional maps defined in the whole plane, the two preimages $T_1^{-1}(x', y')$ and $T_2^{-1}(x', y')$ are not located one on the right and one on the left of LC_{-1}, as both are located in the

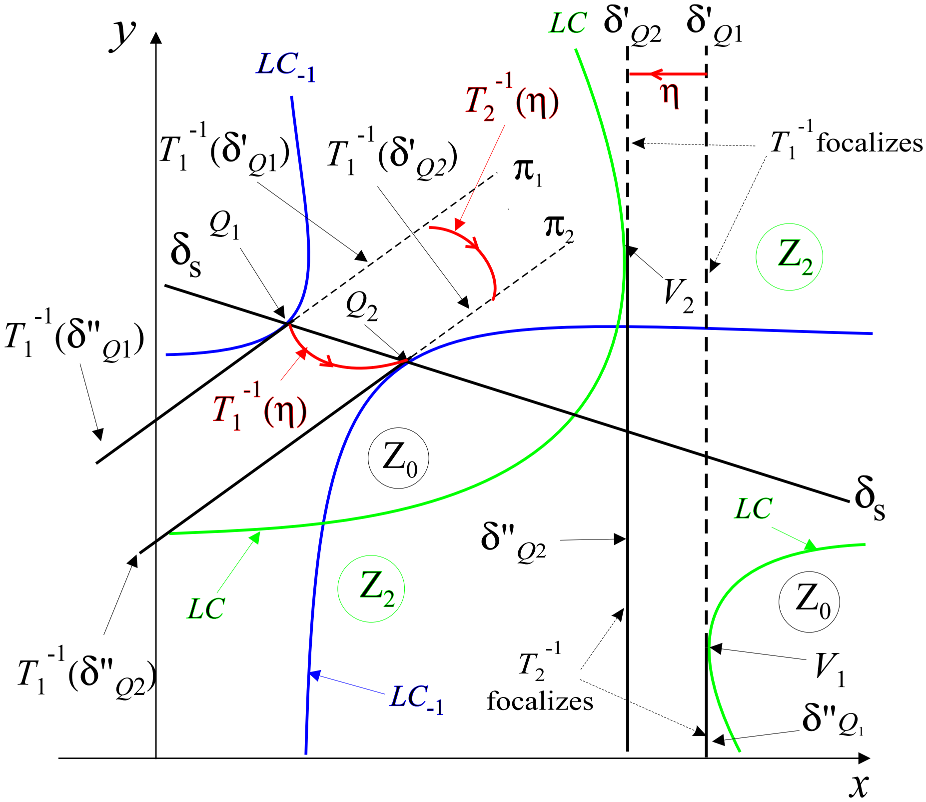
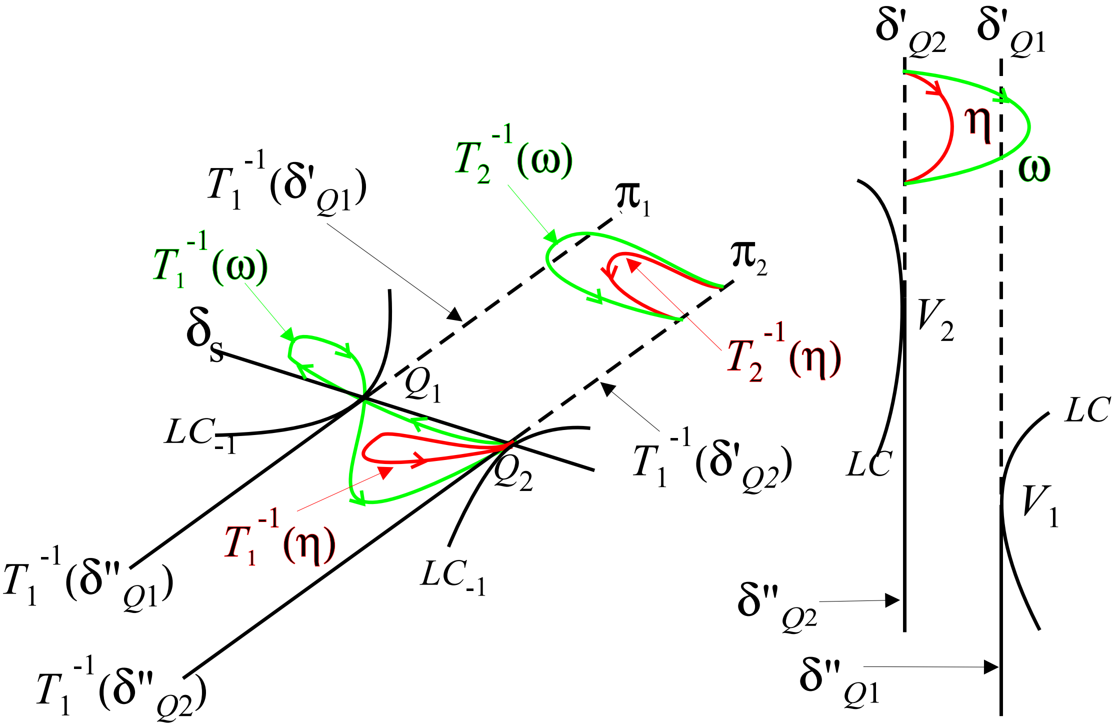


Fig. 4. Qualitative representation of the preimages of an arc crossing through a prefocal set in a situation with two prefocal curves generated by a noninvertible map of type (Z_0–Z_2). The prefocal set δ_Q is made up of two lines $\delta_Q = \delta_{Q_1} \cup \delta_{Q_2}$, $\delta_{Q_1} = \delta'_{Q_1} \cup \delta''_{Q_1}$, $\delta_{Q_2} = \delta'_{Q_2} \cup \delta''_{Q_2}$. Every arc crossing through δ'_{Q_j}, $j = 1, 2$, is mapped by T_1^{-1} into an arc crossing through Q_j (focalization). Every arc crossing through δ''_{Q_j}, $j = 1, 2$, is mapped by T_2^{-1} into an arc crossing through Q_j. Moreover one has $T_2^{-1}(\delta'_{Q_i}) \cup T_1^{-1}(\delta''_{Q_i}) = \pi_i$, $i = 1, 2$. Segments δ'_{Q_j} and δ''_{Q_j} join at the tangency points V_j between the critical curve LC and the prefocal set.

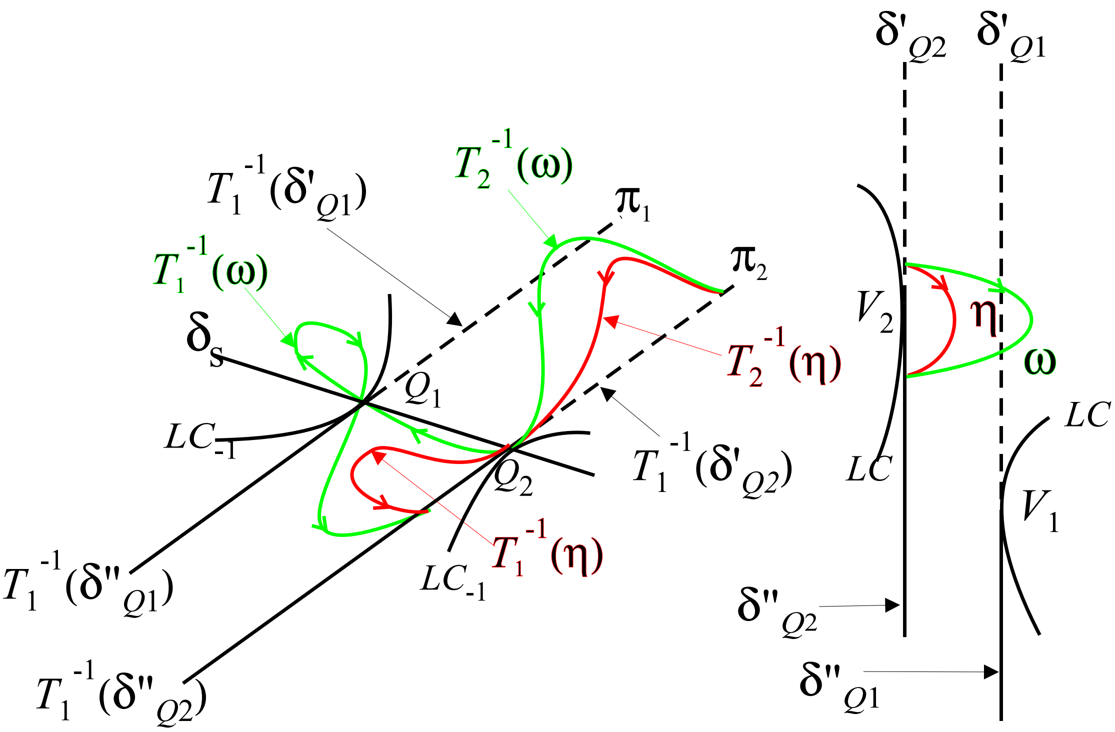
strip bounded by the two lines π_i tangent to LC_{-1} in Q_i which does not include any point of LC_{-1} (see Fig. 4). Moreover, the two rank-1 preimages are separated by the singular line δ_s, i.e. the two inverses of a point (x', y') of the strip between the two prefocal curves are one above and one below δ_s, in the strip between the two π_i. As an example, the two rank-1 preimages of a segment η connecting two points of δ'_{Q_i} are shown in Fig. 4. As T_1^{-1} focalizes the points of δ'_{Q_i} we have that $T_1^{-1}(\eta)$ must be an arc belonging to the strip between π_1 and π_2 and connecting the two focal points Q_1 and Q_2. While T_2^{-1} has no focalizing properties, thus $T_2^{-1}(\eta)$ is an arc belonging to the strip between π_1 and π_2 connecting the two preimages of the extrema of the segment η. It is clear that if we take a point (x', y') close to a prefocal line then one of the two preimages is close to a focal point. Thus we can say that the preimages of that strip are related to the focalization properties, in the sense that if we take an arc η in the strip, or an arc ω crossing the strip, then their preimages T_1^{-1} and T_2^{-1} may have peculiar properties, specific to maps with focal points, as evidenced in the qualitative pictures of Fig. 5.

The bifurcation situations due to a tangential contact between an arc η or ω with a prefocal line δ_{Q_i} have not been evidenced in Fig. 5. However the geometric behavior is quite obvious, except for the fact that the preimage with a focalizing inverse is such that the contact with the corresponding focal point is *nonsmooth* (i.e. at the bifurcation such a preimage has a cusp point at the focal point).

It can be noticed that there are some particular configurations of arcs, crossing a prefocal line at two distinct points, whose two rank-1 preimages have no loops with knot in any focal point (and thus cannot be associated with the formation of "lobes"). This occurs when an arc crosses only one prefocal curve and the intersection points are one above and

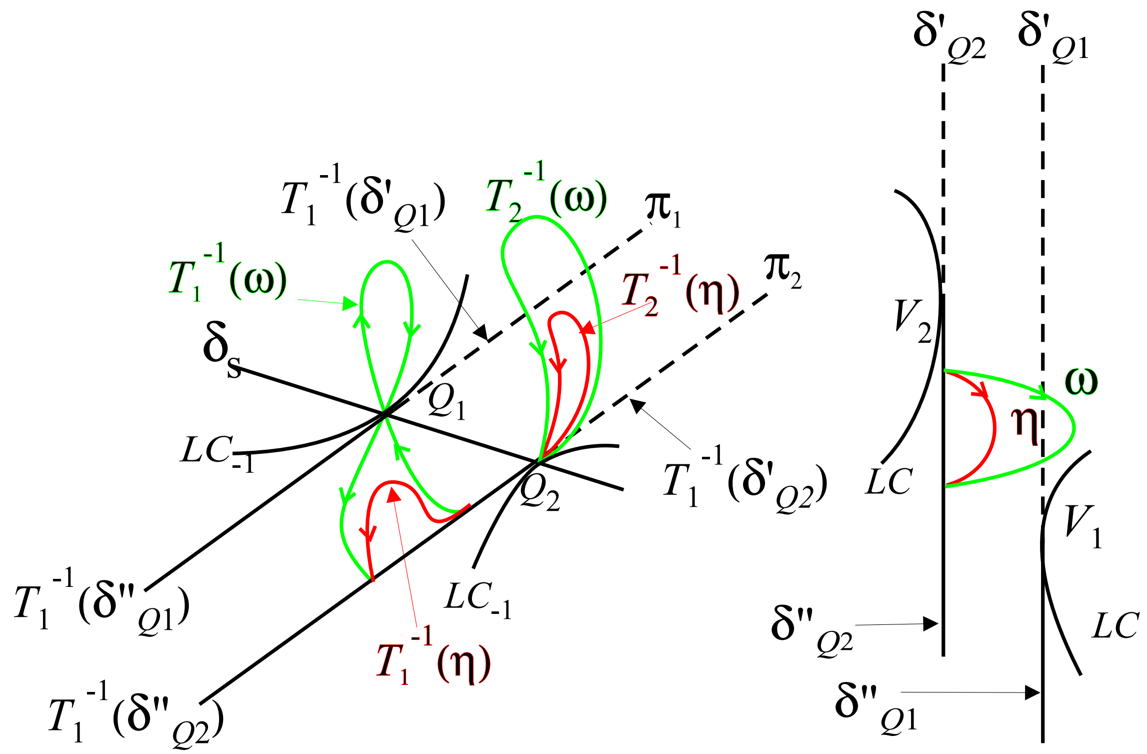


(a)

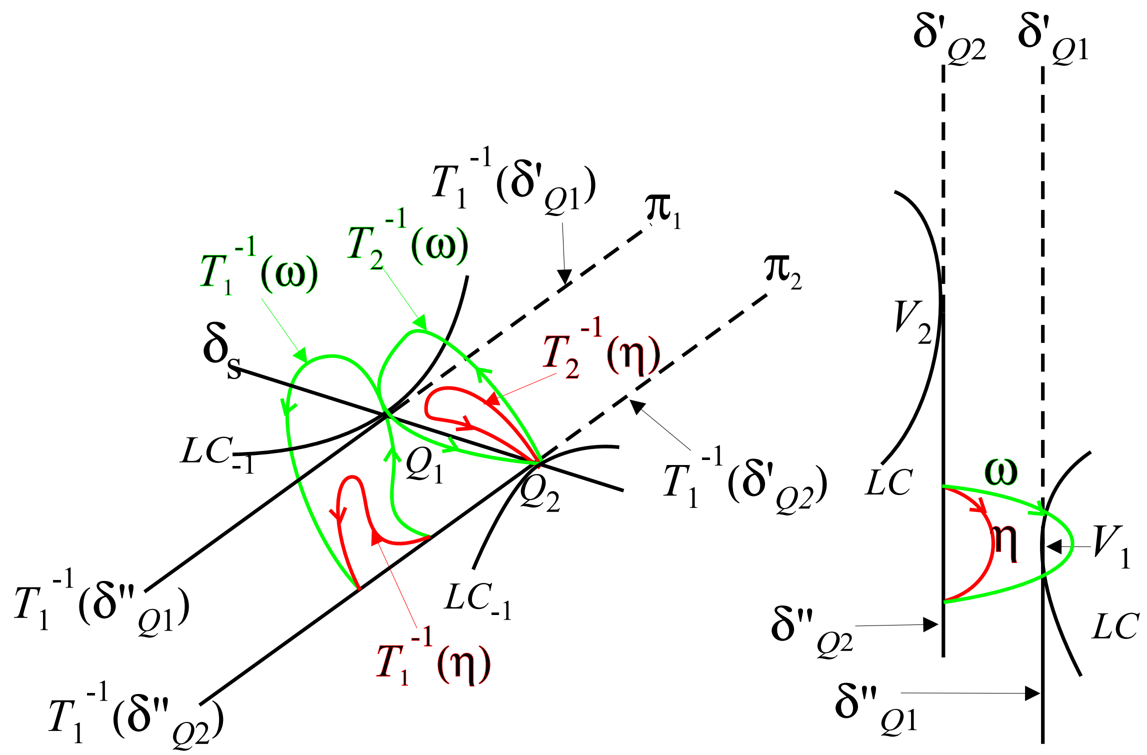


(b)

Fig. 5. Some qualitative sketches showing peculiar properties (specific to noninvertible maps with focal points) of the preimages of arcs crossing one or two prefocal lines. These sketches depend on the intersection locations with respect to the tangency points V_j between the critical curve LC and the prefocal set.

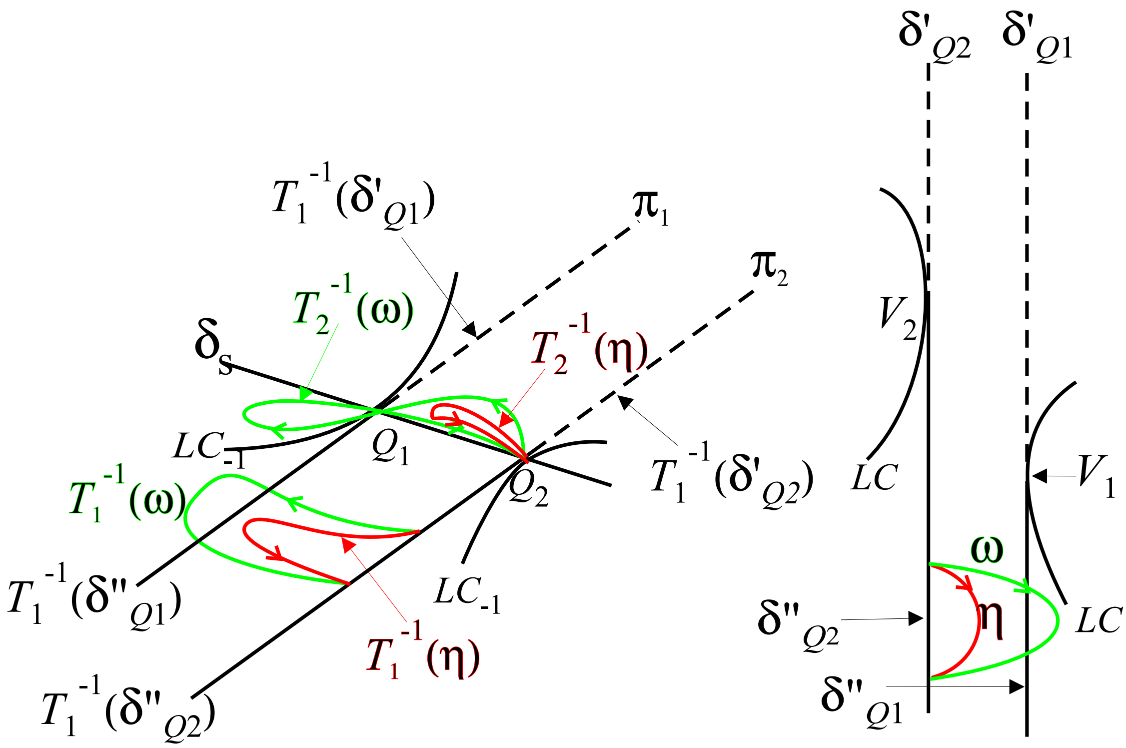


(c)



(d)

Fig. 5. (*Continued*)



(e)

Fig. 5. (*Continued*)

one below the tangency point V, as for the arc η in Fig. 5(b).

It is also worth noticing, however, that in any other situation, as evidenced from Figs. 5(a), 5(c)–5(e), at least one loop in one focal point is necessarily formed, and also that the formations of particular arcs [Figs. 5(a) and 5(e)], called "crescents" in Part I, associated with arcs and lobes crossing two focal points, may occur via a mechanism different from the one shown in Part I. In Part I we have seen that the formation of particular arcs through two focal points, giving rise to the special structure of "crescents", occurs when two distinct loops, or "lobes", approach a critical curve LC_{-1} and have a contact on that curve, so that the reunion of two "lobes" gives rise to one "crescent" (see Fig. 9 of Part I), and now we may think that such a structure results from the merging of two prefocal sets without merging of focal points. Moreover, a new mechanism of formation of crescents is described in this paper. In fact, the preimage of the arc ω in Fig. 5(a) (via T_1^{-1}) and in Fig. 5(e) (via T_2^{-1}) shows how crescents may be formed involving only one of the inverses and no critical point of LC_{-1}. In Fig. 5(a), we see that the preimage $T_1^{-1}(\eta)$ is a lobe issuing from the focal point Q_2 and as η approaches the second prefocal curve the preimage $T_1^{-1}(\eta)$ approaches the second focal point Q_1. Thus the preimage of an arc ω crossing both the two prefocal curves has $T_1^{-1}(\omega)$ which includes a crescent connecting the two focal points Q_1 and Q_2, and a lobe issuing from Q_1. Similarly occurs with T_2^{-1} in Fig. 5(e).

3.3.3. *Bifurcation case*

Taking into account that the focal points Q_i belong to the closure $\overline{LC_{-1}}$ and that the role of the singular set δ_s in the segment between the two focal points is particular (it separates the preimages of the points belonging to the strip between the two prefocal lines) we are led to consider that the segment of δ_s between Q_1 and Q_2 has particular geometric properties as for a segment of LC_{-1} (as stated in Sec. 2.3). This particular geometric role played by the set of nondefinition δ_s is specially evidenced in the bifurcation cases (we shall see an example in Sec. 5.3). That is, with the merging of the two prefocal sets, when $\delta_{Q_1} \to \delta_{Q_2}$ (due to a parameter variation) without merging of the focal points

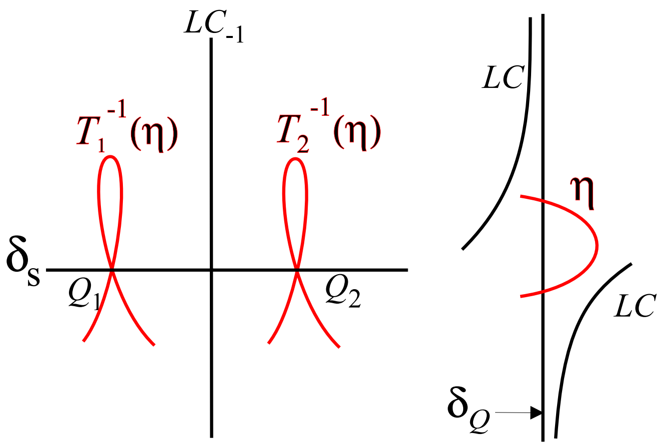


Fig. 6. Qualitative picture concerning the case of two merging prefocal lines without merging of the focal points. The unique δ_Q ($\delta_Q = \delta_{Q_1} \equiv \delta_{Q_2}$) becomes an asymptote for the critical curve LC. A branch of LC_{-1} [$LC = T(LC_{-1})$] reduces to a line intersecting the set of nondefinition δ_s at a point between the two focal points Q_i, $i = 1, 2$, and the second branch degenerates to the line δ_s itself.

(i.e. $Q_1 \neq Q_2$ and $F(Q_1) = F(Q_2)$), the tangency points V_i on the prefocal curves tend to infinity, i.e. the two prefocal lines merge into a unique one δ_Q ($\delta_Q = \delta_{Q_1} = \delta_{Q_2}$) which becomes an asymptote for LC, as it occurs in the case described in Sec. 3.2. This also corresponds to a qualitative change in the foliation of the two-dimensional map, i.e. a qualitative change of LC_{-1} and a change in the role played by the two inverses. At the bifurcation, the merging of the two prefocal lines also causes the disappearance of the strip between them and of the strip between the two tangent lines π_i as well, so that each point $(F(Q), y)$ of δ_Q has now two distinct inverses, both focalizing: $T_1^{-1}(F(Q), y) = Q_1$ and $T_2^{-1}(F(Q), y) = Q_2$. As we shall see in the example in Sec. 5.3, the critical curve LC_{-1} is an hyperbola which degenerates into the asymptotes at the bifurcation of the prefocal curves, reducing to a line intersecting the set of nondefinition δ_s at a point between the two focal points Q_i, $i = 1, 2$, and the second degenerate line is δ_s itself (see the qualitative picture in Fig. 6), which turns out to verify condition (16) in all its points. It is thus clear why the cases described in Sec. 3.2 (as well as several examples in Part I) can be viewed as bifurcation cases of those described in Sec. 3.3, which are the generic ones. An explicit example and the description of some situations will better clarify the mechanisms.

4. Consequences on the Geometric Properties of the Phase Curves

When the presence of a vanishing denominator induces the existence of focal points, important effects on the geometrical and dynamical properties of the map T can be observed. Indeed, a contact between a curve segment γ and the singular set δ_s causes remarkable qualitative changes in the shape of the image $T(\gamma)$. Moreover, a contact of an arc ω with a prefocal curve δ_Q, gives rise to important qualitative changes in the shape of the preimages $T_j^{-1}(\omega)$. When the arcs ω are portions of phase curves of the map T, such as *invariant closed curves*, *stable or unstable sets* of saddles, *basin boundaries*, one has that contacts between singularities of different nature generally induce important qualitative changes, which constitute *new types of global bifurcations* that change the structure of the attracting sets, or of their basins. In particular, this concerns the formation of lobes and crescents for basin boundaries.

As we have reminded in Sec. 3.3.2, the formation of crescents has already been shown in Part I for maps having *focal points not belonging to* $\overline{LC_{-1}}$. In such a case the contact on LC_{-1} of two lobes issuing from two distinct focal points is the only mechanism giving rise to a crescent. That is, the mechanism requires first the formation of two lobes via a contact with a prefocal curve, followed by a contact with a critical curve LC causing on its turn the merging of the two lobes on the critical curve LC_{-1}, thus leading to the formation of a crescent.

In the case of noninvertible maps with *focal points belonging to* $\overline{LC_{-1}}$ a crescent in a basin boundary, or the stable set of a saddle fixed (or periodic) point, may even be formed starting from a single lobe, issuing from one focal point (due to the crossing of some arc with one prefocal curve at two points which are both on the same side with respect to the tangency point, say V_i), which is then conveyed through a different focal point (due to the contact of the same arc with a different prefocal curve on the same side with respect to the new tangency point, say V_j), as it is qualitatively shown in Figs. 5(a) and 5(e). This mechanism may also be frequently repeated giving rise to sequences of crescents issuing from the same two focal points.

In the following section, by using only one family of $Z_0 - Z_2$ maps T with several parameter constellations, we shall see some numerical explorations to illustrate some global bifurcations induced by the presence of a vanishing denominator, and the related properties. In particular, we shall consider the boundary between the basin of divergent trajectories and the basin of the attractor at finite distance.

5. Examples of Situations Generated by a $Z_0 - Z_2$ Map

5.1. *Focal points, prefocal lines, equation of the two inverses*

Consider the following map

$$T : \begin{cases} x' = y + \varepsilon x \\ y' = \dfrac{\alpha x^2 + \gamma x}{y - \beta + \sigma x} \end{cases} \tag{18}$$

not defined in the points of the line δ_s of equation $y-\beta+\sigma x = 0$, on which two focal points exist given by

$$Q_1 = \left(-\frac{\gamma}{\alpha}, \beta + \frac{\gamma\sigma}{\alpha}\right) \quad \text{and} \quad Q_2 = (0, \beta)$$

whose corresponding prefocal curves, of equation $x = F(Q_i)$, $i = 1, 2$, are

$$x = \beta + (\sigma - \varepsilon)\gamma/\alpha(\delta_{Q_1}) \quad \text{and} \quad x = \beta(\delta_{Q_2}) \tag{19}$$

Being $(N_x D_y - N_y D_x) = 2\alpha x + \gamma$ we have that $(\overline{N}_x \overline{D}_y - \overline{N}_y \overline{D}_x)|_{Q_1} = -\gamma$ and $(\overline{N}_x \overline{D}_y - \overline{N}_y \overline{D}_x)|_{Q_2} = \gamma$ so that for $\gamma \neq 0$ both the focal points are simple. The one-to-one correspondence (6) associated with each simple focal point Q_i is given by $y_i(m) = 2\alpha x_{Q_i} + \gamma/\sigma + m$, that is

$$y(m) = -\frac{\gamma}{\sigma + m} \text{ for } \delta_{Q_1}$$

$$y(m) = \frac{\gamma}{\sigma + m} \text{ for } \delta_{Q_2}$$

from which the inverse relations $m(y)$ easily follow.

The map T in (18) is a noninvertible map of (Z_0–Z_2) type, i.e. a point (x', y') can have two or no rank-1 preimages whose coordinates are obtained by solving the algebraic system (18) with respect to the unknowns x and y. The solution of this system is equivalent to

$$\begin{cases} \alpha x^2 + (\gamma - (\sigma - \varepsilon)y')x + y'(\beta - x') = 0 \\ y = x' - \varepsilon x \end{cases} \tag{20}$$

for $y \neq -\sigma x + \beta$, and gives the inverses of the map from its two real and distinct solutions if

$$\Delta(x', y') = (\gamma - (\sigma - \varepsilon)y')^2 - 4\alpha y'(\beta - x') > 0 \tag{21}$$

while no solution exists if the reverse inequality holds. Accordingly, the two regions Z_0 and Z_2 are defined as $Z_0 = \{(x, y)|\Delta(x, y) < 0\}$ and $Z_2 = \{(x, y)|\Delta(x, y) > 0\}$. From each point $(x', y') \in Z_2$ the two distinct rank-1 preimages are computed by the following two inverses (obtained by solving the second degree equation in (20)):

$$T_1^{-1}(x', y') : \begin{cases} x = \dfrac{1}{2\alpha}[(\sigma - \varepsilon)y' - \gamma - \sqrt{\Delta(x', y')}] \\ y = x' - \varepsilon x \end{cases} \tag{22}$$

$$T_2^{-1}(x', y') : \begin{cases} x = \dfrac{1}{2\alpha}[(\sigma - \varepsilon)y' - \gamma + \sqrt{\Delta(x', y')}] \\ y = x' - \varepsilon x \end{cases} \tag{23}$$

where $\Delta(x', y')$ is given in (21).

5.2. *Critical curve and preimages of the prefocal lines*

The boundary which separates the region Z_2 from Z_0, defined by the equation $\Delta(x, y) = 0$, is the critical set LC (locus of points having merging rank-1 preimages), which can be expressed as

$$\overline{LC} : \ x = \frac{4\alpha\beta y - (\gamma - (\sigma - \varepsilon)y)^2}{4\alpha y} \tag{24}$$

so that LC is formed by the two branches of an hyperbola of asymptotes

$$y = 0$$

and

$$\begin{cases} y = \dfrac{-4\alpha x + 4\alpha\beta + 2(\sigma - \varepsilon)\gamma}{(\sigma - \varepsilon)^2} & \text{for } \varepsilon \neq \sigma \\ x = \beta & \text{for } \varepsilon = \sigma \end{cases}$$

and symmetry centre $s = (\beta + ((\sigma - \varepsilon)\gamma/2\alpha), 0)$. The locus of merging rank-1 preimages LC_{-1} has the parametric equation obtained by (22), or (23), with $\Delta = 0$, i.e.

$$x = \frac{1}{2\alpha}((\sigma - \varepsilon)y' - \gamma)$$
$$y = x' - \varepsilon x$$

So, for $\varepsilon \neq \sigma$, LC_{-1} is given by the hyperbola of cartesian equation

$$\overline{LC_{-1}} : \ y = \frac{-\alpha(\sigma+\varepsilon)x^2 + (2\alpha\beta - \varepsilon\gamma)x + \beta\gamma}{\gamma + 2\alpha x} \tag{25}$$

The same equation can also be obtained from the condition det $DT(x, y) = 0$, where

$$DT(x, y) = \begin{bmatrix} \varepsilon & 1 \\ \dfrac{(2\alpha x+\gamma)D(x, y) - \sigma N(x, y)}{D(x, y)^2} & -\dfrac{N(x, y)}{D(x, y)^2} \end{bmatrix}$$

In fact, det $DT(x, y) = (\sigma - \varepsilon)N(x, y) - (2\alpha x + \gamma)D(x, y)/D(x, y)^2$ so that

$$J_0 = \{(x, y) \in \mathbb{R}^2 | (\sigma - \varepsilon)N(x, y) - (2\alpha x + \gamma)D(x, y) = 0\} \cap \{\mathbb{R}^2 \backslash \delta_s\}$$

and for $\varepsilon \neq \sigma$ we have $LC_{-1} = J_0 = J_C$. The slopes of the tangent lines π_i to LC_{-1} in the focal points, given by (14), are $m_1 = m_2 = -\varepsilon$ and the points of tangency between LC and the prefocal lines δ_{Q_i}, via (15), are given by

$$V_1 = \left(\beta + \frac{(\sigma-\varepsilon)\gamma}{\alpha}, -\frac{\gamma}{\sigma - \varepsilon}\right), \quad V_2 = \left(\beta, \frac{\gamma}{\sigma-\varepsilon}\right) \tag{26}$$

The rank-1 preimage of the prefocal lines can be explicitly computed. Let us consider first a generic point of δ_{Q_2}, say (β, y'). Then

$$T_{1,2}^{-1}(\beta, y') : \begin{cases} x = \dfrac{1}{2\alpha}((\sigma-\varepsilon)y' - \gamma \mp |(\sigma - \varepsilon)y' - \gamma|) \\ y = \beta - \varepsilon x \end{cases}$$

so that

if $(\sigma - \varepsilon)y' - \gamma > 0$ then

$$T_1^{-1}(\beta, y') = Q_2 : \begin{cases} x = 0 \\ y = \beta \end{cases}$$

$$T_2^{-1}(\beta, y') \in \pi_2 : \begin{cases} x = \dfrac{(\sigma-\varepsilon)y' - \gamma}{\alpha} \\ y = \beta - \varepsilon x \end{cases}$$

if $(\sigma - \varepsilon)y' - \gamma < 0$ then

$$T_1^{-1}(\beta, y') \in \pi_2 : \begin{cases} x = \dfrac{(\sigma-\varepsilon)y' - \gamma}{\alpha} \\ y = \beta - \varepsilon x \end{cases}$$

$$T_2^{-1}(\beta, y') = Q_2 : \begin{cases} x = 0 \\ y = \beta \end{cases}$$

The condition $(\sigma - \varepsilon)y' - \gamma > 0$ is satisfied by the points of an half line of the prefocal δ_{Q_2} issuing from the tangency point V_2, which one (upper or lower part) depending on the value of the parameters. Similarly, which part of the tangent line π_2 issuing from Q_2 is obtained depends on the parameters. Of course, for $(\sigma - \varepsilon)y' - \gamma < 0$ the complementary half lines are involved.

Analogously, considering a generic point of δ_{Q_1}, say $(\beta + ((\sigma - \varepsilon)\gamma/\alpha), y')$, we have

$$T_{1,2}^{-1}(\beta, y') : \begin{cases} x = \dfrac{1}{2\alpha}((\sigma-\varepsilon)y' - \gamma \mp |(\sigma - \varepsilon)y' + \gamma|) \\ y = \beta + \dfrac{(\sigma-\varepsilon)\gamma}{\alpha} - \varepsilon x \end{cases}$$

so that

if $(\sigma - \varepsilon)y' + \gamma > 0$ then

$$T_1^{-1}(\beta, y') = Q_1 : \begin{cases} x = -\dfrac{\gamma}{\alpha} \\ y = \beta + \dfrac{\sigma\gamma}{\alpha} \end{cases}$$

$$T_2^{-1}(\beta, y') \in \pi_1 : \begin{cases} x = \dfrac{(\sigma-\varepsilon)y'}{\alpha} \\ y = \beta + \dfrac{(\sigma-\varepsilon)\gamma}{\alpha} - \varepsilon x \end{cases}$$

if $(\sigma - \varepsilon)y' + \gamma < 0$ then

$$T_1^{-1}(\beta, y') \in \pi_1 : \begin{cases} x = \dfrac{(\sigma-\varepsilon)y'}{\alpha} \\ y = \beta + \dfrac{(\sigma-\varepsilon)\gamma}{\alpha} - \varepsilon x \end{cases}$$

$$T_2^{-1}(\beta, y') = Q_1 : \begin{cases} x = -\dfrac{\gamma}{\alpha} \\ y = \beta + \dfrac{\sigma\gamma}{\alpha} \end{cases}$$

The condition $(\sigma - \varepsilon)y' + \gamma > 0$ is satisfied by the points of an half line of the prefocal δ_{Q_1} issuing from the tangency point V_1, which one depending on the parameters, and which part of the tangent line π_1 issuing from Q_1 is obtained depends on the parameters as well. Also in this case for $(\sigma - \varepsilon)y' + \gamma < 0$ the complementary half lines are involved.

The map T in (18) has two fixed points, the origin $O = (0, 0)$ and $P = (x_p, (1 - \varepsilon)x_p)$, where $x_p = (\beta(1 - \varepsilon) + \gamma)/((1 - \varepsilon)^2 + \gamma(1 - \varepsilon) - \alpha))$. In Fig. 7(a), obtained with (18) and parameters $\alpha = 0.5$, $\gamma = 0.5$, $\beta = 1.4$, $\sigma = 0.3$, $\varepsilon = -0.2$, we have a situation similar to the one shown in

Fig. 4. The red region represents the basin of the stable fixed point $O = (0, 0)$, and the complementary white region denotes the basin of infinity, i.e. the set of diverging trajectories. In Fig. 7(a), as in Fig. 4, T_1^{-1} (resp. T_2^{-1}) focalizes the points of the prefocal lines δ_{Q_i} above (resp. below) the tangency points V_i.

5.3. *Bifurcation by merging of the prefocal lines, without merging of the focal points*

In the particular case occurring when $\varepsilon = \sigma$ the two prefocal lines δ_{Q_i} merge. Indeed from $x = F(Q_i)$ given in (19) we obtain $\delta_Q = (\delta_{Q_1} \equiv \delta_{Q_2})$, and its equation is $x = \beta$. We obtain the situation of the noninvertible maps of Part I, for which several focal points are associated with a given prefocal curve. The parametric equation of LC_{-1}, obtained by (22) with $\Delta = 0$, reduces to:

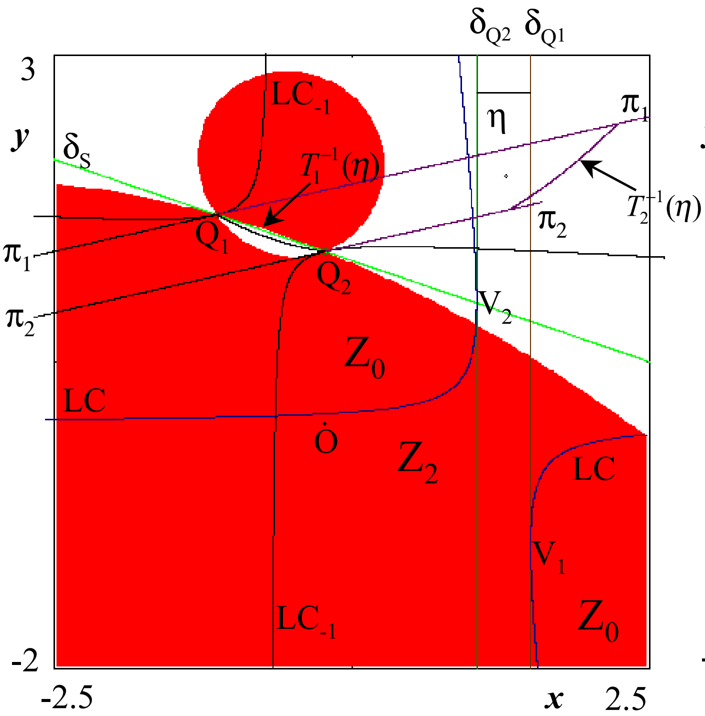
$$x = -\frac{\gamma}{2\alpha}\,,$$

$$y = x' - \varepsilon x$$

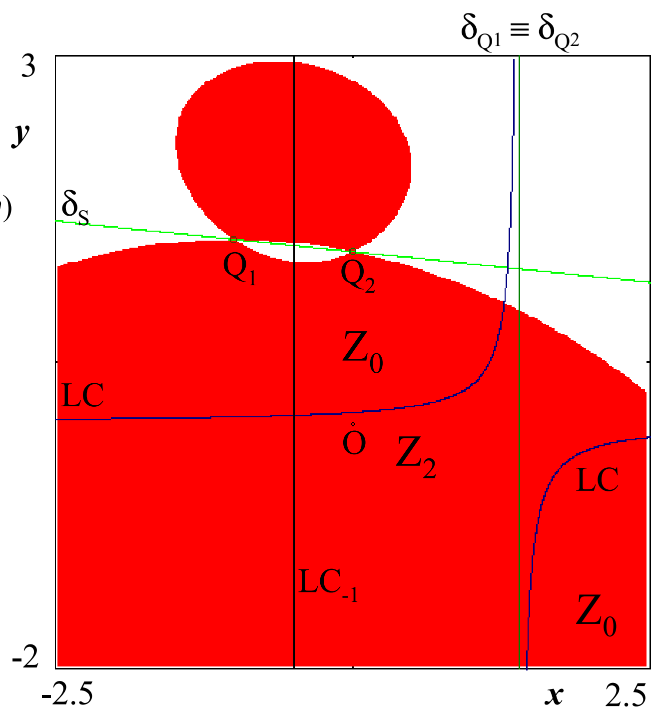
thus in this (bifurcation) case LC_{-1} is the line of equation $x = -\gamma/2\alpha$. The Jacobian determinant becomes $\det DT(x, y) = -(2\alpha x + \gamma)/D(x, y)$, so that we have $LC_{-1} = J_0$ and the hyperbola (25) degenerates into the two asymptotes $x = -\gamma/2\alpha$ and δ_s. Notice that δ_s now belongs to the set denoted by J_C in Sec. 2.3. In fact, the Jacobian determinant changes sign both crossing $x = -\gamma/2\alpha$ (i.e. LC_{-1}) and crossing the line δ_s, at which $D(x, y) = 0$, so that $J_C = \{(x, y)|x = -\gamma/2\alpha\} \cup \delta_s = LC_{-1} \cup \delta_s$. That is, at this bifurcation value we have $LC_{-1} = J_0 \subset J_C$, the resulting "double" prefocal curve δ_Q is an asymptote of LC because the tangency points V_i tend to the Poincaré Equator (as can be seen from the coordinates of V_i given in (26) for $\varepsilon = \sigma$), and the lines π_1 and π_2 degenerate at the two focal points, which now are not located on $\overline{LC_{-1}}$. In Fig. 7(b), obtained with $\varepsilon = \sigma = 0.1$ and the other parameters as in Fig. 7(a), the two focal points are still distinct and simple, and we can see that the two prefocal lines merge into a unique one, which is an asymptote of LC. Note that LC_{-1} is degenerate only for $\varepsilon = \sigma$. For $\sigma < \varepsilon$ as in the case of Fig. 7(c), the two prefocal lines, LC_{-1} and LC have the same structure as for $\sigma > \varepsilon$ [Fig. 7(a)], but the two branches of the hyperbola LC_{-1} have a different geometric configuration, the strip between π_1 and π_2 is rotated and (as it is evident from the inverses of the map) the role of the focalizing inverses is exchanged: now T_2^{-1} focalizes the points of the prefocal lines δ_{Q_i} above the tangency points V_i. Considering a segment η between the two prefocal lines above the points V_i, then $T_2^{-1}(\eta)$ is an arc through the two focal points, while $T_1^{-1}(\eta)$ is not focalized [compare the arc η and its preimages in Figs. 7(a) and 7(c)]. Clearly, the two rank-1 preimages of the strip, say S, between the two prefocal lines are still in the strip between the tangency lines π_i, separated by the segment of δ_s, one being above and one below that segment. In Fig. 7(d) we have labeled the strip between the two prefocal lines as $S = A \cup B \cup C$ (where A, B, C denote the portions of the basins: white, red and white, respectively). In this figure it can be seen that $T_1^{-1}(S)$ is in the strip between π_i on the left of δ_s and $T_2^{-1}(S)$ on the right. Moreover, it can also be seen that the role of the portion of δ_s in the strip, although it separates preimages, has not the unfolding action (characteristic of a segment of LC_{-1}) because the preimages $T_i^{-1}(A)$, $T_i^{-1}(B)$, $T_i^{-1}(C)$ go from left to right for both $i = 1, 2$ [see Fig. 7(d)]. This is due to the particular foliation structure existing in the strip between the two prefocal lines.

5.4. *Foliation generated by the map between two prefocal curves*

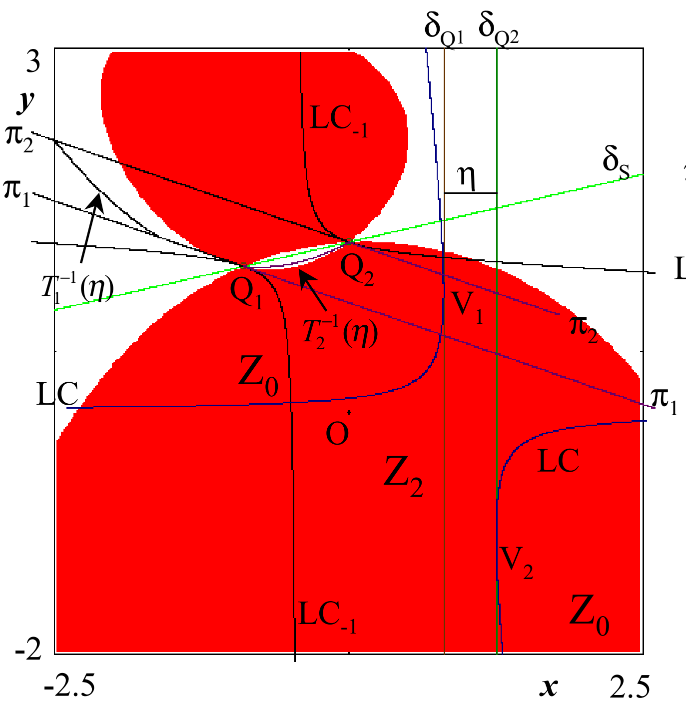
The particular foliation structure generated by the map (18) of (Z_0–Z_2) type, in the cases shown in Fig. 7, is represented by the qualitative pictures in Figs. 8 and 9. The strip between the two lines π_i is separated in two parts, say P_1 and P_2, by the segment of δ_s. We have $T(P_1) = S$ and $T(P_2) = S$ as well, S being the whole strip between the two prefocal lines. The arrows in Fig. 8 show the mechanism of the foliation of the forward image by T, for both cases $\sigma > \varepsilon$ [Fig. 8(a)], and $\sigma < \varepsilon$ [Fig. 8(b)]. The points close to δ_s on both sides, left and right parts are mapped by T into points close to the Poincaré Equator (P.E. henceforth), on opposite sides of S. More precisely for $\sigma > \varepsilon$, points close to δ_s on the left side are mapped into points close to the P.E. on the upper part of S, and points close to δ_s on the right side are mapped into points close to the P.E. on the lower part of S. For $\sigma < \varepsilon$, points close to δ_s on the left side are mapped into points close to the P.E. on the lower part of S, and points close to δ_s on the right side are mapped into points close to the P.E. on the upper part of S.



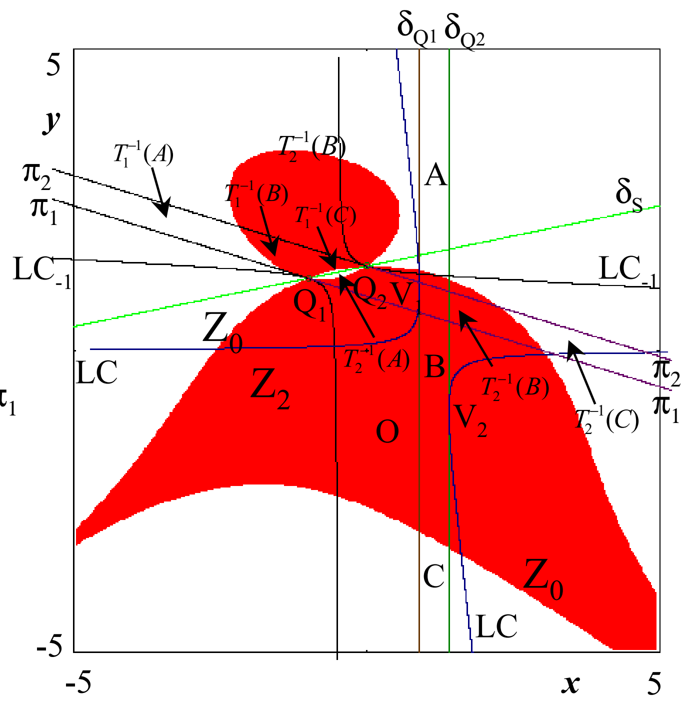
(a)



(b)



(c)



(d)

Fig. 7. Basins of attraction obtained with the map (18). The red region represents the basin of the stable fixed point $O = (0, 0)$, and the complementary white region denotes the basin of infinity, i.e. the set of diverging trajectories. (a) Basins obtained with parameters $\alpha = 0.5$, $\gamma = 0.5$, $\beta = 1.4$, $\sigma = 0.3$, $\varepsilon = -0.2$. The situation is that of the qualitative Fig. 4. (b) The same parameters as in (a) and $\varepsilon = \sigma = 0.1$, the two prefocal lines δ_{Q_1} and δ_{Q_2} merge (see the situation described in the qualitative Fig. 6). (c) The same parameters as in (a) and $\varepsilon = -0.2$, $\sigma = 0.3$. The two branches of LC_{-1} change their location with respect to the four sectors defined by their asymptotes. (d) A larger view of basins of (c).

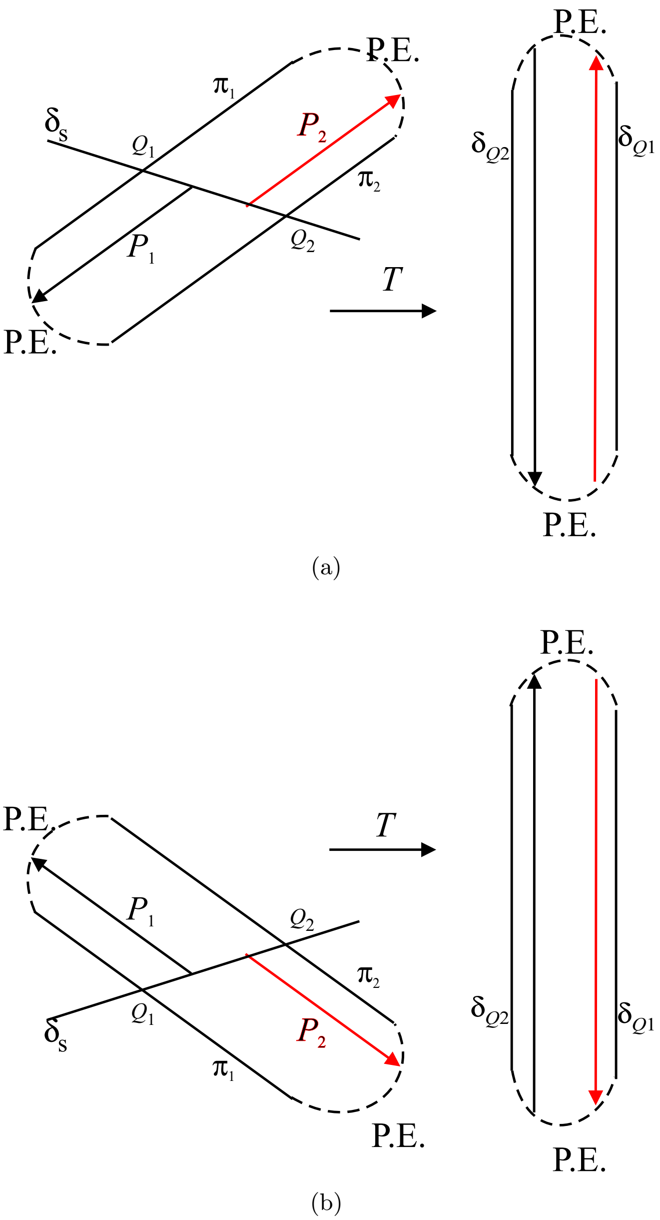


Fig. 8. Qualitative pictures of the foliation structure generated by the map (18) in the region between two prefocal curves. The mechanism of the foliation of the forward image by T is represented for the case (a) $\sigma > \varepsilon$ and (b) $\sigma < \varepsilon$. The points close to δ_s are mapped by T into points close to the Poincaré Equator (P.E.).

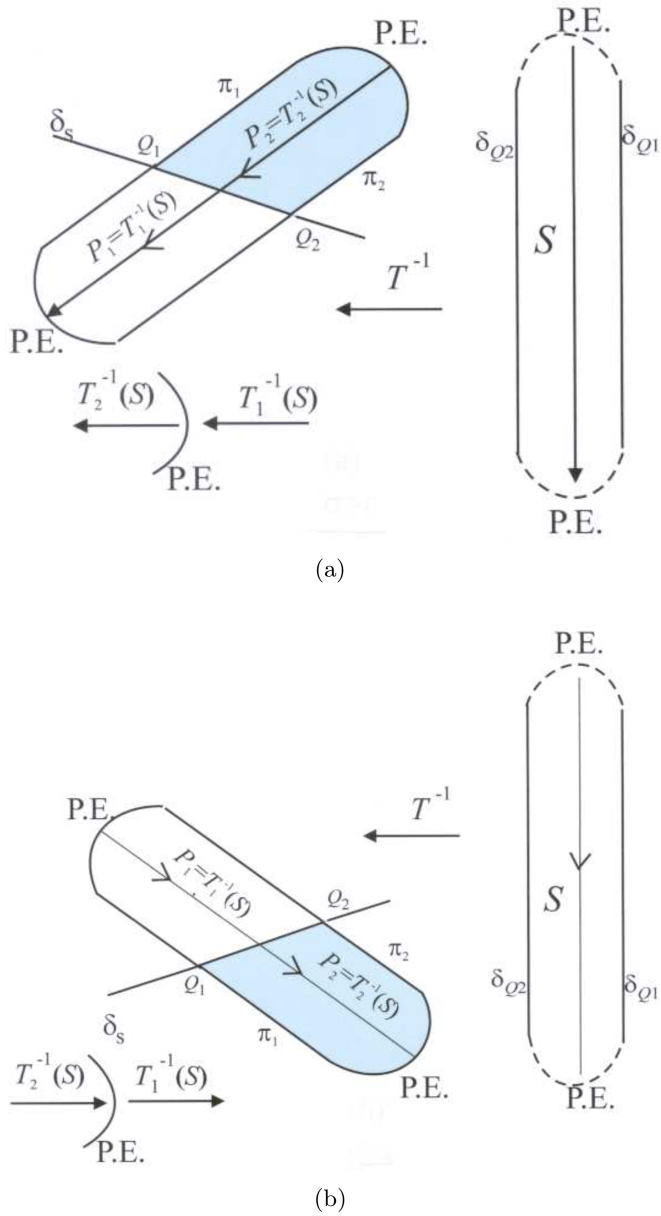


Fig. 9. The mechanism of the preimages of the unbounded strip S is represented by qualitative pictures for the case (a) $\sigma > \varepsilon$ and (b) $\sigma < \varepsilon$.

In Fig. 9, similar pictures show the mechanism of the preimages of the unbounded strip S ($T_1^{-1}(S)$ in white and $T_2^{-1}(S)$ in light blue). The focalizing inverse in the upper and lower parts of S, close to the P.E., is the one giving preimages close to δ_s. Thus in Fig. 9(a), for $\sigma > \varepsilon$, the upper part of the prefocal lines δ_{Q_i} is focalized by T_1^{-1} and the lower part by T_2^{-1}; while the *vice versa* occurs for $\sigma < \varepsilon$, i.e. T_2^{-1} focalizes the upper part and T_1^{-1} the lower one.

5.5. *First example of formation of crescents*

Consider the map T given in (18) with the following parameter values: $\alpha = 0.5$, $\gamma = 0.5$, $\beta = -1.7$, $\sigma = -0.4$, $\varepsilon = 0$; in Fig. 10(a) we show the basin of attraction $\mathcal{B}$ of the stable fixed point O, which is again represented by the red area, and separated by the set of divergent trajectories (white area) by a frontier $\mathcal{F} = \partial\mathcal{B}$. At these parameter values the tangent lines to LC_{-1} in the focal points have slope $\varepsilon = 0$ so that the lines π_i have equation $y = y(Q_i)$ defining a horizontal strip (that is, $y = \beta$ through Q_1 and $y = \beta + (\gamma\sigma/\alpha)$ through Q_2). It is evident that a portion of the frontier is very close to the critical curve LC in the lower part of Fig. 10(a) (see the arrow), and is very close also to the prefocal lines (below the two tangency points V_i, $i = 1, 2$). As α decreases, the frontier crosses LC [see Fig. 10(b)] causing the appearance of a portion of basin $\mathcal{B}$ in the region Z_2, that is, creating a *headland* d_0 (d_0 is bounded by an arc of LC and an arc of $\mathcal{F}$) made up of points having two distinct preimages, which give rise to the appearance of an *island* d_{-1}, a nonconnected part of the basin $\mathcal{B}$, located on the two sides of an arc of LC_{-1} (this situation is due to the fact that $\mathcal{B}\cap\overline{Z}_2$ is nonconnected, as shown in [Mira *et al.*, 1994], see also [Mira *et al.*, 1996a; Abraham *et al.*, 1997]. That is, $d_{-1} = T^{-1}(d_0)$ is given by two components, $d_{-1} = d_{-1,1} \cup d_{-1,2} = T_1^{-1}(d_0) \cup T_2^{-1}(d_0)$, where $T_1^{-1}(d_0)$ is on the left of LC_{-1} in the figure, while $T_2^{-1}(d_0)$ is on the right of the same segment of LC_{-1}. In Fig. 10(b), the island d_{-1} has no contact with the focal point Q_2, neither with the tangent line π_2 but we can see that d_0 is not far from the

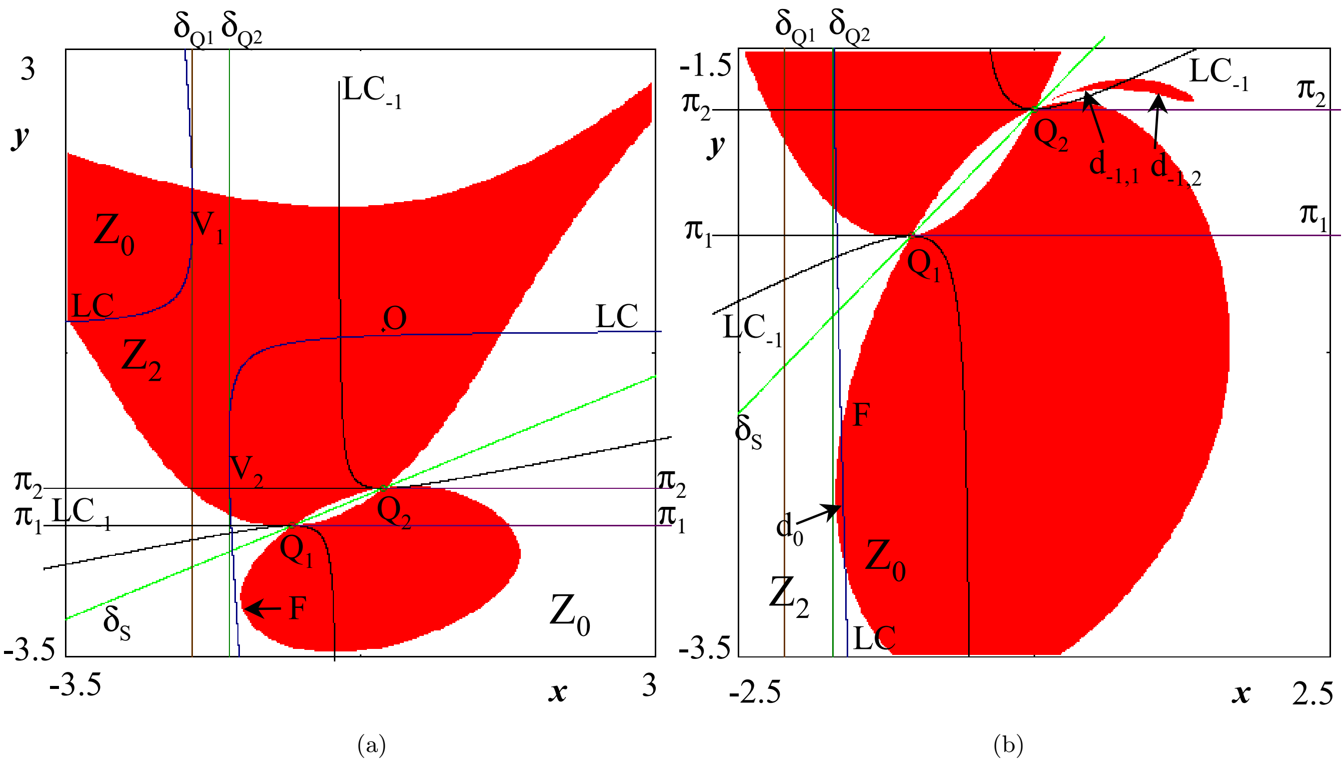
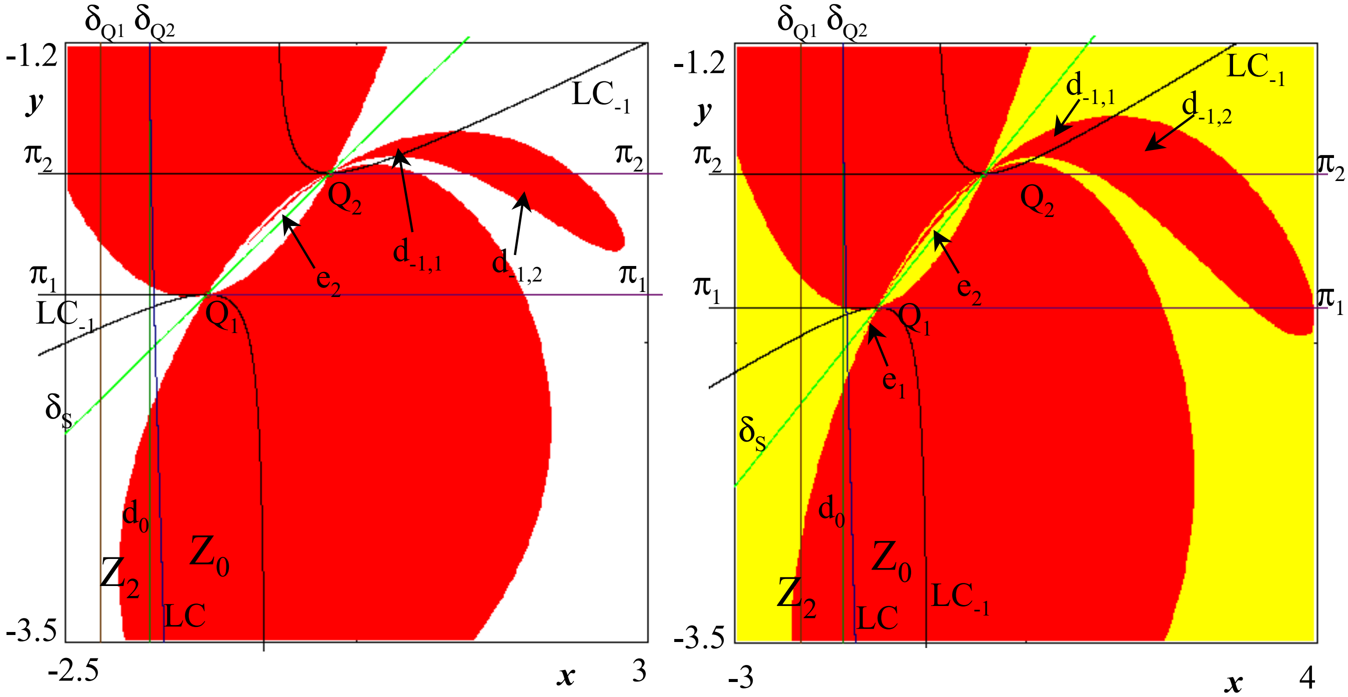
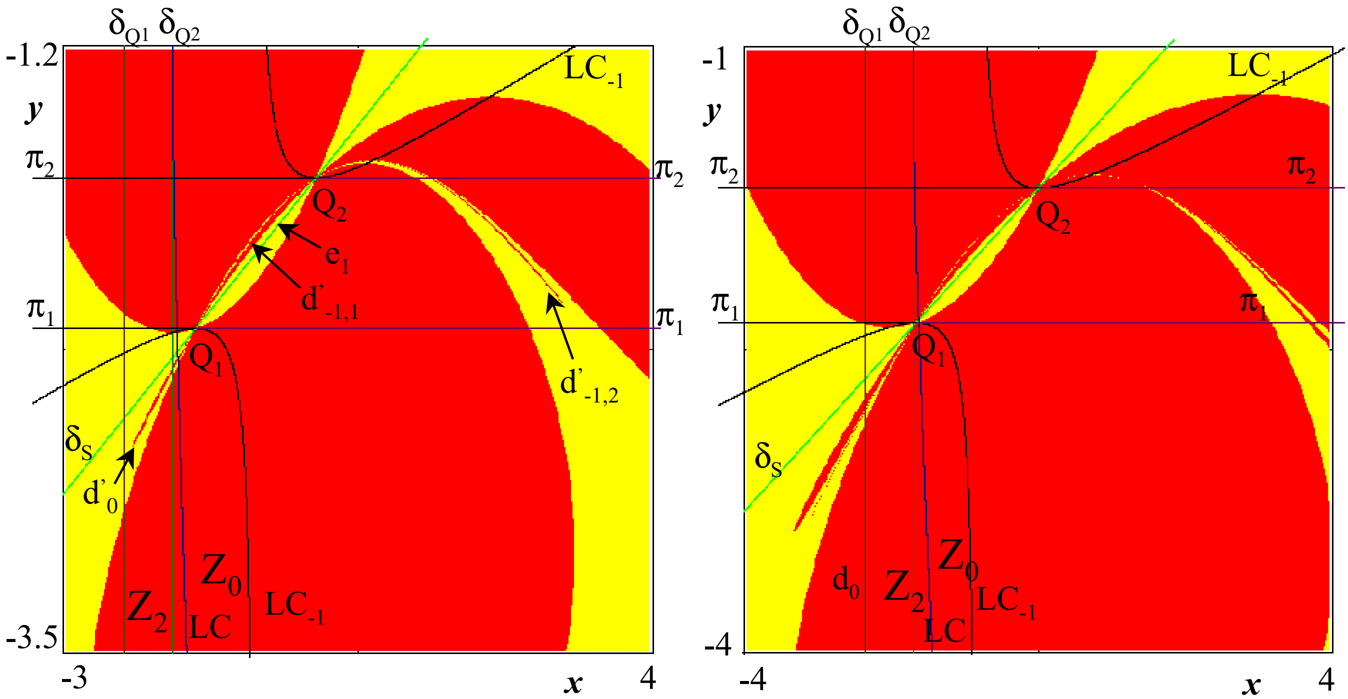


Fig. 10. Lobes and crescent of a basin generated by the map T given in (18) with $\varepsilon = 0$, $\gamma = 0.5$, $\beta = -1.7$, $\sigma = -0.4$, $\varepsilon = 0$. The basin of attraction of the stable fixed point O is red colored, the set of divergent trajectories is the white area, whereas the yellow area in (d)–(f) represents the basin of a stable cycle of period 4. (a) Basins obtained with the parameter $\alpha = 0.5$. (b) Parameter $\alpha = 0.48$. As described in [Mira *et al.*, 1994], d_0 (bounded by a LC segment and an arc of the basin boundary) is a headland, whose preimage $d_{-1} = T^{-1}(d_0)$ is made up of two components $d_{-1} = d_{-1,1} \cup d_{-1,2} = T_1^{-1}(d_0) \cup T_2^{-1}(d_0)$, forming an island (nonconnected part of the red basin). (c) Parameter $\alpha = 0.43$. Just before attaining this value with α decreasing, the contact of the boundary ∂d_0 of d_0 with δ_{Q_2}, causes at the same time the contact between $d_{-1,1}$ and the focal point Q_2 on one side (not smooth, i.e. with a cusp point in Q_2, as described in Part I) and a smooth contact between $d_{-1,2}$ and the line π_2 on the other side. Then for $\alpha = 0.43$ two lobes area is created. The crossing of δ_{Q_2} at two distinct points below V_2 causes the creation of a lobe issuing from the focal point Q_2, $T_1^{-1}(d_0)$ now includes the lobe l_2. (d) Parameter $\alpha = 0.39$. This figure is obtained after the contact between ∂d_0 and δ_{Q_1} which leads to a nonsmooth contact of l_2 has the focal point Q_1, thus creating a crescent c_1 between Q_1 and Q_2, while on the other side $d_{-1,2}$ becomes tangent to π_1. (d) Shows that a d_0 part now crosses the prefocal δ_{Q_1} at two distinct points below V_1. This part has a rank-1 preimage (by T_1^{-1}) which is a lobe l_1 issuing from Q_1. The rank-1 preimage (by T_2^{-1}) is the portion of $d_{-1,2}$ below the tangent line π_1. (e) $\alpha = 0.35$; (f) $\alpha = 0.3$. (e) and (f) Show the birth of a new headland d_0', a new island $d_{-1}' = d_{-1,1}' \cup d_{-1,2}'$, and after crossing through δ_{Q_2} a new lobe l_2'.



(c) (d)

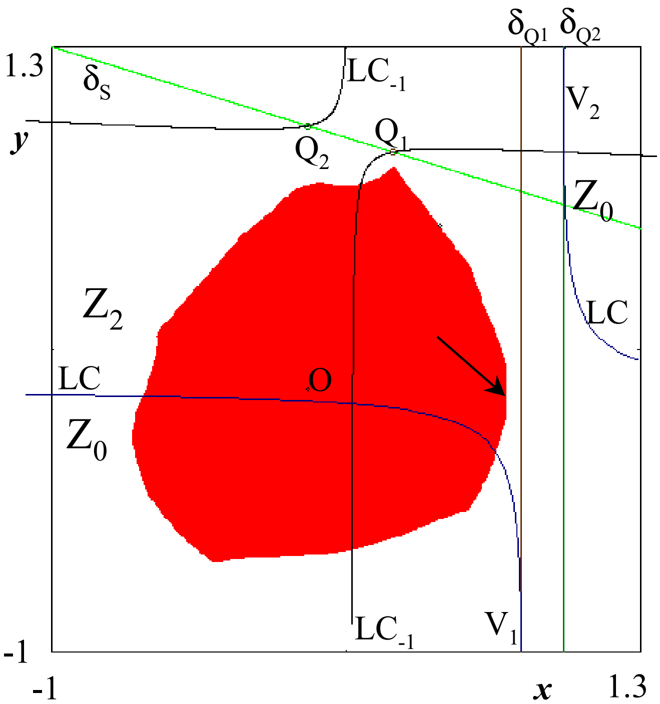


(e) (f)

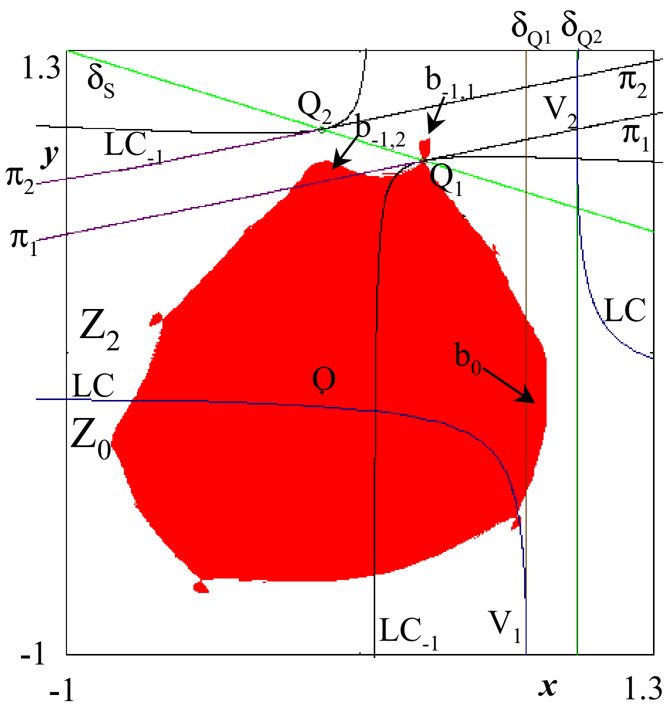
Fig. 10. (*Continued*)

prefocal line δ_{Q_2}, and as the parameter α is further decreased, a contact will occur between ∂d_0 ($\subset \mathcal{F}$) and δ_{Q_2} causing at the same time the contact between $d_{-1,1}$ and the focal point Q_2 on one side (not smooth, i.e. with a cusp point in Q_2, as described in Part I) and a smooth contact between $d_{-1,2}$ and the line π_2 on the other side. The crossing of δ_{Q_2} at two distinct points below V_2 causes the creation of a lobe issuing from the focal point Q_2, $T_1^{-1}(d_0)$ now includes the lobe l_2, as shown in Fig. 10(c). We can also see that ∂d_0 is already close to the other prefocal line δ_{Q_1} (and also here involving points below the tangency point V_1). Clearly, as d_0 approaches δ_{Q_1} the portion l_2 of its preimage is approaching the focal point Q_2 on one side (by $T_1^{-1}(d_0)$) and on the other side (by $T_2^{-1}(d_0)$) a part of $d_{-1,2}$ is approaching the tangent line π_1. At the contact between ∂d_0 and δ_{Q_1}, the arc l_2 has a nonsmooth contact with the focal point Q_1, thus creating a crescent between Q_1 and Q_2, while on the other side $d_{-1,2}$ becomes tangent to π_1. In Fig. 10(d), we can see that a portion of d_0 now crosses the prefocal δ_{Q_1} at two distinct points below V_1, and this portion has a rank-1 preimage (by T_1^{-1}) which is a lobe issuing from Q_1 [see l_1 in Fig. 10(d)]. The rank-1 preimage (by T_2^{-1}) is the portion of $d_{-1,2}$ below the tangent line π_1. As α is further decreased, the lobe l_1 issuing from Q_1 again approaches the critical curve LC and crosses it creating a new *headland* d'_0 and a new *island* d'_{-1}, and after the crossing of δ_{Q_2} a new lobe l'_2, belonging to $T_1^{-1}(d'_0)$ appears, issuing from Q_2. In Fig. 10(e) we can see that d'_0 is quite close to the other prefocal line δ_{Q_1}, so that the lobe l'_2 is approaching the other focal point Q_1 preparing a new crescent and a new lobe issuing from Q_1, and so on, the process is repeated in the same way. In Fig. 10(f) we can also see that the second lobe issuing from Q_1 crosses the critical curve LC and the prefocal lines, thus further lobes and crescents are created. We also note that in Figs. 10(d)–10(f) the frontier $\mathcal{F}$ separates the basin $\mathcal{B}$ of the origin O from the basin (light blue points) of another attractor at finite distance, a stable cycle of period 4 which, along the path followed in the parameter space described in Fig. 10, appeared via a saddle-node bifurcation at $\alpha \cong 4$.

The structure of the basins shown up to now is still quite simple, mainly because the focal points belong to the region Z_0 so that they have no preimages. When the focal points belong to Z_2 we may expect that the contact bifurcations associated with the prefocal curves and the focal points may give rise to more complex structures, as we shall see in the following example.

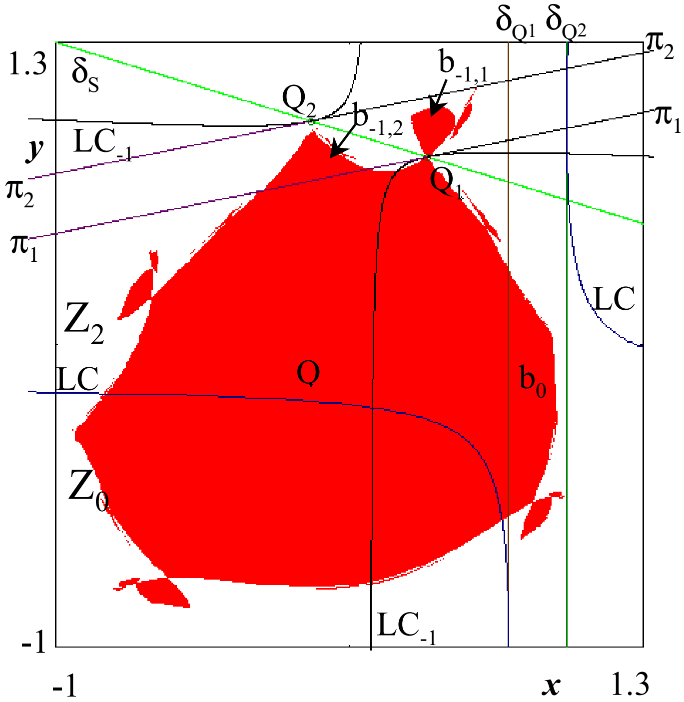


(a)

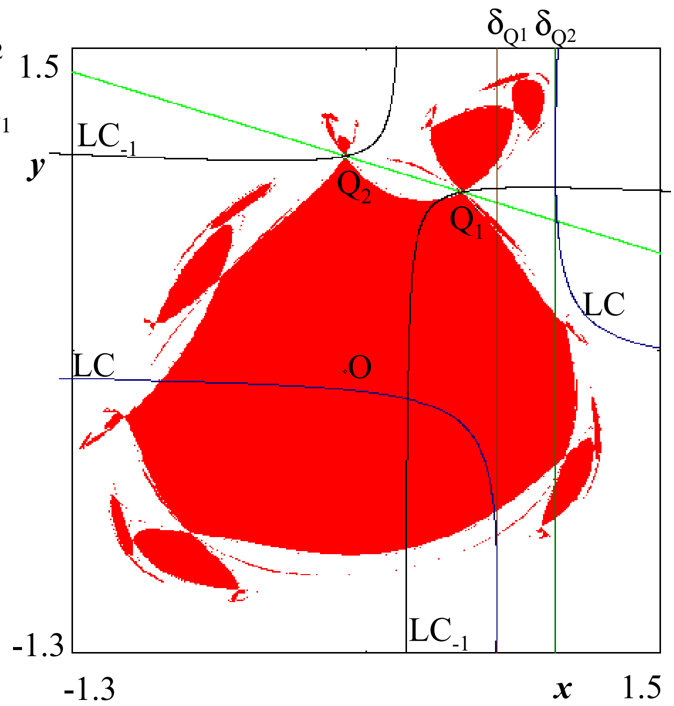


(b)

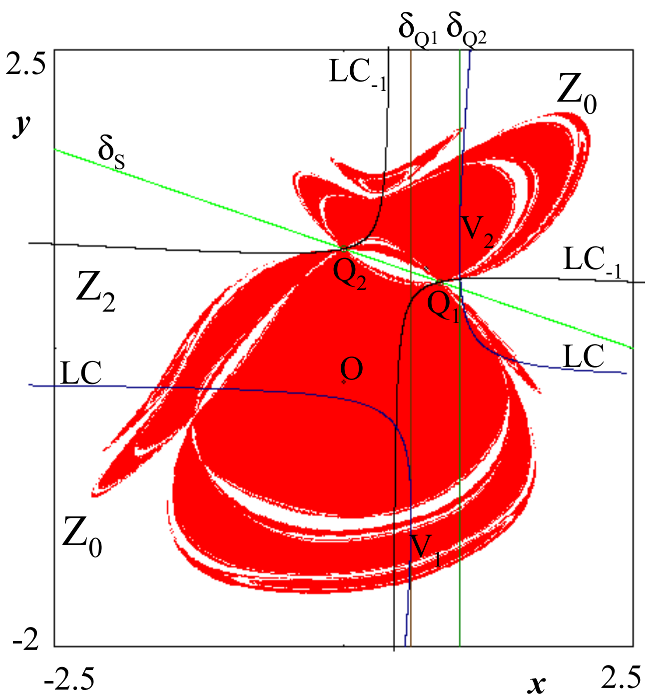
Fig. 11. Another example of lobes and crescents generation by the map T given in (18) with $\varepsilon = -0.2$, $\gamma = 0.5$, $\beta = 1$, $\sigma = 0.3$. The basin of attraction of the stable fixed point O is red colored, the set of divergent trajectories is the white area. (a) Parameter $\alpha = -1.5$. (b) Parameter $\alpha = -1.25$; a lobe appears from Q_1. (c) $\alpha = -1.1$; (d) $\alpha = -0.9$; (e) $\alpha = -0.6$; (f) $\alpha = -0.4$. (c)–(f) show the birth of lobes and crescents, as in Fig. 10.



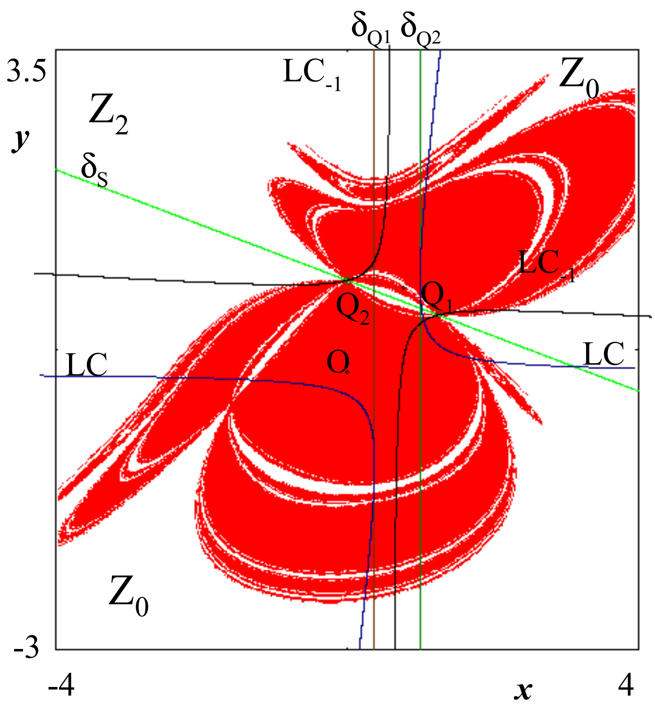
(c)



(d)



(e)



(f)

Fig. 11. (*Continued*)

5.6. *Second example of formation of crescents*

Consider the map T given in (18) with the following parameter values: $\alpha = -1.5$, $\gamma = 0.5$, $\beta = 1$, $\sigma = 0.3$, $\varepsilon = -0.2$; in Fig. 11(a), the basin of attraction $\mathcal{B}$ of the stable fixed point O is again red colored, and separated by the set of divergent trajectories (white area) by a frontier $\mathcal{F} = \partial\mathcal{B}$ which includes the stable set of the unstable fixed point P (located on $\mathcal{F}$). The two focal points belong to the region Z_2. At these parameter values we can see that $\mathcal{F}$ is approaching the prefocal line δ_{Q_1}, and consequently also the focal point Q_1 (and its preimages of any rank as well). As the parameter α is increased, a contact occurs causing the crossing of δ_{Q_1} at two points located between the two tangency points V_1 and V_2. In Fig. 11(b), obtained after the crossing, we can see that a portion of the basin $\mathcal{B}$, denoted by b_0, is between the two prefocal lines (we are in a situation similar to that of the arc η described in the qualitative picture in Fig. 5(c), so that one of its preimages issues from the focal point Q_1, given by $b_{-1,1} = T_1^{-1}(b_0)$, while the other preimage, $b_{-1,2} = T_2^{-1}(b_0)$, is a portion of the same basin $\mathcal{B}$ in the strip above the tangent line π_1 [see the arrow in Fig. 11(b)]. If Fig. 11(c), at a higher value of α, we see that the portion of basin b_0 is approaching also the other prefocal line δ_{Q_2} below the point V_2 (where the inverse T_2^{-1} focalizes), and inside the strip between the lines π_i, $i = 1, 2$, the preimage $T_2^{-1}(b_0)$ approaches the focal point Q_2. It is clear that a crossing of δ_{Q_2} will cause the appearance of a lobe issuing the focal point Q_2, given by the inverse T_2^{-1}, as shown in Fig. 11(d) [we are in a situation similar to that of the arc ω described in the qualitative picture in Fig. 5(c)].

As α is further increased, lobes and crescents are formed, and the structure of the basin is more complex, both when two focal points are in the region Z_2 [as shown in Fig. 11(e)] and when only one focal point belongs to Z_2 [as shown in Fig. 11(f)].

6. Conclusions

The first part [Bischi *et al.*, 1999, Part I] of the study of two-dimensional maps, defined by at least one function with a vanishing denominator, concerned invertible maps in generic case and noninvertible maps in a nongeneric situation. It was shown that the presence of a vanishing denominator requires the introduction of new concepts like the *set of nondefinition*, the *focal points* and the *prefocal curves*. Their definitions imply that at least one inverse, called the "*focalizing inverse*", must exist along a prefocal curve. These concepts allowed to give a geometric characterization to some new bifurcations which change the structure of the basin boundaries, and to describe a new mechanism for the occurrence of homoclinic bifurcations.

With respect to that paper, the present one is essentially devoted to some generic global dynamical properties of noninvertible two-dimensional maps. More particularly, considering the simplest case for which only two inverses exist (case of $(Z_0 - Z_2)$ maps), it is shown that the focalizing inverse is generally only one, but not always the same in the whole prefocal set. Then when two distinct focal points exist, their related prefocal sets may be either different sets (generic case considered in this Part II) or the same set (as considered in Part I). This qualitative difference is associated with a change in structure in the Riemann foliation of the plane. If two focalizing inverses at each point of a prefocal set exist, then this situation may be the consequence of the merging of two prefocal sets. In the generic case, when two distinct focal points exist, the fact that their related prefocal sets are different induces a more complex behavior of the preimage of a curve intersecting the prefocal curves at two points. In other words, the mechanism of formation of lobes and crescents issuing from the focal points, frequently met in basin boundaries, presents a higher level of complexity.

Another difference, between the properties of the noninvertible maps considered in Part I and those of the present paper, lies in the situation of the critical curve LC, and its rank-1 preimage LC_{-1}, with respect to the prefocal curves, and the focal points. In the cases considered in Part I, no focal points belong to the closure $\overline{LC_{-1}}$ of LC_{-1}, and the prefocal curve (common to two focal points) is an asymptote of LC. In this paper we considered the generic case of focal points belonging to $\overline{LC_{-1}}$, which implies that the critical curve LC is tangent to the prefocal curves.

All these properties, characterizing the generic situation of two-dimensional noninvertible maps, lead to new bifurcations which change the structure of the basin boundaries. They also result in consequences for maps defined in the whole plane (for example polynomial maps) due to the presence of a vanishing denominator in at least one inverse map. Part I has shown that for noninvertible maps the locus of points where the denominator of some inverse vanishes may separate regions of the phase plane whose points have a different number of preimages. Such points are the two-dimensional analogue of a horizontal asymptote of a one-dimensional map (which corresponds to a vertical asymptote of at least one inverse) which separates zones with different numbers of preimages. Another property, related to such maps defined in the whole plane, will be a more complex mechanism of formation of *knots* of a chaotic attractor, with respect to the one described in Part I for an invertible map (see Sec. 3.1). From this point of view, the study of examples of noninvertible maps defined in the whole plane, with a vanishing denominator in at least one inverse map in the generic case considered in this paper, presents a real interest.

Important bifurcations, related to two-dimensional maps, have not been considered in the present paper, and will be the object of further studies. It is the case of bifurcations: due to the merging of focal points, the merging of prefocal sets different from the one mentioned in this paper, the merging of focal points and fixed points, the contact between prefocal curves and critical curves. Extensions to the case of m-dimensional noninvertible maps, $m > 2$, also offer a wide field of new results. The fundamental concepts of a *set of nondefinition*, *focal points*, and *prefocal curves*, introduced for $m = 2$, can now be generalized. So the dimension p of a *focal set* may be such that $m - 1 > p \geq 0$, and the dimension q of a *prefocal set* such that $m > q \geq 1$, this depending on the numerator and the denominator forms. It results in a larger richness of the bifurcation situations.

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