International Journal of Bifurcation and Chaos, Vol. 15, No. 2 (2005) 451–496
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PLANE MAPS WITH DENOMINATOR. PART III: NONSIMPLE FOCAL POINTS AND RELATED BIFURCATIONS

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Received March 15, 2004; Revised April 29, 2004

This paper continues the study of the global dynamic properties specific to maps of the plane characterized by the presence of a denominator that vanishes in a one-dimensional submanifold. After two previous papers by the same authors, where the definitions of new kinds of singularities, called focal points and prefocal sets, are given, as well as the particular structures of the basins and the global bifurcations related to the presence of such singularities, this third paper is devoted to the analysis of nonsimple focal points, and the bifurcations associated with them. We prove the existence of a one-to-one relation between the points of a prefocal curve and arcs through the focal point having all the same tangent but different curvatures. In the case of nonsimple focal points, such a relation replaces the one-to-one correspondence between the slopes of arcs through a focal point and the points along the associated prefocal curve that have been proved and extensively discussed in the previous papers. Moreover, when dealing with noninvertible maps, other kinds of relations can be obtained in the presence of nonsimple focal points or prefocal curves, and some of them are associated with qualitative changes of the critical sets, i.e. with the structure of the Riemann foliation of the plane.

Keywords: Focal points; prefocal sets; bifurcations; noninvertible maps; vanishing denominator.

1. Introduction

Two previous papers [Bischi *et al.*, 1999, 2003], denoted as Parts I and II henceforth, have been devoted to the study of some global dynamical properties of two-dimensional maps that include at least a fraction $N(x,y)/D(x,y)$, with denominator $D(x,y)$ that vanishes in a one-dimensional subset of the plane (see also [Bischi *et al.*, 2000]). In Part I the *set of nondefinition* is defined as the locus of points where a denominator vanishes, and the definitions of new kinds of singularities are given, like the *focal points* and the *prefocal curves*. Roughly speaking, a *prefocal curve* is a set of points which are mapped (or "focalized", as we shall say for short) into a single point by the inverse map (if the map is invertible), or by at least one of the inverses (if the map is noninvertible). Such a point, solution of $N(x,y) = D(x,y) = 0$, is called *focal point*. The presence of these singularities may cause the occurrence of some global bifurcations, that change the qualitative structure of the attracting sets or of the basins of attraction, due to contacts of a prefocal curve with other singular sets, such as basins' boundaries or critical curves. In Parts I and II we have evidenced some of these global bifurcations, that cause the creation of basin structures specific to maps with denominator, called *lobes* and *crescents*, and we explained them in terms of contacts between basin boundaries and prefocal curves (see also [Bischi & Gardini, 1997, 1999; Mira, 1999; Bischi *et al.*, 2001a]). These structures have been recently observed in discrete dynamical systems of the plane arising in different contexts, see e.g. [Yee & Sweby, 1994; Billings & Curry, 1996; Bischi & Naimzada, 1997; Brock & Hommes, 1997; Billings *et al.*, 1997; Gardini *et al.*, 1999; Bischi *et al.*, 2001b; Foroni *et al.*, 2003].

The properties of these new singularities have been studied by considering the image of an arc crossing through a *focal point*. In the generic case, given by a focal point which is simple (i.e. located at a transverse intersection of the curve of vanishing denominator and that of vanishing numerator), it has been shown that a one-to-one correspondence is obtained between the slopes of the arcs through a focal point and the points in which their images cross the corresponding prefocal curve. This implies that the preimages of any curve crossing the prefocal set at two points include a loop with a knot in the focal point, and this is the basic mechanism leading to the formation of *lobes*. In the case of noninvertible maps, lobes issuing from distinct focal points may merge, giving rise to particular structures of the basins called *crescents* (see Part I).

We remark that the existence and the localization of the focal points is often visible at first sight by a quick inspection of the structure of the basins, when these are characterized by the presence of lobes and crescents. Moreover, their coordinates can be easily computed by solving the system $N(x, y) = D(x, y) = 0$. Instead, the localization of the prefocal set needs some more steps of analytical determination.

From the definition of focal point, given in Part I, it follows that each focal point is associated with a prefocal curve. As argued in Part II, in the generic case distinct focal points are associated with distinct prefocal curves: In the case of invertible maps, each prefocal curve is "focalized" by the inverse map into the corresponding focal point, and in the case of noninvertible maps one of the inverses "focalizes" (not necessarily the same along the whole prefocal curve), and is called "*focalizing inverse*". So, the role played by the presence of two or more inverses at a point of a prefocal curve is not obvious. In fact, from the definitions of focal points, at least one "focalizing inverse" must exist that maps arcs crossing through the prefocal curve into arcs through the corresponding focal point. In Part II we have shown that the focalizing inverse is generally only one, but not always the same along the whole prefocal curve. However, it may happen that several focal points are associated with the same prefocal curve, one for each different "focalizing inverse". This corresponds to the case of simple focal points associated with a nonsimple (i.e. double, or, more generally, multiple) prefocal curve, and it may be the result of a bifurcation characterized by the merging of two (or more) prefocal curves without merging of corresponding focal points. So, some examples described in Part I, for which several focal points are associated with a given prefocal curve, can be considered as bifurcation cases due to the merging of prefocal curves.

In this paper the assumption of simple focal points is relaxed. Some bifurcations not considered in Parts I and II, related to the merging of focal points, are presented, as well as some bifurcations due to the merging of prefocal curves, that can be associated with qualitative changes of structure in the Riemann foliation of the plane (for the definitions and properties related to the Riemann foliation of two-dimensional noninvertible maps we refer to [Mira *et al.*, 1996a; Mira *et al.*, 1996b]).

Another remarkable bifurcation, that will be briefly discussed in this paper, is related to the merging of a focal point and a fixed point. All these situations will be denoted as *bifurcations of second class*, in order to distinguish them from the bifurcations considered in Part I and in [Bischi *et al.*, 2000], denoted as *bifurcations of first class*, which are related to contacts between a prefocal curve, or a set of nondefinition, with arcs of phase curves (like those constituting basin boundaries, or stable and unstable sets of saddles).

We are rather far from a complete and systematic understanding of the effects of these bifurcations, and the results given in this paper constitute only a first step towards this goal. For this reason, we prefer to illustrate our results and conjectures through a collection of worked examples, a sort of pedagogical tour through the effects of some bifurcations of second class observed in some exemplary families of maps given in the form $T: (x, y) \rightarrow (x', y')$, where $x' = F(x, y)$, $y' = G(x, y) = N(x, y)/D(x, y)$, with $N(x, y)$ and $D(x, y)$

sufficiently smooth functions such that the denominator $D(x,y)$ vanishes in a one-dimensional submanifold of the plane.

The paper is organized as follows. In Sec. 2 we remind some definitions and generic properties. Section 3 is devoted to the study of properties of nonsimple focal points, i.e. solutions of the system $N(x,y) = D(x,y) = 0$ such that $\overline{N}_x\overline{D}_y = \overline{N}_y\overline{D}_x$, and in the several subsections of Sec. 3 we analyze the different situations characterized by the vanishing of two, three or four of the partial derivatives $\overline{N}_x$, $\overline{D}_y$, $\overline{N}_y$ and $\overline{D}_x$ computed at the nonsimple focal point. In Sec. 4 we give some examples of noninvertible maps, as some parameters are varied, such that the merging of focal points and/or prefocal curves occur. These examples allow us to show some consequences of the global bifurcations described in Sec. 3 on the structure of the basins of attraction, as well as some qualitative changes of the critical set (set of points having two merging preimages) induced by these bifurcations. The latter question leads us to state that some of the bifurcations of second class, studied in this paper, may cause important qualitative changes of the Riemann foliation of the plane associated with the structure of the critical sets of a noninvertible map.

2. Some Definitions and Generic Properties

In this section we recall some definitions and properties already given in Parts I and II, and we argue about generic and nongeneric situations for maps of the plane with focal points. In order to simplify the exposition, and to follow the notations used in Parts I and II, we assume that only one of the two functions that define the map T has a vanishing denominator:

$$T: \begin{cases} x' = F(x,y) \\ y' = G(x,y) = \dfrac{N(x,y)}{D(x,y)} \end{cases} \tag{1}$$

where x and y are real variables and the functions $N(x,y)$ and $D(x,y)$ are defined in the whole plane $\mathbb{R}^2$, so that the *set of nondefinition* δ_s of the map T coincides with the locus of points in which the denominator $D(x,y)$ vanishes:

$$\delta_s = \{(x,y) \in \mathbb{R}^2 | D(x,y) = 0\} \tag{2}$$

The recurrence obtained by the iteration of T does not generate terminating trajectories provided that the initial condition belongs to the set E given by

$$E = \mathbb{R}^2 \Big\backslash \bigcup_{k=0}^{\infty} T^{-k}(\delta_s) \tag{3}$$

so that $T: E \to E$.

We recall here the following definition.

Definition 1. A point Q is a focal point of the map (1) if $G(x,y)$ takes the form $0/0$ in Q and there exist smooth simple arcs $\gamma(t)$, with $\gamma(0) = Q$, such that $\lim_{\tau\to 0} T(\gamma(\tau))$ is finite. The set of all such finite values, obtained by taking different arcs $\gamma(t)$ through Q, is the prefocal set δ_Q.

In order to recall the main geometric properties related to the concepts of *focal point* and *prefocal curve*, we consider a smooth simple arc γ transverse to δ_s, represented by the parametric equations

$$\gamma(\tau): \begin{cases} x(\tau) = x_0 + \xi_1\tau + \xi_2\tau^2 + \cdots \\ y(\tau) = y_0 + \eta_1\tau + \eta_2\tau^2 + \cdots \end{cases} \quad \tau \neq 0 \tag{4}$$

where (x_0, y_0) is the point in which γ intersects δ_s. To study the shape of its image $T(\gamma)$ we assume that the arc γ is deprived of (x_0, y_0), so that it can be seen as the union of two disjoint pieces, say $\gamma = \gamma_- \cup \gamma_+$, obtained from (4) with $\tau < 0$ and $\tau > 0$ respectively. As $(x_0, y_0) \in \delta_s$ we have $D(x_0, y_0) = 0$. Let us first assume that $N(x_0, y_0) \neq 0$, then

$$\begin{aligned} &\lim_{\tau\to 0^+} T(\gamma(\tau)) = (F(x_0,y_0), \pm\infty) \quad \text{and} \\ &\lim_{\tau\to 0^-} T(\gamma(\tau)) = (F(x_0,y_0), \pm\infty) \end{aligned} \tag{5}$$

where $F(x_0, y_0)$ is a finite value. This means that the image $T(\gamma)$ is made up of two disjoint unbounded arcs asymptotic to the line of equation $x = F(x_0, y_0)$.

A different situation may occur if also $N(x_0, y_0) = 0$, so that in the limit (5) the second component assumes the form $0/0$. This implies that, in contrast with (5), the limit may give a finite value, so that the image $T(\gamma)$ is a bounded arc crossing the line $x = F(x_0, y_0)$ at the point $(F(x_0, y_0), y)$, where

$$y = \lim_{\tau\to 0} G(x(\tau), y(\tau)) \tag{6}$$

Of course, the value y of the limit (6) depends on the arc γ. According to Definition 1, the set of points $(F(x_0, y_0), y)$, where y is a finite value computed by (6), constitutes the *prefocal curve* δ_Q associated with the focal point Q.

In Parts I and II we have only considered *simple focal points*, defined as focal points $Q = (x_0, y_0)$ such that

$$\overline{N}_x\overline{D}_y - \overline{N}_y\overline{D}_x \neq 0 \tag{7}$$

where $\overline{N}_x = (\partial N/\partial x)(x_0, y_0)$ and analogously for the other partial derivatives. This condition implies that Q is a simple root of the algebraic system

$$\begin{cases} N(x,y) = 0 \\ D(x,y) = 0 \end{cases} \tag{8}$$

i.e. it is located at a transverse intersection of the curves $N(x,y) = 0$ and $D(x,y) = 0$. In Parts I and II we have shown that, in the case of a simple focal point, a one-to-one correspondence can be obtained between the slope m of γ in Q and the point $(F(Q), y)$ in which $T(\gamma)$ crosses δ_Q. Indeed, let us consider an arc γ, with parametric representation (4), crossing through a focal point $Q = (x_0, y_0)$, and assume that the numerator $N(x,y)$ and the denominator $D(x,y)$ of the second component $G(x,y)$ of T are smooth functions. Since both these functions vanish in Q they can be expressed as

$$\begin{aligned} N(x,y) &= \overline{N}_x(x - x_0) + \overline{N}_y(y - y_0) + O_2 \\ D(x,y) &= \overline{D}_x(x - x_0) + \overline{D}_y(y - y_0) + O_2' \end{aligned} \tag{9}$$

where O_2, O_2' represent terms of higher order. If Q is a simple focal point then

$$\lim_{\tau \to 0} G(\gamma(\tau)) = \frac{\overline{N}_x\xi_1 + \overline{N}_y\eta_1}{\overline{D}_x\xi_1 + \overline{D}_y\eta_1}. \tag{10}$$

from which a one-to-one correspondence between the slope $m = \eta_1/\xi_1$ of the arc γ in Q and the point $(F(Q), y)$ in which the image $T(\gamma)$ crosses the prefocal curve δ_Q is obtained, is defined as

$$m \to (F(Q), y(m)), \quad \text{with} \quad y(m) = \frac{\overline{N}_x + m\overline{N}_y}{\overline{D}_x + m\overline{D}_y} \tag{11}$$

or, equivalently

$$(F(Q), y) \to m(y) \quad \text{with} \quad m(y) = \frac{\overline{D}_x y - \overline{N}_x}{\overline{N}_y - \overline{D}_y y}. \tag{12}$$

So, if Q is simple, the point $(F(Q), y)$ spans the whole line $x = F(Q)$ as m varies (the complete proof is in Part II). This is the generic occurrence considered in Parts I and II, where it is stressed that the existence of a prefocal curve may have interesting consequences on the structure of the basins of attraction, due to the action of the inverse (or the inverses, if T is a noninvertible map) applied to an arc which crosses a prefocal curve. In fact, at least one preimage of any arc crossing δ_Q must be "focalized" into an arc through Q. This implies that given an arc crossing δ_Q in two distinct points at least one rank-1 preimage exists which is formed by a loop with knot in Q. This is the basic mechanism which leads to the creation of lobes, a feature that often characterizes the structure of the basins of maps with denominator (see e.g. [Mira, 1999; Bischi & Gardini, 1997, 1999; Bischi & Naimzada, 1997; Billings *et al.*, 1997; Brock & Hommes, 1997; Gardini *et al.*, 1999; Bischi *et al.*, 1999, 2001b]).

In this paper we study the properties of the prefocal sets in the case of nonsimple focal points. First of all we recall that if T is a *noninvertible map* then, by definition, the prefocal set δ_Q cannot belong to a region where no inverses are defined, i.e. $\delta_Q \cap Z_0 = \emptyset$, and any arc crossing δ_Q must be "focalized" through Q by at least one inverse. However, it may occur that several inverses exist which "focalize" δ_Q into distinct (and simple) focal points. That is, when T is noninvertible, several distinct focal points may be associated with the same prefocal curve, and when the focal points are simple each of them has its own one-to-one correspondence (10). Several examples have already been given in Parts I and II. Moreover, in Part II we have seen that prefocal curves having only one focalizing inverse is the generic occurrence, and in the examples given in the present paper we shall see that the existence of at least two distinct inverses which "focalize" in distinct focal points, or in the same focal point, occurs in bifurcation situations, related to the merging of prefocal curves associated with simple focal points, or nonsimple focal points, respectively.

As stressed above, in this paper we investigate the global effects of the merging of focal points or prefocal curves on the qualitative structure of the basins of attraction. Moreover, we shall see that in the case of noninvertible maps with focal points, the presence of nonsimple focal points, as well as the merging of prefocal curves associated with distinct focal points, may also cause important qualitative changes in the structure of the critical curves and, consequently, of the Riemann foliation, by which the properties of the inverses are often represented (see e.g. [Mira *et al.*, 1996a]).

We recall that noninvertible map means "many-to-one", that is, distinct points $p_1 \neq p_2$ may have the same image, i.e. $T(p_1) = T(p_2) = p$. Geometrically, the action of a noninvertible map of the plane can be expressed by saying that it "folds and pleats" the plane, so that the two distinct points p_1 and p_2 are mapped into the same point p. This is expressed by saying that p has several distinct rank-1 preimages, i.e. several inverses are defined in p. In this case, the plane can be subdivided into regions Z_k, $k \geq 0$, whose points have k distinct rank-1 preimages. Generally, as the point (x', y') varies in the plane $\mathbb{R}^2$, pairs of preimages appear or disappear as it crosses the boundaries separating different regions, hence such boundaries are characterized by the presence of at least two coincident (merging) preimages. This leads to the definition of the critical curves, one of the distinguishing features of noninvertible maps. Following the notations of [Gumowski & Mira, 1980] (see also [Mira *et al.*, 1996a; Abraham *et al.*, 1997]), the *critical set* LC (from the French "Ligne Critique") is defined as the locus of points having two, or more, coincident rank-1 preimages, located on a set (*set of merging preimages*) called LC_{-1}. A curve (or a region of the plane) U, that intersects LC_{-1}, is "folded" along LC into the side with more preimages, and the two folded images have opposite orientation. This implies that the map has the Jacobian $\det DT$ with a different sign in the two portions of U separated by LC_{-1}, because T is locally an orientation preserving map near points (x, y) such that $\det DT(x, y) > 0$ and orientation reversing if $\det DT(x, y) < 0$. So, if we denote by J_C the set of points through which we have a change in the sign of the Jacobian of T, we can say that the set LC_{-1} must belong to J_C, i.e. $LC_{-1} \subseteq J_C$.

However, points of LC_{-1} in which the map is differentiable are necessarily points where the Jacobian determinant vanishes: in fact in any neighborhood of a point of LC_{-1} there are at least two distinct points which are mapped by T in the same point, hence the map is not locally invertible in points of LC_{-1}. This implies that for a differentiable map

$$LC_{-1} \subseteq J_0 = \{(x, y) \in \mathbb{R}^2 | \det DT(x, y) = 0\} \quad (13)$$

So, if the map is smooth, $LC_{-1} \subseteq J_C \cap J_0$. However, from the geometric properties of T, we conjecture that the relation $LC_{-1} = J_C \cap J_0$ holds. From the knowledge of the set LC_{-1}, the set LC can be easily obtained as $LC = T(LC_{-1})$. Arcs of LC separate the regions of the phase plane characterized by a different number of real rank-1 preimages. In order to study the unfolding action of the multivalued inverse relation T^{-1} it is useful to consider a region Z_k of the phase plane as the superposition of k sheets, each associated with a different inverse function. Such a representation is known as *Riemann foliation* of the plane. Different sheets are connected by folds joining two sheets, and the projections of such folds on the phase plane are arcs of LC, see e.g. [Mira *et al.*, 1996a, 1996b].

Due to the existence of several different structures of foliations (and related division of the plane into regions Z_k characterized by different numbers, k, of inverses) it is difficult, at this early stage of our study, to give a rigorous and systematic description of the relations between the merging of focal points or of prefocal curves and the qualitative changes induced in the critical curves. We shall give some descriptions of what happens in particular cases, through the examples proposed in the following sections.

For the class of maps (1) considered in this paper, the Jacobian matrix is given by

$$DT(x, y) = \begin{bmatrix} F_x & F_y \\ \dfrac{N_x D - N D_x}{D^2} & \dfrac{N_y D - N D_y}{D^2} \end{bmatrix} \quad (14)$$

and we assume that F_y is not identically zero, i.e. we do not consider the particular case of a triangular map T (whose properties are often related to those of a one-dimensional map), because we are interested in true two-dimensional properties, and in particular we shall assume that $\overline{F}_y$ evaluated at a focal point Q is different from zero. The determinant of the Jacobian matrix is

$$\det DT(x, y) = \frac{A(x, y)}{D^2} \quad (15)$$

where

$$A(x, y) = [F_x(N_y D - N D_y) - F_y(N_x D - N D_x)] \quad (16)$$

the Jacobian $\det DT$ is defined in $\mathbb{R}^2 \backslash \{\delta_s\}$ and LC_{-1} is included in the set of points at which $A(x, y) = 0$. This implies that, in general, *the focal points* (*at which* $N = 0$ *and* $D = 0$), *belong to* LC_{-1} (to be more precise, they belong to the closure $\overline{LC}_{-1}$ of LC_{-1}, as LC_{-1} is not defined at a focal point). The set $\overline{LC}_{-1}$ does not contain focal points when the prefocal set is the result of the merging of two prefocal curves without merging of focal points (see the examples shown in Part I, and the bifurcations described in Part II). By the Implicit Function Theorem it is easy to detect the slope of $\overline{LC}_{-1}$ at each simple focal point Q. In fact, the partial derivative with respect to y, computed at Q, of the expression $A(x, y)$ is $\overline{F}_y(\overline{N}_y\overline{D}_x - \overline{N}_x\overline{D}_y)$, and the partial derivative with respect to x, computed at Q, is $\overline{F}_x(\overline{N}_y\overline{D}_x - \overline{N}_x\overline{D}_y)$. Hence, the slope of $\overline{LC}_{-1}$ at any simple focal point Q is

$$\overline{m}_Q = -\overline{F}_x/\overline{F}_y \tag{17}$$

It is plain that $\overline{m}_Q$ depends on Q, as $\overline{F}_x$ and $\overline{F}_y$ are evaluated at Q. When $F(x, y)$ is a linear or an affine function, then $\overline{F}_x$ and $\overline{F}_y$ are constant, and the slope $\overline{m}$ is the same at each focal point. The property (17) has some implications concerning the critical set $LC = T(LC_{-1})$. In fact, for each simple focal point $Q_i \in LC_{-1}$, the critical curve $LC = T(LC_{-1})$ has a contact with the corresponding prefocal line δ_Q at a point R whose y-coordinate can be computed by (11), with m given in (17):

$$R(Q_i) = (F(Q_i), \overline{y}), \quad \overline{y} = y(\overline{m}) = \frac{\overline{N}_x\overline{F}_y - \overline{F}_x\overline{N}_y}{\overline{D}_x\overline{F}_y - \overline{F}_x\overline{D}_y} \tag{18}$$

The fact that $\overline{LC}_{-1}$ is generally formed by one or more branches which include the focal points, suggests that *a merging of the focal points* (that is, cases with nonsimple focal points) may cause the formation of double points (i.e. loops or crossing of different branches) of LC_{-1}. Of course, such occurrence causes a qualitative change of $LC = T(LC_{-1})$ as well, and consequently a qualitative change of the Riemann foliation of the noninvertible map. Such global bifurcations will be observed in several examples given in the following sections.

From (15) it is also evident that $\overline{LC}_{-1}$ may intersect δ_s at a nonfocal point, i.e. a point (x, y) such that $D(x, y) = 0$ and $N(x, y) \neq 0$, provided that the equality

$$F_yD_x - F_xD_y = 0 \tag{19}$$

holds in (x, y). This suggests that, generally, $\overline{LC}_{-1}$ may intersect δ_s in *isolated* nonfocal points. However, it may occur, in particular cases, that (19) holds at all points of δ_s (i.e. LC_{-1} becomes degenerated, as explained below). From (15)

$$\det DT = \frac{1}{D^2}[D(F_xN_y - F_yN_x) + N(F_yD_x - F_xD_y)],$$

so, if (19) holds in δ_s and D changes sign crossing the set of nondefinition, then the Jacobian $\det DT$ also changes sign across δ_s, as $(F_xN_y - F_yN_x)$ is finite. In this case, even if the map T is not defined at the points of δ_s, we can say that δ_s has properties similar to those of LC_{-1}, and in this sense we can write $\delta_s \subset \overline{LC}_{-1}$ (this will be better explained through the examples in Sec. 4). Such a degeneracy of LC_{-1} occurs when the prefocal set is the result of the merging of two prefocal curves without merging of focal points (examples are given in Parts I and II), and when two sets of nondefinition merge (as we shall see in Examples 4 and 5 in Sec. 4)

3. Nonsimple Focal Points

The first question that we investigate in this paper is what happens when the assumption (7) is relaxed, i.e.

$$\det\begin{bmatrix} \overline{N}_x & \overline{N}_y \\ \overline{D}_x & \overline{D}_y \end{bmatrix} = \overline{N}_x\overline{D}_y - \overline{N}_y\overline{D}_x = 0 \tag{20}$$

so that Q is nonsimple. It is convenient to distinguish different cases in which (20) holds, according to the number and location of zeroes in the matrix of the first-order derivatives of $N(x, y)$ and $D(x, y)$ defined in (20). We shall first consider the case in which (20) holds with all nonvanishing entries of the matrix. Then we shall analyze the cases in which some of these entries vanish. Indeed, if (20) holds and one of the first-order derivatives of $N(x, y)$ or $D(x, y)$ vanishes in Q, then at least another one, belonging to the same row or column of the matrix in (20) must vanish. So, four different cases of two vanishing first-order derivatives will be examined in this section, together with the subcases in which three of them vanish. Finally, the case of all the first-order derivatives of $N(x, y)$ and $D(x, y)$ vanishing in Q will be considered.

Other bifurcation cases, not necessarily related to the presence of a nonsimple focal point, such as the ones related to the merging of prefocal curves not associated with the merging of the corresponding focal points, or those given by the contact of a simple focal point with its prefocal curve, will be discussed in the next sections, throughout the examples.

3.1. $\overline{N}_x\overline{D}_y = \overline{N}_y\overline{D}_x \neq 0$

If (20) holds at a focal point $Q = (x_0, y_0)$ and all the first-order derivatives of $N(x, y)$ and $D(x, y)$ are different from zero, then from (9) $G(x,y)$ can be written as

$$G(x,y) = \frac{\overline{N}_y\left[\dfrac{\overline{N}_x}{\overline{N}_y}(x-x_0)+(y-y_0)\right]+O_2}{\overline{D}_y\left[\dfrac{\overline{D}_x}{\overline{D}_y}(x-x_0)+(y-y_0)\right]+O_2'}$$

where O_2 and O_2' represent higher order terms. In order to determine the prefocal set, let us consider the generic arc $\gamma(\tau)$ given in parametric form (4) with $\xi_1 \neq 0$ and $\eta_1 \neq 0$, so that we have

$$G(\tau) = \frac{\overline{N}_y\left[\dfrac{\overline{N}_x}{\overline{N}_y}\xi_1+\eta_1\right]+\tau(\cdots)}{\overline{D}_y\left[\dfrac{\overline{D}_x}{\overline{D}_y}\xi_1+\eta_1\right]+\tau(\cdots)}$$

It follows that if the arc $\gamma(\tau)$ is not tangent to the singular set δ_s in Q, i.e. the slope $m = \eta_1/\xi_1$ of γ is different from the slope $m_{\delta_s} = -\overline{N}_x/\overline{N}_y = -\overline{D}_x/\overline{D}_y$ of δ_s in Q, then we have

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \frac{\overline{N}_y}{\overline{D}_y}$$

and the image $T(\gamma)$ crosses the line $x = F(Q)$ in the point

$$R = (F(Q), y_R), \quad \text{with} \quad y_R = \frac{\overline{N}_y}{\overline{D}_y} = \frac{\overline{N}_x}{\overline{D}_x} \tag{21}$$

In order to consider the particular case in which the arc γ is tangent to δ_s in Q, let us assume that the functions $N(x,y)$ and $D(x,y)$ are smooth enough, so that we can consider their Taylor expansions with basic point Q

$$G(x,y) = \frac{\overline{N}_x(x-x_0)+\overline{N}_y(y-y_0)+\frac{1}{2}\overline{N}_{xx}(x-x_0)^2+\overline{N}_{xy}(x-x_0)(y-y_0)+\frac{1}{2}\overline{N}_{yy}(y-y_0)^2+O_3}{\overline{D}_x(x-x_0)+\overline{D}_y(y-y_0)+\frac{1}{2}\overline{D}_{xx}(x-x_0)^2+\overline{D}_{xy}(x-x_0)(y-y_0)+\frac{1}{2}\overline{D}_{yy}(y-y_0)^2+O_3'}$$

where O_3 and O_3' are higher order terms, $\overline{N}_{xx} = (\partial^2 N/\partial x\partial x)(x_0,y_0)$, and analogously for the other partial derivatives. If we consider an arc γ, given by (4) with $\xi_1 \neq 0$, $\eta_1 = m\xi_1$, $\xi_2 \neq 0$ and $\eta_2 \neq 0$, tangent to δ_s in Q, i.e. $m = \eta_1/\xi_1 = -\overline{N}_x/\overline{N}_y = -\overline{D}_x/\overline{D}_y$, then we get:

$$G(\gamma(\tau)) = \frac{\overline{N}_x\xi_2+\overline{N}_y\eta_2+\frac{1}{2}\overline{N}_{xx}\xi_1^2+\overline{N}_{xy}\xi_1\eta_1+\frac{1}{2}\overline{N}_{yy}\eta_1^2+\tau(\cdots)}{\overline{D}_x\xi_2+\overline{D}_y\eta_2+\frac{1}{2}\overline{D}_{xx}\xi_1^2+\overline{D}_{xy}\xi_1\eta_1+\frac{1}{2}\overline{D}_{yy}\eta_1^2+\tau(\cdots)}$$

so that

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_x\xi_2+\overline{N}_y\eta_2+\frac{1}{2}\overline{N}_{xx}\xi_1^2+\overline{N}_{xy}\xi_1\eta_1+\frac{1}{2}\overline{N}_{yy}\eta_1^2}{\overline{D}_x\xi_2+\overline{D}_y\eta_2+\frac{1}{2}\overline{D}_{xx}\xi_1^2+\overline{D}_{xy}\xi_1\eta_1+\frac{1}{2}\overline{D}_{yy}\eta_1^2} \tag{22}$$

It follows that, on varying the parameters in the arc γ tangent to δ_s, $T(\gamma)$ spans the whole line $x = F(Q)$. So, this line is the prefocal set related to the nonsimple focal point Q. We now consider the fact that different arcs $\gamma(\tau)$, tangent to δ_s in Q, have different curvatures[1]

$$\chi = \frac{2\xi_1\eta_2 - 2\xi_2\eta_1}{(\xi_1^2+\eta_1^2)^{3/2}} = \frac{2(\eta_2 - m\xi_2)}{\xi_1^2(1+m^2)^{3/2}}. \tag{23}$$

obtained from (4) evaluated at $\tau = 0$ with $\eta_1 = m\xi_1$ and assuming, without loss of generality, $\xi_1 > 0$. From (22) we have

$$\tilde{y} = \frac{\overline{N}_y 2(\eta_2 - m\xi_2) + \xi_1^2(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\overline{D}_y 2(\eta_2 - m\xi_2) + \xi_1^2(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)} = \frac{\overline{N}_y\chi + \dfrac{1}{(1+m^2)^{3/2}}(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\overline{D}_y\chi + \dfrac{1}{(1+m^2)^{3/2}}(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)}$$

[1]We recall that for a curve of the plane parameterized as $x = \varphi(\tau)$, $y = \psi(\tau)$, the curvature is defined as $\chi = (\varphi_\tau\psi_{\tau\tau} - \varphi_{\tau\tau}\psi_\tau)/(\varphi_\tau^2 + \psi_\tau^2)^{3/2}$ where the partial derivatives are evaluated at a given point.

and introducing the angle $\alpha \in [-\pi/2, +\pi/2]$, such that $m = \tan\alpha$, we get

$$\tilde{y}(\chi) = \frac{\overline{N}_y\chi + \cos^3\alpha(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\overline{D}_y\chi + \cos^3\alpha(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)}$$
$$= \frac{\dfrac{\chi\overline{N}_y}{\cos\alpha} + (\overline{N}_{xx}\cos^2\alpha + 2\overline{N}_{xy}\sin\alpha\cos\alpha + \overline{N}_{yy}\sin^2\alpha)}{\dfrac{\chi\overline{D}_y}{\cos\alpha} + (\overline{D}_{xx}\cos^2\alpha + 2\overline{D}_{xy}\sin\alpha\cos\alpha + \overline{D}_{yy}\sin^2\alpha)}\,. \tag{24}$$

It follows that we can define a one-to-one correspondence between the curvatures χ of the arcs through Q with a common slope $m = \tan\alpha$ and the points of the line $x = F(Q)$ expressed by (24), which can easily be inverted. This means that any arc η which crosses the prefocal line at a point $(F(Q), \tilde{y})$ has at least one preimage by T which is an arc through the focal point Q, tangent to the singular set δ_s in Q (with slope m) and curvature $\chi(\tilde{y})$ obtained inverting (24). We have so proved the following proposition:

Proposition 1. *Consider the map (1), and a nonsimple focal point $Q = (x_0, y_0)$ such that $\overline{N}_x\overline{D}_y = \overline{N}_y\overline{D}_x \neq 0$. Then*:

- *the prefocal set δ_Q associated with Q is the line $x = F(Q)$*;
- *any arc γ which is not tangent to the singular set δ_s in Q is mapped by T into an arc crossing the prefocal line at the point R given in (21)*;
- *any arc γ which is tangent to the singular set δ_s in Q is mapped by T into an arc crossing the prefocal line at the point with y-coordinate $\tilde{y}$ given in (22), and there exists a one-to-one correspondence between the curvature χ of such arcs and the points $(F(Q), y)$ of the prefocal line, given by the relation in (24) and its inverse.*

This situation represents a typical bifurcation case, that generally is the consequence of the merging of two focal points related to the creation (or destruction) of a pair of simple focal points, as we shall see in the examples described in the following sections. In fact, a nonsimple focal point Q generally evolves, as some parameters are varied, toward the splitting of Q into a pair of simple focal points, $Q_1 \neq Q_2$, or toward the disappearance of Q. From this point of view, the prefocal line $x = F(Q)$ may be considered, respectively, as the "germ" from which two simple prefocal curves, $x = F(Q_1)$ and $x = F(Q_2)$, are created, or as the "merging" of two simple prefocal curves which disappear.

From a dynamical point of view, Proposition 1 has implications on the qualitative structure of some invariant sets, as those forming the basins' boundaries. In fact, *any arc of the phase plane crossing the prefocal line at a generic point $(F(Q), \tilde{y})$ with $\tilde{y} \neq y_R$ has at least one rank-1 preimage which is an arc tangent to the singular set δ_s in the focal point Q, while an arc crossing the prefocal line in two distinct points has at least one rank-1 preimage which is a loop tangent to the singular set δ_s in the focal point.*

Moreover, when the map is noninvertible, the particular point $R \in \delta_Q$ given in (21) represents the intersection of the prefocal curve with the critical curve LC, because the point $R(Q)$ given in (18) reduces to R given in (21). This also suggests that such a bifurcation situation may be associated with a qualitative change of the critical set of the map, and consequently of its Riemann foliation.

3.2. $\overline{N}_x = \overline{D}_x = 0$

Let us now consider the case $\overline{N}_x = \overline{D}_x = 0$ with both $\overline{N}_y \neq 0$ and $\overline{D}_y \neq 0$, so that the matrix in (20) has the structure

$$\begin{bmatrix} 0 & \overline{N}_y \\ 0 & \overline{D}_y \end{bmatrix}$$

and the slope of δ_s in Q is $m_{\delta_s} = -\overline{D}_x/\overline{D}_y = 0$. As before, we assume the functions $N(x, y)$ and $D(x, y)$ smooth enough, so that we can consider their Taylor expansions with basic point Q

$$G(x,y) = \frac{\overline{N}_y(y-y_0) + \frac{1}{2}\overline{N}_{xx}(x-x_0)^2 + \overline{N}_{xy}(x-x_0)(y-y_0) + \frac{1}{2}\overline{N}_{yy}(y-y_0)^2 + O_3}{\overline{D}_y(y-y_0) + \frac{1}{2}\overline{D}_{xx}(x-x_0)^2 + \overline{D}_{xy}(x-x_0)(y-y_0) + \frac{1}{2}\overline{D}_{yy}(y-y_0)^2 + O_3'}$$

where O_3 and O_3' are higher order terms, $\overline{N}_{xx} = (\partial^2 N/\partial x \partial x)(x_0, y_0)$, and analogously for the other partial derivatives. Then, the possible limit values of $T(\gamma(\tau))$, as $\tau \to 0$, are obtained by considering the generic arc $\gamma(\tau)$ given by (4) with $\xi_1 \neq 0$ and $\eta_1 \neq 0$, so that we have

$$G(\gamma(\tau)) = \frac{\overline{N}_y \eta_1 + \tau(\cdots)}{\overline{D}_y \eta_1 + \tau(\cdots)}$$

It follows that if the arc $\gamma(\tau)$ is not tangent to the singular set δ_s in Q, i.e. $\eta_1 \neq 0$ so that the slope $m = \eta_1/\xi_1$ of γ is different from the slope $m_{\delta_s} = 0$ of δ_s in Q, then

$$\lim_{\tau \to 0} G(\gamma(\tau)) = \frac{\overline{N}_y}{\overline{D}_y}$$

and the image $T(\gamma)$ crosses the line $x = F(Q)$ at the point

$$R = (F(Q), y_R), \quad \text{with} \quad y_R = \frac{\overline{N}_y}{\overline{D}_y} \tag{25}$$

Instead, if we consider an arc γ tangent to δ_s in Q, i.e. $\xi_1 \neq 0$, $\eta_1 = 0$ and $\eta_2 \neq 0$, then

$$G(\gamma(\tau)) = \frac{\overline{N}_y \eta_2 + \frac{1}{2}\overline{N}_{xx}\xi_1^2 + \tau(\cdots)}{\overline{D}_y \eta_2 + \frac{1}{2}\overline{D}_{xx}\xi_1^2 + \tau(\cdots)}$$

so that

$$\lim_{\tau \to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_y \eta_2 + \frac{1}{2}\overline{N}_{xx}\xi_1^2}{\overline{D}_y \eta_2 + \frac{1}{2}\overline{D}_{xx}\xi_1^2} \tag{26}$$

It follows that as the parameters ξ_1 and η_2 vary, and assuming that $\overline{N}_{xx}$ or $\overline{D}_{xx}$ is not vanishing, then $\tilde{y}$ takes all the values, so that the whole line $x = F(Q)$ is the prefocal set associated with the nonsimple focal point Q. Now, let us consider arcs $\gamma(\tau)$ tangent to the singular set δ_s in Q (i.e. with $\eta_1 = 0$) with different curvatures

$$\chi = \frac{2\eta_2}{\xi_1^2}. \tag{27}$$

Such arcs are mapped by T into arcs crossing the line $x = F(Q)$ at different points, and we can define a one-to-one correspondence between the curvatures of the arcs through Q with slope $m = 0$ and the points of the prefocal line, expressed by

$$\chi \to (F(Q), y(\chi)) \quad \text{with} \quad y(\chi) = \frac{\overline{N}_y \chi + \overline{N}_{xx}}{\overline{D}_y \chi + \overline{D}_{xx}}. \tag{28}$$

This relation can also be obtained directly from (24) taking $\alpha = 0$. The inverse relation is easily obtained:

$$(F(Q), y) \to \chi(y) \quad \text{with} \quad \chi(y) = \frac{\overline{N}_{xx} - y\overline{D}_{xx}}{y\overline{D}_y - \overline{N}_y}. \tag{29}$$

This means that any arc η crossing the prefocal line at a point $(F(Q), y)$ has at least one preimage by T which is an arc through the focal point Q, with horizontal tangent in Q and curvature $\chi(y)$ as given in (29). It is worth noting that the above relations also hold in the particular cases of zero or infinite curvature. In fact, if $\eta_2 = 0$ (so that $\chi = 0$), then $\tilde{y} = \overline{N}_{xx}/\overline{D}_{xx}$, while if $\xi_1 = 0$ and $\eta_2 \neq 0$, then we get $\tilde{y} = \overline{N}_y/\overline{D}_y$, i.e. the point R in (25). To complete our analysis we notice that if both $\overline{N}_{xx} = 0$ and $\overline{D}_{xx} = 0$, then the generic arc $\gamma(\tau)$, given in parametric form (4) with $\xi_1 \neq 0$, $\eta_1 = 0$, $\eta_2 = 0$ (i.e. tangent to the singular set with zero curvature) and $\eta_3 \neq 0$, gives

$$G(\gamma(\tau)) = \frac{\overline{N}_y \eta_3 + \frac{1}{3!}\overline{N}_{xxx}\xi_1^3 + \tau(\cdots)}{\overline{D}_y \eta_3 + \frac{1}{3!}\overline{D}_{xxx}\xi_1^3 + \tau(\cdots)}$$

and

$$\lim_{\tau \to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_y \eta_3 + \frac{1}{3!}\overline{N}_{xxx}\xi_1^3}{\overline{D}_y \eta_3 + \frac{1}{3!}\overline{D}_{xxx}\xi_1^3}$$

It follows that as the parameters ξ_1 and η_3 vary, and assuming that $\overline{N}_{xxx}$ or $\overline{D}_{xxx}$ is not vanishing, then $\tilde{y}$ takes all the values, so that the whole line $x = F(Q)$ is the prefocal set. And similar arguments apply to the particular case in which we have $\overline{N}_{xxx} = 0$ and $\overline{D}_{xxx} = 0$, just considering higher order derivatives. We have so proved the following proposition:

Proposition 2. *Consider the map (1), and a nonsimple focal point* $Q = (x_0, y_0)$ *such that* $\overline{N}_x = \overline{D}_x = 0$, *with* $\overline{N}_y \neq 0$ *and* $\overline{D}_y \neq 0$. *Then*:

- *the prefocal set* δ_Q *associated with* Q *is the line* $x = F(Q)$;
- *any arc* γ *which is not tangent to the singular set* δ_s *in* Q *is mapped by* T *into an arc crossing the prefocal line at the point* $R = \big(F(Q), \overline{N}_y/\overline{D}_y\big)$;
- *if* $\overline{N}_{xx} \neq 0$ *or* $\overline{D}_{xx} \neq 0$ *then any arc* γ *which is tangent to the singular set* δ_s *in* Q *is mapped by* T *into an arc crossing the prefocal line at the point*

with y-coordinate $\tilde{y}$ given in (26), and there exists a one-to-one correspondence between the curvature χ of such arcs and the points $(F(Q), y)$ of the prefocal line, given by the relations in (28) and in (29).

Three zero entries: Particular cases of the relations given above concern the simultaneous vanishing of three of the partial derivatives in Q, in the matrix in (20). For example, let

$$\overline{N}_x = \overline{D}_x = \overline{N}_y = 0 \quad \text{and} \quad \overline{D}_y \neq 0 \tag{30}$$

so that the matrix in (20) assumes the structure

$$\begin{bmatrix} 0 & 0 \\ 0 & \overline{D}_y \end{bmatrix}.$$

In this case the point in (25) becomes $R = (F(Q), 0)$, while considering arcs tangent to thesingular set δ_s in Q (i.e. $m = m_{\delta_s} = 0$) we get:

$$\lim_{\tau \to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\frac{1}{2}\overline{N}_{xx}\xi_1^2}{\overline{D}_y\eta_2 + \frac{1}{2}\overline{D}_{xx}\xi_1^2} \tag{31}$$

and the one-to-one correspondence (28) between the curvature of such arcs and points of δ_Q becomes

$$\chi \to (F(Q), y(\chi)), \quad \text{with} \quad y(\chi) = \frac{\overline{N}_{xx}}{\overline{D}_y\chi + \overline{D}_{xx}} \tag{32}$$

provided that $\overline{N}_{xx} \neq 0$. It is worth noting that if $\overline{N}_{xx} = 0$, then arcs $\gamma(\tau)$ tangent to δ_s in Q (i.e. $\eta_1 = 0$) for which $\overline{D}_y\eta_2 + (1/2)\overline{D}_{xx}\xi_1^2 = 0$, i.e. assuming $\eta_2 = -(1/2)(\overline{D}_{xx}/\overline{D}_y)\xi_1^2$ (so that the curvature is $\chi = 2\eta_2/\xi_1^2 = -\overline{D}_{xx}/\overline{D}_y$), give

$$G(\gamma(\tau)) = \frac{\overline{N}_{xy}\xi_1\eta_2 + \frac{1}{3!}\overline{N}_{xxx}\xi_1^3 + \tau(\cdots)}{\overline{D}_y\eta_3 + \overline{D}_{xx}\xi_1\xi_2 + \overline{D}_{xy}\xi_1\eta_2 + \frac{1}{3!}\overline{D}_{xxx}\xi_1^3 + \tau(\cdots)}$$

and

$$\lim_{\tau \to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_{xy}\xi_1\eta_2 + \frac{1}{3!}\overline{N}_{xxx}\xi_1^3}{\overline{D}_y\eta_3 + \overline{D}_{xx}\xi_1\xi_2 + \overline{D}_{xy}\xi_1\eta_2 + \frac{1}{3!}\overline{D}_{xxx}\xi_1^3}.$$

So, also in this case, the $\tilde{y}$ values span the whole line $x = F(Q)$, which is thus the prefocal set.

In the other case

$$\overline{N}_x = \overline{D}_x = \overline{D}_y = 0 \quad \text{and} \quad \overline{N}_y \neq 0 \tag{33}$$

the matrix in (20) assumes the structure

$$\begin{bmatrix} 0 & \overline{N}_y \\ 0 & 0 \end{bmatrix}$$

and the point R goes to infinity, i.e. $R = (F(Q), \pm\infty)$, while arcs $\gamma(\tau)$ tangent to δ_s in Q with $\xi_1 \neq 0$, $\eta_1 = 0$ and $\eta_2 \neq 0$ give

$$\lim_{\tau \to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_y\eta_2 + \frac{1}{2}\overline{N}_{xx}\xi_1^2}{\frac{1}{2}\overline{D}_{xx}\xi_1^2}$$

and the one-to-one correspondence (28) between curvatures of arcs with slope $m = 0$ in Q and points of δ_Q become

$$y(\chi) = \frac{\chi\overline{N}_y + \overline{N}_{xx}}{\overline{D}_{xx}} \tag{34}$$

provided that $\overline{D}_{xx} \neq 0$. It is worth noting that if $\overline{D}_{xx} = 0$, then arcs $\gamma(\tau)$ with slope $m = 0$ in Q (i.e. $\eta_1 = 0$) for which $\overline{N}_y\eta_2 + (1/2)\overline{N}_{xx}\xi_1^2 = 0$, i.e. $\eta_2 = -(1/2)(\overline{N}_{xx}/\overline{N}_y)\xi_1^2$ so that the curvature is $\chi = 2\eta_2/\xi_1^2 = -\overline{N}_{xx}/\overline{N}_y$, give

$$G(\gamma(\tau)) = \frac{\overline{N}_y\eta_3 + \overline{N}_{xx}\xi_1\xi_2 + \overline{N}_{xy}\xi_1\eta_2 + \frac{1}{3!}\overline{N}_{xxx}\xi_1^3 + \tau(\cdots)}{\overline{D}_{xy}\xi_1\eta_2 + \frac{1}{3!}\overline{D}_{xxx}\xi_1^3 + \tau(\cdots)}$$

so that

$$\lim_{\tau \to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_y\eta_3 + \overline{N}_{xx}\xi_1\xi_2 + \overline{N}_{xy}\xi_1\eta_2 + \frac{1}{3!}\overline{N}_{xxx}\xi_1^3}{\overline{D}_{xy}\xi_1\eta_2 + \frac{1}{3!}\overline{D}_{xxx}\xi_1^3}$$

Hence, again, the $\tilde{y}$ values span the whole line $x = F(Q)$, which is thus the prefocal set.

3.3. $\overline{N}_y = \overline{D}_y = 0$

In the case defined by $\overline{N}_y = \overline{D}_y = 0$ with both $\overline{N}_x \neq 0$ and $\overline{D}_x \neq 0$, the matrix in (20) has the structure

$$\begin{bmatrix} \overline{N}_x & 0 \\ \overline{D}_x & 0 \end{bmatrix}$$

and now the slope of δ_s in Q is infinite. This case gives symmetric results with respect to the case considered in the previous subsection, obtained just swapping x and y. So we shall briefly summarize the results. Considering the Taylor expansions

$$G(x,y) = \frac{\overline{N}_x(x-x_0) + \frac{1}{2}\overline{N}_{xx}(x-x_0)^2 + \overline{N}_{xy}(x-x_0)(y-y_0) + \frac{1}{2}\overline{N}_{yy}(y-y_0)^2 + O_3}{\overline{D}_x(x-x_0) + \frac{1}{2}\overline{D}_{xx}(x-x_0)^2 + \overline{D}_{xy}(x-x_0)(y-y_0) + \frac{1}{2}\overline{D}_{yy}(y-y_0)^2 + O_3'}$$

and considering the generic arc $\gamma(\tau)$ given in parametric form in (4) with $\xi_1 \neq 0$ and $\eta_1 \neq 0$, we have

$$G(\gamma(\tau)) = \frac{\overline{N}_x\xi_1 + \tau(\cdots)}{\overline{D}_x\xi_1 + \tau(\cdots)}.$$

It follows that if the arc $\gamma(\tau)$ is not tangent to δ_s in Q, i.e. $\xi_1 \neq 0$ so that the slope $m = \eta_1/\xi_1$ of γ is different from the slope of δ_s in Q, then we have

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \frac{\overline{N}_x}{\overline{D}_x}$$

and the image $T(\gamma)$ crosses the line $x = F(Q)$ at the point

$$R = (F(Q), y_R), \quad \text{with} \quad y_R = \frac{\overline{N}_x}{\overline{D}_x} \tag{35}$$

Instead, considering an arc γ with vertical slope, i.e. tangent to δ_s in Q, obtained by assuming $\xi_1 = 0$, $\xi_2 \neq 0$ and $\eta_1 \neq 0$ in (4), then we get:

$$G(\gamma(\tau)) = \frac{\overline{N}_x\xi_2 + \frac{1}{2}\overline{N}_{yy}\eta_1^2 + \tau(\cdots)}{\overline{D}_x\xi_2 + \frac{1}{2}\overline{D}_{yy}\eta_1^2 + \tau(\cdots)}$$

and

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_x\xi_2 + \frac{1}{2}\overline{N}_{yy}\eta_1^2}{\overline{D}_x\xi_2 + \frac{1}{2}\overline{D}_{yy}\eta_1^2} \tag{36}$$

It follows that as the parameters ξ_2 and η_1 vary, and assuming that $\overline{N}_{yy}$ or $\overline{D}_{yy}$ is not vanishing, $\tilde{y}$ takes all the values, so that the whole line $x = F(Q)$ is the prefocal set associated with this nonsimple focal point Q. Moreover, considering that the generic arc $\gamma(\tau)$ tangent to the singular set δ_s in Q has curvature given by $\chi = -2\xi_2/\eta_1^2$, we get a one-to-one correspondence between the curvatures of the arcs through Q with slope $m = \infty$ and the points of the prefocal line, expressed by

$$\chi \to (F(Q), y(\chi)) \quad \text{with} \quad y(\chi) = \frac{-\overline{N}_x\chi + \overline{N}_{yy}}{-\overline{D}_x\chi + \overline{D}_{yy}} \tag{37}$$

The inverse relation is:

$$(F(Q), y) \to \chi(y) \quad \text{with} \quad \chi(y) = -\frac{\overline{N}_{yy} - y\overline{D}_{yy}}{y\overline{D}_x - \overline{N}_x}. \tag{38}$$

This means that any arc η which crosses the prefocal line at a point $(F(Q), y)$ has at least one preimage by T which is an arc through the focal point Q, with vertical tangent in Q and curvature $\chi(y)$ as given in (38). The above relations also hold in the particular cases of zero or infinite curvature, in fact if $\xi_2 = 0$ (so that $\chi = 0$), then $\tilde{y} = \overline{N}_{yy}/\overline{D}_{yy}$ while if $\eta_1 = 0$ and $\xi_2 \neq 0$, then we get $\tilde{y} = \overline{N}_x/\overline{D}_x$ (i.e. the point R). In the case $\overline{N}_{yy} = 0$ and $\overline{D}_{yy} = 0$ we consider arcs with $\xi_2 = 0$ and $\xi_3 \neq 0$, and again the line $x = F(Q)$ is the prefocal set. We have so proved the following proposition:

Proposition 3. *Consider the map (1), and a nonsimple focal point $Q = (x_0, y_0)$ such that $\overline{N}_y = \overline{D}_y = 0$, with $\overline{N}_x \neq 0$ and $\overline{D}_x \neq 0$. Then:*

- *the prefocal set δ_Q associated with Q is the line $x = F(Q)$;*
- *any arc γ which is not tangent to the singular set δ_s in Q is mapped by T into an arc crossing the prefocal line in the point $R = \left(F(Q), \overline{N}x/\overline{D}_x\right)$;*
- *if $\overline{N}_{yy} \neq 0$ or $\overline{D}_{yy} \neq 0$ then any arc γ with vertical slope, which is tangent to the singular set δ_s in Q, is mapped by T into an arc crossing the prefocal line at the point with y-coordinate $\tilde{y}$ given in (36), and there exists a one-to-one correspondence between the curvature χ of such arcs and the points $(F(Q), y)$ of the prefocal line, given by the relations in (37) and in (38).*

The particular cases in which three partial derivatives vanish are similar to the previous ones and are left as exercises.

As already stated in Sec. 3.1, the consequences of Propositions 2 and 3, from a dynamical point of view, are mainly associated with the structure of the basins' boundaries, because any portion of a basin boundary that crosses the prefocal line in a generic point $(F(Q), \tilde{y})$ has at least one rank-1 preimage which is an arc tangent to the singular set δ_s at the focal point Q.

Moreover, it is easy to see that the particular point $R \in \delta_Q$ given in (25) or in (35) when the map is noninvertible, corresponds to the intersection of the prefocal set with the critical curve LC, because the point $R(Q)$ given in (18) reduces to R given in (25) or in (35). As already stated in Sec. 3.1, this situation may be associated with a bifurcation of the critical set LC of the map.

3.4. $\overline{N}_x = \overline{N}_y = 0$

In the case $\overline{N}_x = \overline{N}_y = 0$ with both $\overline{D}_x \neq 0$ and $\overline{D}_y \neq 0$, the matrix in (20) assumes the structure

$$\begin{bmatrix} 0 & 0 \\ \overline{D}_x & \overline{D}_y \end{bmatrix}$$

and the slope of δ_s in Q is not vanishing, with $m_{\delta_s} = -\overline{D}_x/\overline{D}_y \neq 0$. $G(x, y)$ can be written as

$$G(x,y) = \frac{\frac{1}{2}\overline{N}_{xx}(x-x_0)^2 + \overline{N}_{xy}(x-x_0)(y-y_0) + \frac{1}{2}\overline{N}_{yy}(y-y_0)^2 + O_3}{\overline{D}_x(x-x_0) + \overline{D}_y(y-y_0) + O_2}.$$

Let us consider the generic arc $\gamma(\tau)$ given in parametric form in (4) with $\xi_1 \neq 0$ and $\eta_1 \neq 0$. If it is not tangent to δ_s in Q, i.e. the slope $m = \eta_1/\xi_1$ of γ is different from the slope $m_{\delta_s} = -\overline{D}_x/\overline{D}_y$ of δ_s in Q, so that $\overline{D}_x\xi_1 + \overline{D}_y\eta_1 \neq 0$, then $\lim_{\tau\to 0} G(\gamma(\tau)) = 0$ and we get the single point

$$R = (F(Q), y_R), \quad \text{with} \quad y_R = 0. \tag{39}$$

Instead, if we consider an arc γ tangent to δ_s in Q, so that $\overline{D}_x\xi_1 + \overline{D}_y\eta_1 = 0$, then

$$G(\gamma(\tau)) = \frac{\frac{1}{2}\overline{N}_{xx}\xi_1^2 + \overline{N}_{xy}\xi_1\eta_1 + \frac{1}{2}\overline{N}_{yy}\eta_1^2 + \tau(\cdots)}{\overline{D}_x\xi_2 + \overline{D}_y\eta_2 + \frac{1}{2}\overline{D}_{xx}\xi_1^2 + \overline{D}_{xy}\xi_1\eta_1 + \frac{1}{2}\overline{D}_{yy}\eta_1^2 + \tau(\cdots)}.$$

It follows that if all the second-order partial derivatives of N in Q are vanishing then the limit is again the point $R = (F(Q), 0)$, while assuming that the second-order partial derivatives of N in Q are not all vanishing we have

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\frac{1}{2}\overline{N}_{xx}\xi_1^2 + \overline{N}_{xy}\xi_1\eta_1 + \frac{1}{2}\overline{N}_{yy}\eta_1^2}{\overline{D}_x\xi_2 + \overline{D}_y\eta_2 + \frac{1}{2}\overline{D}_{xx}\xi_1^2 + \overline{D}_{xy}\xi_1\eta_1 + \frac{1}{2}\overline{D}_{yy}\eta_1^2}. \tag{40}$$

As the parameters vary, $\tilde{y}$ takes all the values, hence the whole line $x = F(Q)$ is the prefocal set associated with this nonsimple focal point Q. Moreover, the generic arc $\gamma(\tau)$ tangent to the singular set δ_s in Q has different curvatures given by $\chi = (2\xi_1\eta_2 - 2\xi_2\eta_1)/(\xi_1^2 + \eta_1^2)^{3/2} = 2(\eta_2 - m\xi_2)/\xi_1^2(1+m^2)^{3/2}$. From (40) we have, with $\overline{D}_x = -m\overline{D}_y$,

$$\tilde{y} = \frac{\xi_1^2(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\overline{D}_y 2(\eta_2 - m\xi_2) + \xi_1^2(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)} = \frac{\frac{1}{(1+m^2)^{3/2}}(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\overline{D}_y\chi + \frac{1}{(1+m^2)^{3/2}}(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)}$$

and introducing the angle $\alpha \in [-\pi/2, +\pi/2]$, such that $m = \tan\alpha$, we get

$$\tilde{y}(\chi) = \frac{\cos^3\alpha(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\overline{D}_y\chi + \cos^3\alpha(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)}$$

$$= \frac{(\overline{N}_{xx}\cos^2\alpha + 2\overline{N}_{xy}\sin\alpha\cos\alpha + \overline{N}_{yy}\sin^2\alpha)}{\frac{\chi\overline{D}_y}{\cos\alpha} + (\overline{D}_{xx}\cos^2\alpha + 2\overline{D}_{xy}\sin\alpha\cos\alpha + \overline{D}_{yy}\sin^2\alpha)}\ . \tag{41}$$

This defines a one-to-one correspondence between the curvatures χ of the arcs through Q, with a common slope $m = \tan\alpha$, and the points of the line $x = F(Q)$. We have so proved the following proposition:

Proposition 4. *Consider the map (1) and a non-simple focal point* $Q = (x_0, y_0)$ *such that* $\overline{N}_x = \overline{N}_y = 0$, *with* $\overline{D}_x \neq 0$ *and* $\overline{D}_y \neq 0$. *Then*:

- *any arc* γ *which is not tangent to the singular set* δ_s *in* Q *is mapped by* T *into an arc crossing the line* $x = F(Q)$ *at the point* $R = (F(Q), 0)$;
- *if the second-order partial derivatives of* N *in* Q *are not all vanishing then the prefocal set* δ_Q *associated with* Q *is the line* $x = F(Q)$;
- *any arc* γ *which is tangent to the singular set* δ_s *in* Q *is mapped by* T *into an arc crossing the prefocal line at the point with y-coordinate* $\tilde{y}$ *given in (40), and there exists a one-to-one correspondence between the curvature* χ *of such arcs and the points* $(F(Q), y)$ *of the prefocal line, given by the relation in (41) and its inverse.*

Three zero entries: Again, we have the particular cases in which also another derivative vanishes. For example, in the case

$$\overline{N}_x = \overline{N}_y = \overline{D}_x = 0 \quad \text{and} \quad \overline{D}_y \neq 0$$

if we consider an arc $\gamma(\tau)$ tangent to δ_s in Q we have $\eta_1 = m\xi_1 = -\xi_1\overline{D}_x/\overline{D}_y = 0$, so that the limit in (40) reduces to the one already obtained in (31):

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\frac{1}{2}\overline{N}_{xx}\xi_1^2}{\overline{D}_y\eta_2 + \frac{1}{2}\overline{D}_{xx}\xi_1^2}.$$

So, if $\overline{N}_{xx} \neq 0$, the one-to-one correspondence between the curvature $\chi = 2\eta_2/\xi_1^2$ of γ and the point $(F(Q), \tilde{y})$ in which $T(\gamma)$ intersects the prefocal line, is the one already given in (32):

$$\chi \to (F(Q), y(\chi)), \quad \text{with} \quad y(\chi) = \frac{\overline{N}_{xx}}{\overline{D}_y\chi + \overline{D}_{xx}}.$$

This relation can also be obtained directly by setting $\alpha = 0$ in (41).

Similarly, in the case

$$\overline{N}_x = \overline{N}_y = \overline{D}_y = 0 \quad \text{and} \quad \overline{D}_x \neq 0.$$

an arc $\gamma(\tau)$ tangent to δ_s in Q (that is with vertical slope, with $m_{\delta_s} = -\overline{D}_x/\overline{D}_y$) is obtained setting $\xi_1 = 0$ and $\eta_1 \neq 0$, so that the limit in (40) reduces to the one already obtained in (36) with $\overline{N}_x = 0$

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\frac{1}{2}\overline{N}_{yy}\eta_1^2}{\overline{D}_x\xi_2 + \frac{1}{2}\overline{D}_{yy}\eta_1^2}.$$

If $\overline{N}_{yy} \neq 0$, then we have a one-to-one correspondence between the curvature $\chi = -2\xi_2/\eta_1^2$ of arcs γ with vertical slope, which are tangent to the singular set in Q, and the point $(F(Q), \tilde{y})$ in which $T(\gamma)$ intersects the prefocal line, as given in (37) with $\overline{N}_x = 0$:

$$\chi \to (F(Q), y(\chi)), \quad \text{with} \quad y(\chi) = \frac{\overline{N}_{yy}}{-\overline{D}_x\chi + \overline{D}_{yy}}.$$

3.5. $\overline{D}_x = \overline{D}_y = 0$

Finally, in the case defined by $\overline{D}_x = \overline{D}_y = 0$ with both $\overline{N}_x \neq 0$ and $\overline{N}_y \neq 0$, the matrix in (20) assumes the structure

$$\begin{bmatrix} \overline{N}_x & \overline{N}_y \\ 0 & 0 \end{bmatrix}.$$

$G(x, y)$ can be written as

$$G(x,y) = \frac{\overline{N}_x(x - x_0) + \overline{N}_y(y - y_0) + O_2}{\frac{1}{2}\overline{D}_{xx}(x - x_0)^2 + \overline{D}_{xy}(x - x_0)(y - y_0) + \frac{1}{2}\overline{D}_{yy}(y - y_0)^2 + O_3^{'}}.$$

Considering the generic arc $\gamma(\tau)$ given in parametric form in (4) with $\overline{N}_x\xi_1 + \overline{N}_y\eta_1 \neq 0$, then the limit of $G(\gamma(\tau))$ is equal to infinity, and we get the point at infinity $R = (F(Q), \infty)$.

Instead, considering an arc γ with $\overline{N}_x\xi_1 + \overline{N}_y\eta_1 = 0$, i.e. with slope $m = \eta_1/\xi_1 = -\overline{N}_x/\overline{N}_y$ (the same slope as the set $N(x,y) = 0$ in Q), given by (4) with $\xi_1 \neq 0$, $\eta_1 = m\xi_1$, $\xi_2 \neq 0$ and $\eta_2 \neq 0$, we get:

$$G(\gamma(\tau)) = \frac{\overline{N}_x\xi_2 + \overline{N}_y\eta_2 + \frac{1}{2}\overline{N}_{xx}\xi_1^2 + \overline{N}_{xy}\xi_1\eta_1 + \frac{1}{2}\overline{N}_{yy}\eta_1^2 + \tau(\cdots)}{\frac{1}{2}\overline{D}_{xx}\xi_1^2 + \overline{D}_{xy}\xi_1\eta_1 + \frac{1}{2}\overline{D}_{yy}\eta_1^2 + \tau(\cdots)}$$

It follows that if all the second-order partial derivatives of D in Q are vanishing then the limit is again the point $R = (F(Q), \infty)$, while assuming that the second-order partial derivatives of D in Q are not all vanishing, we have

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_x\xi_2 + \overline{N}_y\eta_2 + \frac{1}{2}\overline{N}_{xx}\xi_1^2 + \overline{N}_{xy}\xi_1\eta_1 + \frac{1}{2}\overline{N}_{yy}\eta_1^2}{\frac{1}{2}\overline{D}_{xx}\xi_1^2 + \overline{D}_{xy}\xi_1\eta_1 + \frac{1}{2}\overline{D}_{yy}\eta_1^2} \tag{42}$$

As the parameters vary $\tilde{y}$ takes all the values, the whole line $x = F(Q)$ is the prefocal set associated with this nonsimple focal point Q. Moreover, considering that the generic arc $\gamma(\tau)$ tangent to the singular set δ_s in Q, has different curvatures $\chi = (2\xi_1\eta_2 - 2\xi_2\eta_1)/(\xi_1^2+\eta_1^2)^{3/2} = 2(\eta_2 - m\xi_2)/\xi_1^2(1+m^2)^{3/2}$, we have (being $\overline{N}_x = -m\overline{N}_y$):

$$\tilde{y} = \frac{\overline{N}_y 2(\eta_2 - m\xi_2) + \xi_1^2(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\xi_1^2(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)} = \frac{\overline{N}_y\chi + \frac{1}{(1+m^2)^{3/2}}(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\frac{1}{(1+m^2)^{3/2}}(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)}$$

Introducing the angle $\alpha \in [-\pi/2, +\pi/2]$, such that $m = \tan\alpha$, we get

$$\tilde{y}(\chi) = \frac{\overline{N}_y\chi + \cos^3\alpha(\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2)}{\overline{D}_y\chi + \cos^3\alpha(\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2)} = \frac{\frac{\chi\overline{N}_y}{\cos\alpha} + (\overline{N}_{xx}\cos^2\alpha + 2\overline{N}_{xy}\sin\alpha\cos\alpha + \overline{N}_{yy}\sin^2\alpha)}{(\overline{D}_{xx}\cos^2\alpha + 2\overline{D}_{xy}\sin\alpha\cos\alpha + \overline{D}_{yy}\sin^2\alpha)}. \tag{43}$$

It follows that we can define a one-to-one correspondence between the curvatures χ of the arcs through Q with a common slope $m = \tan\alpha$ and the points of the line $x = F(Q)$ expressed by (43), which can easily be inverted.

We have so proved the following proposition:

Proposition 5. *Consider the map (1) and a nonsimple focal point $Q = (x_0, y_0)$ such that $\overline{D}_x = \overline{D}_y = 0$, with $\overline{N}_x \neq 0$ and $\overline{N}_y \neq 0$. Then:*

- *any arc γ with a slope $m = \eta_1/\xi_1 \neq -\overline{N}_x/\overline{N}_y$ at Q is mapped by T into an arc crossing the line $x = F(Q)$ at the point $R = (F(Q), \infty)$;*
- *if the second-order partial derivatives of D in Q are not all vanishing then the prefocal set δ_Q associated with Q is the line $x = F(Q)$;*
- *any arc γ with a slope $m = \eta_1/\xi_1 = -\overline{N}_x/\overline{N}_y$ at Q is mapped by T into an arc crossing the prefocal line at the point with y-coordinate $\tilde{y}$ given in (42), and there exists a one-to-one correspondence between the curvature χ of such arcs and the points $(F(Q), y)$ of the prefocal line, given by the relation in (43) and its inverse.*

Three zero entries: As before, we have the particular cases in which also another derivative vanishes. For example, in the case

$$\overline{D}_x = \overline{D}_y = \overline{N}_x = 0 \quad \text{and} \quad \overline{N}_y \neq 0\,,$$

if we consider an arc $\gamma(\tau)$ with $\eta_1 = 0$ (i.e. an arc with $\alpha = 0$ at Q) the limit (42) reduces to the one already obtained in (26) with $\overline{D}_y = 0$

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_y\eta_2 + \frac{1}{2}\overline{N}_{xx}\xi_1^2}{\frac{1}{2}\overline{D}_{xx}\xi_1^2}.$$

So, if $\overline{D}_{xx} \neq 0$, then we have a one-to-one correspondence between the curvature $\chi = 2\eta_2/\xi_1^2$ of arcs γ through Q and the point $(F(Q), \tilde{y})$ in which $T(\gamma)$ intersects the prefocal line, as already given in (28) with $\overline{D}_y = 0$ (see also (34))

$$\chi \to (F(Q), y(\chi)), \quad \text{with} \quad y(\chi) = \frac{\overline{N}_y\chi + \overline{N}_{xx}}{\overline{D}_{xx}}.$$

This relation can also be directly obtained taking $\alpha = 0$ in (43). Similarly, in the case

$$\overline{D}_x = \overline{D}_y = \overline{N}_y = 0 \quad \text{and} \quad \overline{N}_x \neq 0$$

if we consider an arc $\gamma(\tau)$ with $\xi_1 = 0$ and $\eta_1 \neq 0$ (i.e. $\alpha = \pi/2$) the limit (42) reduces to the one already obtained in (36) with $\overline{D}_x = 0$

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \tilde{y} = \frac{\overline{N}_x\xi_2 + \frac{1}{2}\overline{N}_{yy}\eta_1^2}{\frac{1}{2}\overline{D}_{yy}\eta_1^2}.$$

If $\overline{D}_{yy} \neq 0$, then we have a one-to-one correspondence between the curvature $\chi = -2\xi_2/\eta_1^2$ of arcs γ through Q and the point $(F(Q), \tilde{y})$ in which $T(\gamma)$ intersects the prefocal line, as already given in (37) with $\overline{D}_x = 0$:

$$\chi \to (F(Q), y(\chi)), \quad \text{with} \quad y(\chi) = \frac{-\overline{N}_x\chi + \overline{N}_{yy}}{\overline{D}_{yy}}. \tag{44}$$

3.6. $\overline{N}_x = \overline{N}_y = \overline{D}_x = \overline{D}_y = 0$

If $N_x = N_y = D_x = D_y = 0$ when computed at a point $Q = (x_0, y_0)$ where $G(Q)$ assumes the form $0/0$, then the expansions (8) of the functions $N(x, y)$ and $D(x, y)$ start from the second-order terms:

$$\begin{aligned} N(x,y) &= \frac{1}{2}\overline{N}_{xx}(x-x_0)^2 + \overline{N}_{xy}(x-x_0)(y-y_0) \\ &\quad + \frac{1}{2}\overline{N}_{yy}(y-y_0)^2 + O_3 \\ D(x,y) &= \frac{1}{2}\overline{D}_{xx}(x-x_0)^2 + \overline{D}_{xy}(x-x_0)(y-y_0) \\ &\quad + \frac{1}{2}\overline{D}_{yy}(y-y_0)^2 + O_3'. \end{aligned} \tag{45}$$

If we consider an arc γ with the parametric representation (4) then the limit (6) assumes the form:

$$\lim_{\tau\to 0} G(\gamma(\tau)) = \frac{\overline{N}_{xx}\xi_1^2 + 2\overline{N}_{xy}\xi_1\eta_1 + \overline{N}_{yy}\eta_1^2}{\overline{D}_{xx}\xi_1^2 + 2\overline{D}_{xy}\xi_1\eta_1 + \overline{D}_{yy}\eta_1^2} \tag{46}$$

If the partial derivatives at the denominator in (46) are not all vanishing, then the limit assumes in general finite values, depending on the slope $m = \eta_1/\xi_1$ of the arc γ in Q, so that the prefocal set still belongs to the line $x = F(Q)$. Moreover, we have the following correspondence between the slope of an arc γ through Q and the y-coordinate of the point $(F(Q), y(m))$ in which $T(\gamma)$ intersects the prefocal line

$$m \to y(m) = \frac{\overline{N}_{xx} + 2\overline{N}_{xy}m + \overline{N}_{yy}m^2}{\overline{D}_{xx} + 2\overline{D}_{xy}m + \overline{D}_{yy}m^2}. \tag{47}$$

It is worth to note that now the relation $y(m)$ is generally two-to-one (instead of one-to-one as it occurs in simple focal points). In fact, given a point $(F(Q), y) \in \delta_Q$, from (47) we obtain two distinct values of m,

$$m_\pm(y) = \frac{-(\overline{N}_{xy} - y\overline{D}_{xy}) \pm \sqrt{\Delta}}{(\overline{N}_{yy} - y\overline{D}_{yy})}, \tag{48}$$

provided that $\Delta > 0$, where $\Delta = \left[(\overline{N}_{xy} - y\overline{D}_{xy})^2 - (\overline{N}_{yy} - y\overline{D}_{yy})(\overline{N}_{xx} - y\overline{D}_{xx})\right]$. Furthermore, the prefocal set δ_Q may be a proper subset of the line $x = F(Q)$. In fact, as m changes from $-\infty$ to $+\infty$ the values taken by $y(m)$ depend on the second derivatives of $N(x, y)$ and $D(x, y)$ appearing in (47). We have so proved the following proposition:

Proposition 6. *Consider the map (1), and a non-simple focal point $Q = (x_0, y_0)$ such that $\overline{D}_x = \overline{D}_y = \overline{N}_x = \overline{N}_y = 0$. Then:*

- *the prefocal set δ_Q associated with Q belongs to the line $x = F(Q)$;*
- *if the second-order partial derivatives of $D(x, y)$ in Q are not all vanishing then there exists a correspondence between the slope m of an arc γ through Q and the y-coordinate of the point $(F(Q), y(m))$, given in (47), and this correspondence is generally two-to-one.*

We finally remark that if all the second-order partial derivatives of N and D at Q are equal to zero up to the order $p - 1$, then we can proceed as above, to get the relation

$$m \to y(m) = \frac{\overline{N}_{x^p} + p\overline{N}_{x^{p-1}y}m + \cdots + p\overline{N}_{xy^{p-1}}m^{p-1} + \overline{N}_{y^p}m^p}{\overline{D}_{x^p} + p\overline{D}_{x^{p-1}y}m + \cdots + p\overline{D}_{xy^{p-1}}m^{p-1} + \overline{D}_{y^p}m^p}$$

where the coefficients of $\overline{N}_{x^{p-j}y^j}m^{p-j}$ and $\overline{D}_{x^{p-j}y^j}m^{p-j}$, $j = 2, \ldots, p-2$, are the binomial ones. This corresponds to more complex bifurcations for the focal points, given by the merging of more than two focal points, and more than two prefocal curves.

4. Examples

4.1. *Example 1*

Let us consider the map

$$T: \begin{cases} x' = y + \varepsilon x \\ y' = \dfrac{\mu y + \alpha x^2 + \gamma x}{y - \beta + \sigma x} \end{cases} \tag{49}$$

It is not defined at the points of the line δ_s of equation

$$D(x, y) = y - \beta + \sigma x = 0. \tag{50}$$

The focal points of the map (49) are located at the intersections, if any, of the line (50) with the parabola $N(x, y) = \mu y + \alpha x^2 + \gamma x = 0$. If

$$(\gamma - \mu\sigma)^2 > 4\alpha\beta\mu \tag{51}$$

then two simple focal points exist, given by

$$Q_i = (x_{Q_i}, \beta - \sigma x_{Q_i}), \quad i = 1, 2 \tag{52}$$

where

$$x_{Q_1} = \frac{\mu\sigma - \gamma - \sqrt{(\gamma - \mu\sigma)^2 - 4\alpha\beta\mu}}{2\alpha},$$
$$x_{Q_2} = \frac{\mu\sigma - \gamma + \sqrt{(\gamma - \mu\sigma)^2 - 4\alpha\beta\mu}}{2\alpha}$$

The corresponding prefocal curves δ_{Q_i} have equations

$$x = F(Q_i) = \beta + (\varepsilon - \sigma)x_{Q_i} \quad i = 1, 2 \tag{53}$$

and the one-to-one correspondences (11) associated with the simple focal points Q_i, $i = 1, 2$, are

$$y_i(m) = \frac{2\alpha x_{Q_i} + \gamma + \mu m}{\sigma + m} = \mu \mp \frac{\sqrt{(\gamma - \mu\sigma)^2 - 4\alpha\beta\mu}}{\sigma + m}. \tag{54}$$

For

$$(\gamma - \mu\sigma)^2 = 4\alpha\beta\mu \tag{55}$$

the two focal points merge:

$$Q_1 \equiv Q_2 = Q = \left(\frac{\mu\sigma - \gamma}{2\alpha}, \beta - \sigma\frac{\mu\sigma - \gamma}{\alpha}\right) \tag{56}$$

and Q is a nonsimple focal point. In fact, being $\overline{N}_x = 2\alpha x_Q + \gamma$; $\overline{N}_y = \mu$; $\overline{D}_x = \sigma$; $\overline{D}_y = 1$, it is easy to check that the condition (20) is satisfied when (55) holds, and we are in the case considered in Sec. 3.1. No focal points exist for $(\gamma - \mu\sigma)^2 < 4\alpha\beta\mu$, thus the bifurcation condition (55) marks the appearance/disappearance of two simple focal points.

The map (49) is a noninvertible map of Z_0–Z_2 type, i.e. a point (x', y') can have up to two rank-1 preimages, whose coordinates are obtained by solving the algebraic system (49) with respect to the unknowns x and y. If $(x, y) \notin \delta_s$ such system is equivalent to

$$\begin{cases} \alpha x^2 + (\gamma - \sigma y' + \varepsilon y' - \mu\varepsilon)x + \mu x' + \beta y' - x'y' = 0 \\ y = x' - \varepsilon x \end{cases} \tag{57}$$

which has two real and distinct solutions if

$$\Delta(x', y') = (\gamma - \sigma y' + \varepsilon y' - \mu\varepsilon)^2 - 4\alpha(\mu x' + \beta y' - x'y') > 0 \tag{58}$$

and no solutions if the reverse inequality holds. Hence, the regions Z_0 and Z_2 are defined as $Z_0 = \{(x, y)|\Delta(x, y) < 0\}$ and $Z_2 = \{(x, y)|\Delta(x, y) > 0\}$

respectively. For each point $(x', y') \in Z_2$ two distinct inverses are defined, given by:

$$T_1^{-1}: \begin{cases} x = \frac{1}{2\alpha}(\sigma - \varepsilon)y' + \mu\varepsilon - \gamma - \sqrt{\Delta(x', y')} \\ y = x' - \varepsilon x \end{cases}$$

$$T_2^{-1}: \begin{cases} x = \frac{1}{2\alpha}(\sigma - \varepsilon)y' + \mu\varepsilon - \gamma + \sqrt{\Delta(x', y')} \\ y = x' - \varepsilon x \end{cases} \quad (59)$$

The two regions Z_0 and Z_2 are separated by the critical set LC, locus of points having merging preimages, defined by the equation $\Delta(x, y) = 0$, which can be expressed as

$$x = \Psi(y) = \frac{((\varepsilon - \sigma)y + \gamma - \mu\varepsilon)^2 - 4\alpha\beta y}{4\alpha(\mu - y)} \quad (60)$$

So, LC is formed by the two branches of an hyperbola of asymptotes

$$y = \mu \quad \text{and} \quad \begin{cases} y = -\frac{4\alpha}{(\sigma - \varepsilon)^2}x + \frac{4\alpha\beta + 2(\sigma - \varepsilon)(\gamma - \mu\varepsilon)}{(\sigma - \varepsilon)^2} - \mu & \text{for } \varepsilon \neq \sigma \\ x = \beta & \text{for } \varepsilon = \sigma \end{cases} \quad (61)$$

and symmetry centre $S = (\beta + (\sigma - \varepsilon)((\gamma - \mu\varepsilon) - \mu(\sigma - \varepsilon))/2\alpha, \mu)$.

The locus of merging preimages, LC_{-1}, can be obtained from (59) with $\Delta(x', y') = 0$:

$$\begin{cases} x = \frac{(\sigma - \varepsilon)y' + \mu\varepsilon - \gamma}{2\alpha} \\ y = x' - \varepsilon x \end{cases}$$

where $x' = \Psi(y')$ and, for $\sigma \neq \varepsilon$, $y' = (2\alpha x + \gamma - \mu\varepsilon)/(\sigma - \varepsilon)$. Thus, for $\sigma \neq \varepsilon$, LC_{-1} is given by the hyperbola of equation

$$y = \frac{\alpha(\sigma + \varepsilon)x^2 - (2\alpha\beta - \varepsilon\gamma + \mu\sigma\varepsilon)x - \beta(\gamma - \mu\varepsilon)}{\mu\sigma - \gamma - 2\alpha x} \quad (62)$$

This can also be obtained from the condition $\det DT(x, y) = 0$, where

$$DT(x, y) = \begin{bmatrix} \varepsilon & 1 \\ \frac{(2\alpha x + \gamma)D(x, y) - \sigma N(x, y)}{D(x, y)^2} & \frac{\mu D(x, y) - N(x, y)}{D(x, y)^2} \end{bmatrix} \quad (63)$$

denotes the Jacobian matrix of the map T. In the particular case $\sigma = \varepsilon$, LC_{-1} reduces to the line

$$x = \frac{\mu\varepsilon - \gamma}{2\alpha}. \quad (64)$$

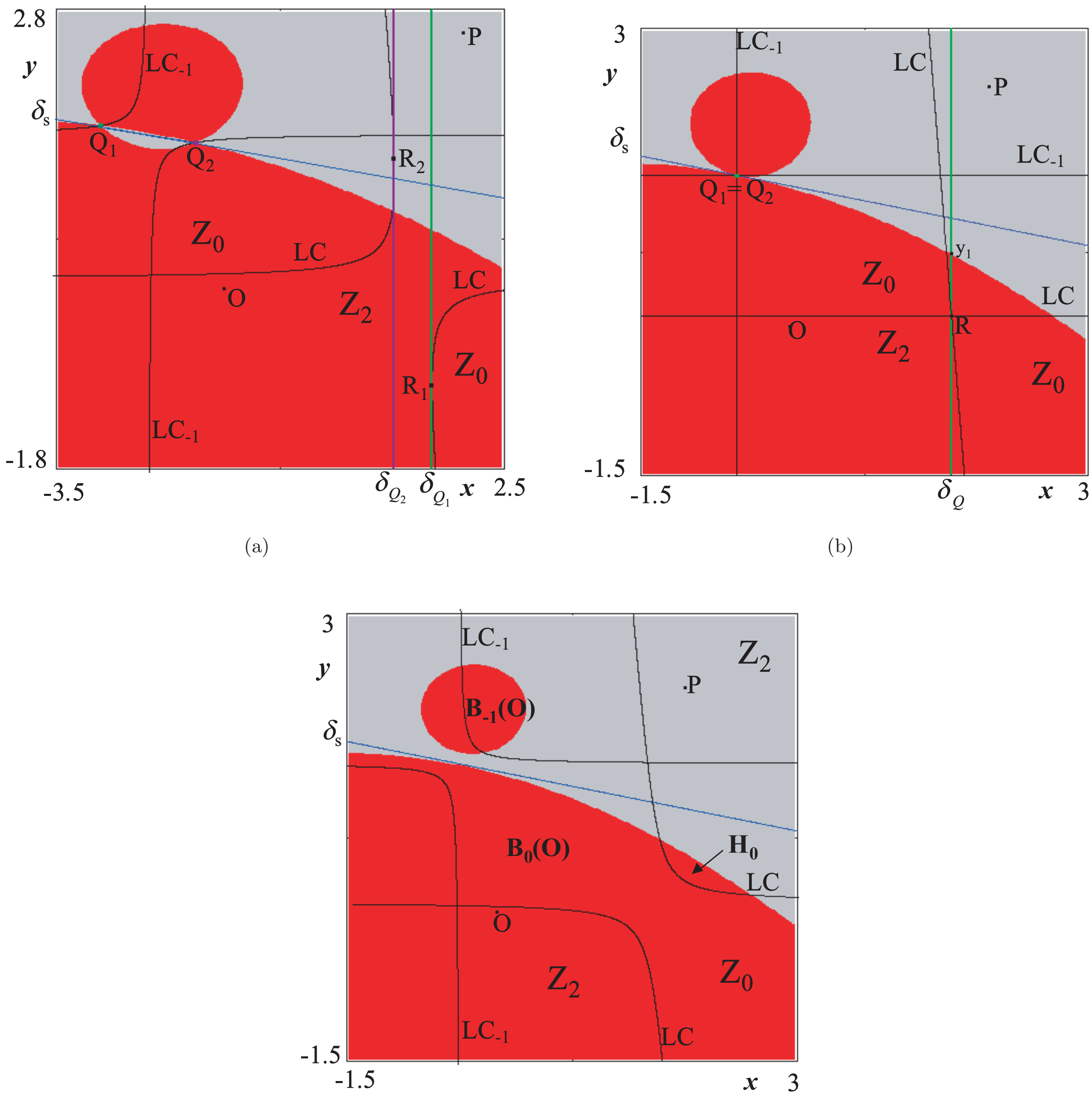
as we shall see below.

The map (49) has two fixed points:

$$O = (0, 0) \quad \text{and} \quad P = (x_P, (1 - \varepsilon)x_P) \quad (65)$$

where $x_P = ((\beta + \mu)(1 - \varepsilon) + \gamma)/((1 - \varepsilon)^2 + \sigma(1 - \varepsilon) - \alpha)$. In Fig. 1(a), obtained with parameters $\mu = 0.1$, $\alpha = 0.5$, $\gamma = 0.7$, $\beta = \sqrt{2}$, $\sigma = 0.2$, $\varepsilon = -0.2$, the fixed point O is a stable focus, whose basin $\mathcal{B}(O)$, represented by the red region, is bounded by the stable set of a saddle cycle of period 2. The gray region represents the basin of infinity $\mathcal{B}(\infty)$, defined as the set of points which generate diverging trajectories.[2] For the set of parameters used in Fig. 1(a), two simple focal points Q_1 and Q_2 exist, being $(\gamma - \mu\sigma)^2 > 4\alpha\beta\mu$, associated with the two prefocal curves δ_{Q_1} and δ_{Q_2}. The critical curve LC_{-1}, the hyperbola of Eq. (62), includes the two focal points Q_1 and Q_2, when they exist, and it has symmetry centre $K = ((\mu\varepsilon - \gamma)/2\alpha, \beta - \sigma(\mu\sigma - \gamma)/2\alpha)$, midpoint of the segment Q_1Q_2. The slope of LC_{-1} in both the focal points is given, according to (17),

[2]Both the red and gray regions include a set of points of zero measure, which are not numerically visible, which do not belong to the respective basins, such as the periodic points of unstable cycles. For example, the unstable fixed point P is included in the gray region even if, of course, $P \notin \mathcal{B}(\infty)$.



(a) (b)

(c)

Fig. 1. Critical sets LC_{-1} and LC (black), singular line δ_s (blue), prefocal curves δ_{Q_1} and δ_{Q_2} (green and violet respectively), corresponding focal points Q_1 and Q_2 (green and violet respectively) and basins of attraction for the map (49). The red region represents the basin $\mathcal{B}(O)$ of the stable fixed point O, the gray region represents the basin of infinity, $\mathcal{B}(\infty)$, defined as the set of points which generate diverging trajectories. The parameters are $\mu = 0.1$, $\alpha = 0.5$, $\beta = \sqrt{2}$, $\sigma = 0.2$, $\varepsilon = -0.2$ and: (a) two simple focal points at $\gamma = 0.7$; (b) one nonsimple focal point at $\gamma = 0.551829$; (c) no focal points at $\gamma = 0.4$.

by $\overline{m} = -\varepsilon$, and $LC = T(LC_{-1})$ has tangential contacts with δ_{Q_i} at the points of y-coordinate

$$\overline{y}_{1,2} = \mu \mp \frac{\sqrt{(\gamma - \mu\sigma)^2 - 4\alpha\beta\mu}}{\sigma - \varepsilon} \tag{66}$$

obtained from (54) with $m = -\varepsilon$. In fact, in this map the critical curve LC cannot cross the simple prefocal lines because branches of LC separate the regions Z_0 and Z_2, and, as stated in Sec. 2, no portions of a prefocal curve can belong to Z_0.

As stressed in Part II, if one moves along a prefocal line, at a tangency point between δ_Q and LC the two inverses, the one which focalizes and the one which does not, merge and then swap. So, the two prefocal curves, as well as the whole strip between them, are entirely included in the region Z_2. For each point $(x_{Q_i}, y) \in \delta_{Q_i}$ one rank-1 preimage belongs to the line $y = \beta + (\varepsilon - \sigma)x_{Q_i} - \varepsilon x$, tangent to LC_{-1} in Q_i, and the other one is "focalized" in the focal point Q_i.

Now, let us analyze what happens in the bifurcation case defined by the identity (55), which implies that a nonsimple focal point exists. It is important to notice that when (55) holds LC_{-1} degenerates into a pair of lines crossing at the nonsimple focal point Q, given by (56), and LC also degenerates into a pair of lines crossing through the point $R = (F(Q), \mu) = (\beta + (\varepsilon - \sigma)x_Q, \mu)$, as remarked in Sec. 3.1 [see Fig. 1(b)]. This is the situation shown in Fig. 1(b), obtained for a value of γ such that $(\gamma - \mu\sigma)^2 = 4\alpha\beta\mu$, so that we are at the bifurcation. In fact, for the set of parameters used in Fig. 1(b) we have $\overline{N}_x = \mu\sigma$; $\overline{N}_y = \mu$; $\overline{D}_x = \sigma$; $\overline{D}_y = 1$, so that the situation described in Sec. 3.1 is obtained. This implies that as a parameter is changed so that two simple focal points merge and disappear, or appear, a remarkable structural change also occurs in the Riemann foliation of the map, because the positions of the two branches of LC switch with respect to the asymptotes.

The prefocal line δ_Q crosses the boundary $\partial\mathcal{B}(O)$ at one point denoted by y_1 in Fig. 1(b). Consequently the rank-1 preimage of a small arc $\gamma_1 \in \partial\mathcal{B}(O)$ which includes y_1 is an arc γ_1^{-1} tangent to the singular set δ_s in Q, with curvature $\chi(y_1)$ obtained by inverting Eq. (24), that is $\chi(y_1) = 2\alpha\cos^3\alpha/(y_1 - \mu)$, where α is the angle in $[-\pi/2, \pi/2]$ such that $-\varepsilon = \tan\alpha$.

If we further increase the parameter γ, then the focal points disappear. For example, in Fig. 1(c) no focal points exist, being $(\gamma - \mu\sigma)^2 < 4\alpha\beta\mu$, and the basin $\mathcal{B}(O)$ is formed by the union of two disjoint regions: the bigger one [partially visible in Fig. 1(c)] is the immediate basin $\mathcal{B}_0(O)$, as it includes O, and the smaller portion, denoted by $\mathcal{B}_{-1}(O)$, is the set of the rank-1 preimages of $\mathcal{B}_0(O)$. More precisely, only the portion $\mathcal{B}_0(O) \cap Z_2$ has preimages, so $\mathcal{B}_{-1}(O) = T_1^{-1}(H_0) \cup T_2^{-1}(H_0)$, where H_0 is the portion of $\mathcal{B}_0(O) \cap Z_2$ bounded by LC, indicated by an arrow in Fig. 1(c). The preimages of H_0 are "unfolded" at opposite sides of LC_{-1} and are joined along it, with LC_{-1} as the locus of merging preimages.

The effects, on the structure of the basins, of the bifurcation which leads to the creation of the two focal points is rather evident from the sequence in Fig. 1. In particular, Fig. 1(a) shows a structure of the basins which is typical of the maps with focal points, called *crescent* in Part I. It can also be noticed that the disappearance/creation of the focal points, and the related prefocal curves, is associated with a change of the position of the two branches of LC_{-1} and LC with respect to their asymptotes.

Similar situations can be observed when the bifurcation condition (20) holds with some vanishing entries in the matrix of first-order derivatives of N and D. For example, let us consider the map (49) with $\sigma = 0$ and $\mu \neq 0$, so that at the bifurcation condition the situation described in Sec. 3.2 is obtained. In Fig. 2(a) we first consider a situation with $\sigma = 0$ and $\gamma^2 < 4\alpha\beta\mu$, so that no focal points exist. The enlargement of Fig. 2(a) shows that the basin $\mathcal{B}(O)$ is nonconnected, i.e. the union of disjoint portions: $B_{-1}(O)$ is formed by the rank-1 preimages of the immediate basin $B_0(O)$, $B_{-2}(O)$ is formed by the rank-1 preimages of $B_{-1}(O)$ [which are rank-2 preimages of $B_0(O)$] and so on. In Fig. 2(b), obtained with the same values of the parameters μ, α, γ, σ, ε and with $\beta = \gamma^2/(4\alpha\mu)$, we are at the bifurcation: A nonsimple focal point Q exists, associated with the prefocal set δ_Q, given by the line $x = F(Q) = \beta - \gamma\varepsilon/2\alpha$, according to the relation (28) between the points of δ_Q and the curvatures of arcs tangent to the singular set in Q. The prefocal line δ_Q crosses the boundary $\partial\mathcal{B}(O)$ at four points, denoted by y_i, $i = 1, \dots, 4$ in Fig. 2(b), and consequently the preimages of small arcs $\gamma_i \in \partial\mathcal{B}(O)$ which include y_i are arcs γ_i^{-1} crossing through Q all with the same slope $m = 0$ and different curvatures [see the enlargement of Fig. 2(b)]. In particular, $B_{-2}(O)$ is now formed by two *lobes* issuing from Q, because the segment y_3y_4 is "focalized" into the focal point. Notice that, again, at the bifurcation value, the critical curves LC_{-1} and LC degenerate in two lines, and the two branches of LC intersect at the point R, which is also the intersection with the prefocal line δ_Q.

If β is further decreased, i.e. after the bifurcation, the nonsimple focal point Q splits into two simple focal points Q_1 and Q_2 and the prefocal line δ_Q, characterized by the relation (29) between points and curvatures of arcs having horizontal tangent in Q, splits into two simple prefocal lines δ_{Q_1} and δ_{Q_2}, related to the respective focal points Q_1 and Q_2 by the one-to-one correspondences (11) between points and slopes of arcs through the simple focal points. This is the situation shown in Fig. 2(c), where both the prefocal lines δ_{Q_1} and δ_{Q_2} intersect the boundary of $B_{-1}(O)$, and consequently $B_{-2}(O)$ includes a crescent through Q_1 and Q_2. The situation shown in Fig. 2(d), obtained with a smaller value of β, is qualitatively different because only δ_{Q_1} intersects $B_{-1}(O)$, and

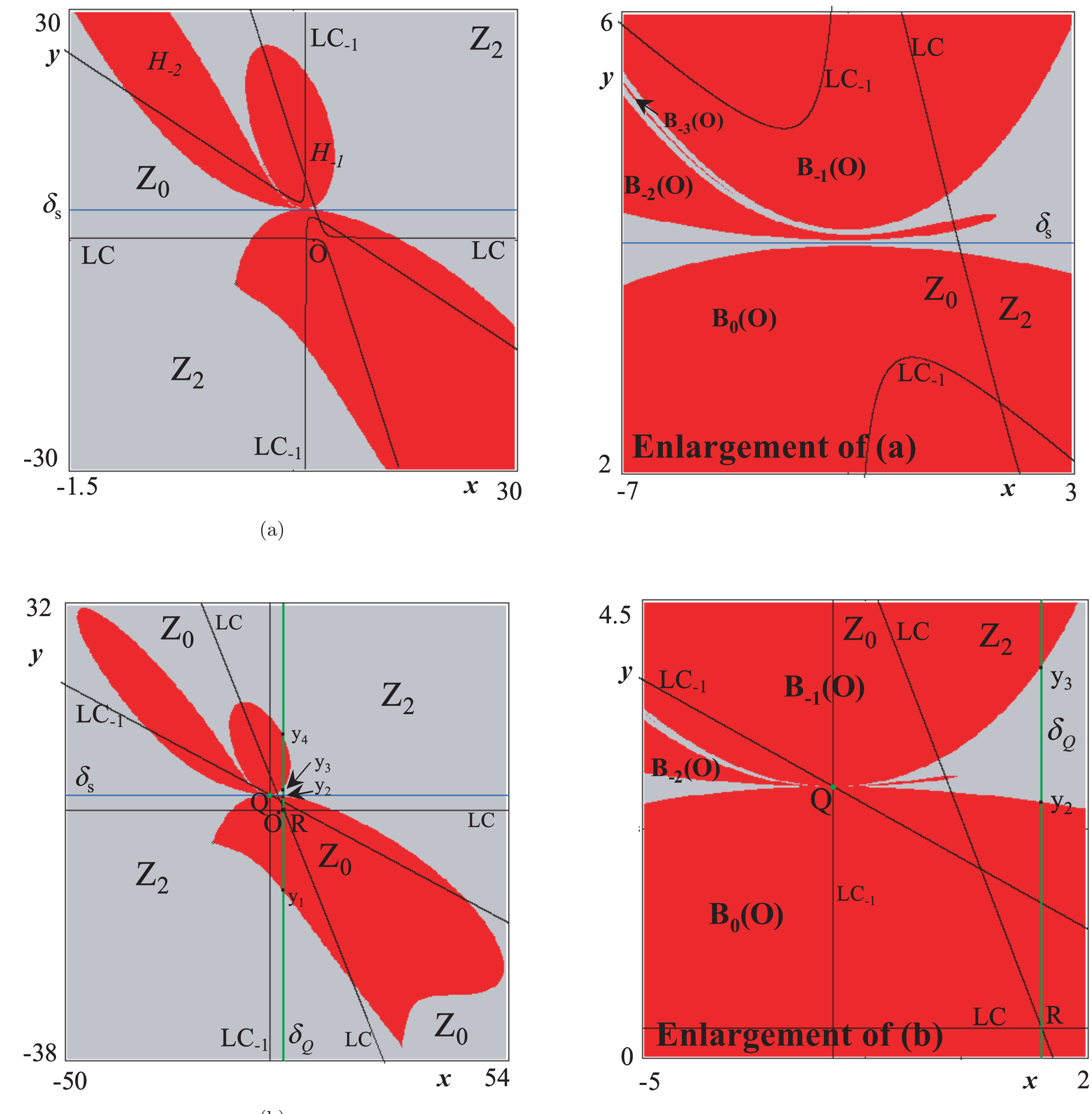


Fig. 2. Critical sets, singular line, prefocal curves and basins for the map (49). The meaning of the colors is the same as in Fig. 1. The parameters are $\mu = 0.3$, $\alpha = 0.2$, $\gamma = 0.8$, $\sigma = 0$, $\varepsilon = -0.7$ and: (a) no focal points at $\beta = 4$; (b) one nonsimple focal point at $\beta = 8/3$; (c) two simple focal points and a crescent between them at $\beta = 2$; (d) two simple focal points and a lobe between them at $\beta = 1.5$.

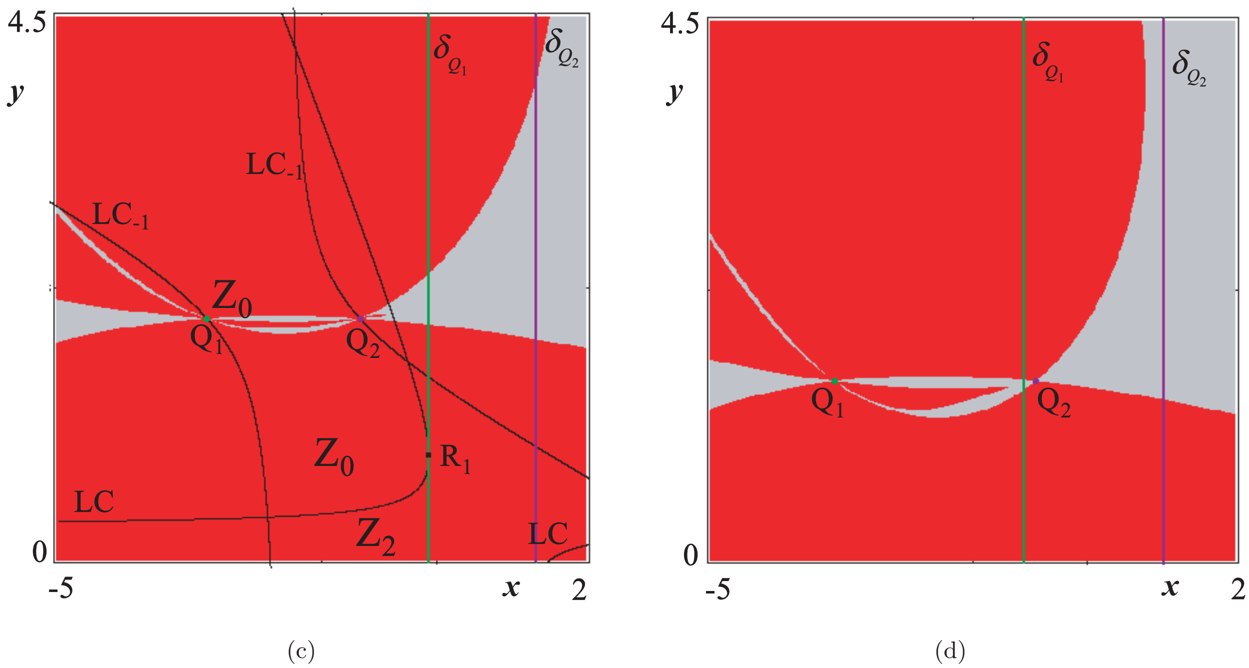


Fig. 2. (*Continued*)

consequently $B_{-2}(O)$ only includes a lobe issuing from Q_1.

After the creation of the two simple focal points, the two branches of LC_{-1} cross through Q_1 and Q_2 the singular set δ_s with the same slope $\overline{m} = -\varepsilon$, and the two corresponding branches of $LC = T(LC_{-1})$ are tangent to the prefocal curves δ_{Q_1} and δ_{Q_2} at the points $\overline{y}_1$ and $\overline{y}_2$, computed according to (66).

As we have seen, the merging of two focal points also implies the merging of the two corresponding prefocal curves. *However, the merging of two prefocal lines may even occur without any merging of focal points.* When this occurs we have a degeneracy on the critical set LC_{-1}. As we have seen above, the merging of two focal points gives rise to a nonsimple focal point, and at the bifurcation the prefocal set includes a particular point R which is also the intersection of the two branches of critical curve LC (which degenerates in two lines). Instead, when two prefocal lines merge maintaining two simple focal points, as we have already seen in Part II, only the critical set LC_{-1} degenerates, while LC becomes asymptotic to δ_Q. For the map (49) this happens for $(\gamma - \mu\sigma)^2 > 4\alpha\beta\mu$ and $\sigma = \varepsilon$. Indeed, for $(\gamma - \mu\sigma)^2 > 4\alpha\beta\mu$ and $\sigma \neq \varepsilon$, LC_{-1} crosses δ_s only at the two simple focal points, and we have $F_y D_x - F_x D_y = \varepsilon - \sigma$, so that (19) is not satisfied if $\sigma \neq \varepsilon$. Instead, for $\varepsilon = \sigma$ (19) holds, so δ_s may be considered as a part of LC_{-1}, in the sense discussed at the end of Sec. 2. This is consistent with the fact that as $\varepsilon \to \sigma$ the two asymptotes of LC_{-1}

$$x = \frac{\mu\sigma - \gamma}{2\alpha} \quad \text{and}$$

$$y = -\frac{\sigma + \varepsilon}{2}x + \beta + \frac{(\varepsilon - \sigma)(\mu\sigma - \gamma)}{4\alpha}$$

tend to the line (64) and to the line of nondefinition (50), respectively. We also remark that the tangency points of LC with δ_Q go to infinity as $\varepsilon \to \sigma$, thus the prefocal line becomes an asymptote of LC. At the bifurcation the two prefocal lines merge into a unique one, given by $x = \beta$, which also constitutes an asymptote for the hyperbola LC [see (61)] and this unique prefocal line is associated with two simple focal points, each one with its own one-to-one correspondence (11).

The effects of such bifurcation are shown numerically in the sequence in Fig. 3, where two coexisting attractors are present: one located around the fixed point O (or the fixed point O itself when it is stable), whose basin is represented

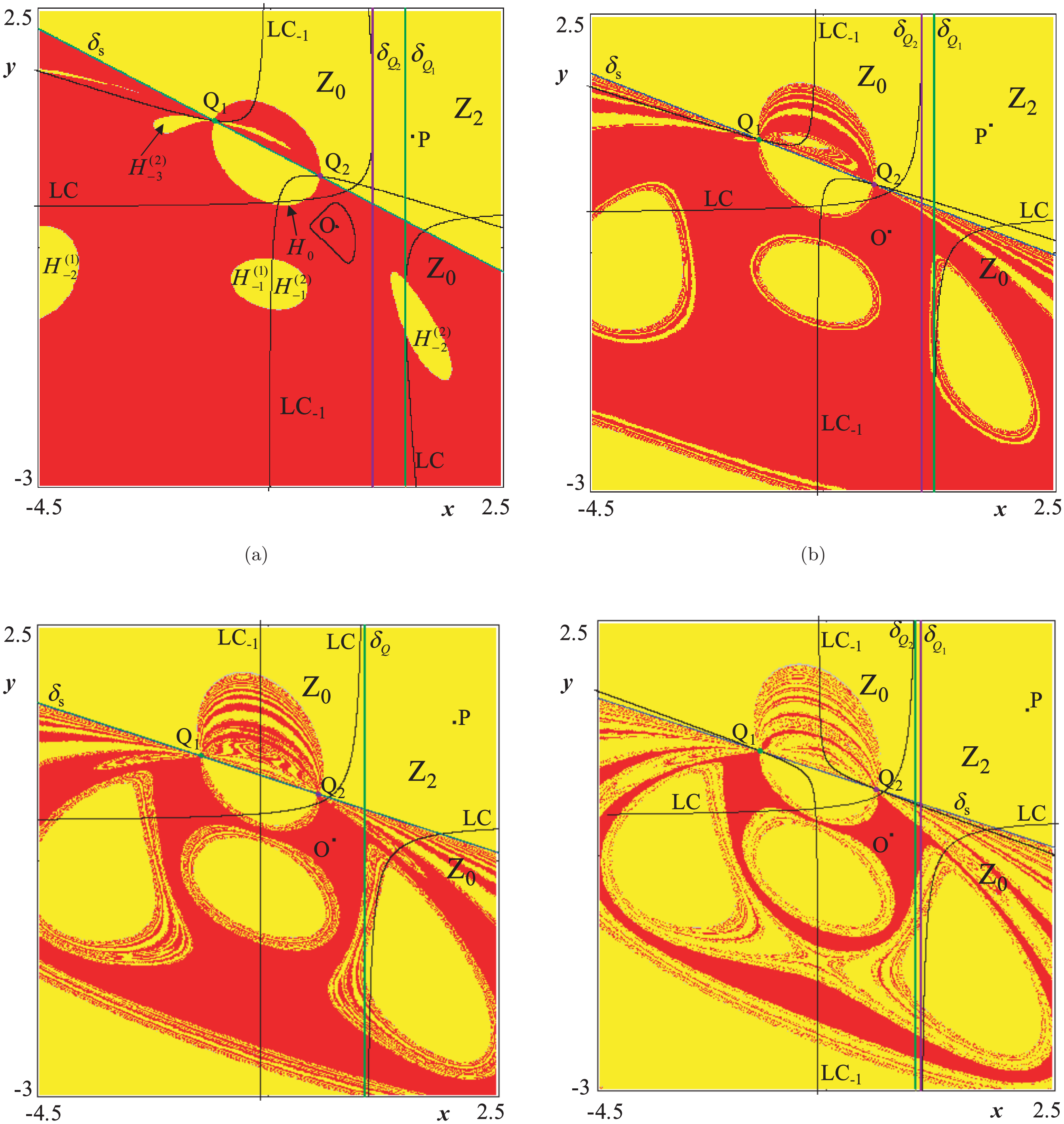


Fig. 3. Critical sets, singular line, prefocal curves and basins for the map (49). The parameters are $\mu = 0.2$, $\alpha = 0.2$, $\gamma = 0.5$, $\beta = 0.47$ and: (a) $\sigma = 0.4$, $\varepsilon = 0.1$; (b) $\sigma = 0.3$, $\varepsilon = 0.2$; (c) bifurcation of merging prefocal curves at $\sigma = 0.25$, $\varepsilon = 0.25$; (d) $\sigma = 0.25$, $\varepsilon = 0.3$.

by the red region, and a stable cycle C_2 of period two, whose basin is represented by the yellow region. *The boundary that separates the two basins is given by the line of nondefinition δ_s and its preimages of any rank.* This is a common occurrence in maps not defined everywhere, as stressed in Part I, even for one-dimensional maps.[3] The structure of the basins shown in Fig. 3(a) is typical for a noninvertible map with denominator, since nonconnected portions (or *holes*) are present together with *lobes*. The presence of both these structures of the basins can be easily explained by looking at the critical curves, the focal points and the corresponding prefocal lines. In fact, the yellow hole H_{-1}, nested inside the red basin, is located across LC_{-1}, and is formed by the rank-1 preimages of the portion H_0 of the basin of C_2, indicated by an arrow in Fig. 3(a). H_0 belongs to the region Z_2 and it is bounded by LC, so its two rank-1 preimages, denoted by $H_{-1}^{(1)}$ and $H_{-1}^{(2)}$, are unfolded along LC_{-1}. Since H_{-1} is included in Z_2 as well, it has two rank-1 preimages whose points belong to the basin of C_2, denoted by $H_{-2}^{(1)}$ and $H_{-2}^{(2)}$ in Fig. 3(a). One of them, $H_{-2}^{(2)}$, is partially included in Z_2 and crosses δ_{Q_1}, so its two rank-1 preimages are given by the two lobes issuing from Q_1.

Also in this case, we notice that the two branches of LC_{-1} cross through the two focal points Q_1 and Q_2 with the same slope $-\varepsilon$, and the two corresponding branches of LC are tangent to δ_{Q_1} and δ_{Q_2} respectively, at the two points computed according to (66).

A more complex structure of lobes and crescents is shown in Fig. 3(b), obtained with closer values of ε and σ, but still with $\varepsilon < \sigma$. The merging of the two prefocal curves is shown in Fig. 3(c), obtained for $\varepsilon = \sigma$, where a *unique prefocal line* is associated with two distinct *simple focal points* Q_1 and Q_2. In this case, the two branches of LC_{-1} degenerate into a pair of lines even if, as explained above, one merges with δ_s, so that we should say that LC_{-1} is formed by a unique line (intersecting δ_s in a nonfocal point) and δ_s has properties which recall those of LC_{-1} (that it, crossing the set δ_s the Jacobian determinant changes sign). As stressed above, LC is asymptotic to δ_Q. In Fig. 3(d), where $\varepsilon > \sigma$, the two branches of LC_{-1} are swapped with respect to these in Fig. 3(b). For $\varepsilon = \sigma$, LC is given by an equilateral hyperbola with asymptotes $y = \mu$ and $x = \beta$, obtained as $LC = T(LC_{-1}) = T(\{x = (\mu\varepsilon - \gamma)/2\alpha\})$, made up of two disjoint unbounded branches because $\overline{LC}_{-1}$ intersects the line δ_s at a point which is nonfocal. So, the structure of LC, and hence the Riemann foliation of the noninvertible map, does not show qualitative changes. This explains why, as it can be seen from a comparison of the Figs. 3(b)–3(d) the merging and swapping of the prefocal lines do not have important effects on the qualitative structure of the basins.

4.2. *Example 2*

Situations similar to the ones described in Example 1 can be observed in maps with more than two focal points, whenever pairs of focal points, or pair of prefocal curves, merge. For example, let us consider the map

$$T: \begin{cases} x' = y \\ y' = \dfrac{\mu y + \alpha x^3 + \gamma x}{y - \beta + \sigma x} \end{cases} \tag{67}$$

The set of nondefinition δ_s is again the line (50), whereas the numerator $N(x, y)$ vanishes along the curve $y = -(x/\mu)(\gamma + \alpha x^2)$, which is a bimodal function for $\gamma/\alpha < 0$. The two curves $D(x, y) = 0$ and $N(x, y) = 0$ always intersect at least at one point, and other two intersections may exist, created or destroyed through tangencies between the cubic curve $N = 0$ and the line $D = 0$. The points where the second component of (67) assumes the form 0/0 are given by $Q = (x_Q, \beta - \sigma x_Q)$, with x_Q solution of the cubic equation $\alpha x^3 + (\gamma - \mu\sigma)x + \mu\beta = 0$. When Q is a simple focal point, the corresponding prefocal line δ_Q has equation

$$x = \beta - \sigma x_Q \tag{68}$$

The map (67) is a noninvertible map of $Z_1 - Z_3$ type, whose set of merging preimages LC_{-1}, obtained from the condition $\det DT = 0$, is given by

$$LC_{-1}: \quad y = \frac{-2\alpha\sigma x^3 + 3\alpha\beta x^2 + \beta\gamma}{3\alpha x^2 + \gamma - \sigma\mu} \quad \text{if } \sigma \neq 0$$

or by the pair of lines

$$LC_{-1}: \quad x = \pm\sqrt{-\frac{\gamma}{3\alpha}} \quad \text{if } \sigma = 0 \text{ and } \frac{\gamma}{\alpha} < 0 \tag{69}$$

[3]Vertical asymptotes which constitute the boundary of basins are frequently encountered in the study of one-dimensional iterated maps.

while for $\sigma = 0$ and $\gamma/\alpha > 0$ the map (67) is an invertible map, i.e. Z_1 covers the whole plane. The critical curves LC_{-1} and $LC = T(LC_{-1})$ are represented in Fig. 4 for a set of parameters such that three simple focal points exist, represented by the points Q_i, $i = 1, 2, 3$, with the corresponding prefocal lines denoted by δ_{Q_i}, $i = 1, 2, 3$, respectively. We stress again that in this generic situation the three branches of LC_{-1}, say $LC_{-1}^{(i)}$, $i = 1, 2, 3$, cross δ_s through the focal points Q_i, $i = 1, 2, 3$, respectively, all with the same slope computed according to (17), which now reads $\overline{m} = 0$, and the corresponding branches of LC, say $LC^{(i)} = T(LC_{-1}^{(i)})$, are tangent to δ_{Q_i}, $i = 1, 2, 3$, at the points of y coordinates $y_i(0)$ obtained from the respective one-to-one correspondences (11). However, differently from what occurs in $Z_0 - Z_2$ maps, in this case LC may also have transverse intersections with the prefocal curves, such as the point A in Fig. 4, as well as the points B and C. This has a simple explanation. In fact, along any simple prefocal line δ_{Q_i} just one inverse focalizes, and the point where LC is tangent to δ_{Q_i} represents the merging between the inverse which focalizes into Q_i and another (nonfocalizing) inverse. The two merging preimages, which must belong to LC_{-1}, are located in Q_i because the preimage, by the focalizing inverse, of any point of δ_{Q_i} is in Q_i. Instead, a point where LC has a transverse intersection with δ_{Q_i} is related to the merging of the two inverses which do not focalize. So, the merging preimages of the point $A \in \delta_{Q3} \cap LC$ belong to LC_{-1}, but are out of Q_3. The portion of δ_{Q3} below A belongs to Z_1, since the two inverses which merge in A disappear below A, and only the inverse which focalizes exists in $\delta_{Q3} \cap Z_1$.

Also for this map, the bifurcation related to the merging of two simple focal points implies a structural change of the foliation. This can be clearly seen in Figs. 5(a)–5(c). In these figures the red region represents the basin $\mathcal{B}(O)$ of the stable fixed point $O = (0, 0)$, and the gray region represents the basin $\mathcal{B}(\infty)$ of diverging trajectories. Figure 5(a) is obtained for a set of parameters such that three simple focal points Q_i, $i = 1, 2, 3$, exist, with the

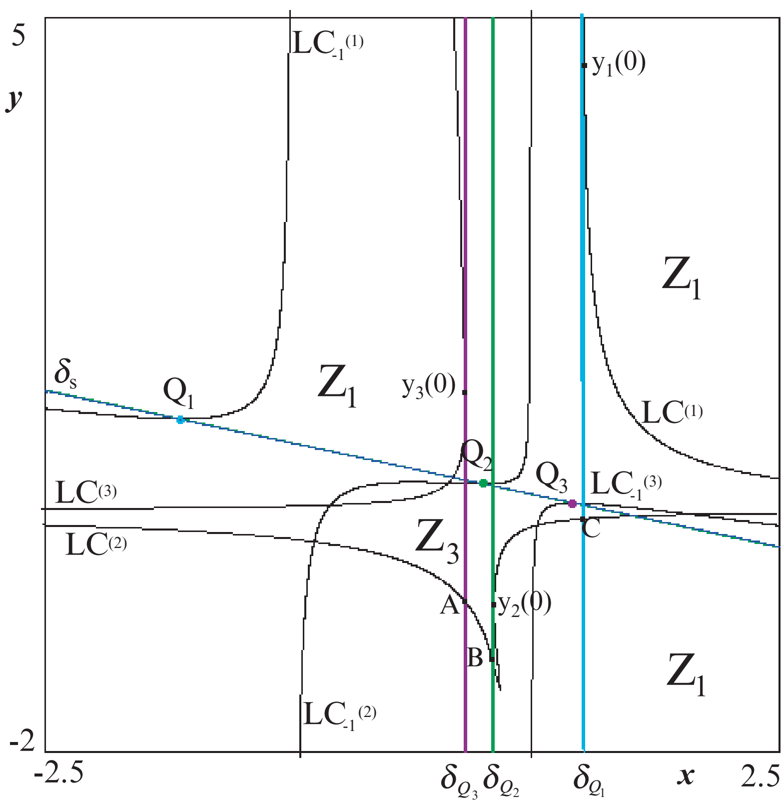


Fig. 4. Critical sets LC_{-1} and LC (black), singular line δ_s (blue), prefocal curves δ_{Q_1}, δ_{Q_2} and δ_{Q_3} (pale blue, green and violet, respectively) and corresponding focal points Q_1, Q_2 and Q_3 (pale blue, green and violet, respectively) for the map (67). The figure has been obtained for $\mu = 0.3$, $\alpha = 0.25$, $\gamma = -0.4$, $\beta = 0.7$, $\sigma = 0.3$.

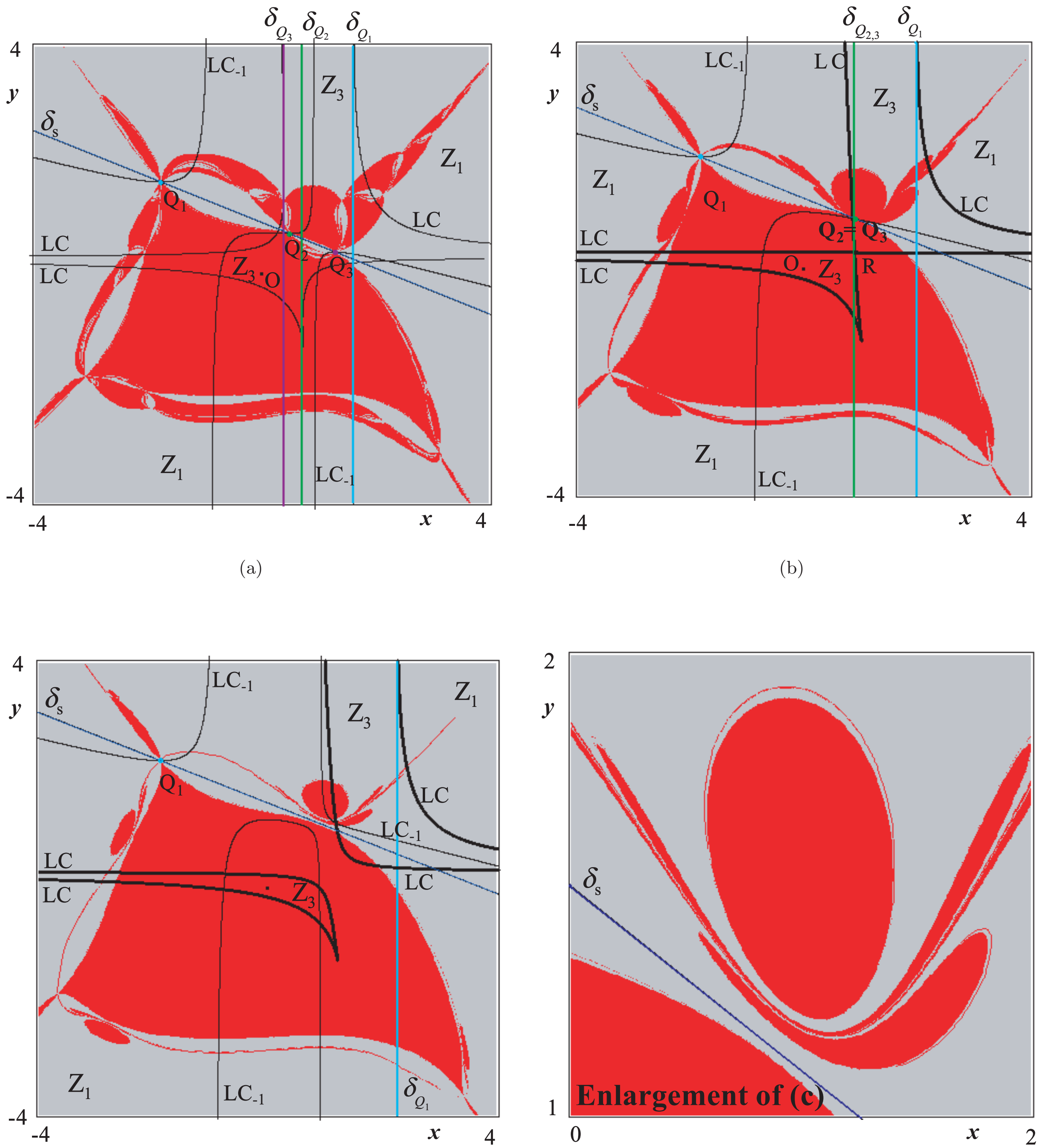


Fig. 5. Critical sets, singular line δ_s, prefocal curves, focal points and basins for the map (67). The meaning of the colors is the same as in Fig. 4, the red region represents the stable fixed point $O = (0, 0)$, the gray region represents the basin of diverging trajectories. (a) $\mu = 0.3$, $\alpha = 0.25$, $\gamma = -0.5$, $\beta = 0.9$, $\sigma = 0.4$. Three simple focal points exist. (b) Merging of the focal points Q_2 and Q_3 for $\beta = 1.2525$. (c) $\beta = 1.5$, only one simple focal point exists.

corresponding prefocal lines δ_{Q_i}, $i = 1, 2, 3$, given by (68). The basin $\mathcal{B}(O)$ is characterized by the presence of lobes and crescents issuing from focal points and their preimages, and the critical curves have a structure similar to the one shown in Fig. 4. If the parameter β is increased, the focal points Q_2 and Q_3 become closer and closer, and the same holds for the corresponding prefocal curves δ_{Q_2} and δ_{Q_3}, until they merge, at $\beta = (2(\sigma\mu - \gamma)/3\mu)/\sqrt{(\sigma\mu - \gamma)/3\alpha}$, as shown in Fig. 5(b), where Q_2 and Q_3 merge giving rise to a nonsimple focal point, and the two prefocal lines merge into a unique one, as described in Sec. 3.1. The two points where, before the bifurcation, LC is tangent to δ_{Q_2} and δ_{Q_3}, at the bifurcation merge at the point R, so that two branches of LC now intersect at this point R [Fig. 5(b)]. So, as already observed in Example 1, the bifurcation associated with merging focal points is followed by a qualitative change of the critical set LC.

After the bifurcation, only the focal point Q_1 exists, and the structure of LC is qualitatively different with respect to the one in Fig. 5(a) (see Fig. 5(c), obtained by increasing β). The disappearance of two focal points also causes a qualitative change in the structure of the basins, since the lobes of $\mathcal{B}(O)$ issuing from Q_2 and Q_3 are now replaced by "islands" of $\mathcal{B}(O)$ surrounded by points of $\mathcal{B}(\infty)$ [see the enlargement of a portion of Fig. 5(c)].

4.3. *Example 3*

The presence of nonsimple focal points and/or nonsimple prefocal curves, which have been considered as bifurcation situations in the previous examples, may be "structural" for some particular maps, in the sense that the particular properties observed at the bifurcation values in the examples discussed above are "persistent" properties (several examples can be found in Part I). Let us consider, for example, the map

$$T: \begin{cases} x' = y \\ y' = y - \lambda x + \dfrac{\alpha x^2 - \gamma}{y - \beta} \end{cases} \tag{70}$$

whose set of nondefinition δ_s is the line $y = \beta$. If $\gamma/\alpha > 0$ then two simple focal points exist,

$$Q_1 = \left(-\sqrt{\frac{\gamma}{\alpha}}, \beta\right) \quad \text{and} \quad Q_2 = \left(\sqrt{\frac{\gamma}{\alpha}}, \beta\right), \tag{71}$$

associated with the same prefocal line δ_Q of equation $x = \beta$ through the correspondences between slopes and points

$$y_1(m) = \beta + \lambda\sqrt{\frac{\gamma}{\alpha}} - \frac{2\alpha\sqrt{\dfrac{\gamma}{\alpha}}}{m};$$

$$y_2(m) = \beta - \lambda\sqrt{\frac{\gamma}{\alpha}} + \frac{2\alpha\sqrt{\dfrac{\gamma}{\alpha}}}{m}$$

respectively. The map (70) is a noninvertible map of Z_0–Z_2 type, and the regions Z_0 and Z_2 are defined as $Z_0 = \{(x, y)|\Delta(x, y) < 0\}$ and $Z_2 = \{(x, y)|\Delta(x, y) > 0\}$ respectively, where

$$\Delta(x, y) = \lambda^2(x - \beta)^2 - 4\alpha(x - \beta)(x - y) + 4\alpha\gamma.$$

The critical set LC, which separates the regions Z_0 and Z_2, is defined by the equation $\Delta(x, y) = 0$. For each point $(x', y') \in Z_2$ two distinct preimages exist given by $T^{-1}(x', y') = T_1^{-1}(x', y') \cup T_2^{-1}(x', y')$, where:

$$\begin{aligned} T_1^{-1}&: \begin{cases} x = \dfrac{1}{2\alpha}\left(\lambda(x' - \beta) - \sqrt{\Delta(x', y')}\right) \\ y = x' \end{cases} \\ T_2^{-1}&: \begin{cases} x = \dfrac{1}{2\alpha}\left(\lambda(x' - \beta) + \sqrt{\Delta(x', y')}\right) \\ y = x' \end{cases} \end{aligned} \tag{72}$$

It is immediate to verify that if $\gamma/\alpha > 0$ and $(x', y') \in \delta_Q$, i.e. $x' = \beta$, then (72) gives $x = \pm\sqrt{\gamma/\alpha}$ and $y = \beta$, that is, both the inverses "focalize" δ_Q into the respective focal points: $T_1^{-1}(\delta_Q) = Q_1$ and $T_2^{-1}(\delta_Q) = Q_2$.

The critical set LC is an hyperbola with asymptotes of equation $x = \beta$ (i.e. δ_Q) and $y = (1 - \lambda^2/4\alpha)x + \beta\lambda^2/4\alpha$. The set of merging preimages LC_{-1} is the line of equation $2\alpha x - \lambda y + \lambda\beta = 0$, which is easily obtained from (72) with $\Delta = 0$ or from the condition $\det DT = 0$. As we have seen in Example 1, when a prefocal curve δ_Q is related to two simple focal points Q_1 and Q_2, then LC has δ_Q as an asymptote, and LC_{-1} includes a line crossing δ_s at the point in between Q_1 and Q_2.

We can also notice that, for the map (70), we have $F_x = 0$ and $D_x = 0$, hence the relation (19) is always satisfied. So, the line of nondefinition δ_s has properties similar to those of LC_{-1} (that it, crossing the set δ_s the Jacobian determinant changes sign), as discussed at the end of Sec. 2.

For $\gamma = 0$ a unique nonsimple focal point $Q = (0, 0)$ exists, associated with the prefocal line δ_Q of equation $x = \beta$, according to the situation described in Sec. 3.2. In this case LC degenerates into the asymptotes, so that the prefocal line δ_Q becomes part of LC and separates zones with different number of preimages.

The effects of these occurrences are shown in Fig. 6, obtained for $\lambda = 0.8$, $\alpha = -0.2$, $\beta = \sqrt{3.5}$, and different values of γ. Figure 6(a) is obtained for a set of parameters such that no focal points exist, with $\gamma/\alpha < 0$, and there are two fixed points, say $P_1^* = (x_1^*, x_1^*)$ and $P_2^* = (x_2^*, x_2^*)$, with $x_1^* = (-\beta + \sqrt{\beta^2\lambda^2 + 4\gamma(\alpha - \lambda)})/2(\alpha - \lambda)$ and $x_2^* = (-\beta - \sqrt{\beta^2\lambda^2 + 4\gamma(\alpha - \lambda)})/2(\alpha - \lambda)$, stable focus and saddle point respectively. The red region represents the basin of P_1^*, the gray region the basin of infinity. In Fig. 6(a) the line $x = \beta$ is entirely included inside Z_0. In Fig. 6(b), obtained for $\gamma = 0$, we have a nonsimple focal point $Q = (0, 0) \in LC_{-1}$ and the line $x = \beta$, which is the corresponding prefocal curve, satisfies the equation $\Delta(x, y) = 0$ and it is part of LC: it separates regions having a different number of preimages. Its merging preimages are focalized in Q, i.e. $T_1^{-1}(\delta_Q) = T_2^{-1}(\delta_Q) = Q$. It can be noticed that the structure of the basins is qualitatively different from the one shown in Fig. 6(a), since lobes and crescents are now present.

As γ is slightly decreased, so that $\gamma/\alpha < 0$, two simple focal points Q_1 and Q_2 appear, and the related prefocal curve δ_Q is now entirely included inside the region Z_2, due to the change of the position of LC with respect to the asymptotes. The two preimages of δ_Q are focalized into the distinct focal points, i.e. $T_1^{-1}(\delta_Q) = Q_1$, $T_2^{-1}(\delta_Q) = Q_2$. The presence of the two focal points is also evident from the structure of the basins, which include lobes and crescents issuing from the focal points and their preimages (with both Q_1 and Q_2 inside the region Z_2).

We end the discussion of this example by observing that $Q_2 \in \delta_Q$ for $\gamma = \alpha\beta^2$. *The merging of a focal point with the related prefocal curve also implies the merging of the focal point with a fixed point of the map.* In this particular case, it is easy to check such occurrence, since for $\gamma = \alpha\beta^2$ we have $Q_2 = P_2^*$. For the set of parameters used in Fig. 6 this occurs for $\gamma = -0.7$ [see Fig. 6(d)]. Of course, at this bifurcation P_2^* is no longer to be considered a fixed point, since the map is not defined in it, but the focal point which takes its place "inherits" part of the properties, because at least for one of the inverses of T that point is a fixed point. In fact, being $T_2^{-1}(\delta_Q) = Q_2$ and $Q_2 \in \delta_Q$, we have $T_2^{-1}(Q_2) = Q_2$. In this case, for T_2^{-1} the fixed point Q_2 is a saddle, with an eigenvalue equal to zero and the other one greater than one in absolute value. For the map T, this focal point that belongs to its prefocal set, resulting from the merging of a fixed point and a focal point, may be seen as a "generalized saddle" with one eigenvalue at infinity and the other inside the unit circle.

4.4. *Example 4*

The bifurcation situations described in Sec. 3, associated with nonsimple focal points, are not only related to the "appearance–disappearance" of focal points, but also to the "merging and crossing" of simple focal points. In order to explore such occurrences, as well as their effects on the structure of the basins, let us consider the map

$$T: \begin{cases} x' = y + \varepsilon x \\ y' = \dfrac{\alpha x^2 + \gamma x}{(y - \beta_1)(y - \beta_2)} \end{cases} \tag{73}$$

This map is not defined at the points of the set $D(x, y) = 0$, given by $\delta_s = \delta_s^1 \cup \delta_s^2$, where δ_s^1 and δ_s^2 are the lines $y = \beta_1$ and $y = \beta_2$, respectively. As the numerator $N(x, y)$ vanishes along the two vertical lines $x = 0$ and $x = -\gamma/\alpha$, if $\gamma \neq 0$ and $\beta_1 \neq \beta_2$ then the map (73) has four simple focal points, associated with four simple prefocal curves, given by:

$$\begin{aligned} &Q_1 = (0, \beta_1) && \text{with } \delta_{Q_1} \text{ of equation } x = \beta_1 \\ &Q_2 = (0, \beta_2) && \text{with } \delta_{Q_2} \text{ of equation } x = \beta_2 \\ &Q_3 = \left(-\frac{\gamma}{\alpha}, \beta_1\right) && \text{with } \delta_{Q_3} \text{ of equation } x = \beta_1 - \frac{\gamma\varepsilon}{\alpha} \\ &Q_4 = \left(-\frac{\gamma}{\alpha}, \beta_2\right) && \text{with } \delta_{Q_4} \text{ of equation } x = \beta_2 - \frac{\gamma\varepsilon}{\alpha} \end{aligned} \tag{74}$$

In fact, with $N_y = D_x = 0$, at each Q_i the condition (7) for a simple focal point becomes

$$\gamma(\beta_1 - \beta_2) \neq 0 \tag{75}$$

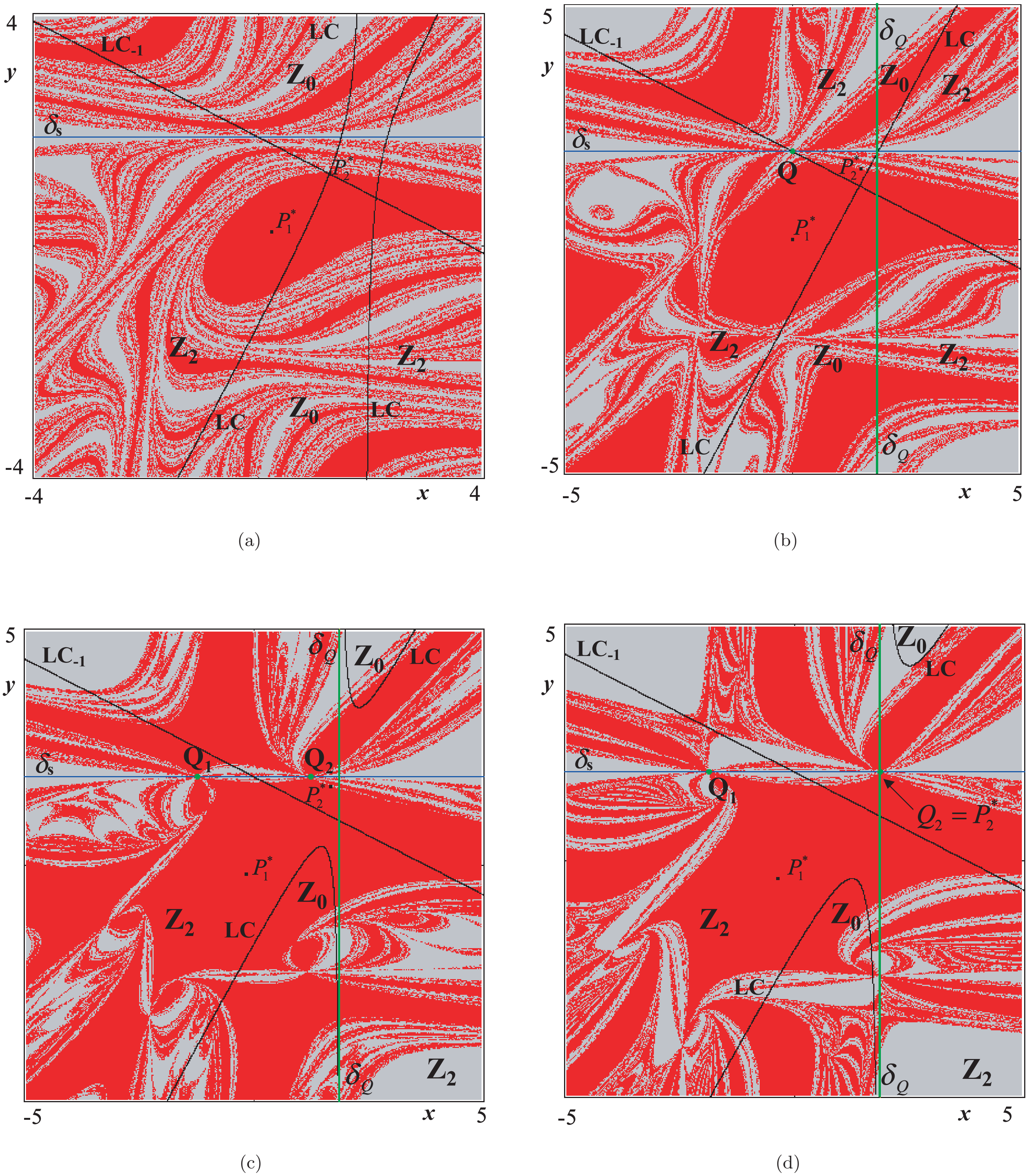


Fig. 6. Critical sets, singular line δ_s, prefocal curves, focal points and basins for the map (70). The red region represents the stable fixed point P_1^*, bounded by the stable set of the unstable fixed point P_2^*, the gray region represents the basin of diverging trajectories. (a) $\lambda = 0.8$, $\alpha = -0.2$, $\gamma = 0.3$, $\beta = \sqrt{3.5} = 1.870829$. (b) Nonsimple focal point for $\gamma = 0$. (c) $\gamma = -0.3$. (d) $Q_2 = P_2^*$ for $\gamma = -0.7$.

If (75) is true, then the one-to-one correspondences (11) are readily obtained: for Q_1 and Q_4 we have

$$y_1(m) = \frac{\gamma}{m(\beta_1 - \beta_2)} \tag{76}$$

and for Q_2 and Q_3 we have

$$y_2(m) = -\frac{\gamma}{m(\beta_1 - \beta_2)}. \tag{77}$$

The map T is a noninvertible map of $Z_0 - Z_2$ type. In fact, the preimages of a point (x', y') are the real solutions of the second degree algebraic system

$$\begin{cases} (\varepsilon^2 y' - \alpha)x^2 + (\varepsilon y'(\beta_1 + \beta_2) - 2\varepsilon x' y' - \gamma)x \\ \qquad + y'(x'^2 - (\beta_1 + \beta_2)x' + \beta_1\beta_2) = 0 \\ y = x' - \varepsilon x \end{cases} \tag{78}$$

equivalent to the system (73) for $y \neq \beta_1$ and $y \neq \beta_2$. Hence a point (x', y') has two distinct preimages, i.e. $(x', y') \in Z_2$ if

$$\begin{aligned} \Delta(x, y) = {} & (\varepsilon(\beta_1 + \beta_2)y - 2\varepsilon xy - \gamma)^2 \\ & - 4(\varepsilon^2 y - \alpha)y(x - \beta_1)(x - \beta_2) > 0 \end{aligned} \tag{79}$$

and in this case the two inverses are given by $T^{-1}(x', y') = T_1^{-1}(x', y') \cup T_2^{-1}(x', y')$, where

$$T_1^{-1}\colon \begin{cases} x = \dfrac{\gamma - \varepsilon(\beta_1 + \beta_2)y' + 2\varepsilon x' y' - \sqrt{\Delta(x', y')}}{2(\varepsilon^2 y' - \alpha)} \\ y = x' - \varepsilon x \end{cases}$$

$$T_2^{-1}\colon \begin{cases} x = \dfrac{\gamma - \varepsilon(\beta_1 + \beta_2)y' + 2\varepsilon x' y' + \sqrt{\Delta(x', y')}}{2(\varepsilon^2 y' - \alpha)} \\ y = x' - \varepsilon x \end{cases} \tag{80}$$

The set LC_{-1} is defined by the equation

$$\begin{aligned} & \frac{1}{(y - \beta_1)^2(y - \beta_2)^2}[\varepsilon(\alpha x^2 + \gamma x)(2y - \beta_1 - \beta_2) \\ & \qquad + (2\alpha x + \gamma)(y - \beta_1)(y - \beta_2)] = 0 \end{aligned} \tag{81}$$

obtained from (80) with $\Delta = 0$ or from the condition $\det DT = 0$. The set LC_{-1} is generally formed by three disjoint branches, which cross the set of nondefinition at the focal points and includes the point $(-\gamma/2\alpha, (\beta_1 + \beta_2)/2)$ located at the center of the quadrilateral $Q_1Q_2Q_3Q_4$. From (17) we get that the tangents to LC_{-1} in all the four focal points have slope $\overline{m} = -\varepsilon$ (because $F(x, y)$ is linear in this example).

A generic representation of the critical curves, obtained for $\varepsilon \neq 0$, $\beta_1 \neq \beta_2$ and $\gamma \neq 0$, is shown in Fig. 7, where the three branches of LC_{-1} are represented together with the corresponding three branches of $LC = T(LC_{-1})$, tangent to the four prefocal lines δ_{Q_i}, $i = 1, 2, 3, 4$, at the points of y coordinates $y_1(-\varepsilon)$ and $y_2(-\varepsilon)$ computed according to (76) and (77) respectively.[4] In such a generic situation LC_{-1} does not intersect δ_s at nonfocal points, as it can be deduced from (19) which becomes, both along δ_s^1 and δ_s^2,

$$\varepsilon(\beta_1 - \beta_2) = 0. \tag{82}$$

The point $O = (0, 0)$ is always a fixed point of (73), and two other fixed points, say E and P, exist if the following system

$$\begin{cases} y = (1 - \varepsilon)x \\ \lambda x^2 - ((\beta_1 + \beta_2)\lambda + \alpha)x + \lambda\beta_1\beta_2 - \gamma = 0 \end{cases}$$

has real solutions. In the examples considered in this section, the fixed point O is the unique attractor (stable fixed point), and we denote by $\mathcal{B}(O)$ its basin of attraction, represented by the red regions in all the figures, while the gray points denote $\mathcal{B}(\infty)$, i.e. the locus of points having divergent trajectories.

Starting from the situation depicted in Fig. 7, we can change the value of several parameters in order to analyze several kinds of bifurcations. For example, by decreasing γ, at $\gamma = 0$ we see that the relation in (75) does not hold, and in fact we have the merging of Q_1 and Q_3, as well as the merging of Q_2 and Q_4, in two nonsimple focal points. However, this does not lead to the disappearance of the focal points, but only to an exchange of their positions. If we decrease ε, when $\varepsilon = 0$ we have four simple focal points, associated with only two prefocal curves, $\delta_{Q_1} = \delta_{Q_3}$ and $\delta_{Q_2} = \delta_{Q_4}$, as the relation in (82) is satisfied. Moreover, it is immediate to see that when $\beta_1 = \beta_2$ then the relation (82) is satisfied, and (75) does not hold. In fact, we have the merging of two pairs of focal points, namely $Q_1 = Q_2$ and

[4] We recall that LC can be obtained from the equation $\Delta(x, y) = 0$, with Δ given in (79) or, equivalently, as $LC = T(LC_{-1})$.

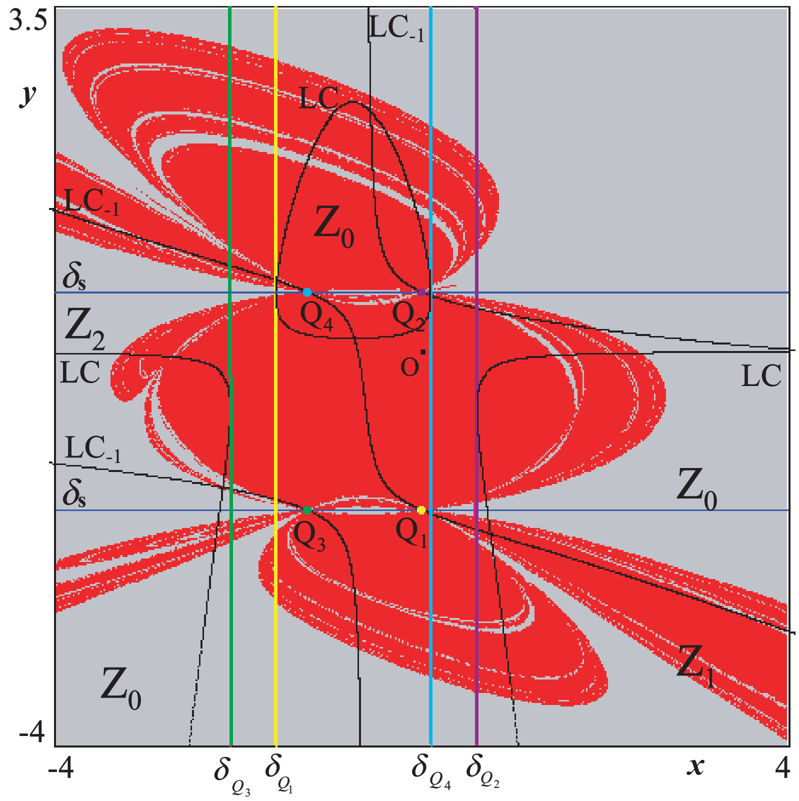


Fig. 7. Critical sets LC_{-1} and LC (black), prefocal curves δ_{Q_1}, δ_{Q_2}, δ_{Q_3} and δ_{Q_4} (yellow, violet, green and pale blue, respectively), corresponding focal points Q_1, Q_2, Q_3 and Q_4 (yellow, violet, green and pale blue, respectively) and basins of attraction for the map (73) with parameters $\alpha = 0.4$, $\gamma = 0.5$, $\beta_1 = -1.6$, $\beta_2 = 0.6$, $\varepsilon = 0.4$. The two straight lines through the points Q_2, Q_4 and Q_1, Q_3 are the lines of nondefinition δ_s. The red region represents the basin $\mathcal{B}(O)$ of the stable fixed point O, the gray region represents the basin of infinity.

$Q_3 = Q_4$, in two nonsimple focal points, together with a bifurcation in the foliation of the plane. Also in this case the focal point will not disappear, but they only exchange their positions.

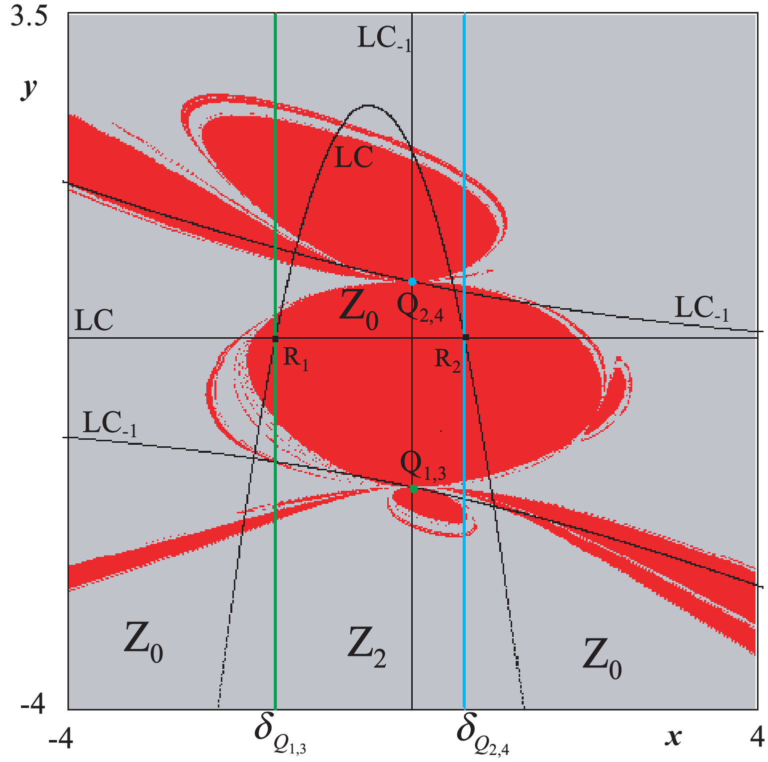
Let us first consider the situation obtained for $\varepsilon \neq 0$, $\beta_1 \neq \beta_2$ and $\gamma = 0$ [Fig. 8(a)]. In this case, we have Q_1 and Q_3 merging at the point $Q_{1,3} = (0, \beta_1)$, with related prefocal line $\delta_{Q_{1,3}}$ of equation $x = \beta_1$, and Q_2 merges at Q_4 giving the point $Q_{2,4} = (0, \beta_2)$, with related prefocal line $\delta_{Q_{2,4}}$ of equation $x = \beta_2$. In $Q_{1,3}$ we have $\overline{N}_x = \overline{N}_y = \overline{D}_x = 0$ and $\overline{D}_y = \beta_1 - \beta_2$, and in $Q_{2,4}$ we have $\overline{N}_x = \overline{N}_y = \overline{D}_x = 0$ and $\overline{D}_y = \beta_2 - \beta_1$. So, in both cases, we have the situation described in one of the particular cases considered in Sec. 3.2, hence we expect that, in this bifurcation situation, the preimages of arcs crossing through the prefocal lines are arcs through the focal points with horizontal tangent (i.e. tangent to both $\delta_s^{1,2}$) and different curvatures according to (29), as can be seen in Fig. 8(a). The preimages of the prefocal lines are given by $T^{-1}(\delta_{Q_{1,3}}) = Q_1 \cup (\{y = \beta_1 - \varepsilon x\})$ and $T^{-1}(\delta_{Q_{2,4}}) = Q_2 \cup (\{y = \beta_2 - \varepsilon x\})$, with a change of role of the focalizing inverse at the points R_1 and R_2, respectively. We also observe that at this value of γ the set LC_{-1} reduces to the union of a straight line and an hyperbola

$$LC_{-1} = \{x = 0\} \cup \left\{ x = \frac{2}{3} \frac{(y-\beta_1)(y-\beta_2)}{2y - \beta_1 - \beta_2} \right\}$$

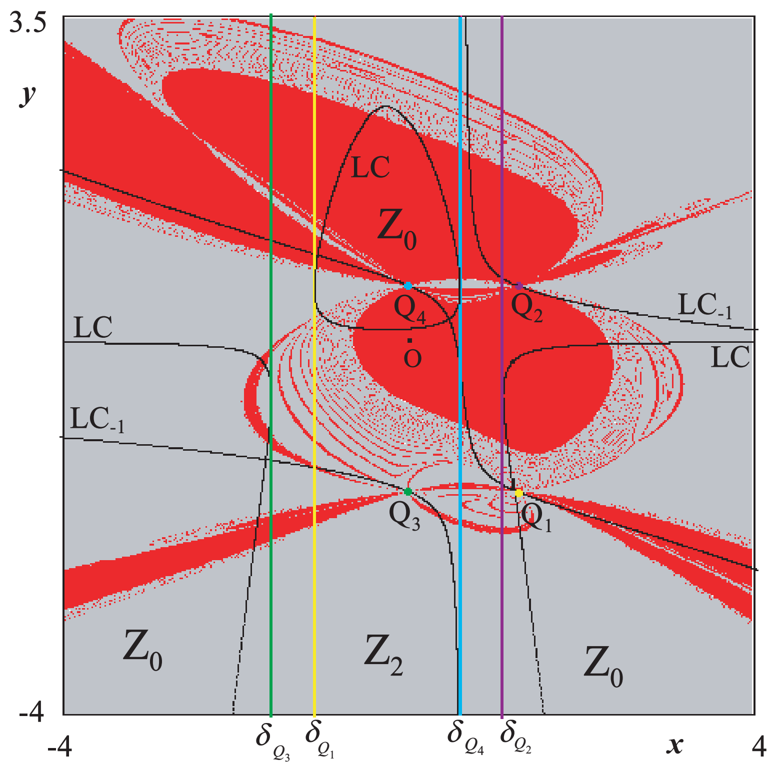
and LC reduces to the union of a straight line and a parabola

$$LC = \{y = 0\} \cup \left\{ y = -\frac{4\alpha(x-\beta_1)(x-\beta_2)}{\varepsilon^2(\beta_1 - \beta_2)} \right\}$$

This drastic change in the structure of the critical curves seems to suggest some qualitative change on the foliation of the map, but it is not the case,



(a)



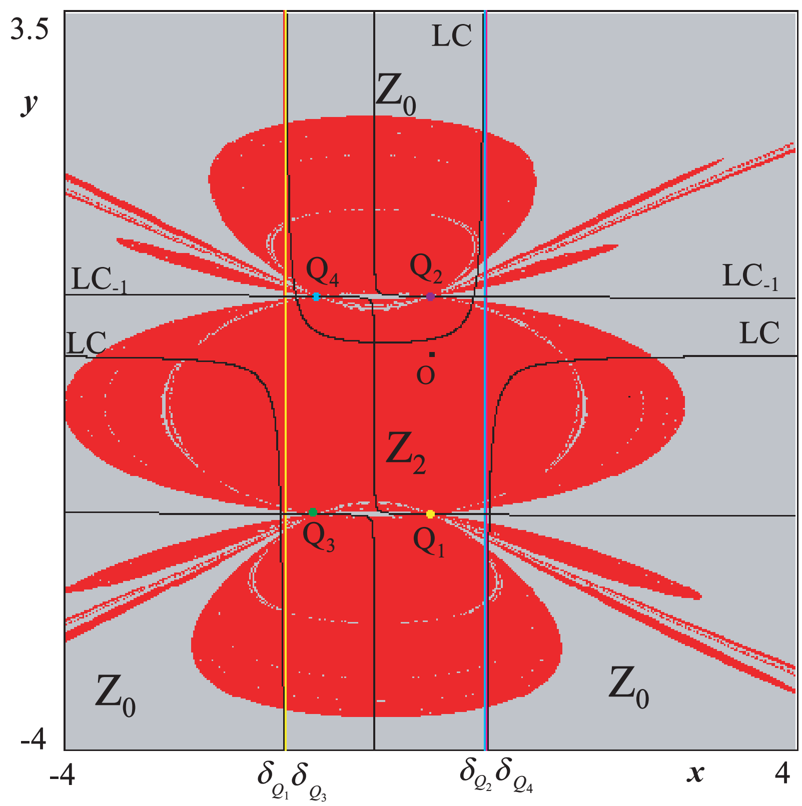
(b)

Fig. 8. Critical sets, prefocal curves, focal points and basins for the map (73). The meaning of the colors is the same as in Fig. 7. (a) Nonsimple focal points $Q_{2,4} = Q_2 = Q_4$ and $Q_{1,3} = Q_1 = Q_3$ for $\gamma = 0$, the other parameters are the same as in Fig. 7. (b) $\gamma = -0.5$.

because when the parameter γ changes from positive to negative, the shape of the critical curves becomes again as in Fig. 7 [see Fig. 8(b)]. Indeed, in this case the relation (82) is not satisfied.

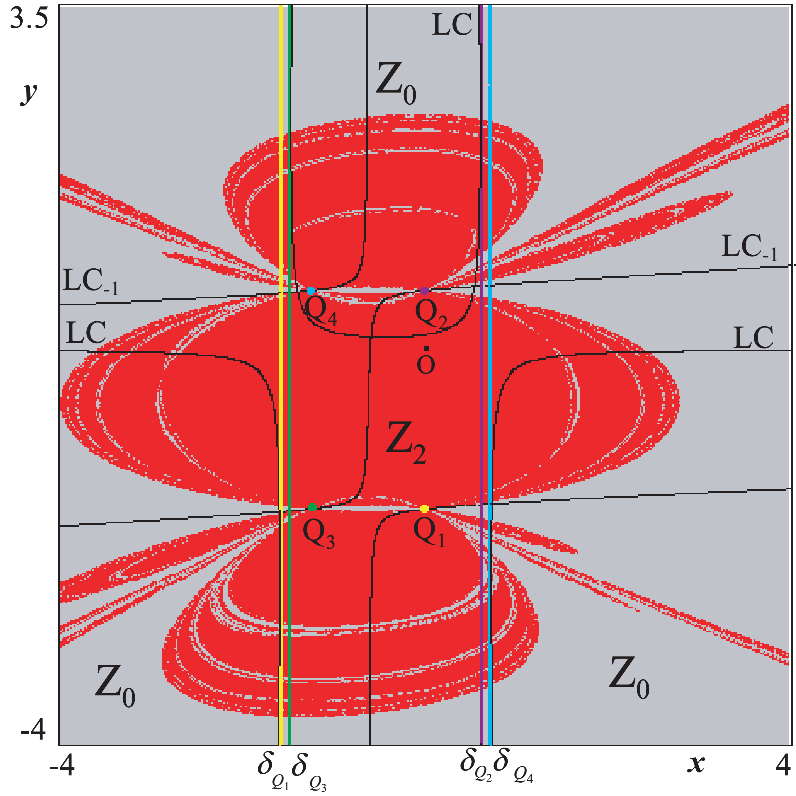
Instead, for $\varepsilon = 0$ and $\beta_1 \neq \beta_2$, $\det DT$ changes sign along the line $x = -\gamma/2\alpha$ and along each of the two lines δ_s^1 and δ_s^2. In fact, in this case (82) holds and, as argued at the end of Sec. 2, these two lines behave as a portion of LC_{-1}, even if the map (73) is not defined along such lines. Indeed, as $\varepsilon \to 0$ the three branches of LC_{-1} tend to join and approach more and more closely the lines $y = \beta_1$, $y = \beta_2$ and $x = -\gamma/2\alpha$, as shown in Fig. 9(a), obtained for $\varepsilon = 0.01$. It is clear that at the bifurcation $\varepsilon = 0$ we have the merging of the two prefocal lines δ_{Q_1} and δ_{Q_3} into a unique prefocal line $\delta_{Q_{1,3}}$ of equation $x = \beta_1$, and the merging of the two prefocal lines δ_{Q_2} and δ_{Q_4} into a unique prefocal line $\delta_{Q_{2,4}}$ of equation $x = \beta_2$. Moreover, the coordinates $y_1(-\varepsilon)$ and $y_2(-\varepsilon)$ of the tangency points go to infinity, which means that the two prefocal lines become asymptotes for LC. This can also be seen from $LC = T(\{x = -\gamma/2\alpha\})$, since the line $x = -\gamma/2\alpha$ intersects both δ_s^1 and δ_s^2 in nonfocal points, or it can be directly deduced from the equation $\Delta(x, y) = 0$, which for $\varepsilon = 0$ becomes $y = -\gamma^2/4\alpha(x - \beta_1)(x - \beta_2)$. Differently from what occurred in the previous case, now the change in sign of the parameter ϵ shall cause a qualitative change in the foliation. In fact, for $\epsilon < 0$ [see Fig. 9(b)], the two nonsimple prefocal curves separate again into four simple prefocal curves, but now the set LC_{-1} is "on the other side" of the line $x = -\gamma/2\alpha$.

Another standard situation, when $\varepsilon \neq 0$, $\beta_1 \neq \beta_2$ and $\gamma \neq 0$ is shown in Fig. 10(a): four simple focal points are associated with four prefocal curves. The crossing of invariant phase curves (as those belonging to the basin boundary) with the prefocal lines, implies the crossing of phase curves through all the focal points, all having a known slope (due to the relations (76), (77), and their inverses), as shown in the enlargement of Fig. 10(a). As $\beta_1 \to \beta_2$ the three branches of LC_{-1} shown in Fig. 10(a)



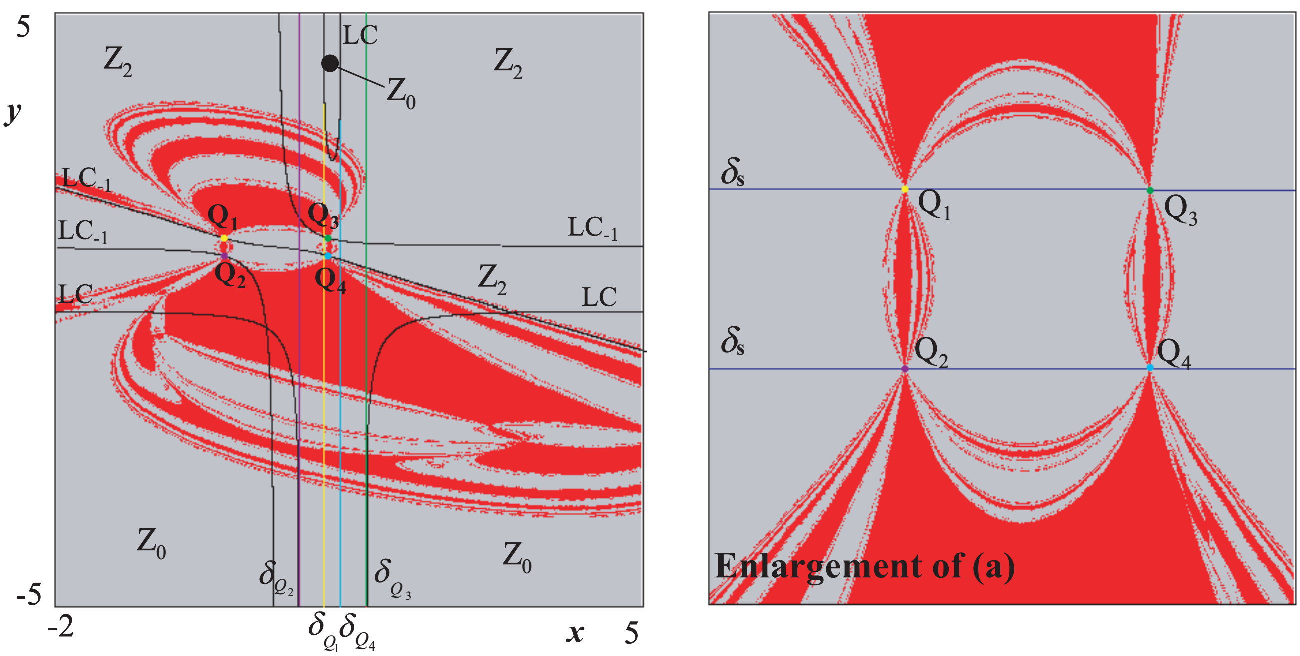
(a)

Fig. 9. Critical sets, prefocal curves, focal points and basins for the map (73). (a) $\varepsilon = 0.01$, the other parameters are the same as in Fig. 7. (b) $\varepsilon = -0.1$.



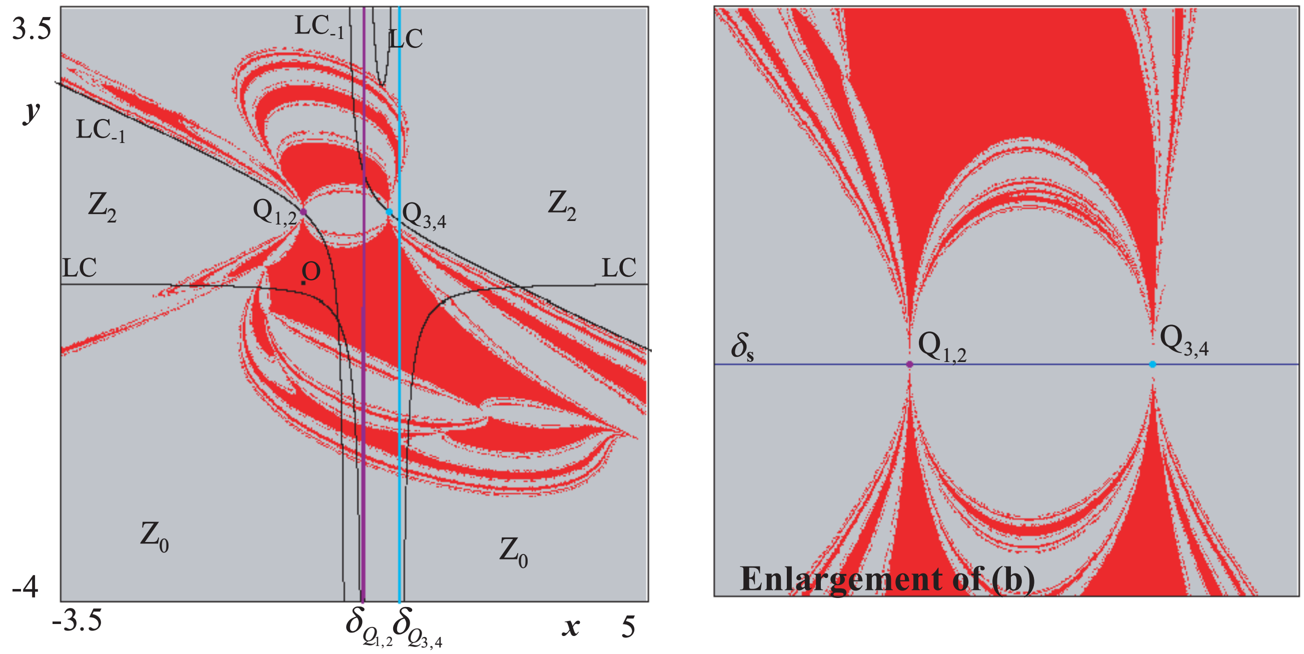
(b)

Fig. 9. (*Continued*)



(a)

Fig. 10. Critical sets, prefocal curves, focal points and basins for the map (73). (a) $\gamma = -0.5$, $\beta_1 = 1.2$, $\beta_2 = 0.9$, the other parameters are the same as in Fig. 7. (b) Nonsimple focal points $Q_{1,2} = Q_1 = Q_2$ and $Q_{3,4} = Q_3 = Q_4$ for $\beta_1 = \beta_2 = 0.9$.



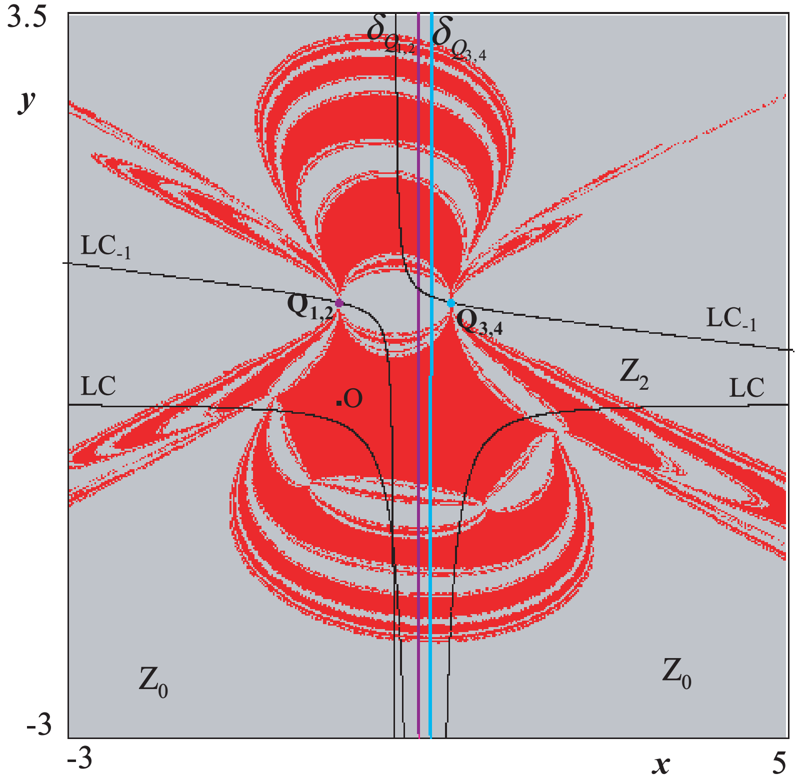
(b)

Fig. 10. (*Continued*)

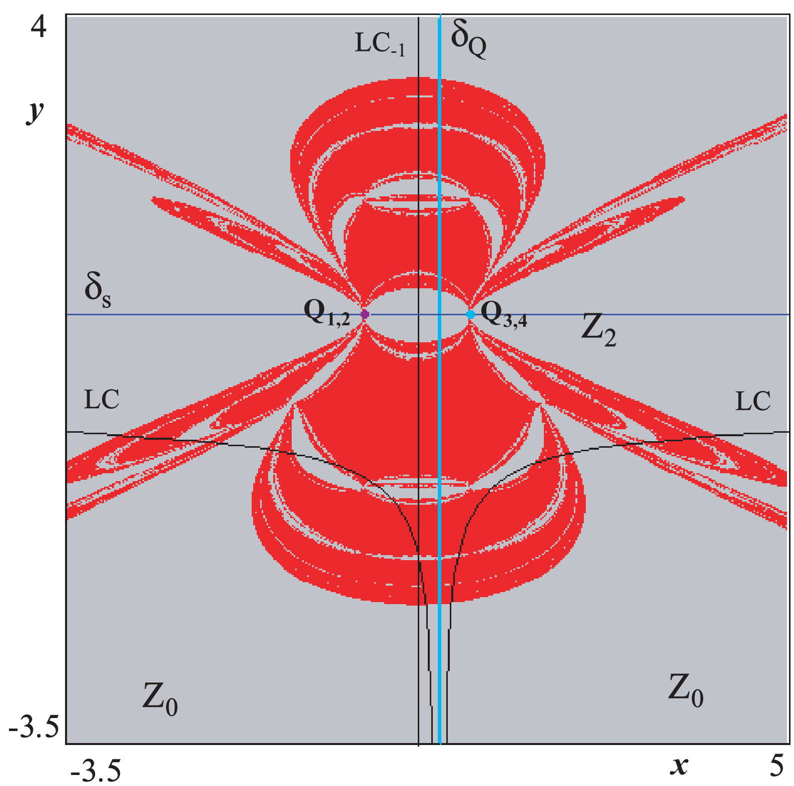
approach the set of nondefinition, and for $\beta_1 = \beta_2 = \beta$ the two lines merge into a unique line of nondefinition, $y = \beta$ [Fig. 10(b)]. For $\beta_1 = \beta_2 = \beta$, from (81) it is easy to see that $\det DT$ changes sign across the line $y = \beta$ and across the two branches of the hyperbola of equation $y = \beta - (2\varepsilon x(\alpha x + \gamma))/(2\alpha x + \gamma)$ (constituting LC_{-1}) as well. It is clear that the merging $Q_1 = Q_2 = Q_{1,2} = (0, \beta)$ occurs, as well as $Q_3 = Q_4 = Q_{3,4} = (-\gamma/\alpha, \beta)$. We also observe that at $Q_{1,2}$ we have $\overline{N}_y = \overline{D}_x = \overline{D}_y = 0$ and $\overline{N}_x = \gamma$, and at $Q_{3,4}$ we have $\overline{N}_y = \overline{D}_x = \overline{D}_y = 0$ and $\overline{N}_x = -\gamma$. So, in both cases, we have a situation described in Sec. 3.3, hence we expect that, in this bifurcation situation, the preimages of arcs crossing through the prefocal lines are arcs through the focal points with vertical tangent (i.e. orthogonal to the set of nondefinition) and different curvatures according to (38), as can be seen from the enlargement of Fig. 10(b).

As a consequence of the merging of the focal points, also the corresponding prefocal lines merge, i.e. $\delta_{Q_1} = \delta_{Q_2} = \delta_{Q_{1,2}}$ of equation $x = \beta$ and $\delta_{Q_3} = \delta_{Q_4} = \delta_{Q_{3,4}}$ of equation $x = \beta - \gamma/\alpha$. If we apply the inverses (80) to the points $(\beta, y') \in \delta_{Q_{1,2}}$ it is easy to realize that only one inverse focalized these points in $(0, \beta)$ and the other maps them into the points of the line $y = b - \varepsilon$. Analogously, if we apply the inverses (80) to the points $(\beta - \gamma\varepsilon/\alpha, y') \in \delta_{Q_{3,4}}$ then one inverse focalizes in $(-\gamma/\alpha, \beta)$ and the other one maps the points along the line $y = \beta - \gamma\varepsilon/\alpha - \varepsilon$. This explains why, even if a merging of focal points (and related prefocal lines) is obtained for $\beta_1 = \beta_2$ the prefocal curves are not part of LC. We note that also in this case, as $\beta_1 \to \beta_2$ the coordinates $y_1(-\varepsilon)$ and $y_2(-\varepsilon)$ of the tangency points between LC and the prefocal curves go to infinity, i.e. the two prefocal lines become asymptotes for LC.

Considering this last case, shown in Fig. 10(b), we can investigate the simultaneous occurrence of several bifurcations. For example, as $\varepsilon \to 0$ we expect that the two branches of LC_{-1} shall approach the line of nondefinition, as in fact occurs, see Fig. 11(a). At the bifurcation, when $\varepsilon = 0$ and $\beta_1 = \beta_2 = \beta$, LC_{-1} reduces to the vertical line $x = -\gamma/\alpha$, but the points of the line $y = \beta$ (i.e. the set of nondefinition δ_s) have properties similar to LC_{-1}, as discussed at the end of Sec. 2. As in the previous cases, $LC = T(LC_{-1})$ is formed by two branches asymptotic to δ_Q, given by $y = -\gamma^2/4\alpha(x - \beta)^2$. While no effect can be observed in the focal points $Q_{1,2} = (0, \beta)$ and $Q_{3,4} = (-\gamma/\alpha, \beta)$, this bifurcation causes the merging of the two prefocal curves into a unique prefocal curve δ_Q of equation $x = \beta$, which is focalized, by the two inverses, in the two nonsimple focal points respectively.



(a)



(b)

Fig. 11. Map (73). (a) Nonsimple focal points and related nonsimple prefocal lines for $\alpha = 0.4$, $\gamma = -0.5$, $\beta_1 = \beta_2 = 0.9$, $\varepsilon = 0.1$. (b) Merging of the prefocal lines for $\varepsilon = 0$.

Another bifurcation can be observed starting from the situation shown in Fig. 10(b) for $\gamma \to 0$. The two focal points approach each other [see Fig. 12(a)] and for $\beta_1 = \beta_2 = \beta$ and $\gamma = 0$ we get a unique focal point $Q = (0, \beta)$, obtained from the merging of four focal points [see Fig. 12(b)]. Of course Q is nonsimple, and since $\overline{N}_x = \overline{N}_y = \overline{D}_x = \overline{D}_y = 0$ the case described in Sec. 3.4 is obtained, where the relation between slopes of arcs through Q and points of the prefocal set is given by the second-order terms of the expansion of N and D. In this case we generally get a relation which is two-to-one (two slopes associated with the same point of δ_Q) and the prefocal set may be a half-line. In fact, the two-to-one relation described in (47) now reads as

$$m \to y(m) = \frac{\alpha}{m^2}$$

thus the prefocal set is a half-line: the portion of the line of equation $x = \beta$ included inside Z_2, and both the inverses focalize it into Q. Any arc crossing through the prefocal at a point (β, y) has two rank-1 preimages crossing through the focal point Q [see the enlargement of Fig. 9(d)] with slopes given by

$$y \to m(y) = \pm\sqrt{\frac{\alpha}{y}}.$$

It can be noticed that a drastic change in the critical set occurs at this bifurcation: the set LC_{-1} is an hyperbola in Fig. 12(a), while it reduces to a vertical line in Fig. 12(b). In fact, the hyperbola degenerates into its asymptotes, however at the bifurcation value only the vertical line $x = 0$ belongs to the critical set LC_{-1}, the second branch, of equation $y = -\varepsilon x + \beta$, does not belong to the critical set, even if the determinant of the Jacobian vanishes in it. This is due to the fact that this line is mapped by T into one single point (the focal point Q), denoting that this line is a prefocal set for an inverse of T, with a focal point at Q. This occurrence is not a surprise in maps with a vanishing denominator, as already remarked in Part I. We can so conclude that in this situation, Q is a focal point also for an inverse, and for that inverse we have that the focal point belongs to its prefocal set.

4.5. *Example 5*

This example concerns the generation of a nonconnected basin from the merging of focal points. The map is:

$$T: \begin{cases} x' = y + \varepsilon x \\ y' = \dfrac{\alpha x^2 + \gamma}{(y - \beta_1)(y - \beta_2)} \end{cases} \tag{83}$$

This map is not defined at the points of the set $D(x, y) = 0$, given by $\delta_s = \delta_s^1 \cup \delta_s^2$, where δ_s^1 and δ_s^2 are the lines $y = \beta_1$ and $y = \beta_2$, respectively. The numerator $N(x, y)$ vanishes for $x = \pm\sqrt{-\gamma/\alpha}$, then the map T has either four, two or zero simple focal points Q_i, according to the sign of $(\beta_i + \gamma)/\alpha$, $i = 1, 2$, associated with four simple prefocal curves of equation $x = F(Q_i)$, given by:

$$Q_1 = \left(-\sqrt{-\gamma/\alpha}, \beta_1\right) \quad \text{if } \gamma/\alpha < 0; \quad \delta_{Q_1}: x = \beta_1 - \varepsilon\sqrt{-\gamma/\alpha}$$

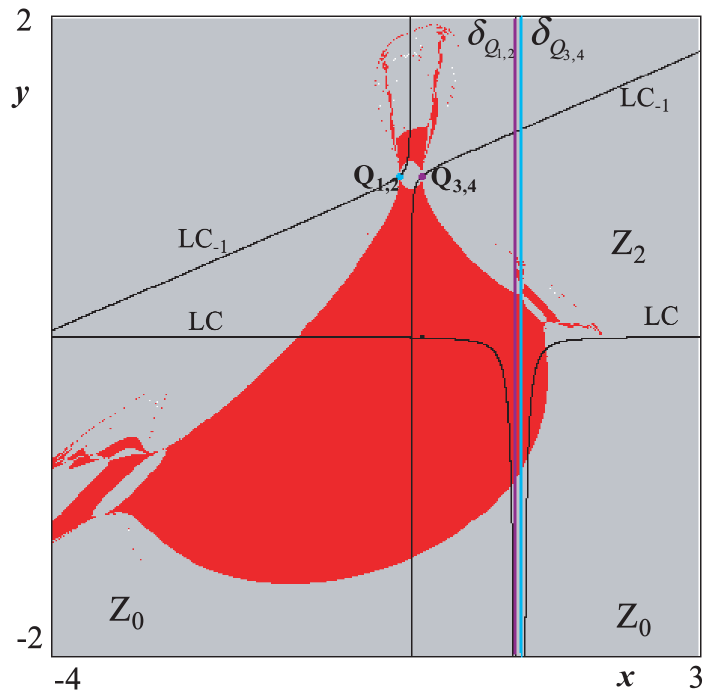
$$Q_2 = \left(\sqrt{-\gamma/\alpha}, \beta_1\right) \quad \text{if } \gamma/\alpha < 0; \quad \delta_{Q_2}: x = \beta_1 + \varepsilon\sqrt{-\gamma/\alpha}$$

$$Q_3 = \left(-\sqrt{-\gamma/\alpha}, \beta_2\right) \quad \text{if } \gamma/\alpha < 0; \quad \delta_{Q_3}: x = \beta_2 - \varepsilon\sqrt{-\gamma/\alpha}$$

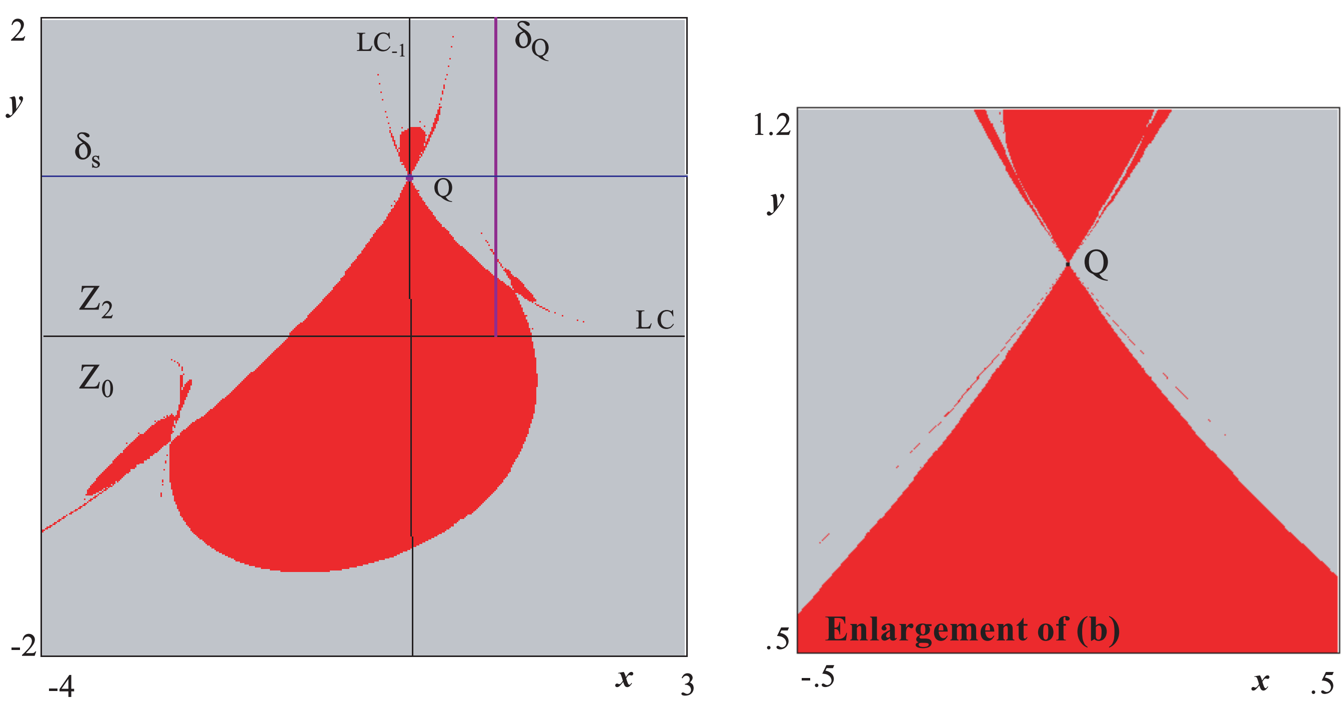
$$Q_4 = \left(\sqrt{-\gamma/\alpha}, \beta_2\right) \quad \text{if } \gamma/\alpha < 0; \quad \delta_{Q_4}: x = \beta_2 + \varepsilon\sqrt{-\gamma/\alpha}$$

Being $N_y = D_x = 0$, at each Q_i the condition (7) for a simple focal point becomes

$$\sqrt{-\gamma/\alpha})(\beta_1 - \beta_2) \neq 0,$$



(a)



(b)

Fig. 12. Map (73). (a) $\alpha = 0.4$, $\gamma = 0.1$, $\beta_1 = \beta_2 = 1$, $\varepsilon = -0.25$. (b) For $\gamma = 0$, a unique focal point $Q = (0, 1)$ exists.

and the one-to-one correspondences (11) are readily obtained: for Q_1, Q_2, Q_3 and Q_4 we have

$$y_1(m) = y_4(m) = \frac{-2\alpha\sqrt{-\gamma/\alpha}}{m(\beta_1 - \beta_2)},$$

$$y_2(m) = y_3(m) = \frac{2\alpha\sqrt{-\gamma/\alpha}}{m(\beta_1 - \beta_2)}$$

The map T is a noninvertible map of $Z_0 - Z_2$ type. In fact, the preimages of a point (x', y') are the real solutions of the second degree algebraic system

$$\begin{cases} (\alpha - \varepsilon^2 y')x^2 + \varepsilon[2x'y' - (\beta_1 + \beta_2)y']x + \gamma \\ \qquad + (\beta_1 + \beta_2)x'y' - \beta_1\beta_2 y' - x'^2 y' = 0 \\ y = x' - \varepsilon x \end{cases}$$

Hence a point (x', y') has two distinct preimages, i.e. $(x', y') \in Z_2$, if

$$\begin{aligned} \Delta(x', y') = \varepsilon^2 y'^2 [2x' - (\beta_1 + \beta_2)]^2 \\ - 4(\alpha - \varepsilon^2 y')[\gamma - (x' - \beta_1)(x' - \beta_2)y'] \\ > 0 \end{aligned}$$

and in this case the two inverses are given by $T^{-1}(x', y') = T_1^{-1}(x', y') \cup T_2^{-1}(x', y')$, where

$$T_1^{-1}\colon \begin{cases} x = \dfrac{-2\varepsilon x'y' + \varepsilon(\beta_1 + \beta_2)y' - \sqrt{\Delta(x', y')}}{2(\alpha - \varepsilon^2 y')} \\ y = x' - \varepsilon x \end{cases}$$

$$T_2^{-1}\colon \begin{cases} x = \dfrac{-2\varepsilon x'y' + \varepsilon(\beta_1 + \beta_2)y' + \sqrt{\Delta(x', y')}}{2(\alpha - \varepsilon^2 y')} \\ y = x' - \varepsilon x \end{cases}$$

The set LC_{-1} of rank-1 merging preimages is defined by the equation

$$\begin{aligned} \varepsilon\alpha(2y - \beta_1 - \beta_2)x^2 + 2\alpha x D(y) \\ + \varepsilon\gamma(2y - \beta_1 - \beta_2) = 0, \\ D(y) = (y - \beta_1)(y - \beta_2) \end{aligned}$$

obtained from the condition $\det DT = 0$ given in (15) and (16). The set LC_{-1} is generally formed by three disjoint branches, say $LC_{-1} = L_{-1} \cup L'_{-1} \cup L''_{-1}$, which cross the set of nondefinition at the focal points. LC_{-1} also includes the point $(\varepsilon/2\alpha, (\beta_1 + \beta_2)/2)$, located at the center of the quadrilateral $Q_1Q_2Q_3Q_4$. From (17) we get that the tangents to LC_{-1} in all the four focal points have slope $\overline{m} = -\varepsilon$. The curve LC_{-1} of merging preimages is made up of the arcs given by

$$x = \frac{-\alpha D(y) \pm \sqrt{\alpha^2 D(y)^2 - \varepsilon^2\alpha\gamma(2y - \beta_1 - \beta_2)^2}}{\varepsilon\alpha(2y - \beta_1 - \beta_2)}$$

Arcs related to the sign "+" (resp. "−") are of pink color (resp. black color) in the illustrations in this subsection. The critical set LC is given either by $LC = T(LC_{-1})$ or by $\Delta(x, y) = 0$. It is generally formed by three disjoint branches $LC = L \cup L' \cup L''$. The slope of the tangent to LC is $p = -\Delta_x/\Delta_y$, Δ_x and Δ_y being the partial derivatives of Δ. The slope $p = 0$ occurs at the two points $(\Delta_x = 0) \cap LC$, i.e. at $x = (\beta_1 + \beta_2)/2$, $y = \alpha/\varepsilon^2$, $y = -4\gamma/(\beta_1 - \beta_2)^2$. The slope $p = \infty$ occurs at the two points $(\Delta_y = 0) \cap LC$, i.e. at the points $R_i = R(Q_i)$, $i = 1, 2, 3, 4$ (see (18)), having coordinates $y(Q_i) = \pm 2\alpha x(Q_i)/[\varepsilon(\beta_1 - \beta_2)]$, where $x(Q_i) = \beta_j \pm \varepsilon\sqrt{-\gamma/\alpha}$, $j = 1, 2$. For the parameter values related to the figures in this subsection the map has three fixed points, only one (not far from the point $x = y = 0$) being stable with a basin D (boundary ∂D), represented in the figures.

Figure 13 represents the situation of four simple focal points Q_1, Q_2, Q_3, Q_4, the three arcs of LC_{-1} passing though these points, the three arcs of LC, and the basin D. The lines of nondefinition (blue colored) are δ_s^1 (passing through Q_1 and Q_2) and δ_s^2 (passing through Q_3 and Q_4). The arcs of basin boundary passing through the focal points limit a hole $\tilde{H}_1$. It is the preimage of a region called $\tilde{H}_0$ (out of the figure framework) having points at infinity located on the Poincaré equator. The hole $\tilde{H}_1 = T^{-1}(\tilde{H}_0)$ has rank-1 and rank-2 preimages $\tilde{H}_2^1 \cup \tilde{H}_2^2 = T^{-1}(\tilde{H}_1)$, $\tilde{H}_3^{11} \cup \tilde{H}_3^{12} = T^{-1}(\tilde{H}_2^1)$.

If $\beta_1 = \beta_2 = \beta$, $\gamma \neq 0$ one has the bifurcation described in Sec. 3.5, characterized by $\overline{D}_x = \overline{D}_y = \overline{N}_y = 0$, $\overline{N}_x \neq 0$, with $Q_1 \equiv Q_3$, $Q_2 \equiv Q_4$ and two double prefocal sets $\delta_{Q_{13}}$ and $\delta_{Q_{24}}$. According to (44), the common slope of arcs, focalizing at each of the two nonsimple focal points, is $m = \infty$. LC_{-1} is made up of the double line of nondefinition δ_s $(y = \beta)$ and the two arcs of the hyperbola $y = \beta - \varepsilon x - \varepsilon\gamma/(\alpha x)$. The critical set

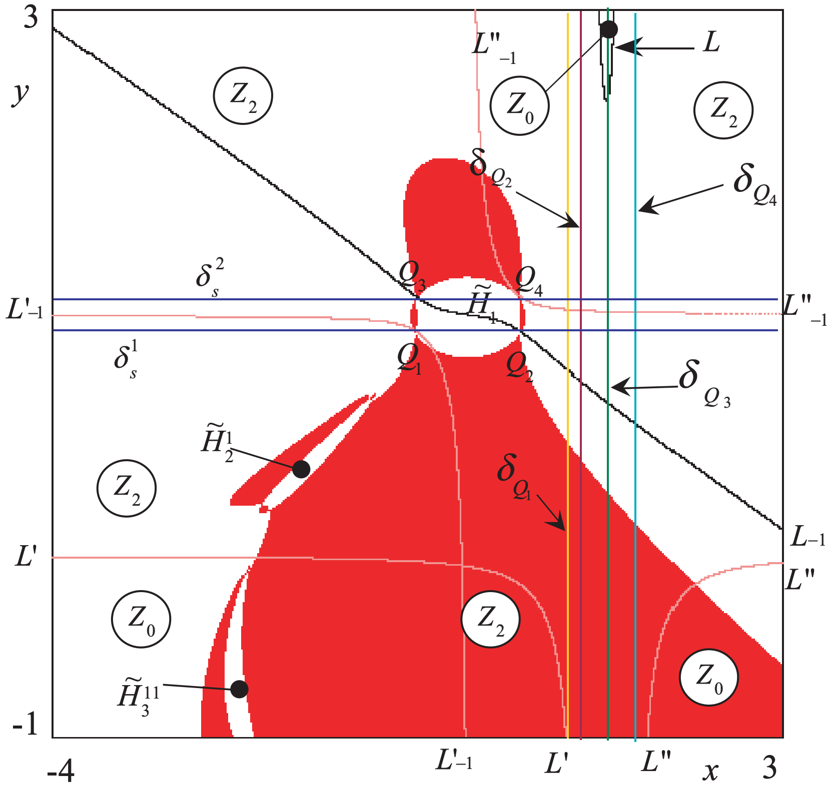


Fig. 13. Critical sets $LC_{-1} = L_{-1} \cup L'_{-1} \cup L''_{-1}$ and $LC = T(LC_{-1}) = L \cup L' \cup L''$, singular lines δ_s^i, $i = 1, 2$, prefocal curves δ_{Q_i}, $i = 1, \ldots, 4$ and corresponding focal points Q_i and basins of attraction for the map (83) with parameters $\alpha = 0.4$, $\gamma = -0.1$, $\beta_1 = \sqrt{1.5}$, $\beta_2 = \sqrt{2}$, $\varepsilon = 0.4$. The red region represents the basin of the stable fixed point, the white region represents the basin of infinity.

LC is $\varepsilon^2 y^2 (x-\beta)^2 - (\alpha - \varepsilon^2 y)[\gamma - y(x-\beta)^2] = 0$, i.e. $x = \beta \pm \sqrt{-\gamma\varepsilon^2/\alpha + \gamma/y}$. Due to $\beta_1 = \beta_2$ the point $(x = (\beta_1+\beta_2)/2,\ y = -4\gamma/(\beta_1-\beta_2)^2)$ of LC with the slope $p = 0$ (as the points R_i with $p = \infty$) is at infinity. Figure 14 represents this bifurcation situation.

If $\beta_1 \neq \beta_2$, $\gamma = 0$ one has the bifurcation considered in Sec. 3.4, characterized by $\overline{D}_x = \overline{N}_y = \overline{N}_x = 0$ and $\overline{D}_y \neq 0$, with $Q_1 \equiv Q_2$ (point Q_{12}), $Q_3 \equiv Q_4$ (point Q_{34}), and two double prefocal sets, say $\delta_{Q_{12}}$ and $\delta_{Q_{34}}$. The common slope of arcs, focalizing at each of the two nonsimple focal points Q_{12} and Q_{34}, is $m = 0$. The set of merging preimages $LC_{-1} = L_{-1} \cup L'_{-1} \cup L''_{-1}$ is made up of the branch $x = 0$ and the two arcs of the curve $x = -2\alpha D(y)/[\varepsilon\alpha(2y - \beta_1 - \beta_2)]$. The critical set $LC = L \cup L' \cup L''$ is made up of the line $y = 0$ and the curve $\varepsilon^2 y[2x - (\beta_1+\beta_2)]^2 + 4(\alpha - \varepsilon^2 y)(x-\beta_1)(x-\beta_2) = 0$. The two points R_{12} and R_{34} [see (18)] of contact between LC and $\delta_{Q_{12}} \cup \delta_{Q_{34}}$ have the coordinates $x = \beta_j$, $j = 1, 2$, $y_1 = -2\alpha\beta_1/[\varepsilon(\beta_1-\beta_2)]$ and $y_2 = 2\alpha\beta_2/[\varepsilon(\beta_1-\beta_2)]$. At these points LC has an inflection point with a vertical tangent.

Figures 15 and 16 represent LC_{-1} and LC at the bifurcation situation ($\overline{D}_x = \overline{N}_y = \overline{N}_x = 0$, $\overline{D}_y \neq 0$) and the basin D. This situation marks the limit case of islands existence. Indeed, Δ_0 and Δ'_0 are two limits of two headlands giving rise to two sets of five regions $\Delta_1 = T^{-1}(\Delta_0)$, $\Delta_2^1 \cup \Delta_2^2 = T^{-1}(\Delta_1)$, $\Delta_3^{11} \cup \Delta_3^{12} = T^{-1}(\Delta_2^1)$, and $\Delta'_1 = T^{-1}(\Delta'_0)$, $\Delta_2'^1 \cup \Delta_2'^2 = T^{-1}(\Delta'_1)$, $\Delta_3'^{11} \cup \Delta_3'^{12} = T^{-1}(\Delta_2'^1)$. Because $\Delta_2^2 \cup \Delta_2^{11} \cup \Delta_3^{12} \cup \Delta_3^{11} \subset Z_0$ and $\Delta_2'^2 \cup \Delta_2'^{11} \cup \Delta_3'^{12} \cup \Delta_3'^{11} \subset Z_0$ these regions have no preimage.

For sufficiently small values of $\gamma > 0$ the basin D is nonconnected (as long as the headlands Δ_0 and Δ'_0 exist) with two sets of five islands Δ_1, Δ_1^1, Δ_1^2, Δ_1^{11}, Δ_1^{12} and Δ'_1, $\Delta_1'^1$, $\Delta_1'^2$, $\Delta_1'^{11}$, $\Delta_1'^{12}$, see Fig. 17. If γ increases the headland Δ'_0 disappears with its related islands Δ'_1, $\Delta_1'^1$, $\Delta_1'^2$, $\Delta_1'^{11}$, $\Delta_1'^{12}$ with different structures of LC_{-1} (compare Figs. 18 and 19).

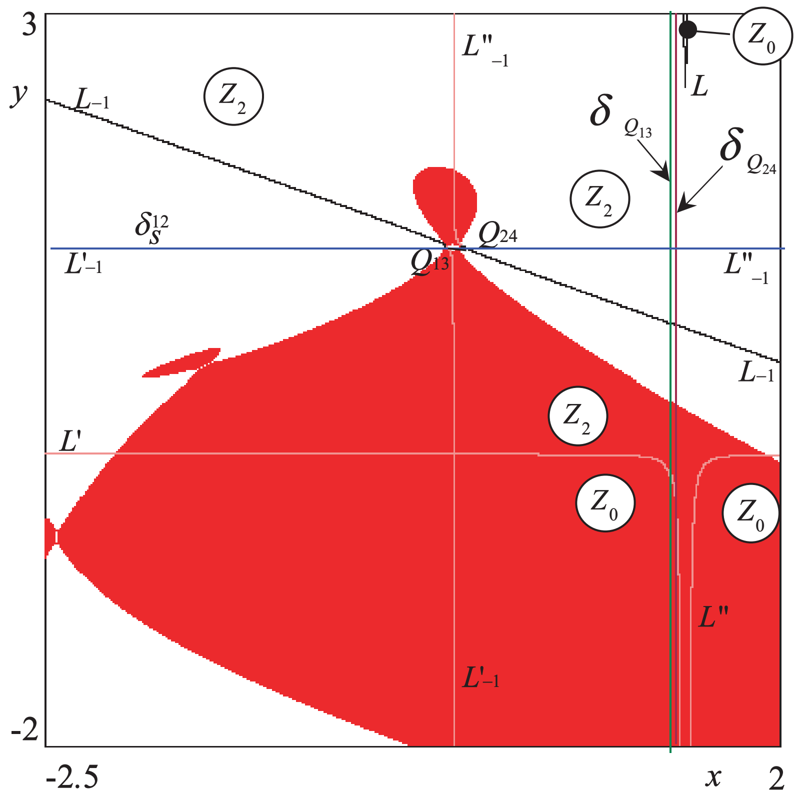


Fig. 14. Map (83) with parameters $\alpha = 0.4$, $\gamma = -0.1$, $\beta_1 = \beta_2 = \sqrt{2}$, $\varepsilon = 0.4$.

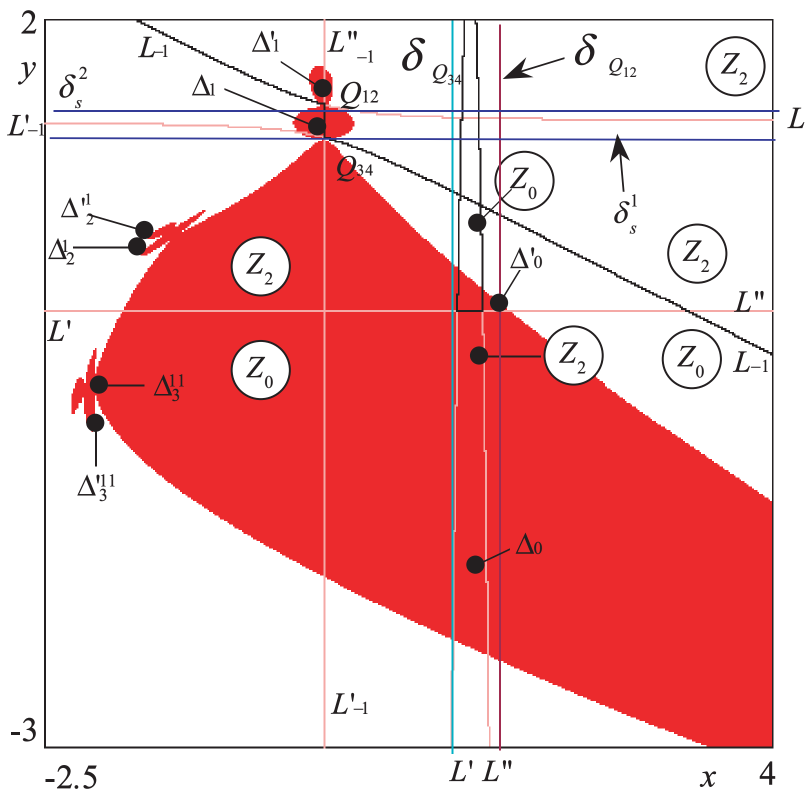


Fig. 15. Map (83) with parameters $\alpha = 0.4$, $\gamma = 0$, $\beta_1 = \sqrt{1.41}$, $\beta_2 = \sqrt{2}$, $\varepsilon = 0.4$.

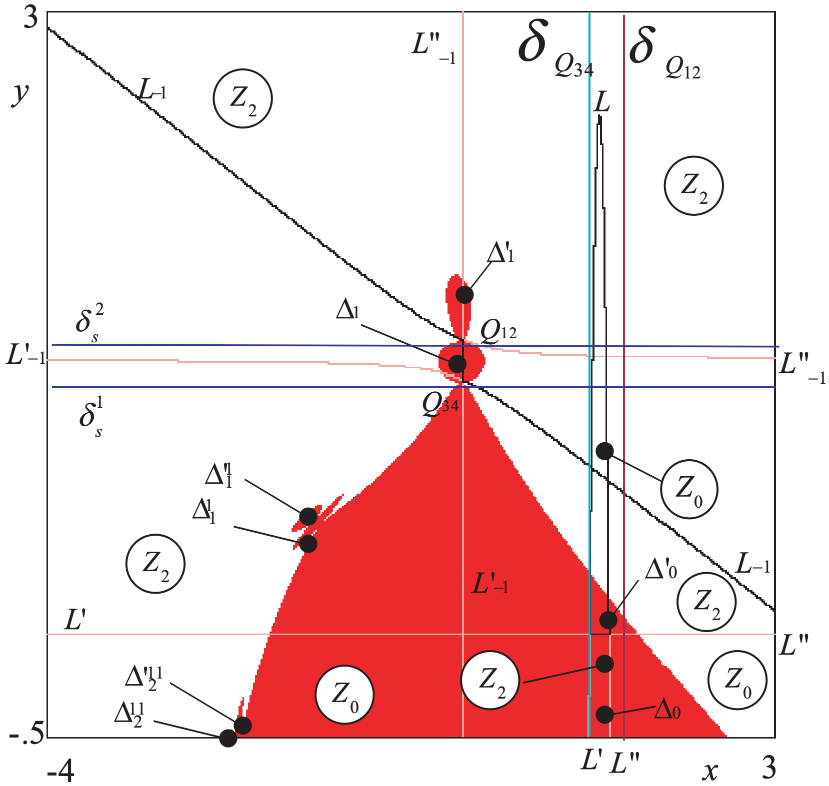


Fig. 16. Map (83) with parameters $\alpha = 0.4$, $\gamma = 0$, $\beta_1 = \sqrt{1.5}$, $\beta_2 = \sqrt{2}$, $\varepsilon = 0.4$.

As long as Δ_0 exists the basin D is nonconnected with five islands Δ_1, Δ_1^1, Δ_1^2, Δ_1^{11}, Δ_1^{12}.

These islands and Δ_0 disappear, so that D becomes connected, when $\gamma > \gamma_b \simeq 0.0286$ (Fig. 20). At the bifurcation value $\gamma = \gamma_b$ the boundary of the region Z_2', made up of an arc of LC, is tangent to the basin boundary ∂D.

If $\beta_1 = \beta_2 = \beta$ and $\gamma = 0$ we have the bifurcation situation considered in Sec. 3.6, characterized by $\overline{N}_x = \overline{N}_y = \overline{D}_x = \overline{D}_y = 0$, with $Q = Q_1 \equiv Q_2 \equiv Q_3 \equiv Q_4$. As proved in Sec. 3.6, in this case there is a two-to-one correspondence between the slopes m of arcs through the focal point and the points of the prefocal set δ_Q (the portion of straight line $x = \beta$ belonging to Z_2). In this example, the relation is given by $y = \alpha/m^2$ and $m = \pm\sqrt{\alpha/y}$. In Fig. 21, there is only one arc of the basin's boundary that crosses the prefocal line, at a point with y-coordinate $\overline{\mathbf{y}}$ so that the boundary must also include two arcs through the focal point with slopes $m_1 = +\sqrt{\alpha/\overline{\mathbf{y}}}$ and $m_2 = -\sqrt{\alpha/\overline{\mathbf{y}}}$.

The set of merging preimages $LC_{-1} = L_{-1} \cup L'_{-1} \cup L''_{-1}$ is made up of the double line of nondefinition δ_s ($y = \beta$) constituted by one arc of L'_{-1} and one arc of L''_{-1}, the straight line $x = 0$ constituted by the complementary arc of L'_{-1} and the complementary arc of L''_{-1}, and the straight line $y = \beta - \varepsilon x$ (L). The critical set LC is given by $\varepsilon^2 y^2 (x-\beta)^2 + (\alpha - \varepsilon^2 y)(x-\beta)^2 y = 0$ i.e. by the three straight lines $y = 0$ and $x = \beta$ (which is also the result of the merging δ_Q of the four prefocal sets δ_{Qi}, $i = 1, 2, 3, 4$) and the line $y = 1 + (\alpha/\epsilon^2)$. Figure 21 represents this bifurcation situation, also limit case of existence islands. Indeed Δ_0 is a limit of an headland giving rise to five regions $\Delta_1 = T^{-1}(\Delta_0)$, $\Delta_2^1 \cup \Delta_2^2 = T^{-1}(\Delta_1)$, $\Delta_3^{11} \cup \Delta_3^{12} = T^{-1}(\Delta_2^1)$. Because $\Delta_2^2 \cup \Delta_2^{11} \cup \Delta_3^{12} \cup \Delta_3^{11} \subset Z_0$ these regions have no preimage. For $0 < \gamma < \gamma_b \simeq 0.0117$ the basin D is nonconnected (as long as the headland Δ_0 exists) with five islands Δ_1, Δ_1^1, Δ_1^2, Δ_1^{11} and Δ_1^{12} (see Fig. 18). At the bifurcation value $\gamma = \gamma_b$ the boundary of the headland Δ_0, made up of an LC arc is tangent to the basin boundary ∂D. Due to $\beta_1 = \beta_2$ the point ($x = (\beta_1 + \beta_2)/2$, $y = -4\gamma/(\beta_1 - \beta_2)^2$) of LC, with the slope $p = 0$, is at infinity.

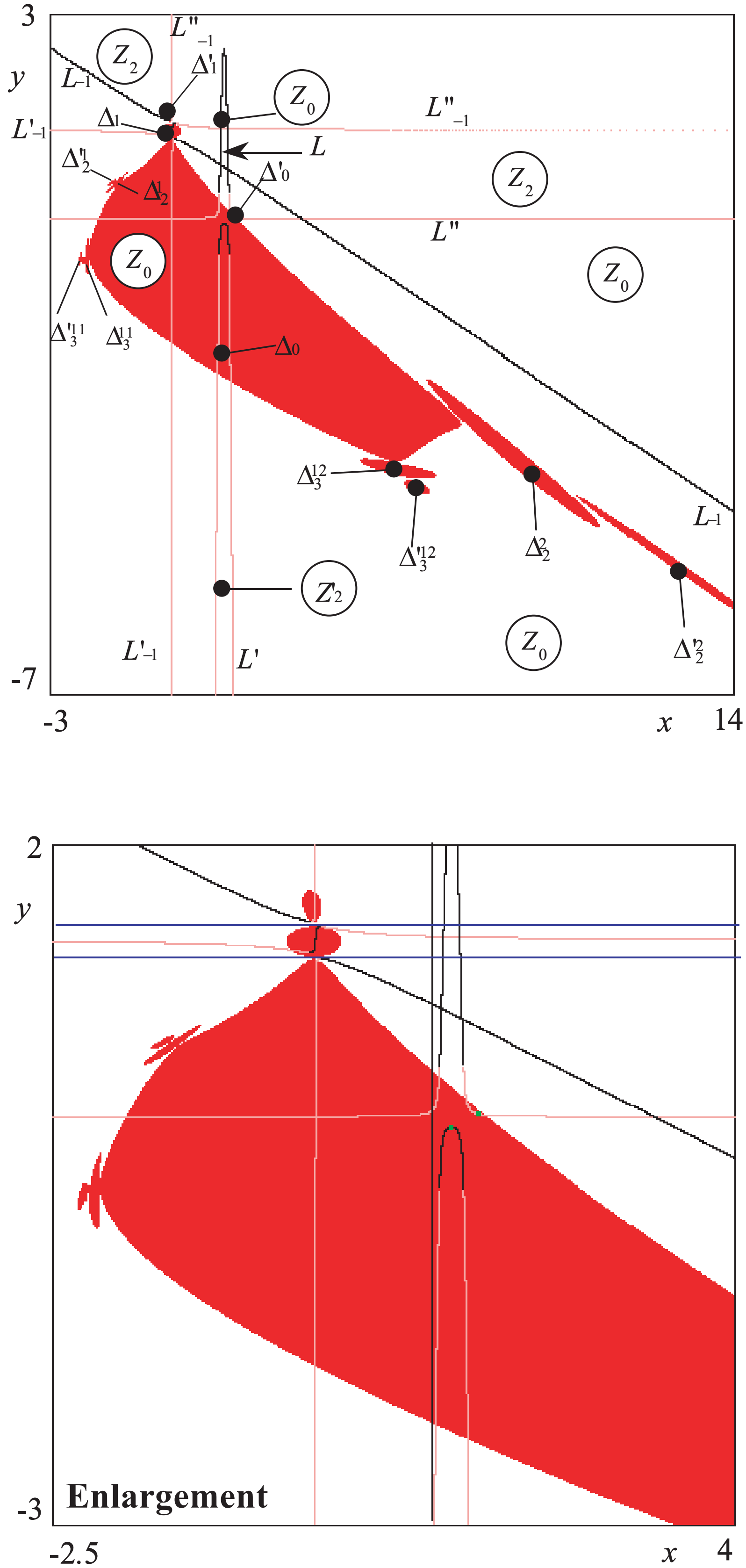
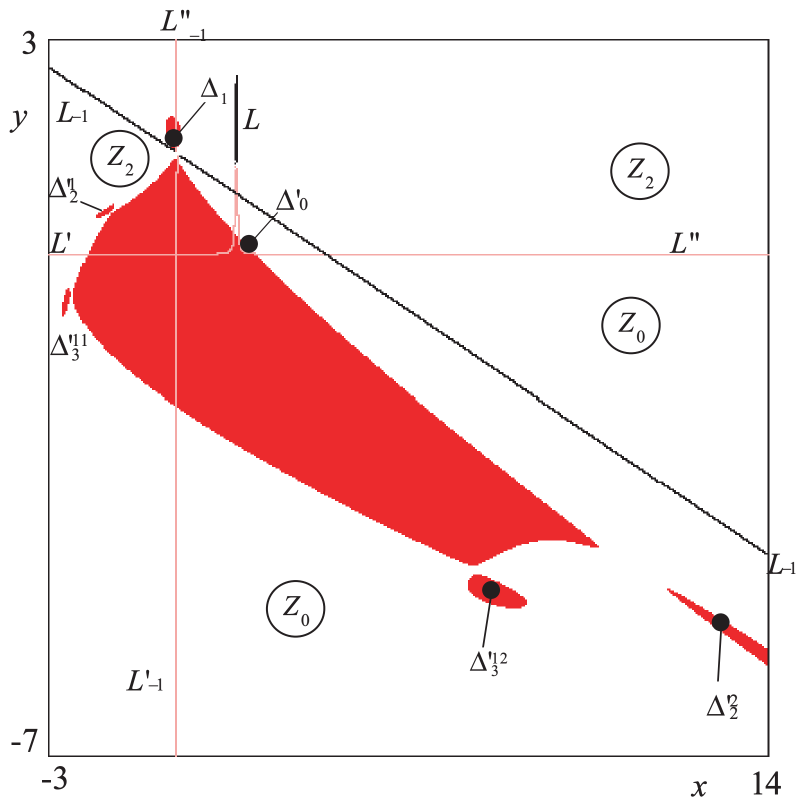


Fig. 17. Map (83) with parameters $\alpha = 0.4$, $\gamma = 0.001$, $\beta_1 = \sqrt{1.41}$, $\beta_2 = \sqrt{2}$, $\varepsilon = 0.4$.



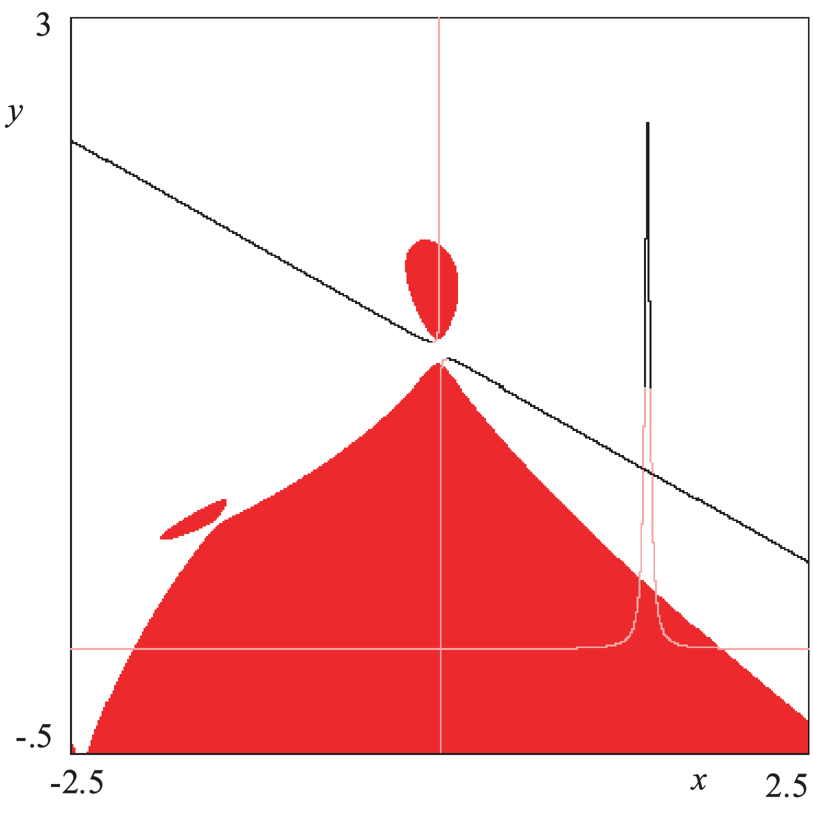


Fig. 18. Map (83) with parameters $\alpha = 0.4$, $\gamma = 0.001$, $\beta_1 = \beta_2 = \sqrt{2}$, $\varepsilon = 0.4$.

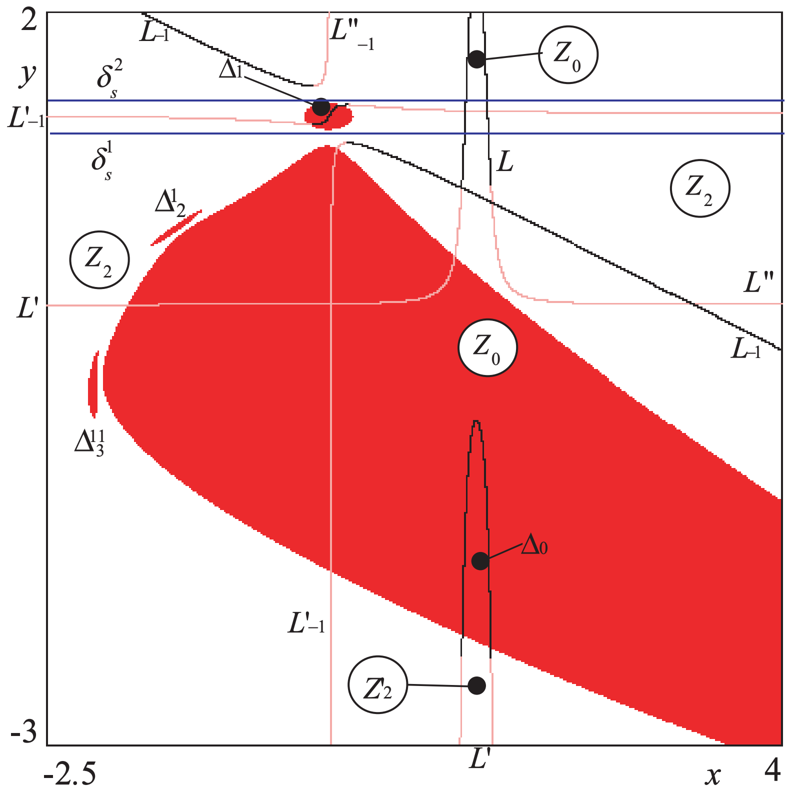


Fig. 19. Map (83) with parameters $\alpha = 0.4$, $\gamma = 0.01$, $\beta_1 = \sqrt{1.41}$, $\beta_2 = \sqrt{2}$, $\varepsilon = 0.4$.

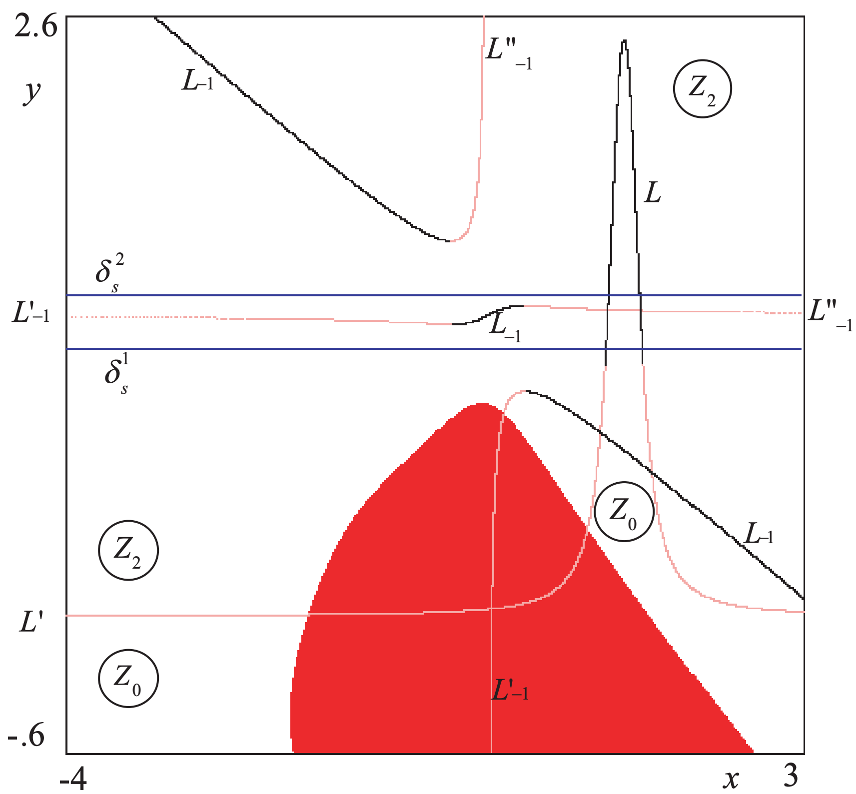


Fig. 20. Map (83) with parameters $\alpha = 0.4$, $\gamma = 0.05 > \gamma_b \simeq 0.0286$, $\beta_1 = \sqrt{1.41}$, $\beta_2 = \sqrt{2}$, $\varepsilon = 0.4$.

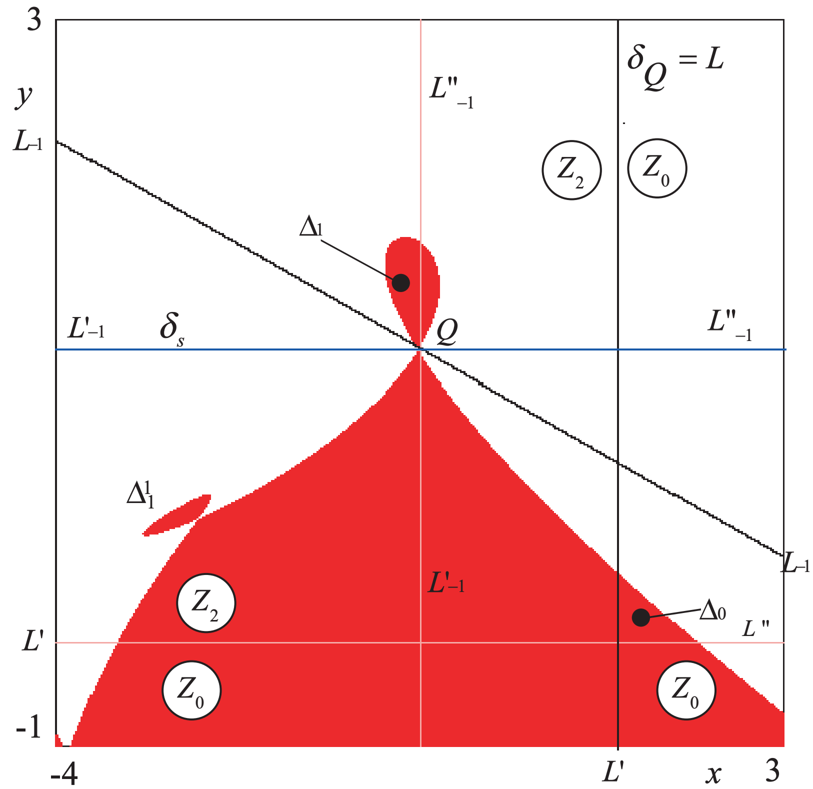


Fig. 21. Map (83) with parameters $\alpha = 0.4$, $\gamma = 0$, $\beta_1 = \beta_2 = \sqrt{2}$, $\varepsilon = 0.4$.

Acknowledgments

This work has been performed within the activity of the national research project "Nonlinear models in economics and finance: complex dynamics, disequilibrium, strategic interactions," MIUR, Italy, and under the Joint Research Grant (0382): "Reconsideration of economic dynamics from a new perspective of nonlinear theory," Chuo University, Japan.

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