Symmetry-breaking bifurcations and representative firm in dynamic duopoly games

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In this paper, we investigate the question of whether the assumption of the "representative agent", often made in economic modeling, is innocuous or whether it may be misleading under certain circumstances. In order to obtain some insight into this question, two dynamic Cournot duopoly games are considered, whose dynamics are represented by discrete-time dynamical systems. For each of these models, the dynamical behavior of the duopoly system with identical producers is compared to that with quasi-identical ones, in order to study the effects of small heterogeneities between the players. In the case of identical players, such dynamical systems become symmetric, and this implies that synchronized dynamics can be obtained, governed by a simpler one-dimensional model whose dynamics summarizes the common behavior of the two identical players. In both the examples, we show that a negligible difference between the parameters that characterize the two producers can give dynamic evolutions that are qualitatively different from that of the symmetric game, i.e. a breaking of the symmetry can cause a noticeable effect. The presence of such bifurcations suggests that economic systems with quasi-identical agents may evolve quite differently from systems with truly identical agents. This contrasts with the assumption, very common in the economic literature, that small heterogeneities of agents do not matter too much.

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1. Introduction

A common assumption, often made in economic modeling, is that the behavior of a system with many identical agents can be summarized by that of a "representative agent" (see e.g. [14,15,17,31]). This approach is generally followed even when heterogeneous agents are present, since it is often argued that small heterogeneities of agents do not matter too much. In other words, unless the agents have very different characteristics, it seems reasonable to expect that each individual will behave in more or less the same qualitative way (though not necessarily identically) so that the behavior of the aggregate system is still summarized by the behavior of the representative agent (see e.g. [31]). This point has recently been criticized by some authors [3,10,17,30].

In this paper, we investigate the question of the effects of small heterogeneities, i.e. small deviations from the condition of identical agents, on the basis of some recent results on the properties of symmetric dynamical systems (see [11] and references therein).

Suppose, for example, that a dynamical system is used to model the interaction between two identical economic agents. Such a dynamical system is symmetric because it remains the same by interchanging the agents. This symmetry property implies that an invariant one-dimensional subspace exists (see [11]). This invariance property corresponds to the obvious statement that identical agents, starting from identical initial conditions, behave identically for each time. Such synchronized dynamics are governed by a simpler one-dimensional dynamical system, given by the restriction of the two-dimensional system to the invariant subspace on which the synchronized dynamics occur, which can be seen as the model of a representative agent, whose dynamic behavior summarizes the common behavior of the two identical agents.

Some recent papers appearing in the mathematical and physical literature show that the behavior of symmetric dynamical systems is often non-generic, since a breaking of the symmetry due, for example, to a slight modification of the parameters with respect to the symmetric situation may lead to a qualitatively different dynamic evolution (see e.g. [19,24,28,32,33]). In fact, the introduction of small asymmetries, that may be due to the presence of noise or of small departures from homogeneity of the agents, may destroy the invariant submanifold and cause important qualitative changes in the dynamical behavior of the model. In other words, the destruction of the invariance submanifold, on which synchronized dynamics takes place, implies that the attractors that characterize the long-run behavior of the one-dimensional model of the representative agent are substituted by new (two-dimensional) attractors that may be very different from those existing when symmetry is present.

In order to obtain some insight into this question, in this paper we study two particular duopoly games with bounded rationality. For both models, we address the question of whether, in the presence of quasi-identical producers, characterized by parameters with very small relative differences, the observed time evolution is similar to that obtained with truly identical players.

In section 2, the bounded rationality adjustment process, on which the Cournot games considered in this paper are based, is briefly described.

In sections 3 and 4, two particular duopoly models, obtained by the same adjustment mechanism but with different demand functions, are considered. In both examples, we show that a negligible difference between the parameters of the producers can give dynamic evolutions that are qualitatively and quantitatively different from that of the representative firm, and the synchronization property can be completely lost.

Such qualitative changes in the system dynamics, due to a very small relative variation with respect to the situation of identical firms, will be called *symmetry-breaking bifurcations*. The methods used in this paper for the study of the effects of small asymmetries in two particular models can be easily applied to the study of other dynamical models describing the behavior of identical (or quasi-identical) interacting agents, and we believe that the bifurcations described in this paper, due to the sudden destruction of the invariant subset in which synchronized dynamics take place, can be met in many other symmetric dynamic models. This suggests that the concept of representative firm should be used very carefully in situations of interacting agents, as already stressed in [4,10,17,18] on the basis of qualitative arguments.

2. Duopoly models

Dynamic duopoly games, in which two players choose their strategies (x_1, x_2) $\mathbb{R}^2$ at discrete-time periods $t = 0, 1, 2, \ldots$, are often modeled by the iteration of a two-dimensional map $T : (x_1(t), x_2(t)) \to (x_1(t+1), x_2(t+1))$. As stressed in section 1, in the case of identical players the map T remains the same by interchanging the players, i.e. $T \circ S = S \circ T$, where $S : (x_1, x_2) \to (x_2, x_1)$ is the reflection through the diagonal $x_1 = x_2$. This symmetry property implies that the diagonal (*line of equal productions*)

$$\Delta = \{(x_1, x_2) \mid x_1 = x_2\} \tag{1}$$

is an invariant submanifold for the map T, i.e. $T(\Delta) \subseteq \Delta$. This means that two identical players, starting with identical initial strategies $x_1(0) = x_2(0)$, behave identically for each $t \geq 0$. The *synchronized trajectories*, belonging to Δ, are characterized by

$$\{(x_1(t), x_2(t)) = T^t(x_1(0), x_2(0)) \mid x_1(t) = x_2(t) \quad t \geq 0\} \tag{2}$$

and are governed by the one-dimensional map $x' = g(x)$, where $g(x)$ represents the restriction of the two-dimensional map T to Δ:

$$g = T|_\Delta : \Delta \to \Delta. \tag{3}$$

The simpler model $x' = g(x)$ can be seen as the model of a *representative player*, whose dynamics summarize the common behavior of the two identical players.

As an example, we consider a duopoly game which describes a market where two quantity-setting firms, producing homogeneous goods, update their productions $q_1(t)$ and $q_2(t)$ at discrete-time periods. The dynamic game is based on the assumption that the two producers have no knowledge of the market. Hence, they are not able to reach a Nash equilibrium in one shot and, consequently, they behave adaptively following a bounded rationality adjustment process based on a local estimate of the marginal profit $\partial \Pi_i / \partial q_i$ obtained, for example, through market experiments (see [9] and references therein). With this kind of information, the firms behave as local (or myopic) profit maximizers: at each time period, a firm decides to increase its production if it perceives a positive marginal profit, and decreases its production if the marginal profit is negative:

$$q_i(t+1) = q_i(t) + \alpha_i(q_i(t)) \frac{\partial \Pi_i}{\partial q_i}(q_1(t), q_2(t)); \quad i = 1, 2;\ t = 0, 1, 2, \ldots, \tag{4}$$

where $\alpha_i(q_i)$ is a positive function which gives the extent of production variation of the ith firm following a given profit signal. In the following, we assume

$$\alpha_i(q_i) = \upsilon_i q_i, \quad i = 1, 2, \tag{5}$$

where υ_i is a positive parameter which represents the relative speed of adjustment. As usual in duopoly models, the price of the good is determined by the total supply $Q(t) = q_1(t) + q_2(t)$ through a given inverse demand function $p = f(Q)$, so that the one-period profit for firm i is given by

$$\Pi_i(q_1, q_2) = q_i f(q_1 + q_2) - c_i q_i, \quad i = 1, 2, \tag{6}$$

where the positive constants c_1 and c_2 represent the marginal costs of the two firms. With assumptions (5) and (6), the time evolution of the dynamic game (4) is determined by the iteration of the following two-dimensional map:

$$T: \begin{cases} q_1' = q_1 + \upsilon_1 q_1 \left[f(q_1 + q_2) + q_1 \dfrac{\partial f}{\partial q_1} - c_1 \right], \\ q_2' = q_2 + \upsilon_2 q_2 \left[f(q_1 + q_2) + q_2 \dfrac{\partial f}{\partial q_2} - c_2 \right], \end{cases} \tag{7}$$

where $'$ denotes the unit-time advancement operator: if the right-hand side variables are productions of period t, then the left-hand ones represent production decisions for period $(t+1)$.

Starting from nonnegative initial productions

$$(q_1(0), q_2(0)) = (q_{10}, q_{20}), \tag{8}$$

the iteration of (7) uniquely defines the time evolution of the production choices, represented by the trajectory $\{(q_1(t), q_2(t)) = T^t(q_{10}, q_{20}),\ t \geq 0\}$ of the two-dimensional discrete-time dynamical system (7). In this model, the producer labeled by i is characterized by the two parameters υ_i and c_i, representing the relative speed of adjustment and the marginal cost, respectively. The case of identical producers is obtained for

$$c_1 = c_2 = c \quad \text{and} \quad \upsilon_1 = \upsilon_2 = \upsilon. \tag{9}$$

In this case, as explained above, every duopoly game starting with equal productions $q_{10} = q_{20}$ has a synchronized trajectory, embedded into the invariant line (1).

In the following sections, we consider two particular examples, obtained with two different demand functions, and for each of them we compare the dynamical behavior of the duopoly system with identical producers to that with *quasi-identical producers*, characterized by a parameter mismatch

$$\nu_2 = \nu_1 + \varepsilon \quad \text{and/or} \quad c_2 = c_1 + \delta, \tag{10}$$

where the constants ε and δ are small in comparison with the values of the respective parameters, i.e. $|\varepsilon|/\nu_1 \ll 1$, $|\delta|/c_1 \ll 1$. If $(\nu_1, c_1) \neq (\nu_2, c_2)$, then the line of equal productions Δ is no longer invariant because the symmetry property of the map is lost. In other words, even starting from identical initial productions, $q_{10} = q_{20}$, the synchronization $q_1(t) = q_2(t)$ $\forall t$ is no longer true. In this case, we can ask if the time evolution of the duopoly game with quasi-identical producers is similar, or not, to that with truly identical producers.

3. Linear demand function

If we assume a linear demand function $p = f(Q) = a - b(q_1 + q_2)$, with a, b positive constants, the model (7) gives rise to the following nonlinear discrete dynamical system:

$$T: \begin{cases} q_1' = q_1[1 + \nu_1(a - c_1) - 2b\nu_1 q_1 - b\nu_1 q_2], \\ q_2' = q_2[1 + \nu_2(a - c_2) - 2b\nu_2 q_2 - b\nu_2 q_2]. \end{cases} \tag{11}$$

This is a noninvertible map of the plane, that is, starting from some nonnegative initial production strategy (8), the iteration of (11) uniquely defines the trajectory $(q_1(t), q_2(t)) = T^t(q_{10}, q_{20})$, $t \geq 0$, whereas the backward iteration of (11) is not uniquely defined because a point (q_1', q_2') of the plane may have several preimages, obtained by solving the fourth degree algebraic system (11) with respect to q_1 and q_2 (see [1] or [22] for a description of the properties of noninvertible maps of the plane). A detailed analysis of the map (11), in the non-symmetric case, is given in [9]. The map (11) has four fixed points: three boundary equilibria, located on the invariant coordinate axes, given by $E_0 = (0, 0)$, $E_1 = ((a - c_1)/2b, 0)$, $E_2 = (0, (a - c_2)/2b)$, and the fixed point

$$E_* = \left(\frac{a + c_2 - 2c_1}{3b}, \frac{a + c_1 - 2c_2}{3b} \right), \tag{12}$$

which is positive provided that

$$2c_1 < a + c_2 \quad \text{and} \quad 2c_2 < a + c_1. \tag{13}$$

It is easy to verify that when (13) are satisfied, the fixed point E_* represents the unique Nash equilibrium for the duopoly game.

In the case (9) of identical producers, the duopoly map (11) can be written as

$$T_S : \begin{cases} q_1' = h(q_1, q_2), \\ q_2' = h(q_2, q_1), \end{cases} \tag{14}$$

where

$$h(x, y) = (1 + \nu(a - c))x - 2b\nu x^2 - b\nu x y. \tag{15}$$

Under assumption (9), the Nash equilibrium (12) becomes

$$E_* = \left(\frac{a - c}{3b}, \frac{a - c}{3b} \right) \in \Delta \tag{16}$$

and the two boundary equilibria E_1 and E_2 are in symmetrical positions with respect to the line Δ. The dynamics of T_S restricted to this invariant line are governed by the one-dimensional map $q' = g(q) = T_S|_\Delta = h(q, q)$, given by

$$g(q) = (1 + \nu(a - c))q - 3b\nu q^2. \tag{17}$$

This is a logistic map in nonstandard form, conjugate to the standard logistic $x' = \mu x(1 - x)$, with parameter $\mu = 1 + \nu(a - c)$, by the linear transformation

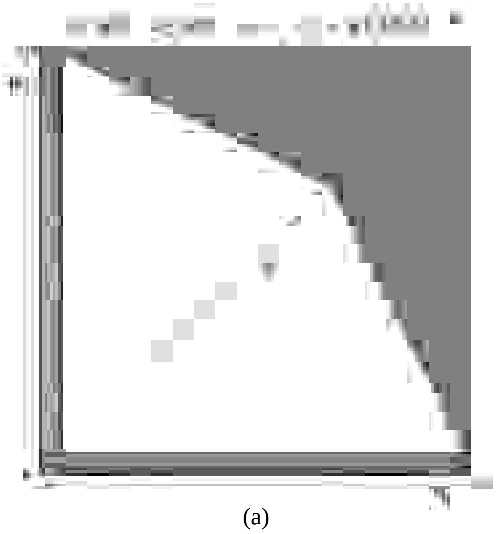
$$q = \frac{1 + \nu(a - c)}{3b\nu} x.$$

Thus, the dynamical behavior of the representative producer can be obtained from the well-known behavior of the standard logistic map by an homeomorphism. The unique positive fixed point of (17) is stable for $0 < \nu(a - c) < 2$ and its basin of attraction is given by

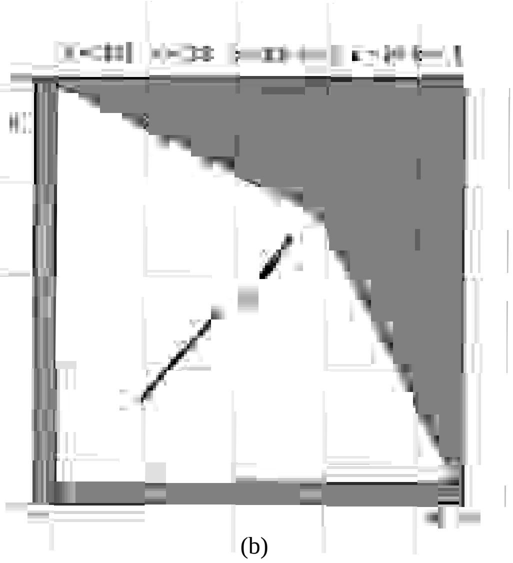
$$q_0 = q_{10} = q_{20} \in \left(0, \frac{1 + \nu(a - c)}{3b\nu} \right).$$

Of course, the positive fixed point of (17) on the diagonal Δ coincides with the Nash equilibrium (16) of the duopoly game. At $\nu(a - c) = 2$, the Nash equilibrium loses stability through a flip (or period doubling) bifurcation, and for $2 < \nu(a - c) < \sqrt{6}$, an attracting cycle of period two exists around E_*. Also this cycle undergoes a flip bifurcation at $\nu(a - c) = \sqrt{6}$ that creates an attracting cycle of period four, and so on, and a sequence of flip bifurcations, known as Myrberg (or Feigenbaum) cascades, leads to chaotic behavior (see e.g. [6,12]). For $\nu(a - c) > \mu^* - 1$, with $\mu^* \simeq 3.57$, the ω-limit set of the generic trajectory of the synchronized dynamics of identical producers starting with $q_{10} = q_{20} \in (0, (1 + \nu(a - c))/(3b\nu))$ is either an attracting cycle or cyclic-invariant chaotic intervals, or a Cantor set belonging to trapping intervals bounded by critical points (see e.g. [21,29]). For $\nu(a - c) > 3$, the generic trajectory of (17) is divergent (see e.g. [13]).

We now examine the effect of a symmetry breaking due to the introduction of a small parameter mismatch, according to (10). If such a difference is very small, we may expect that the trajectories of the duopoly game are not far from the line Δ, i.e. a



(a)



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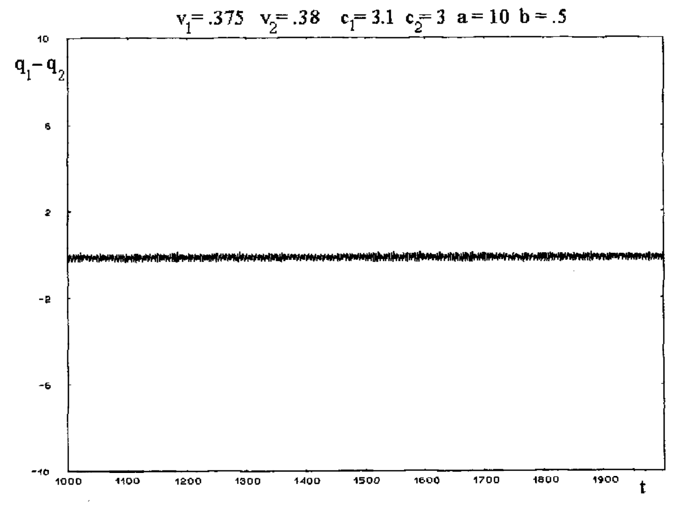


(c)

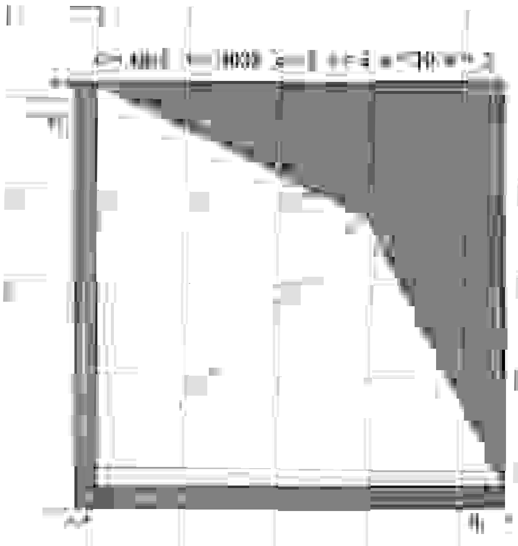
Figure 1.

quasi-synchronized behavior should be obtained. For example, in figure 1, two trajectories are represented in the strategy plane (q_1, q_2), both starting from the same initial strategy (q_{10}, q_{20}) Δ, one obtained in the case of identical producers with $\upsilon_1 = \upsilon_2 = 0.375$ and $c_1 = c_2 = 3$ (figure 1(a)) and the other with $\upsilon_1 = 0.375$, $\upsilon_2 = 0.38$, $c_1 = 3.1$, $c_2 = 3$, i.e. in the case (10) of quasi-identical producers with $|\delta|/c_1 \simeq 0.03$ and $|\varepsilon|/\upsilon_1 \simeq 0.01$ (figure 1(b)). In these figures, the shape of the attractor is obtained, as usual, by representing many points $(q_1(t), q_2(t))$ of a trajectory after the transient part, constituted by the early iterates, has been discarded. The white region represents the basin of attraction of the bounded attractor, whereas the shaded region represents the set of points that generate unbounded trajectories, i.e. the basin of infinity. The attractor shown in figure 1(a) is a two-cyclic chaotic interval on Δ, where the dynamics are governed by the one-dimensional map (17), whereas the attractor shown in figure 1(b) is given by a two-cyclic chaotic area on which the dynamics is governed by the two-dimensional map (11). It can be noticed that this two-dimensional attractor is close to the line Δ, even if such a line is not invariant, and the dynamical behavior of the duopoly game is similar to that obtained by the simpler one-dimensional model (17). In other words, the two quasi-identical producers show quasi-synchronized production decisions, i.e. characterized by small differences $|q_1(t) - q_2(t)|$ between the two productions. This is clearly shown in figure 1(c), where the difference $q_1 - q_2$ is represented versus time.

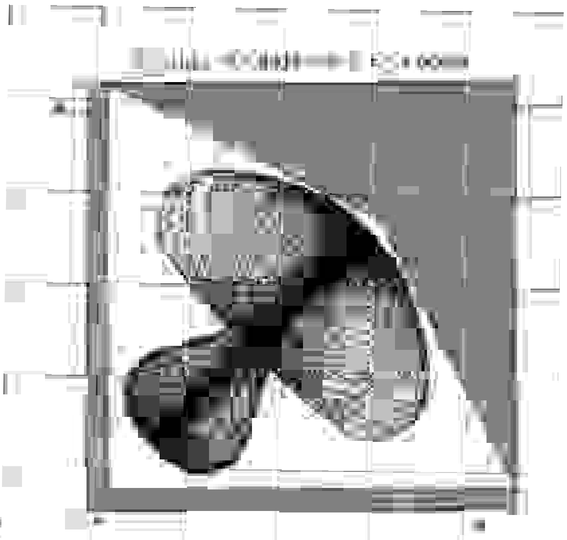
However, this conclusion cannot be applied in general, since for different sets of parameters the situation appears to be very different. For example, the trajectory shown in figure 2(a), obtained in the case of identical producers with $\upsilon_1 = \upsilon_2 = 0.4035$ and $c_1 = c_2 = 3$, is completely different from that shown in figure 2(b), obtained after the introduction of a very small difference between the marginal costs, namely $c_1 = 3$ and $c_2 = 3.00001$, i.e. $\delta/c_1 \simeq 3 \times 10^{-6}$. Both trajectories shown in figures 2(a),(b) have been obtained starting with equal initial productions: the one obtained with identical producers is characterized by a synchronized chaotic dynamics governed by the map (17), whereas the one obtained with quasi-identical producers is characterized by an erratic dynamics inside a large two-dimensional chaotic area. Despite the very small difference between the parameters of the two producers, the synchronization between the productions $q_1(t)$ and $q_2(t)$ is completely lost in the trajectory shown in figure 2(b). Indeed, it can be noticed that inside the large chaotic area, the points near the line of equal productions Δ are more frequently visited than those far from it, i.e. for quasi-identical producers, similar productions are more probable. However, many time periods exist which are characterized by very different production decisions, represented by the points far from Δ. This is clearly seen in figure 2(c), where the difference $q_1 - q_2$ is represented versus time. In this case, even if the difference between the parameters of the two producers is negligible, the simplified model (17) of the representative firm loses any practical meaning, because there are time periods in which the behavior of the two firms is very different. Furthermore, due to the chaotic behavior of the duopoly, the time period in which the "asynchronous" production decisions occur cannot be forecasted.

In order to understand the different effects of a parameter mismatch (10) in the two cases shown in figures 1 and 2, we consider the local stability properties of the attractors of the symmetric map T_S located on the invariant submanifold Δ. The Jacobian matrix of the map (14), computed on the line Δ, assumes the structure

$$DT_S(x, x) = \begin{bmatrix} l(q) & m(q) \\ m(q) & l(q) \end{bmatrix} \tag{18}$$



(a)



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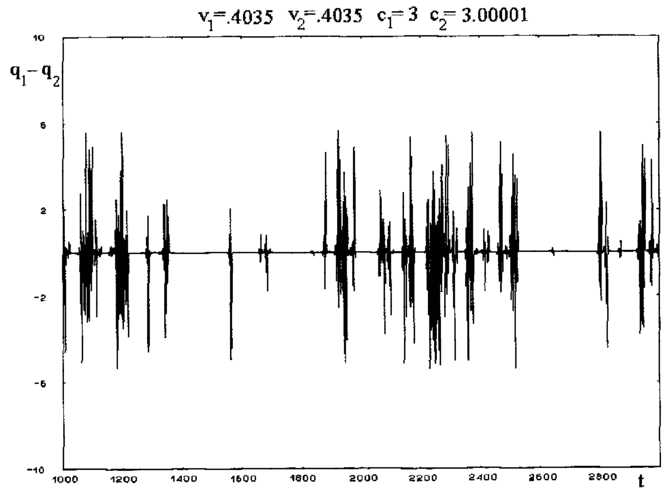


(c)

Figure 2.

with $l(q) = 1 + \nu(a - c) - 5\nu bq$ and $m(q) = -\nu bq$. The eigenvalues are

$$\begin{aligned} \lambda_{\parallel} &= l(q) + m(q) = 1 + \nu(a - c) - 6\nu bq, \quad \text{with eigenvector } \boldsymbol{r}_{\parallel} = (1,1), \\ \lambda_{\perp} &= l(q) - m(q) = 1 + \nu(a - c) - 4\nu bq, \quad \text{with eigenvector } \boldsymbol{r}_{\perp} = (1,-1). \end{aligned} \tag{19}$$

Of course, the eigenvalue $\lambda_{\parallel}$, associated with the invariant manifold along the line Δ, coincides with the multiplier of the restriction $T_S|_{\Delta}$ given by the map (17). The eigenvector associated with the other eigenvalue is always orthogonal to Δ and independent of q.

For a k-cycle $\{(\alpha_1, \alpha_1),\ldots,(\alpha_k, \alpha_k)\}$ of (14), corresponding to the cycle $\{\alpha_1,\ldots,\alpha_k\}$ of the one-dimensional map (17), the two multipliers are

$$
\begin{aligned}
\lambda_{\parallel}^{(k)} &= \prod_{i=1}^{k} (l(\alpha_i) + m(\alpha_i)) = \prod_{i=1}^{k} (1 + \nu(a - c) - 6b\nu\alpha_i), \\
\lambda^{(k)} &= \prod_{i=1}^{k} (l(\alpha_i) - m(\alpha_i)) = \prod_{i=1}^{k} (1 + \nu(a - c) - 4b\nu\alpha_i).
\end{aligned}
\tag{20}
$$

Also for the cycles, the conditions for stability and local bifurcations along Δ are the same as for the one-dimensional map (17). Hence, we focus our attention on the transverse stability.

For $\nu(a-c) \in [0, 3]$, the ω-limit sets of the restriction $g = T_s|_\Delta$ belong to the attracting set

$$
I = [c_1, c] = \left[g\left(\frac{1 + \nu(a - c)}{4}\right), \frac{1 + \nu(a - c)}{4} \right] \subset \Delta,
$$

inside which only one attractor A exists (a periodic cycle or a cycle of chaotic intervals or a Cantor set, which is a weak attractor). The one-dimensional invariant set $I \subset \Delta$ is also an asymptotically attracting set for the two-dimensional map T_s if all the cycles of T_s inside it are transversally attracting (we recall that a set is asymptotically stable if it attracts all the trajectories starting around it: more precisely, a set A is asymptotically stable if it is Lyapunov stable, i.e. for every neighborhood U of A, there exists a neighborhood V of A such that $T^t(V) \subset U \ \forall t$, and the basin $\mathcal{B}(A)$ contains a neighborhood of A). Indeed, for sufficiently small values of $\nu(a-c)$, any attractor of the restriction $T_s|_\Delta$ is also an asymptotically stable attractor for the two-dimensional map T.

For example, a numerical computation, performed with $a = 10$, $b = 0.5$, $c = 3$, shows that the conditions for asymptotic stability of the attractor A along Δ are fulfilled for $\nu < 0.37562$ (we remark that for this set of parameters, the Feigenbaum point is given by $\nu^* = 0.36713\ldots$). Hence, at the parameter values used in figure 1, all the cycles existing inside I are transversally attracting and the attractor shown in figure 1(a) is asymptotically stable. In this case, all the trajectories starting inside a two-dimensional neighborhood U of I synchronize and their ω-limit set is the attractor $A \subseteq I$ of the restriction g.

As ν reaches the bifurcation value $\nu_b \simeq 0.37562$, a 2-cycle located on Δ becomes transversally unstable with $\lambda^{(2)} < -1$. For $\nu < \nu_b$, just before the bifurcation, this 2-cycle is a saddle, repelling along Δ and transversally attracting, with $-1 < \lambda^{(2)} < 0$.

At $\nu = \nu_b$, it becomes a repelling node via a flip bifurcation, at which a saddle cycle of period 4 is created out of Δ, with periodic points located symmetrically with respect to the diagonal. The bifurcation occurring at $\nu = \nu_b$, called riddling bifurcation, has recently been studied in [2,5,19].

For $\nu > \nu_b$, if we consider a sufficiently small neighborhood U of I, we have that the local unstable set of the transversally unstable 2-cycle intersects ∂U. Moreover, repelling "tongues" exist around this transverse unstable manifold such that the trajectories starting from the points of these "tongues" exit U after a finite number of iterations, and similar "tongues" also exist in correspondence with the infinite preimages, along Δ, of the points of the 2-cycle (see [2,11,19,23]). This implies that no neighborhood $V \subset U$ exists such that the definition of Lyapunov stability holds, so that the invariant interval I is no longer an asymptotically stable set for the map T_s. In other words, even if the interval $I \subset \Delta$ attracts the synchronized trajectories embedded into Δ, it is a repelling set for the two-dimensional map T_S.

Nevertheless, numerical explorations show that the bounded attractor $A \subseteq I$ of the map g continues to capture almost all the bounded trajectories of the symmetric two-dimensional map T_S. This suggests that *A is an attractor in the weak Milnor sense* (see [2,5,20]). This is the situation for the parameter set used to obtain figure 2(a). The generic trajectory, starting from a point of the white region, converges to the submanifold Δ where synchronized dynamics take place, even if the attractor $A \subseteq \Delta$ is not an attractor in the usual topological sense. This is the local explanation of the difference between the two different behaviors numerically evidenced by the parameter mismatches shown in figures 1 and 2. For $\upsilon > \upsilon_b$, as in figure 2, most initial conditions close to $I \subseteq \Delta$ are attracted to I, but in any neighborhood of I, there exists a dense set that is locally repelled in a direction transverse to Δ.

A similar "explosive" effect of a small parameter mismatch, called *hard bubbling transition* in [32], has recently been observed in symmetric dynamical systems characterized by the presence of a one-dimensional Milnor (transversally unstable) chaotic attractor (see [32,33]). In this case, we have that both the long-run behavior of identical and quasi-identical producers is erratic. The main difference is that for truly identical agents we have synchronized chaos, so that the common behavior can be modeled by the simpler model of the representative agent, whereas for quasi-identical agents no common behavior can be observed.

In the case of quasi-identical firms, an estimate of the maximum value of the difference $|q_2 - q_1|$ between the two productions, proportional to the distance between the phase point (q_1, q_2) and the line of equal productions Δ, can be obtained by a study of the global properties of the map (11), characterized by its critical curves. In fact, after the symmetry-breaking, which destroys the invariance of Δ, the fate of the locally repelled trajectories depends on the global properties of the non-invertible map T, and in particular on the folding action of its critical curves (see e.g. [1,16,22]).

The notion of critical curve is one of the distinguishing features of non-invertible maps. We recall that the critical curve of rank-1, denoted by LC, is defined as the locus of points having two, or more, coincident rank-1 preimages, located on a set called LC_{-1}. LC is the two-dimensional generalization of the notion of critical value, local minimum or maximum, of a one-dimensional map, LC_{-1} is the generalization of the notion of critical point (local extremum point). Arcs of LC separate the regions of the plane characterized by a different number of real preimages.

Since the map (11) is a continuously differentiable map, LC_{-1} belongs to the locus of points where the Jacobian determinant of T vanishes (i.e. the points where T is not locally invertible). In our case,

$$LC_{-1} = \{x \in \mathbb{R}^2 \mid \det DT = 0\}$$

and LC is the rank-1 image of LC_{-1} under T, i.e. $LC = T(LC_{-1})$.

For the map (11), the condition $\det DT = 0$ becomes

$$q_1^2 + q_2^2 + 4q_1q_2 - \alpha_1 q_1 - \alpha_2 q_2 + \beta = 0$$

with

$$\alpha_i = \frac{4(1 + \upsilon_j(a - c_j)b\upsilon_i) + 1 + \upsilon_i(a - c_i)b\upsilon_j}{4b^2\upsilon_1\upsilon_2}, \quad i = 1, 2;\ j \neq i,$$

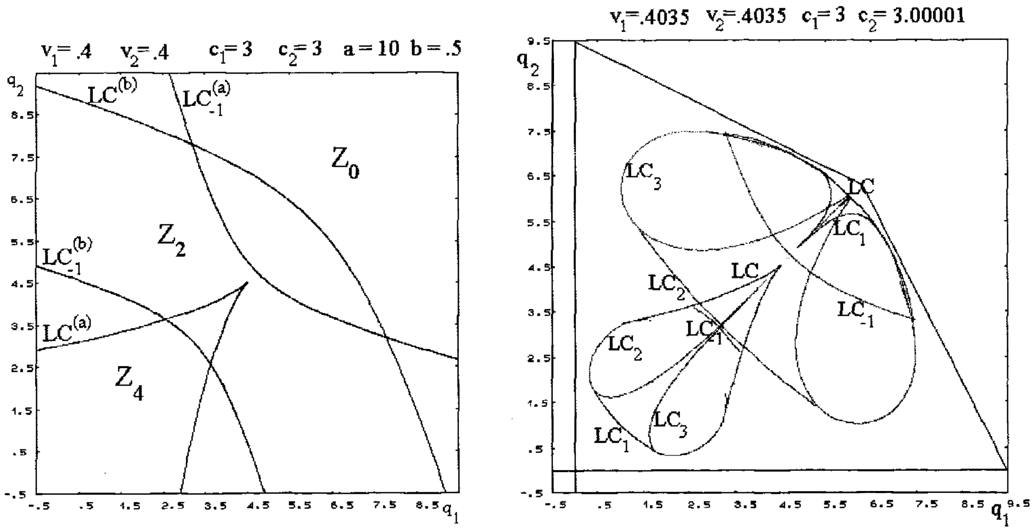
and

$$\beta = \frac{(1 + \upsilon_1(a - c_1)b\upsilon_1)(1 + \upsilon_2(a - c_2)b\upsilon_2)}{4b^2\upsilon_1\upsilon_2}.$$

This is the equation of a hyperbola in the plane (q_1, q_2). Thus, LC_{-1} is formed by two branches, denoted by $LC_{-1}^{(a)}$ and $LC_{-1}^{(b)}$ in figure 3(a). This implies that also LC is the union of two branches, denoted by $LC^{(a)} = T(LC_{-1}^{(a)})$ and $LC^{(b)} = T(LC_{-1}^{(b)})$ (see figure 3(a)). Each branch of the critical curve LC separates the phase plane of T into regions whose points have the same number of distinct rank-1 preimages. In the case of the map (11), $LC^{(b)}$ separates the region Z_0, whose points have no preimages, from the region Z_2, whose points have two distinct rank-1 preimages, and $LC^{(a)}$ separates the region Z_2 from Z_4, whose points have four distinct preimages (for more details, see [9]). The images of rank k of LC_{-1} give the critical sets of rank k, denoted by $LC_{k-1} = T^k(LC_{-1}) = T^{k-1}(LC)$.

Portions of critical curves of increasing rank can be used to bound absorbing and chaotic areas of non-invertible maps of the plane (see e.g. [1,16,22]). We recall that an absorbing area $\mathcal{A}$ is a region of the plane, bounded by critical curve segments of finite rank, such that the successive images of the points of a neighborhood of $\mathcal{A}$, say $\mathcal{U}(\mathcal{A})$, enter inside $\mathcal{A}$ after a finite number of iterations, and never exit, being $T(\mathcal{A}) \subseteq \mathcal{A}$. A chaotic area is an absorbing area $\mathcal{A}$ such that $T(\mathcal{A}) = \mathcal{A}$ and chaotic dynamics occur inside $\mathcal{A}$. The boundary of the chaotic area $\mathcal{A}$ appearing in figure 2(b) can be obtained by taking the images of the two segments of LC_{-1} included inside $\mathcal{A}$. This statement is based on the following general procedure for the determination of the boundary of an absorbing or a chaotic area given in [22, chap. 4]. Let $\gamma = LC_{-1} \cap \mathcal{A}$: then $\partial\mathcal{A}$ is made up of arcs belonging to $T^k(\gamma)$, $k = 1, \ldots, m$, for some suitable integer m.

In our case, $m = 4$, as shown in figure 3(b). From this figure, it can be seen that whenever LC_{-1} intersects LC, say in a point a_0, the images of a_0, given by $a_k = T^k(a_0)$, are tangential points between critical curves, and the union of such tangential segments of critical curves defines the boundary of a trapping region.



(a) (b)

Figure 3.

We remark that $\mathcal{A}$ includes the invariant interval *I*, being $\mathcal{A} \cap \Delta = I$, and all the trajectories starting inside a neighborhood of *I* cannot go out of $\mathcal{A}$. Loosely speaking, $\partial\mathcal{A}$ behaves as a bounded vessel for the trajectories starting from the "tongues" located around the local unstable sets of the transversally repelling cycles.

4. Isoelastic demand function

In this section, we consider a unit-elastic demand function

$$p = f(Q) = \frac{1}{q_1 + q_2}. \tag{21}$$

This demand function is often met in economic modeling. For example, it has been used in [25] and [26] in order to obtain an explicit form of the reaction function in a class of dynamic Cournot games. Instead, in this paper we propose such a demand function in order to show a different kind of symmetry-breaking bifurcation, which is definitely not typical, which gives us the opportunity of remarking the non-generic behavior of a symmetric dynamical system and, consequently, that noticeable effects of small heterogeneities should be expected.

With function (21), model (7) gives rise to the map

$$T: \begin{cases} q_1' = q_1 \left(1 - c_1 v_1 + v_1 \dfrac{q_2}{(q_1+q_2)^2}\right), \\ q_2' = q_2 \left(1 - c_2 v_2 + v_2 \dfrac{q_1}{(q_1+q_2)^2}\right). \end{cases} \tag{22}$$

This is a non-invertible map which is not defined in the whole plane, because the denominator vanishes along the line of equation $q_1 + q_2 = 0$. The general properties concerning the attractors and local bifurcations of (22) are analyzed in [8], and the global bifurcations of the basin boundaries are studied in [7]. It is easy to see that (22) has only one fixed point, given by

$$P^* = \left(\frac{c_2}{(c_1 + c_2)^2}, \frac{c_1}{(c_1 + c_2)^2} \right), \tag{23}$$

which is the unique Nash equilibrium. Its local stability analysis, based on the localization, in the complex plane, of the eigenvalues of the Jacobian matrix $DT(P^*)$, shows that P^* is stable as long as the following stability conditions hold:

$$c_1 c_2 \upsilon_1 \upsilon_2 - 4 \frac{c_1 c_2}{c_1 + c_2} (\upsilon_1 + \upsilon_2) + 4 > 0, \tag{24}$$

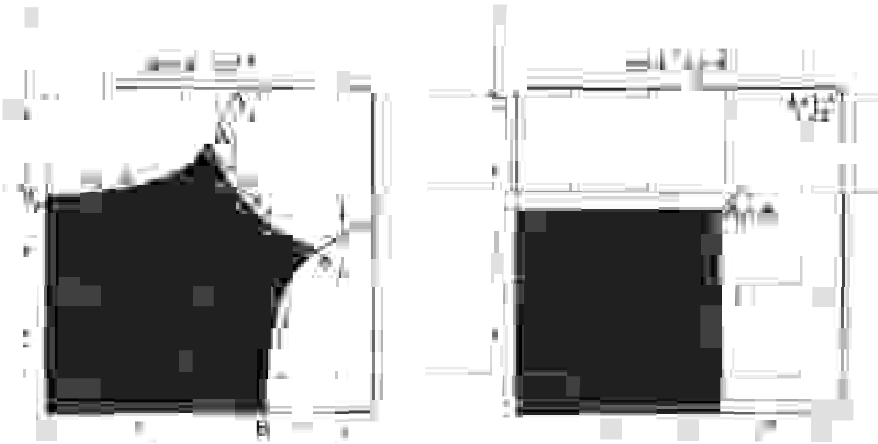
$$c_1 c_2 \upsilon_1 \upsilon_2 - 2 \frac{c_1 c_2}{c_1 + c_2} (\upsilon_1 + \upsilon_2) > 0. \tag{25}$$

The equality given by the vanishing of the left-hand side of (24) gives the condition for a flip bifurcation, whereas the equality given by the vanishing of the left-hand side of (25) gives the condition for a Neimark–Hopf bifurcation of the Nash equilibrium. For fixed values of the parameters c_1 and c_2, such equations represent hyperbolae in the parameter plane (υ_1, υ_2). These hyperbolae intersect in the points

$$A_1 = \left(\frac{2}{c_1}, \frac{2}{c_2} \right), \quad A_2 = \left(\frac{2}{c_2}, \frac{2}{c_1} \right), \quad B_1 = \left(\frac{c_1 + c_2}{c_1 c_2}, 0 \right), \quad B_2 = \left(0, \frac{c_1 + c_2}{c_1 c_2} \right) \tag{26}$$

and bound the stability region S of the Nash equilibrium in the parameter plane (υ_1, υ_2), represented by the shaded area of figure 4. The two lines l_1 and l_2 appearing in figure 4(a) bound a cone, say $\mathcal{F}$, inside which P^* is a focus, a stable focus for the portion of $\mathcal{F}$ belonging to the shaded region, an unstable focus for the part of $\mathcal{F}$ out of the shaded region. If the marginal costs c_1 and c_2 are fixed, the shape of the stability region S remains the same and by increasing υ_1 and/or υ_2, the point $V = (\upsilon_1, \upsilon_2)$ can move out of it. If V crosses the boundary of S along the arc $A_1 A_2$ (belonging to the hyperbola whose equation is given by the vanishing of (25), i.e. det $DT(P^*) = 1$), then the fixed point P^* changes from a stable focus to an unstable focus via a Neimark–Hopf bifurcation, whereas if V exits the region S by crossing one of the arcs $B_1 A_1$ or $B_2 A_2$ (both belonging to the other hyperbola, whose equation is given by the vanishing of (24)), the fixed point P^* is changed from an attracting node to a saddle point through a flip bifurcation.

Similar arguments apply if the marginal costs (c_1, c_2) are varied. For example, if c_1 and c_2 are increased, the stability region S becomes smaller, as can be easily deduced from (26), and this can cause the exit of V from S even if the speeds of adjustment υ_1



(a) (b)

Figure 4.

and υ_2 are held constant. Also in this case, the loss of stability can occur via a Neimark–Hopf or a flip bifurcation, depending on the boundary arc which is crossed by point V.

We observe that if $c_1 > c_2$, the positions of the vertices A_1 and A_2 are exchanged with respect to those appearing in figure 4(a) (obtained with $c_1 < c_2$), and if the difference between the marginal costs of the two firms is increased, the region $\mathcal{F}$ enlarges and, consequently, the arc A_1A_2, representing the curve where Hopf bifurcations occur, becomes larger. If $c_1 = c_2$, the lines l_1 and l_2 merge, so that the cone $\mathcal{F}$ disappears, the stability region S becomes a square, like that shown in figure 4(b), and the possibility of Hopf bifurcations is lost. In the case (9) of identical producers, the map (22) assumes the symmetric form (14) with

$$h(x, y) = x\left(1 - c\upsilon + \upsilon\frac{y}{(x+y)^2}\right). \tag{27}$$

In this case, the restriction $T_S|_\Delta$, which models the dynamics of the representative firm on the invariant line Δ, is given by

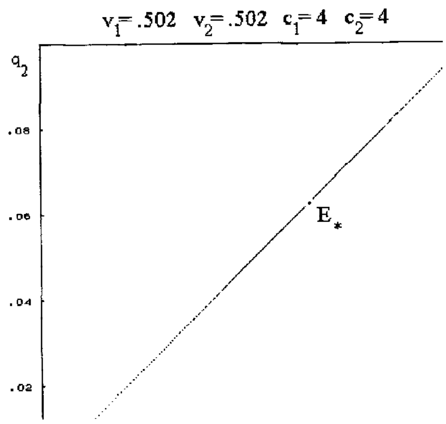
$$q' = (1 - c\upsilon)q + \frac{\upsilon}{4}. \tag{28}$$

This is a linear map with fixed point $q^* = 1/(4c)$, corresponding to the Nash equilibrium $P^* = (1/(4c), 1/(4c))$ of the symmetric game. This fixed point attracts every synchronized trajectory if $c\upsilon < 2$, whereas diverging (and oscillatory) trajectories are obtained along the invariant line Δ if $c\upsilon > 2$. These are also the conditions to have stability or instability of the Nash equilibrium for the two-dimensional symmetric map (14) with (27). In fact, the Jacobian matrix of such map, computed at any point of the invariant line Δ, becomes

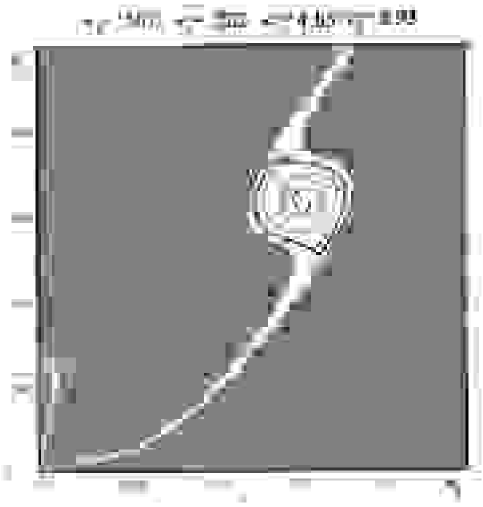
$$DT_S(q, q) = \begin{bmatrix} 1 - c\upsilon & 0 \\ 0 & 1 - c\upsilon \end{bmatrix},$$

so that the fixed point $P^* = (1/(4c), 1/(4c))$ is a stable (unstable) star node if $c\upsilon < 2$ ($c\upsilon > 2$). This implies that for the duopoly model (22), a slight perturbation in the parameter values with respect to the case of identical producers does not change the stability property of the Nash equilibrium. Furthermore, differently from the model analyzed in section 2, the Nash equilibrium is the only possible attractor of the symmetric map on the invariant line Δ because the restriction $T_S|_\Delta$, being linear, can have neither attracting cycles nor chaotic attractors.

Nevertheless, a particular symmetry-breaking bifurcation, causing very different long-run dynamics of quasi-identical producers with respect to the corresponding case of identical ones, can be obtained also in this case. Figures 5(a) and 5(b) show that a



(a)

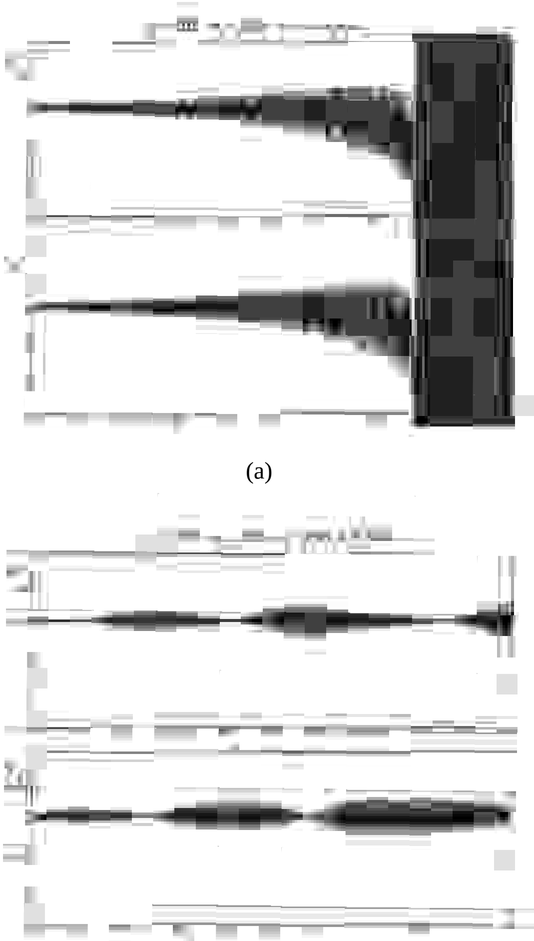


(b)

Figure 5.

small change in the marginal costs, with $\delta/c_1 = (c_2 - c_1)/c_1 \simeq 0.02$, can transform divergent dynamics obtained in the case of identical producers (figure 5(a)) into bounded oscillatory dynamics that asymptotically approach an invariant closed curve, around the Nash equilibrium, in the case of quasi-identical firms. The trajectories shown in figure 5 are obtained starting from the same initial condition. In this figure, also the transient part is represented.

Figure 5(a) is obtained with $\upsilon_1 = \upsilon_2 = 0.502$ and $c_1 = c_2 = 4$. An initial condition belonging to Δ and close to the Nash equilibrium generates a diverging oscillatory trajectory, governed by the linear map (28). In this case, the stability region is given by the square $[0, 0.5\,] \times [0, 0.5]$ shown in figure 4(b), so that the parameter values used to obtain such a trajectory are out of the stability region. The trajectory shown in figure 5(b) is obtained starting from the same initial condition and with the same values of υ_1 and υ_2, but a small difference has been introduced between the two marginal



(a)

(b)

Figure 6

costs. This slight perturbation opens a narrow cone $\mathcal{F}$ bounded by the lines l_1 and l_2 (figure 4), so that the Nash equilibrium is transformed from an unstable star-node into an unstable focus, i.e. the coincident real eigenvalues $\lambda_1 = \lambda_2 = 1 - c\upsilon < -1$ become a pair of complex conjugate eigenvalues, located out of the unitary circle of the complex plane. This causes the sudden appearance of a closed invariant and attracting orbit of finite amplitude. In other words, the divergent dynamics obtained in the symmetric case of identical producers is replaced, after the parameter mismatch, by a bounded behavior (at least for the trajectories starting sufficiently close to the Nash equilibrium). This is shown in figure 6, where the two trajectories shown in figure 5 are represented versus time.

It can be noticed that the creation of the closed invariant curve, caused by the symmetry-breaking bifurcation, is not related to a change of stability of the fixed point.

In other words, the variation of the parameters used to obtain the situation of figure 5(b), starting from the symmetric situation of figure 5(a), does not cause any crossing of the eigenvalues through the unit circle. However, the existence of the invariant closed orbit can easily be revealed if the same point in the parameter space is reached following a different path in which the point (υ_1, υ_2) reaches the point (0.502, 0.502), passing through the arc A_1A_2 at which a Neimark–Hopf bifurcation occurs.

The mechanism that makes the closed attracting orbit appear without the occurrence of a Neimark–Hopf bifurcation, like in the case of the symmetry-breaking bifurcation shown in figure 5, can be explained as follows. As is well known, the occurrence of a supercritical Neimark–Hopf bifurcation causes, as some parameter is made to vary, the transformation of the fixed point from a stable to an unstable focus and the creation of a stable closed invariant curve Γ around the unstable focus. Such a stable orbit Γ attracts all the trajectories starting from a given basin of attraction $\mathcal{B}(\Gamma)$. If the parameter continues to change, the closed orbit Γ becomes larger and when it has contact with the boundary of its basin, $\partial\mathcal{B}(\Gamma)$, it disappears ($\Gamma$ loses the invariance property). This is a global bifurcation, i.e. not related to the eigenvalues of $DT(P^*)$. If now we reverse the direction of the parameter variation, we shall see the sudden appearance of the attracting orbit Γ, with its basin $\mathcal{B}(\Gamma)$, around the unstable fixed point P^*, and this creation of Γ is not related to any change of stability of the fixed point. In our case, due to the symmetry breaking, the line Δ loses its invariance property and the sudden appearance of nonlinear terms causes the creation of a closed invariant orbit, around P^*, tangent to its boundary of attraction (the white region of figure 5(b)).

We conclude, also in this case, that the linear model of the representative firm is no longer meaningful for the description of the long-run behavior of the duopoly model characterized by quasi-identical producers.

5. Conclusions

We have considered symmetric dynamical systems that model economic systems with identical agents, and we have argued that such systems exhibit non-generic properties, in the sense that the introduction of asymmetries, due to small heterogeneities (always present in real systems) can have noticeable effects on the qualitative behavior of the model. This contrasts with the common assumption, usually made in economic modeling, that if the agents have slightly different characteristics they behave more or less in the same way.

To support this statement, we have considered a dynamic duopoly game with identical producers, modeled by a symmetric discrete-time dynamical system of the plane. The symmetry property implies that the diagonal (line of equal productions) is invariant. The dynamics on this invariant line is governed by a one-dimensional map that can be seen as the model of a *representative firm*, in the sense that the productions obtained by the one-dimensional model summarize the common (synchronized) production choices of the two identical producers.

In order to show two different kinds of symmetry-breaking bifurcations, we have studied two duopoly games, and for each of them compared the dynamical behavior of the duopoly system with identical producers to that with quasi-identical ones. In both examples, we have shown that a negligible difference between the parameters can give dynamic evolutions that are qualitatively and quantitatively different from that of the representative firm.

The presence of bifurcations like those shown in this paper suggests that a slight structural modification of the symmetrical model of identical firms can produce noticeable qualitative changes in the dynamics of the duopoly model. Indeed, this statement is quite general in the theory of dynamical systems: dynamic models with symmetries often show a non-generic behavior, i.e. a breaking of the symmetry generally leads to a qualitatively different dynamic evolution. This means that the model of the representative firm, which correctly summarizes the common choices of identical firms, may be completely inadequate to represent the dynamics of quasi-identical firms, as stressed in [17].

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