9 Chaos Synchronization and Intermittency in a Duopoly Game with Spillover Effects

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1 Introduction

Discrete-time dynamic oligopoly games which exhibit chaotic time patters of the competitors' strategic choices are at the center of a flourishing literature, including the seminal paper by Rand (1978) and several papers after it (see e.g. Dana and Montrucchio, 1986, Puu, 1991, Kopel, 1996), where simple microeconomic situations have been proposed which lead to duopoly games with chaotic dynamics. The main concern, in this stream of literature, is to emphasize the features of the duopoly games which are responsible for the transition from regular to chaotic dynamics and to analyze the bifurcations which cause the loss of stability of the equilibria and the appearance of more complex attractors (see e.g. Bischi et al., 2000).

In this chapter, we stress some peculiar features which characterize different kinds of chaotic behaviors. In particular, given a duopoly game characterized by a chaotic behavior of both the competitors, we try to distinguish different degrees of correlation between the choices of the two players. With the term "correlated chaos" we mean that even if both the players behave chaotically, at each time period their choices, say x_t and y_t, are approximately related by a function $y_t = f(x_t)$ or $x_t = f(y_t)$. This means that the chaotic attractor, in the two-dimensional state space (x, y), is approximately located around a portion of a one-dimensional curve. Instead, "uncorrelated chaos" means that a generic chaotic trajectory fills up a two-dimensional region, so that no relations can be evidenced, at a given time, between the decisions of the two players. With other words, provided that in both cases each player behaves chaotically (so that it is impossible to forecast her/his next period decision) in the presence of *correlated chaos* if one observes the choice of a player then the behavior of the other player at the same time period can be approximately deduced, whereas in the case of *uncorrelated* *chaos*, even if one observes what a player does at a given time period, nothing can be inferred about the simultaneous choice of the competitor.

An extreme form of correlation is represented by "chaos synchronization", which means that the chaotic trajectories are embedded into a one-dimensional invariant submanifold, say M, and are governed by the one-dimensional dynamical system given by the restriction of the two-dimensional one to M. Indeed, dynamical systems with chaotic trajectories embedded into an invariant submanifold of lower dimensionality than the total phase space have raised an increasing interest in the scientific community (see e.g. Fujisaka and Yamada, 1983, Pecora and Carrol, 1990, Ott and Sommerer, 1994, Ashwin et al, 1996, Buescu, 1997, Hasler and Maistrenko, 1997, just to cite a few).

Another kind of chaotic behavior, which can be considered as intermediate between correlated and uncorrelated chaos, is the so-called *on-off intermittency*, characterized by chaotic time patterns which are synchronized (or quasi-synchronized) for several time periods, but sometimes clusters of large asynchronous fluctuations occur, i.e. sudden bursts away from the submanifold where synchronized dynamics take place. The distribution, over time, of such asynchronous bursts is quite random, but something can be said about their maximum amplitude (see Bischi and Gardini, 2000, Kopel et al., 2000).

In the mathematical and physical literature these phenomena have mainly been studied for coupled chaotic oscillators, where a coupling parameter exists which only influences the dynamics in a direction which is orthogonal with respect to the invariant submanifold where synchronized dynamics take place (a so called *normal parameter*). Indeed, due to the sensitive dependence on initial conditions which characterizes chaotic systems, two identical and independent chaotic oscillators cannot be, in general, synchronized, whereas it is possible to synchronize them, in the long run, provided that some coupling (or interaction) is introduced (Fujisaka and Yamada, 1983, Pecora and Carrol, 1990).

Chaos synchronization and intermittency are generally associated with symmetric dynamical systems, and such a situation is commonly met in dynamic games with identical or quasi-identical competitors, as recently stressed in Bischi et al., 1998, Bischi et al., 1999, Kopel et al., 2000, Bischi and Gardini, 2000. In these models a perfect symmetry (i.e. an absolute identity of the parameters which characterize the players' behaviors) is a very demanding condition, and even the presence of a normal parameter is not so common. However, as we shall see through the numerical explorations presented in this chapter, intermittency phenomena can also be observed with heterogeneous interacting players. Indeed, intermittency phenomena may be seen as a prelude of chaos synchronization as the degree of homogeneity between the players is gradually increased. However, this is not a general rule, as we shall see in the following.

In this chapter we try to investigate how, starting from a condition of heterogeneous players, an increase of the degree of symmetry, gradually leading to conditions of quasi-identical or fully identical players, may induce the appearance of more and more correlated chaotic behaviors. In order to investigate this issue, we consider a Cournot duopoly game, recently proposed by Bischi and Lamantia (2001), where the interdependence between the quantity-setting firms is not only related to the selling price, determined by the total production through a given demand function, but also on positive cost externalities due to the effects of know-how spillovers, caused by the ability of a firm to take advantage, for free, of the competitors' Research and Development ($R\&D$) results. In this duopoly game, in the case of identical chaotic players, chaos synchronization can occur, due to the presence of a one-dimensional chaotic attractor embedded into the invariant diagonal. The study of this particular Cournot duopoly game allows us to give examples of different kinds of chaotic behaviors according to the degree of chaos correlation, and we observe the gradual transition between these different kinds as the heterogeneity in the spillover parameters, i.e. the asymmetry in the ability to take advantage of the competitor $R\&D$ results, is varied. However, the spillover parameters are not normal ones, because they influence both the dynamics along the diagonal and the dynamics transverse to it. This implies that it is not possible to use many of the results given in the literature on chaos synchronization. However, following Bischi and Gardini (1998), a global characterization of the phenomena is still possible by the method of critical curves, which allow us to obtain the delimitation of an absorbing area surrounding the one-dimensional attractor on which synchronization occurs (which is often only an attractor in Milnor sense, see Ashwin et al., 1996, Buescu, 1997). In fact, as the time evolution of the duopoly game analyzed in this chapter is represented by the iteration of a noninvertible map, the dynamic phenomena observed, such as chaos synchronization, intermittency and uncorrelated chaos, are confined inside a given absorbing area, whose boundary can be obtained by segments of critical curves, and behaves as a bounded vessel inside which the asymptotic dynamics are trapped (see Mira et al., 1996, Bischi and Gardini, 1998).

The reminder of this chapter is organized as follows: in section 2 we describe the duopoly game, in section 3 we consider the same game in the symmetric case of identical players, in section 4 we describe some numerical explorations through which the concepts outlined above are illustrated.

2 A Cournot duopoly game with spillovers

To illustrate the concepts outlined in the introduction, we consider a Cournot duopoly game, proposed in Bischi and Lamantia (2001), where the interdependence between the quantity-setting firms is not only related to the selling price, determined by the total production through a given demand function, but also on cost-reduction effects related to the presence of the competitor. Such cost reductions are introduced to model the effects of technological and intellectual spillovers among companies, caused by the ability of a firm to take advantage, for free, of the competitors' $R\&D$ results, due to the difficulties to protect know-how or to avoid the movements of skilled workers among competing firms, see e.g. Audretsch and Feldman (1996) Aitken et al. (1997). The results of $R\&D$ are generally assumed to lead to costs reductions (see e.g. D'Aspremont and Jaquemin, 1988). So, spillover effects can be seen as a positive cost externality, which we model by assuming the following cost function of firm i:

$$C_i(q_1, q_2) = \frac{c(q_i) + R\&D_i}{1 + \gamma_{ii} g_i\left(R\&D_i\right) + \gamma_{ij} h_j\left(R\&D_j\right)};\ i, j = 1, 2\ ;\quad j \neq i \tag{1}$$

where $c\left(q_i\right)$ represents the production costs of firm i, an increasing function of its own production q_i, $R\&D_i$ represents the $R\&D$ expenses of firm i, g_i and h_j are increasing functions and the positive parameters γ_{ii} and γ_{ij} give a measure of the cost reduction related to its own and competitor's $R\&D$ respectively. A particular choice is proposed in Bischi and Lamantia (2001), where a very simple cost function is obtained by assuming linear production costs, $c_i(q_i) = k_i q_i$, $R\&D$ expenses proportional to the production, $R\&D_i = s_i q_i$, h_i linear and $g_i \equiv 0$. The last assumption captures the fact that only external spillovers are considered cost-reducing, because the benefits from its own $R\&D$ are assumed to be balanced by induced costs, such as higher salaries required by more skilled workers or expenses of a firm to avoid spillovers. However, we assume that $R\&D$ are necessary in a high-tech market, where without $R\&D$ the produced goods become obsolete. With these assumptions, and a linear demand function $D(p) = a - b\,(q_1 + q_2)$, the profit of firm i becomes

$$\pi_i(q_1, q_2) = q_i\,[a - b\,(q_1 + q_2)] - \frac{c_i q_i}{1 + \gamma_{ij} q_j}\ ;\ i, j = 1, 2, i \neq j \qquad (2)$$

In a fully rational Cournot duopoly game, each player decides its own production in order to maximize the expected profit, on the basis of the following two assumptions:

(i) each firm knows beforehand its rival's production decision;

(ii) each firm has a complete knowledge of the profit function.

From the first order conditions $\partial\pi_i/\partial q_i = 0$, we get

$$q_i = r_i(q_j) = \frac{1}{2b}\left(a - bq_j - \frac{c_i}{1 + \gamma_{ij} q_j}\right)\ i, j = 1, 2\ ;\quad j \neq i. \qquad (3)$$

A simple check of the second derivatives testifies that these solutions represent profit maxima, provided that the quantities are non negative. Hence the portions inside the positive orthant of the functions $q_1 = r_1(q_2)$ and $q_2 = r_2(q_1)$, are the two reaction curves. Every intersection between the two reaction curves, being an optimal choice for both firms, is a Cournot-Nash Equilibrium, characterized by the fact that no firm has an incentive to unilaterally deviate from its chosen strategy given the choice of its rival. It is immediate to realize that the introduction of spillover effects in the cost functions has the effect of changing the reaction curves from lines to strictly concave curves, which are unimodal for sufficiently high values of spillover parameters. In Bischi and Lamantia (2001) it is proved that at most one Nash Equilibrium exists, and in order to investigate its stability the Nash Equilibrium is considered as the outcome of a dynamic adjustment process occurring when less that fully rational players play the game repeatedly (see e.g. Fudenberg and Levine, 1998, or Binmore, 1992, ch.9 for such evolutionary interpretation of the stability of a Nash equilibrium). This means that the players generally do not reach a Nash equilibrium immediately, but play the game repeatedly in order to approach it. Several kinds of boundedly rational adjustment processes may be considered, all sharing the same Nash equilibrium but with different methods to update productions when the system is out of it. One kind of dynamic adjustment, proposed by Cournot (1838), is based on the assumption that the two firms have a global knowledge of the profit function, so that they are able to compute their best reply to the expected production choice of the competitor, but the two firms are not so rational to be able to know in advance the competitor's choices, and at each time step they adopt a very simple (or *naive*) expectation, by guessing that the production of the other firm will remain the same as in the previous period. So the repeated game is defined by the recurrence

$$q_i(t+1) = r_i\left(q_j(t)\right) \;\; i,j = 1,2 \;\; i \neq j \quad . \tag{4}$$

As proved in Bischi and Lamantia (2001) this adjustment mechanism, with reaction functions (3), leads to global stability of the Nash equilibrium, i.e. an asymptotic convergence to it for any initial condition in the strategy space. A second kind of dynamic adjustment is proposed in Bischi and Lamantia (2001) where the firms are assumed to be even less rational, in the sense that they don't have a complete knowledge of the profit function, and consequently they use a simpler (and less expensive) "rule of thumb" (see e.g. Baumol and Quandt, 1964) in their decision-making processes, known in the literature as gradient dynamics (or myopic adjustment, see e.g. Sacco, 1991, Varian, 1992, Flam, 1993). This gives rise, for certain sets of parameters, to periodic and chaotic dynamics around the Nash equilibrium, and this will be the object of our studies.

According to this kind of dynamic adjustment, the two players are assumed to update their production strategies at discrete time periods on the basis of a local estimate of the marginal profit $\partial\pi_i/\partial q_i$: At each time period t a firm decides to increase (decrease) its production for period $t+1$ if it perceives positive (negative) marginal profit on the basis of information held at time t, according to the following dynamic adjustment mechanism (see e.g. Bischi and Naimzada, 1999)

$$q_i(t+1) = q_i(t) + \alpha_i(q_i(t))\frac{\partial\pi_i}{\partial q_i}\left(q_1(t), q_2(t)\right) \quad ; \qquad i = 1,2 \tag{5}$$

where $\alpha_i(q_i)$ is a positive function which gives the extent of production variation of ith firm following a given profit signal. Notice that the two producers are not requested to have a complete knowledge of the demand and cost functions, since they only need to infer how the market will respond to small production changes by an estimate of the marginal profit, which may be obtained by brief experiments of small (or local) production variations performed at the beginning of period t (Varian, 1992). Of course, this local estimate of expected marginal profits is much easier to be obtained than a global knowledge of the demand function (involving values of q_i that may be very different from the current ones). In the following we assume linear functions $\alpha_i(q_i) = v_i q_i$, $i = 1, 2$, since this assumption captures the fact that *relative* production variations are proportional to marginal profits, i.e.

$$\frac{q_i(t+1) - q_i(t)}{q_i(t)} = v_i \left(\frac{\partial \pi_i}{\partial q_i} \right).$$

With these assumptions, and the profit functions given in (2), we obtain a discrete dynamical system of the form $(q_1(t+1), q_2(t+1)) = T(q_1(t), q_2(t))$, where the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T : \begin{cases} q_1' = q_1 + v_1 q_1 \left[a - 2bq_1 - bq_2 - \frac{c_1}{1+\gamma_{12} q_2} \right] \\ \\ q_2' = q_2 + v_2 q_2 \left[a - 2bq_2 - bq_1 - \frac{c_2}{1+\gamma_{21} q_1} \right] \end{cases} \tag{6}$$

3 Gradient dynamics with identical players

We now consider the symmetric case of identical players, i.e. firms which have the same speeds of reaction, the same production costs and the same ability to take advantage from $R\&D$ spillover:

$$v_1 = v_2 = v; \;\; c_1 = c_2 = c; \;\; \gamma_{12} = \gamma_{21} = \gamma \tag{7}$$

Under this assumption, the map (6) remains the same if the variables q_1 and q_2 are interchanged, i.e. after a reflection through the diagonal (line of identical productions)

$$\Delta = \left\{ (q_1, q_2) \in \mathbb{R}^2 | q_1 = q_2 \right\} . \tag{8}$$

This symmetry property implies that the diagonal is mapped into itself, i.e., $T(\Delta) \subseteq \Delta$, which corresponds with the obvious statement that, in a deterministic framework, identical competitors, starting from identical initial conditions, behave identically for each time, and the trajectories embedded into Δ, characterized by $q_1(t) = q_2(t)$ for every t, are governed by the one-dimensional map (restriction of T to the invariant submanifold Δ) $f = T|_\Delta : \Delta \to \Delta$, given by:

$$q' = f(q) = q \left(1 + av - 3vbq - \frac{vc}{1 + \gamma q} \right) \tag{9}$$

In Bischi et al. (1999) this one-dimensional model has been considered as the model of a *representative agent* whose dynamics summarize the common behavior of the two synchronized competitors. The map (9) is a unimodal map whose iteration generates chaotic trajectories for sufficiently high values of the common speed of adjustment v. So, synchronized chaos occurs. Moreover, a trajectory starting out of Δ, i.e. with $q_1(0) \neq q_2(0)$, is said to synchronize if $|q_1(t) - q_2(t)| \to 0$ as $t \to +\infty$.

A question which naturally arises is whether trajectories starting from different initial conditions will synchronize in the long run, so that the asymptotic behavior is governed by the simpler one-dimensional model. This question can be reformulated as follows. Let A_s be an attractor of the one-dimensional map: Is it also an attractor for the two-dimensional map? Of course, an attractor A_s of (9) is stable with respect to perturbations along Δ, so an answer to the question raised above can be given through a study of the stability of A_s with respect to perturbations transverse to S (*transverse stability*).

A second question concerns the behavior of the dynamical system when quasi-identical players are considered, i.e. with small differences among the parameters. Of course, in this case the invariance property of the diagonal Δ is lost, and some different dynamic scenarios may replace synchronization. For example, the attractor A_s embedded inside Δ may be replaced by another attractor close to it, where correlated chaos occurs, or a greater attractor may suddenly appear, surrounding the portion of Δ where A_s was located, inside which endless intermittency occurs. To distinguish between these two dynamic scenarios we need at least two steps:

a) a study of the *transverse Lyapunov exponents* for the symmetric system, by which the "average" local behavior of the trajectories in a neighborhood of the invariant set A_s can be understood (see below);

b) a study of the global dynamic behavior of the two-dimensional map, in order to see, in the case A_s is a Milnor (but not asymptotic) attractor if a greater (two-dimensional) Lyapunov attractor exists around A_s, where transient dynamics and intermittency phenomena are trapped (see Bischi et al., 1998, or Bischi and Gardini, 2000, for details).

In order to compute the transverse Lyapunov exponents for the model (6) with identical players we consider the Jacobian matrix along the invariant diagonal

$$DT(q,q) = \begin{bmatrix} 1+av-5vbq-\frac{vc}{1+\gamma q} & vq\left(\frac{c\gamma}{(1+\gamma q)^2}-b\right) \\ vq\left(\frac{c\gamma}{(1+\gamma q)^2}-b\right) & 1+av-5vbq-\frac{vc}{1+\gamma q} \end{bmatrix}$$

whose eigenvalues are $\lambda_{||} = 1+av-6vbq-vc/\left(1+\gamma q\right)^2$, with eigenvector along Δ, and $\lambda_{\perp} = 1+av-4vbq-vc\left(1-2\gamma q\right)/\left(1+\gamma q\right)^2$, with eigenvector orthogonal to Δ. So, the transverse Lyapunov exponents, computed along a generic trajectory embedded into Δ, are:

$$\Lambda_{\perp} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N} \ln\left|\lambda_{\perp}(q(t))\right|. \tag{10}$$

where $\left\{q(t) = f^t(q(0)), t \geq 0\right\}$ is a trajectory embedded in A_s. Indeed, as in a chaotic attractor A_s infinitely many periodic points are nested (all unstable along Δ), for a chaotic set $A_s \subset \Delta$, infinitely many transverse Lyapunov exponents can be obtained, because if $q(0)$ is a $k-$periodic point then $\Lambda_{\perp} = \ln\left|\lambda_{\perp}^k\right|$ and the $k-$cycle is transversely stable if $\Lambda_{\perp} < 0$, whereas if $q(0)$ belongs to a generic aperiodic trajectory embedded inside the chaotic set A_s then $\Lambda_{\perp}$ is the *natural transverse Lyapunov exponent* $\Lambda_{\perp}^{nat}$. Hence a spectrum of transverse Lyapunov exponents can be defined, $\Lambda_{\perp}^{\min} \leq ... \leq \Lambda_{\perp}^{nat} \leq ... \leq \Lambda_{\perp}^{\max}$ (see e.g. Ashwin et al., 1996, Buescu, 1997) where $\Lambda_{\perp}^{nat}$ expresses a sort of "weighted balance" between the transversely repelling and transversely attracting cycles. If $\Lambda_{\perp}^{\max} < 0$, i.e. all the cycles embedded in A_s are transversely stable, then A_s is asymptotically stable, in the usual Lyapunov sense, for the two-dimensional map T. However, it may occur that some cycles embedded in the chaotic set A_s are transversely unstable, i.e. $\Lambda_{\perp}^{\max} > 0$, while $\Lambda_{\perp}^{nat} < 0$. In this case, A_s is no longer Lyapunov stable, but it continues to be a *Milnor attractor* i.e. it attracts a positive (Lebesgue) measure set of points of the two-dimensional phase space. In the latter case intermittency phenomena can occur, according to the global dynamic properties of the map. In fact, the trajectories that are locally repelled along (or near) the local unstable manifolds of the transversely repelling cycles may be reinjected towards A_s by the global (nonlinear) action of the map, so that the dynamics of such trajectories are characterized by some bursts far from Δ before synchronizing on it. This is a rough explanation of the origin of intermittency phenomena (see Ott and Sommerer, 1994, or Bischi and Gardini, 1998, 2000, for more detailed explanations).

The distinction between asymptotic stability and stability in the weaker Milnor sense may also help to understand the effect of a symmetry breaking given by the introduction of small heterogeneities between the two players, i.e. some parameters' mismatch starting from the homogeneity condition (7). Indeed, as conjectured in Bischi et al.(1999) starting from a condition where chaos synchronization occurs, i.e. $\Lambda_{\perp}^{nat} < 0$, a parameter mismatch which breaks the symmetry (and hence destroys the invariance of Δ) can lead to different situations according to the sign of a $\Lambda_{\perp}^{\max}$, i.e. according to the fact that A_s is an asymptotic attractor or only an attractor in Milnor sense. In particular, if A_s is an asymptotic attractor ($\Lambda_{\perp}^{\max} < 0$) then we expect that after a parameter's mismatch the one-dimensional "synchronized attractor" A_s is replaced by a similar attractor, on which quasi-synchronized dynamics occur, i.e. correlated chaos. Instead, if A_s is attractor only in Milnor sense ($\Lambda_{\perp}^{\max} > 0$, i.e. at least a transversely repelling cycle is embedded inside A_s) then a bigger two-dimensional chaotic area may suddenly appear, which is an attractor in the usual Lyapunov sense, inside which endless on-off intermittency is observed. When the time evolution of the duopoly game is obtained by the iteration of a noninvertible map, as generally occurs in problems of chaos synchronization, the boundary of such chaotic area can be often obtained by the method of *critical curves*, as described in Bischi and Gardini, 1998, 2000, see also Chapter 3 of this book.

Indeed, the map T defined in (6) is a noninvertible map, because given a point $(q_1', q_2') \in \mathbb{R}^2$ its preimages are computed by solving (6) with respect to q_1 and q_2, which gives a sixth degree algebraic system which may have up to six real solutions. For a given set of parameters, the critical curves of the map (6) can be easily obtained numerically. In fact, being the map (6) continuously differentiable, the set LC_{-1} can be obtained numerically as the locus of points (q_1, q_2) for which the Jacobian determinant $\det DT$ vanishes, and the critical curves LC, which separate regions Z_k whose points have different numbers of preimages, are obtained as $LC = T(LC_{-1})$.

4 Players' heterogeneity, correlated chaos, intermittency and synchronization.

In this section we consider the dynamic duopoly game described in section 2 in order to illustrate, through numerical explorations guided by the theoretical background of chaos synchronization and critical curves, some of the topics outlined above, such as uncorrelated, correlated and synchronized chaos, intermittency phenomena and the related concepts of homogeneity and heterogeneity of the two players.

In the numerical explorations given below, the parameters $a = 10$, $b = 0.5$, $v_1 = v_2 = v = 0.32$, $c_1 = c_2 = c = 2$ are fixed, and identical for the two players, and different dynamic situations are obtained by tuning the two spillover parameters γ_{12} and γ_{21}. First of all, we consider the case of homogeneous players (7): in fig. 1a we show the bifurcation diagram, for γ ranging from 0.05 to 0.55, for the restriction (9) of the map T to the invariant diagonal; in fig. 1b is the plot of a numerical computation of the transverse Lyapunov exponent (10) $\Lambda_\perp$ as a function of γ in the same range. Each point of the graph is obtained by iterating the map (starting from an initial condition on the diagonal) 10,000 times to eliminate transient behavior, and then averaging over another 50,000 iterations. Of course, we cannot say that the graph in fig. 1b represents $\Lambda_\perp^{nat}$ because when the parameter γ is inside a periodic window of the bifurcation diagram, the corresponding trajectory is captured by the stable cycle, so the computation of (10) gives the transverse Lyapunov exponent of that cycle. However, the global shape of the graph in fig. 1b can give us a qualitative idea of the values of $\Lambda_\perp^{nat}$ as a function of the parameter γ, because the value $\Lambda_\perp^{nat}$ is well approximated by the value of the transverse Lyapunov exponent computed along a cycle, provided that the period of the cycle is sufficiently high. It is plain that the periodic windows of the cycles of period 5 and 3 and other low-period stable cycles, which are clearly visible in the bifurcation diagram, correspond to peaks of the graph in fig. 1b which cannot be peaks of $\Lambda_\perp^{nat}$. Such a difficulty is due to the fact that γ is not a normal parameter, so that as γ varies also the dynamics along Δ change, as clearly shown in the bifurcation diagram of fig. 1a. However, we can guess that around $\gamma = 0.14$ we have $\Lambda_\perp^{nat} < 0$ and probably also $\Lambda_\perp^{\max} < 0$, so that the chaotic attractor $A_s \subset \Delta$, on which synchronized dynamics occur, is asymptotically stable. Moreover, also around $\gamma = 0.4$ a small neighborhood exists where $\Lambda_\perp^{nat} < 0$, but in this case $\Lambda_\perp^{\max} > 0$, i.e. transversely unstable cycles exist embedded inside chaotic attractor A_s, which is, consequently, only an attractor in the weaker Milnor sense.

Our first numerical simulation is shown in fig. 2, where we consider a situation with a marked heterogeneity (or asymmetry) in the spillover parameters, i.e. only player 2 has the ability to take advantage from the $R\&D$ results of the competitor, being $\gamma_{12} = 0$ and $\gamma_{21} = 0.8$. For this set of parameters the time evolution is chaotic, as clearly appears by looking at the plot, in the phase space (q_1, q_2), of a generic trajectory starting from an initial condition in the white region (the grey region represents the set of initial conditions which generate unbounded trajectories).

The iterated points of such a trajectory erratically fill up a quite big chaotic area, and the corresponding time series $q_1(t)$ and $q_2(t)$ are represented versus time in fig. 2c for $0 \leq t \leq 200$.

This is a typical example of uncorrelated chaos, because no correlation can be evidenced between the two simultaneous choices $q_1(t)$ and $q_2(t)$. This can be better appreciated if we represent the difference $(q_1(t) - q_2(t))$ versus time, as in fig. 2d. It can be seen that, often, $(q_1(t) - q_2(t)) < 0$, i.e. $q_2(t) > q_1(t)$, as a consequence of the greater ability of firm 2 in taking advantage from spillover effects. However, some periods such that $q_1(t) > q_2(t)$ exists, and no rules seem to correlate production variations of producer 1 to production variations of producer 2 and vice-versa.

Let us remark that, as the time evolution of the repeated duopoly game is represented by the iteration of a noninvertible map, the boundary of the chaotic attractor can be obtained by segments of critical curves, as shown in fig.2b (see Mira et al., 1996, Bischi and Gardini, 1998, Puu, 2000). This trapping region (also called absorbing area) gives an upper bound for the oscillations of $q_1(t)$ and $q_2(t)$. A practical procedure to obtain such boundaries makes use of the concept of critical curves and can be outlined as follows: starting from a portion of LC_{-1}, say ω, approximately taken in the region occupied by the area of interest, its images by T of increasing rank are computed until a closed region is obtained. The length of the initial segment must be taken, in general, by a trial and error method, although several suggestions are given in Mira et al., 1996. In order to obtain the boundary of the chaotic area $\mathcal{A}$ shown in fig. 2a, eight images of the generating arc $\omega = \mathcal{A} \cap LC_{-1}$ are sufficient, hence in fig. 2b we have $\partial\mathcal{A} \subset \bigcup_{k=1}^{8} T^k(\omega)$.

In fig. 3 we consider a more homogeneous situation with respect to the firms' ability to take advantage from the spillover effects. Indeed, in the numerical simulations performed in fig. 3 we used the same set of parameters as in fig.2, but more homogenous spillover parameters, namely $\gamma_{12} = 0.35$, $\gamma_{21} = 0.45$. As expected, the chaotic attractor shown in fig.3a is more symmetric with respect to the diagonal, i.e. time periods with $q_2(t) > q_1(t)$ are balanced by more or less equally probable periods characterized by $q_2(t) < q_1(t)$. Moreover, the fact that the chaotic attractor is larger than the one shown in fig.2, suggests that greater differences between the two production choices should be expected at a given time period. However, this statement is quite misleading, because the density of the iterated points inside the chaotic area is mainly concentrated along the diagonal $q_1 = q_2$, i.e. a generic trajectory inside the chaotic area visits much more often the region

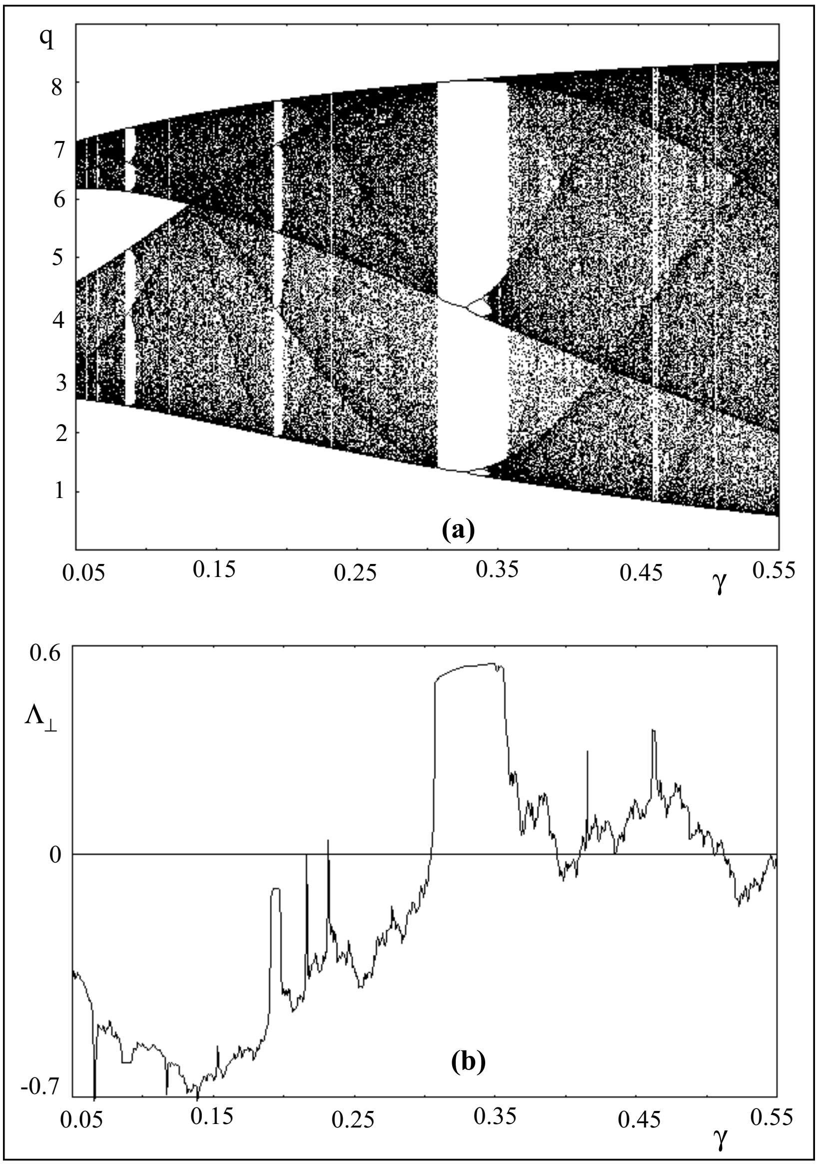

Figure 1: *Symmetric case of identical players, with parameters* $a = 10$*,* $b = 0.5$*,* $v_1 = v_2 = v = 0.32$*,* $c_1 = c_2 = c = 2$*. (a) Bifurcation diagram for the restriction of the map* T *to the invariant diagonal (b) numerical computation of the transverse Lyapunov exponent* $\Lambda_\perp$ *for* γ *ranging from* 0.05 *to* 0.55*.*

around the diagonal with respect to the portions of the chaotic area which are far from it.

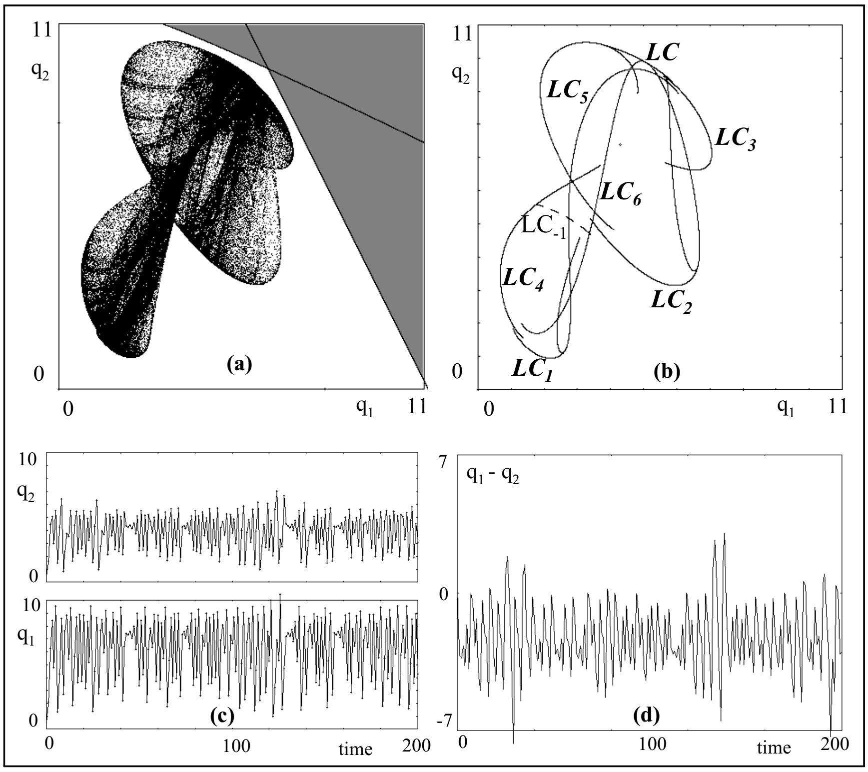

Figure 2: *Numerical simulation obtained with the same set of parameters as in fig. 1 and* $\gamma_{12} = 0$ *and* $\gamma_{21} = 0.8$. *(a) chaotic attractor represented in the phase space* (q_1, q_2)*; (b) boundary of the chaotic attractor obtained by segments of critical curves; (c)* $q_1(t)$ *and* $q_2(t)$ *are represented versus time for* $0 \leq t \leq 200$*; (d) the difference* $(q_1(t) - q_2(t))$ *is represented versus time for the same time range as in (c).*

This property reveals the occurrence of *intermittency* phenomena, as clearly appears from the representation of $(q_1(t) - q_2(t))$ versus time, shown in fig. 3b, where it is evident that several time periods exist at which the difference $(q_1(t) - q_2(t))$ is close to zero, i.e. the production choices of the two firms are similar (both low or both high). However, very frequent “bursts” of asynchronous productions occur along the time evolution. With other words, a typical time evolution is characterized by sequences of time periods, of the order of e.g. 20 or 30 time periods, at which the two firms produce almost the same output, interspersed with clusters of large asynchronous fluctuations.

If homogeneity is further increased, the time intervals of synchronized behavior become longer, i.e. the bursts of asynchronous productions are more rare, and when the two firms become perfectly homogenous, so that the diagonal Δ becomes an invariant submanifold, then chaos synchronization can be observed if $\Lambda_{\perp}^{nat} < 0$. More precisely, if $\Lambda_{\perp}^{nat} < 0$ and also $\Lambda_{\perp}^{\max} < 0$ a fast chaos synchronization occurs, as a consequence of the fact that A_s is an asymptotic attractor, whereas if $\Lambda_{\perp}^{nat} < 0$ and $\Lambda_{\perp}^{\max} > 0$ then synchronization is still obtained in the long run, but after a transient characterized by several bursts. Instead, if $\Lambda_{\perp}^{nat} > 0$ then such bursts never stop (i.e. asymptotic synchronization does not occur).

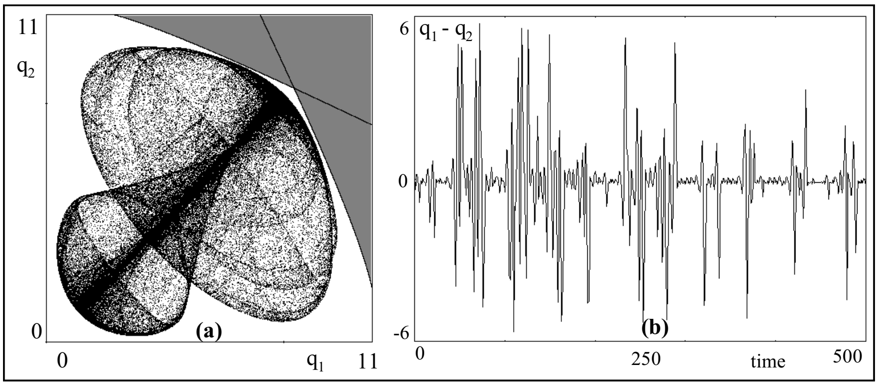

Figure 3: *Numerical simulation obtained with the same set of parameters as in fig.2, but more homogenous spillover parameters,* $\gamma_{12} = 0.35$*,* $\gamma_{21} = 0.45$ *(a) A trajectory is represented in the phase space* (q_1, q_2)*; (b) the difference* $(q_1(t) - q_2(t))$ *is represented versus time for* $0 \leq t \leq 500$.

In the case shown in fig. 4, obtained with $\gamma_{12} = \gamma_{21} = 0.4$, we have $\Lambda_{\perp}^{nat} < 0$ and $\Lambda_{\perp}^{\max} > 0$. In this case, a generic trajectory, starting in the basin of bounded trajectories, synchronizes along the chaotic one-dimensional attractor $A_s \subset \Delta$ after a transient characterized by some asynchronous bursts. The presence of this transient is seen in the phase space (fig. 4a) where several points out of Δ can be seen, and it is more clearly evidenced in the representation, versus time, of $q_1(t)$ and $q_2(t)$ (fig. 4c) and of the difference $(q_1(t) - q_2(t))$ (fig. 4d).

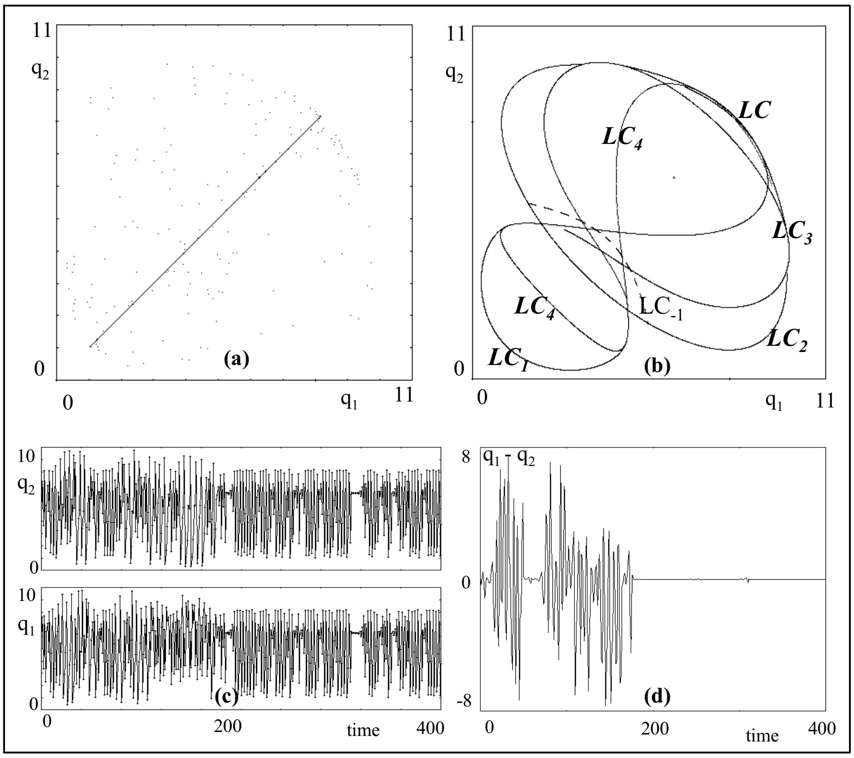

Figure 4: *Numerical simulation obtained with the same set of parameters as in fig. 1 and* $\gamma_{12} = \gamma_{21} = 0.4$ *(a) A typical synchronizing trajectory is represented in the phase space* (q_1, q_2)*; (b) boundary of the chaotic attractor obtained by segments of critical curves; (c)* $q_1(t)$ *and* $q_2(t)$ *are represented versus time for* $0 \leq t \leq 400$*; (d) the difference* $(q_1(t) - q_2(t))$ *is represented versus time for the same time range as in (c).*

These time sequences clearly show that after the early 200 iterations the two players' production choices become perfectly synchronized, even if each of them exhibit a chaotic behavior. Of course, such an asymptotic synchronization occurs because the natural transverse Lyapunov exponent is negative for $\gamma = 0.4$ (see fig. 1b), i.e. the one-dimensional chaotic attractor embedded into the invariant diagonal is an attractor. However, being $\Lambda_{\perp}^{\max} > 0$ it is only an attractor in Milnor sense, i.e. transversely repelling cycles are embedded in the chaotic attractor (even if the attracting ones prevail) and this gives an explanation of the intermittent transient.We observe, however, that the locally repelled trajectories cannot reach the basin of infinity due to the presence of an absorbing area $\mathcal{A}$, bounded by segments of critical curves (shown in fig. 4b). Loosely speaking $\partial\mathcal{A}$ behaves as a bounded vessel for the intermittency phenomena, and the local unstable sets of these transversely repelling cycles embedded inside A_s are folded back (reinjected) by the folding action of the critical curves that form $\partial\mathcal{A}$. Moreover, as remarked above, the presence of such an absorbing area gives us the possibility to define an upper bound for the asynchronous bursts. Instead, the length of the transients and the kind of intermittency which characterizes the trajectories before synchronization, cannot be, in general, forecasted. To explain this statement, in fig.5 we show the versus time representations of the early 400 points of trajectories obtained with the same set of parameters as in fig. 4, but starting from different initial conditions. The trajectory shown in figures 4a,c,d was obtained starting from the initial condition $(q_1(0), q_2(0)) = (5, 6)$, whereas starting from other initial conditions different transient behaviors are observed, characterized by longer or shorter asynchronous bursts, even if synchronization always occurs in the long run. For example, the trajectories used to obtain the $(q_1(t) - q_2(t))$ sequences shown in figures 5a and 5b are obtained starting from $(q_1(0), q_2(0)) = (5.7, 6.2)$ and $(q_1(0), q_2(0)) = (5.5, 6.5)$ respectively. It is interesting to note that sometimes synchronization seems to be reached, but after several time periods other bursts still occur. For example, in fig. 5b, after about 90 time periods the evolution of the system seems to have reached almost complete synchronization. Instead, after 80 time periods of almost perfect synchronization, the trajectory then moves again far away from the diagonal, and the two competitors now act again in a different fashion. Such an intermittent behavior is typical of the convergence to a non-topological Milnor attractor. The pattern of the time series resembles that of a system which is subject to exogenous random shocks, even if the dynamical system that generates such a pattern is completely deterministic.We now consider what happens if we introduce a *small heterogeneity*, due to a small parameter mismatch. For example, if we change one of the spillover

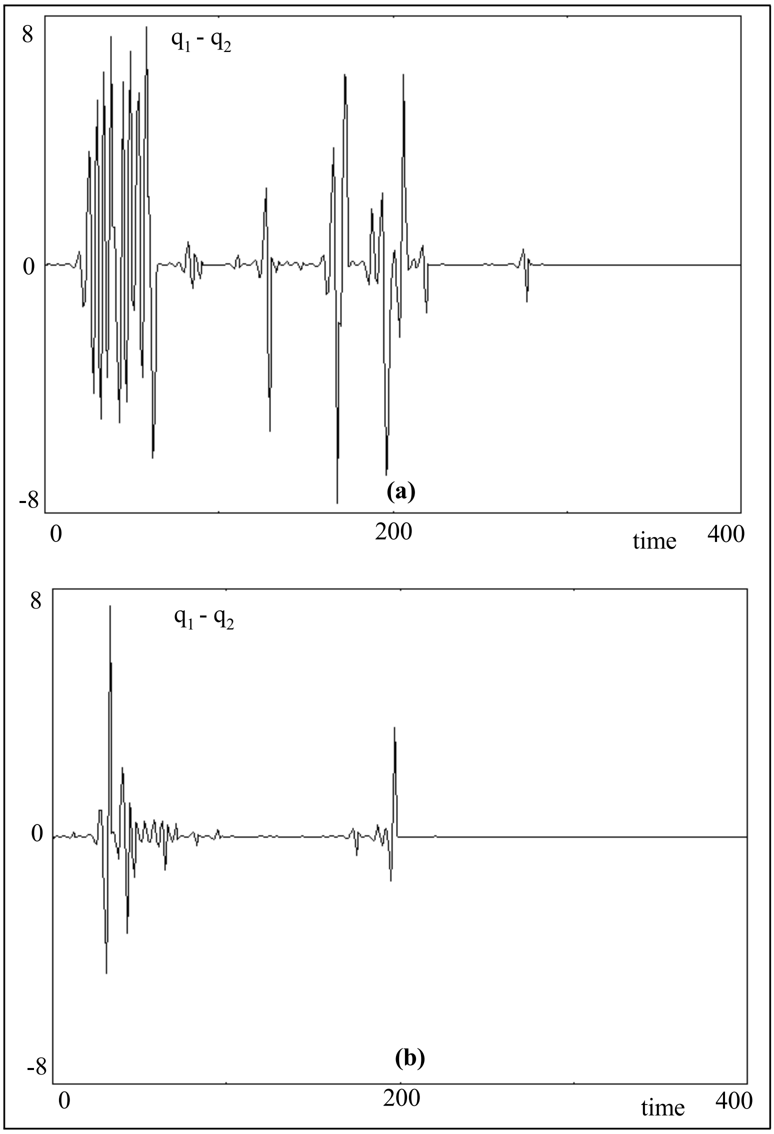

Figure 5: *With the same set of parameters as in fig.4, two sequences* $(q_1(t) - q_2(t))$*, are shown, during the transient with* $0 \leq t \leq 400$*, obtained starting from different initial conditions: (a)* $(q_1(0), q_2(0)) = (5.7, 6.2)$*; (b)* $(q_1(0), q_2(0)) = (5.5, 6.5)$

parameters of just 1% with respect to the homogenous situation of fig.4, i.e.

$$\gamma_{21} = \gamma_{12} + 0.005 \tag{11}$$

where $\gamma_{12} = 0.4$, we obtain the result that bursts never stop, and an endless on-off intermittency is obtained, with bursts whose amplitude is determined by the absorbing area located around the diagonal (fig. 6).

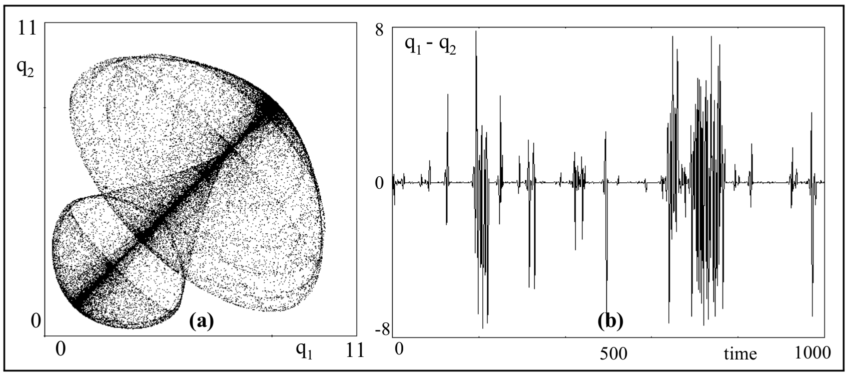

Figure 6: *Symmetry breaking obtained starting from the set of parameters used in fig.4, and with quasi-identical spillover parameters:* $\gamma_{12} = 0.4$*;* $\gamma_{21} = 0.405$*. (a) Chaotic attractor in the phase space; (b) for the same trajectory shwn in (a), the difference* $(q_1(t) - q_2(t))$ *is represented versus time for* $0 \le t \le 1000$*.*

Such a parameters' mismatch causes the destruction of the invariance of Δ, due to the fact that the map is no longer symmetric (this kind of perturbation has been called *symmetry breaking* in Bischi et al. 1999). The fact that the diagonal is no longer an invariant set causes the disappearance of the one-dimensional Milnor attractor A_s along the diagonal, and such a small perturbation may lead to quite different dynamics, since after the symmetry breaking synchronization can no longer occur, and the bursts never stop, and the generic trajectory fills up the absorbing area. However, if the attractor A_s existing along Δ before the parameters' mismatch is a *topological attractor*, that is $\Lambda_{\perp}^{\max} < 0$, then the introduction of small heterogeneities does not have such a disruptive effect. In this case the symmetric model still serves as a good approximation of the behavior of the two firms, and correlated chaos is obtained. Such a situation is shown in fig. 7. In fig.7a identical firms are considered, with the same values of a, b, c, v as in the previous numerical simulations, but with a lower value of γ, $\gamma = 0.15$. As stated before, while commenting fig.1, this set of parameters belongs to a region of the parameter space where the one-dimensional chaotic attractor embedded into the diagonal is an asymptotic attractor, i.e. $\Lambda_{\perp}^{\max} < 0$. We remark that this is not easy to be proved in general, since the cycles included in a chaotic attractor are infinitely many. However, we claim the fact that the natural transverse Lyapunov exponent has a strong negative value, about -0.6, as shown in fig. 1, and the periodic cycles of lower period are transversely stable. This last point constitutes a well known conjecture, based on the fact that if $\Lambda_{nat} < 0$ and $\Lambda_{\max} > 0$, then some low period cycles should be transversely unstable, because if a cycle of high period is transversely unstable, i.e. its transverse Lyapunov exponent is positive, also Λ_{nat} should be positive, see e.g. Maistrenko et al. 1998.

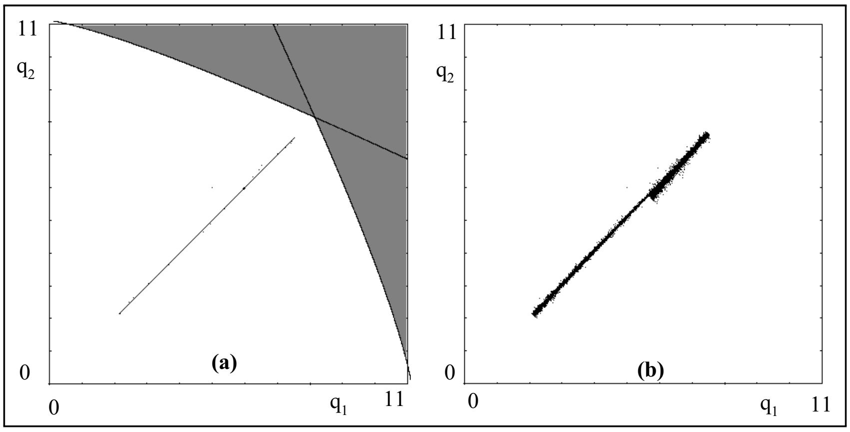

Figure 7: (a) W*ith the same set of parameters as in fig. 1, and* $\gamma_{12} = \gamma_{21} = 0.15$*,* a *typical synchronizing trajectory is represented in the phase space* (q_1, q_2)*; (b) chaotic attractor after a symmetry breaking obtained with* $\gamma_{12} = 0.15$*;* $\gamma_{21} = 0.17$.

The fact that the chaotic set on which synchronized dynamics occur is an asymptotic attractor implies two things: first, the synchronization of trajectories starting out of it (in its basin) is very fast; second, if we introduce a symmetry breaking due to a small parameter mismatch (such as $\gamma_{12} = 0.15$ and $\gamma_{12} = 0.17$) the resulting trajectories are "almost synchronized" (see fig.7b), i.e. small heterogeneities imply small production differences during the time evolution. This is a typical example of correlated chaos. A comparison of figures 6 and 7 leads us to the conjecture that only if the attractor of the symmetric model is a topological attractor, then the introduction of a small heterogeneity would still lead to almost synchronized trajectories. Otherwise, in the case of a Milnor (not asymptotic) attractor endless intermittency phenomena characterize the chaotic evolutions of the two players.

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