Macroeconomic Dynamics, **5**, 2001, 148–179. Printed in the United States of America.

ARTICLES

GLOBAL STABILITY OF INFLATION TARGET POLICIES WITH ADAPTIVE AGENTS

GIAN-ITALO BISCHI
Istituto di Scienze Economiche, University of Urbino

RAMON MARIMON
Madrid Ministry of Science and Technology, European University Institute, Universitat Pompeu Fabra, CEPR, and NBER

We study a dynamic equilibrium model in which agents have adaptive expectations and monetary authorities pursue an inflation target. We show how alternative monetary stabilization policies become more effective when fiscal constraints on deficits are implemented, although they are not binding at the equilibrium target. In particular, we show that the inflation target equilibrium can be locally, or even globally, stable for a large class of adaptive learning schemes. We also compare alternative stabilization policies in terms of their stability properties. Commonly postulated conditional Taylor-type rules tend to be dominated by other rules, such as an unconditional Friedman type.

Keywords: Inflation Targeting, Adaptive Expectations, Stability, Global Dynamics

1. INTRODUCTION

As monetary policy design enters the twenty-first century, the more than half-century-old Friedman (1948) dictum, "rules rather than discretion," seems to define the predominant view among academics and many central bankers. More specifically, a goal of price stability has become the norm and, to this aim, two policy options dominate the debate. One is the need for fiscal constraints (at least constraints on seignorage) as a way to force monetary authorities to pursue price stability. The second is the more or less explicit implementation of an *inflation target rule*. The former is seen as a commitment device whereas the second is seen, once commitment has been granted, as a stabilization policy. We focus on

We want to thank Emilio Barucci for his instrumental role at the start of this joint project. Previous versions of this paper were presented at Stokholm (IIES), Trieste, and in the 1999 SED Meeting in Alghero. We also want to thank seminar participants for their comments. Some of the local stability results reported here were first described by Barucci et al. (1998). Address correspondence to: Bischi Gian Italo, Istituto di Scienze Economiche, University of Urbino, I-61029 Urbino, Italy; e-mail: bischi@econ.uniurb.it.

one (usually neglected) aspect of such policies: the role of expectations formation in the design of policy rules. In particular, we investigate how fiscal constraints can help to achieve price stability (even when there are no credibility problems) and how different inflation target policies can be ranked according to their stability properties in economies in which private agents form their expectations adaptively.

In taking into account the role of fiscal (seignorage) constraints in economies with adaptive learning agents, we follow up on the recent work of Evans et al. (2000). However, in contrast to that work, we consider alternative policies for the central bank (they only consider fixed seignorage financing) and a wider class of (deterministic) learning rules for private agents. In particular, our analysis of alternative stabilization policy rules aims at shedding some light on the discussion of how inflation target policies should be designed. Our analysis of a wide class of learning rules aims at taking into account the fact that, when observed inflation differs from the fixed (trivially stationary) target, private agents are likely to place more weight on recent data. Taking this broader perspective allows us to study how different parameters affect the price stability under alternative rules. For example, we show how fiscal constraints may enhance price stabilization in ways that could not be captured either by rational expectations models or by adaptive learning models with decreasing gain [such as least-squares learning, as studied by Evans et al. (2000b)].

We show how different monetary instruments are equivalent to the use of a single intermediate instrument determining the *ex-post* real return on money. In setting the value of such an instrument (e.g., what would correspond to setting the current interbank rate), the central bank may condition on current information (i.e., deviations from an output target), but also has to forecast the demand for money, which, in our model, reduces to forecast "private agents' expectations." Different inflation target policies differ on how the government conditions on past data and on its beliefs regarding private agents' expectations. The policy that we identify as "optimal" is the one that uses all available information and, therefore, conditions on observed deviations. Such a policy is consistent with rational expectations, in the sense that the monetary authority, assumed to be fully committed to its policy, forecasts that private agents expect that the target will be achieved in the short run. Such a policy is of the form of the inflation target policies proposed by Svensson (1997) and others. However, under our policy, the target is only one of many possible rational expectations equilibria. In fact, as Benhabib et al. (1999) have recently shown, Taylor-rule policies may result in indeterminacy and, in particular, in paths that diverge from the target (when policy is "active"; see Section 2). Along these paths, as often happens with observed series, inflation is autocorrelated and deviations from target cannot be accounted for as simple stochastic innovations.

What should inflation target policy be when deviations from target are not innovations? A first possibility is to think that the optimal policy remains in place. Implicitly, this is the view adopted by the existing literature on Taylor rules [see, e.g., McCallum (1997), Mishkin and Posen (1997), and Clarida et al. (1997a)]. A second possibility is to go back to Friedman's recommendation and postulate an unconditional policy consistent with the long-run objectives. Finally, the central bank can try to "forecast how private agents forecast." This, of course, is not a closed—or well-defined—possibility and it raises a number of interesting issues. We find that, if the central bank succeeds at such forecasting game, then, as in the rational expectations case, the target should prevail in the short run and the best forecast of private agents' beliefs is the same target (see Section 2). However, central banks may not be that farsighted; they may simply postulate a certain amount of inertia on how private agents forecast. As a canonical example, we postulate a simple (fixed) adaptive rule as a conditional inflation target rule. Studying and comparing the performance of the three rules, in an economy in which agents' expectations are adaptive, is the central theme of this paper. For all three rules, there is, of course, a *misspecification* problem: The central bank does not implement a rule that is fully consistent with how private agents learn, nor do private agents postulate learning rules fully consistent with the actual law of motion implied by the central bank policy.[1] Nevertheless, we show that for a wide range of parameters the inflation target is a stable equilibrium of the corresponding adaptive process.

We find that, when policy is "active," under learning the inflation target is more stable when the stationary rational expectations equilibrium is a (locally unique) *determinate* equilibrium. In this respect our work reinforces and complements the contemporaneous work of Bullard and Mitra (1999), who also study the E-stability of inflation target policies.[2] We study a somewhat narrower set of policies than they do, and we provide a full characterization of stability results, not only by considering local stability of a wide class of constant gain rules, but also by considering associated global stability properties.

It is in the *global analysis* that this paper breaks more novel ground: first, by showing how fiscal constraints may affect the global stability of the target, and, second, by making use of some new results on *global bifurcations*.[3]

Our exercise provides a better understanding of how three basic parameters interact with and affect price stability. Two are, to a large extent, policy parameters: (1) how low the inflation target is set in relation to the inflation level at which there is no demand for money; and (2) the tightness of the fiscal constraint. The remaining parameter is endogenous to agents' learning process: (3) how much weight they place on previous-period observed information (i.e., the size of the gain or tracking parameter). In addition, we show how the three, seemingly similar, policies can result in quite different dynamics. As a result, we can provide local and global stability rankings. We show that, in these stability rankings, what appears to be the optimal policy on other grounds actually tends to be dominated by the alternative policies. In particular, Friedman's unconditional rule performs remarkably well as stabilization policy. This may provide a rationale for the observed fact [see Clarida et al. (1997b)] that central banks appear to react much less aggressively to incoming information than standard analyses of Taylor rules suggest.

The paper is divided into two important sections. Section 2 develops the model while Section 3, the bulk of the paper, contains the local and global stability results.

2. INFLATION TARGET POLICIES

In this section, we first consider a general monetary model of inflation targeting. In the next subsection, we provide a specific cash-in-advance interpretation of the model.

The consolidated intertemporal government budget constraint takes the form

$$M_{t+1}^s + B_{t+1}^s = p_t g_t - p_t \tau_t + M_t^s + B_t^s I_t, \quad (1)$$

where g_t is government expenditures; $p_t \tau_t$ is tax revenues; M_{t+1}^s and B_{t+1}^s are the supplies of money and government bonds, respectively, at the end of period t; and I_t is the nominal rate of return on bonds (contracted in period $t-1$ at that rate). It is assumed that the sequence of intertemporal budget constraints satisfies a transversality condition and, therefore, that the government satisfies its present-value budget constraint. It is convenient to express (1) as

$$M_{t+1}^s - M_t^s = p_t d_t,$$

where

$$d_t = g_t - \tau_t + \frac{B_t^s}{p_t} I_t - \frac{B_{t+1}^s}{p_t} \equiv g_t - \tau_t + b_t^s R_t^b - b_{t+1}^s. \quad (2)$$

In the last equality, debts and rates of return are specified in real terms. In particular, R_t^b is the *realized* real rate of return on bonds. With this compact formulation, d_t can be identified as the instrument used to implement the target, although, in practice, changes on the right-hand side of (2) correspond to open-market operations, interbank rate interventions, etc. Although it may be important for policy design, in our model the exact form through which d_t changes is not relevant for the dynamic effects of the policy.[4]

The money-market equilibrium is simply given by $M_{t+1}^d = M_{t+1}^s$. Denoting real balances by $m_{t+1}^d = M_{t+1}^d / p_t$ and gross inflation by $\pi_{t+1} = p_{t+1}/p_t$ the intertemporal equilibrium condition reduces to

$$m_{t+1}^d = \frac{m_t^d}{\pi_t} + d_t. \quad (3)$$

We consider economies in which the demand for real balances takes the form

$$m_{t+1}^d = m^d\left(\pi_{t+1}^e\right),$$

where π_{t+1}^e is the agents' expected inflation.

2.1. Introducing Inflation Target Policies

An *inflation target policy* specifies a desired level of inflation together with a level of d_t as a function of the available information in period t. We consider recursive policies. More specifically, consistent with the intertemporal equilibrium map (3), we consider policies of the form $d_t = d^{(P)}(m_t^d)$. Furthermore, if demand functions are known, these policies take the form $d_t = d^P(\pi_t^e)$. It follows that realized inflation is given by

$$\pi_t = \phi^{(P)}\left(\pi_t^e, \pi_{t+1}^e\right) \equiv \frac{m^d\left(\pi_t^e\right)}{m^d\left(\pi_{t+1}^e\right) - d^P\left(\pi_t^e\right)}. \tag{4}$$

Notice that, with the assumption that private agents have rational expectations, equation (4) reduces to $\pi_t = \phi^{(P)}(\pi_t, \pi_{t+1})$. That is, we can derive an equilibrium map, $\psi^{(P)}$, such that rational expectations equilibrium paths are those satisfying

$$\pi_t = \psi^{(P)}(\pi_{t+1}). \tag{5}$$

Using equation (3), inflation target policies take the form

$$d^{(P)}\left(m_t^d\right) = E_t^g m_{t+1}^d - m_t^d/\pi^*, \tag{6}$$

where $E_t^g m_{t+1}^d$ denotes the (government) expected demand for real balances conditional on the available information at the beginning of period t. That is, the resulting policy is *conditional* on past and expected future real balances.

To see the sense in which these policies are of the type of those proposed by Taylor (1993) and Svensson (1997), and estimated by Clarida et al. (1997a,b), let $R^* = 1/\pi^*$, $m^* = m^d(\pi^*)$, and $d^* = [(\pi^* - 1)/\pi^*]m^*$. Then, equation (6) takes the form

$$d^{(P)}\left(m_t^d\right) = d^* + \left(E_t^g m_{t+1}^d - m^*\right) + R^*\left(m^* - m_t^d\right). \tag{7}$$

That is, the central bank's optimal reaction is to increase the money supply if either the expected demand for real balances is above the target or the realized one is below the target, so as to adapt to any expected deviation from target or adjust for any experienced deviation from target. More specifically, in the special (linear) case $m^d(\pi_{t+1}^e) = b - \pi_{t+1}^e$, equation (7) can be written as

$$d^{(P)}\left(\pi_t^e\right) = d^* + \left(\pi^* - E_t^g \pi_{t+1}^e\right) + R^*\left(\pi_t^e - \pi^*\right),$$

showing that the government reaction should be to increase the money supply above the target level if it expects the private sector's forecasted inflation to be below the target or if past expectations of inflation were too high. Notice that, as long as higher expected inflation results in lower output, a positive deviation $[\pi_t^e - \pi^*]$ corresponds to a realized value of output below the target. In other words, under $d^{(P)}$ rules, monetary authorities adapt to forecasted money demands and to realized output gaps.

However, as can be seen from equation (6), with such a feedback rule the rate of return on money ($R_t = 1/\pi_t$) satisfies $R_t - R^* = (m_{t+1}^d - E_t^g m_{t+1}^d)/m_t$. In other words, realized inflation differs from target inflation only if the government miscalculates the private sector's demands. In fact, when the government knows the money demand function, the target is achieved—immediately—as long as the government accurately forecasts the private sector's expectations of inflation. This also means that the forecast consistent with rational expectations is $E_t^g \pi_{t+1} = \pi^*$, which results in the optimal target policy

$$d_t^O = d^O\left(m_t^d\right) \equiv m^d(\pi^*) - R^* m_t^d = d^* + R^*\left(m^* - m_t^d\right),$$

where the money supply is constant except for deviations of realized real balances from their target level (or output deviations, in the constant-velocity case). Furthermore, consistency with rational expectations also implies that $E_{t-1}[d^O(m_t^d)] = d^*$. In other words, the expected money growth must be the constant growth implied by the desired inflation target. The constant growth of money rule d^* is, in fact, the rule proposed by Milton Friedman, who explicitly advocated "rules rather than discretion" and also advocated designing short-run rules in terms of long-term objectives and not in terms of discretionary reactions to economic fluctuations [e.g., Friedman (1948)]. For this reason, we refer to the constant policy d^* as the Friedman policy d^F, given by

$$d_t^F = d^F \equiv m^d(\pi^*) - R^* m^d(\pi^*) = d^*.$$

Such a policy is not optimal in the sense that it does not make use of all available information as the conditional policy $d^O(m_t^d)$ does. But, as we have seen, the conditional policy should only react to unexpected deviations of m_t^d. In particular, if the government has been following such a policy and private agents have rational expectations, then it should be the case that $m^d(\pi_t^e) = m^d(\pi^*) = m^*$ and, if there are no other sources of uncertainty, this implies that $d^O(m_t^d) = d^*$.

2.1.1. Indeterminacy, policy activism, and consistency with rational expectations. Under both policies, O and F, there is, in general, a continuum of rational expectations equilibria (REE) and two stationary rational expectations equilibria (SREE), that is, two fixed points of $\psi^{(P)}$. In particular, under the O policy the two SREE are π^* and $b/(1+\pi^*)$, while under the F policy the two SREE are π^* and b/π^*. Notice that F corresponds to the standard hyperinflation model of a constant deficit financed through seignorage, and the two SREE reflect the existence of two inflation-tax levels raising the same revenues (i.e., a version of the Laffer curve). Furthermore, π^* should be the lower steady-state inflation rate, otherwise the target policy cannot be optimal. In fact, these models have a Laer curve, and the two SREE generate the same revenues, but higher inflation is associated with lower savings and lower welfare. For the policy F, this requires $b > \pi^{*^2}$. Similarly, π^* is the lower SREE inflation under the policy O if and only if $b > \pi^*(1+\pi^*)$, a more stringent condition than under F.

It is convenient to consider the inverse map of equation (5), say $\varphi \equiv \psi^{-1}$. In fact, provided that $\varphi^{(P)\prime}(\pi) > 0$, if $\bar{\pi}$ is a SREE and $\varphi^{(P)\prime}(\bar{\pi}) > 1$, then the corresponding target policy is called *active* and the corresponding SREE is *determinate*, whereas if $\varphi^{(P)\prime}(\bar{\pi}) < 1$, then the policy is called *passive* and there is *indeterminacy*, in the sense that a continuum of REE has a long-run inflation of $\bar{\pi}$, that is a continuum of solutions of (5) with $\pi_t \to \bar{\pi}$ [see, e.g., Leeper (1991) or Benhabib et al. (1999)].

It is easy to see that, under any of the two policies, we have $\varphi^{(P)\prime}(\pi) > 0$ and, provided that π^* is the lower-inflation SREE, $\varphi^{(P)\prime}(\pi^*) > 1$. The high SREE is, in contrast, *indeterminate* and, correspondingly, the O policy is *passive* at $b/(1+\pi^*)$ while the F policy is *passive* at b/π^*. However, at high-inflation SREE, as well as along the REE hyperinflationary paths approaching them, the government should realize that its target policy is not being achieved and, therefore, the rationality of the policy should be questioned. In other words, these paths are not fully consistent with rational expectations on the part of the government.

What should the government do if it observes $m_t^d \neq m^*$?[5] In the following, we explore several plausible options, but we do not provide a complete answer to this question. We first consider the case in which the government simply follows the optimal policy O even when output (i.e., real balance) deviations are autocorrelated. However, Friedman's implicit criticism of conditional policies as possibly being too "overreactive" may apply to this case and, therefore, we also consider the unconditional policy F.

2.1.2. Policies based on forecasts of private agents' forecasts. Facing deviations from rational expectations, the government may want to infer how private agents forecast inflation. As we have said, if the government succeeds at "learning how private agents learn," then the resulting inflation must be the target, but then private agents' forecasts (forecasting rules) may be affected by the corresponding shift to the announced target. This problem is similar to that of using "good predictors" of inflation as a guide for monetary policy. As Woodford (1994) has argued, such "nonstandard indicators" suffer from the Lucas critique problem: As much as they are "good predictors," if they are used in the design of policy, then they should cease to be good indicators.

Let us assume that government's ability to accurately predict how private agents forecast is limited. In particular, since a broad class of learning rules show some degree of *inertia*,[6] a benchmark option to consider is that the government postulates that inertia persists; that is, $E_t^g m_{t+1}^d = m_t^d$.

Inertia in private agents' forecasts results in autocorrelated deviations from target. In particular, notice that if agents update their estimates of inflation according to an adaptive rule of the form

$$\pi_{t+1}^e = \pi_t^e + \alpha_t\left(\pi_{t-1} - \pi_t^e\right), \tag{8}$$

with $\alpha_t \in (0,1), \alpha_t \approx 0$ (or $\alpha_t \searrow 0$ as is the case when they use standard OLS techniques), then the government is almost right (in the limit) in postulating that inertia persists, although they could choose better predictors of private agents' forecasts—namely, the same rule (8)! Postulating that (one-period) inertia persists, we get an inflation target policy of the form

$$d^I\left(m_t^d\right) \equiv m_t^d - R^* m_t^d = \left(\frac{\pi^* - 1}{\pi^*}\right) m_t^d = d^* + (R^* - 1)\left(m^* - m_t^d\right).$$

For $\pi^* > 1$ (i.e., $R^* < 1$), whenever real balances (output) are below the target, this policy recommends reducing the money supply below the target because it adapts to the expected low money demand. Such a recommendation is the opposite of the recommendation under the optimal policy d^O, which only takes into account the current period downturn, but expects demand to be at the target level the following period.

The REE under the I policy is characterized by the $\psi^{(I)}$ map (5): There is only one SREE corresponding to the target π^* and there is a continuum of REE paths with the property that, in the long run, money loses its value. Notice that when $\pi^* = 1$, I is equivalent to F. Of course, along nonstationary REE paths, there is an element of irrationality on the part of the government because its *inertia* assumption is not satisfied.

In summary, we consider the three alternative stabilization policies, O, F, and I. However, it should be clear from our discussion that, within our class of models, other policies may be considered, reflecting central bank perceptions of how the private sector will forecast inflation, given its announced policy. Nevertheless, a careful stability analysis of our benchmark policies may help us to understand how policies should be modified in order to enhance stability properties. In particular, we are interested in contrasting the performance of the so-called optimal policy with the other two policies. To do this, in what follows, we describe the dynamics of the model with adaptive private agents and a linear demand

$$m^d\left(\pi^e_{t+1}\right) = b - \pi^e_{t+1}. \tag{9}$$

As we will see, although the design of an optimal fiscal and monetary mix, under rational expectations, does not place any restriction on $b - \pi^*$, other than $b - \pi^* > 0$, the saturation value b may determine the success of the inflation target π^*. The fact that the stability of the inflation target may be affected by the point of *currency collapse*, even if a collapse never occurs, is a general feature of our results. Our linear demand formulation simplifies the corresponding analysis.

2.2. Introducing Fiscal Constraints

Nonnegative prices require $m^d_{t+1} - d_t \geq 0$. Here, we follow Evans et al. (2000) in considering constrained policies that satisfy $m^d_{t+1} - d^{(P)}(m^d_t) \geq 0$. In particular, we consider a constraint on the ratio of seignorage to (private) GDP,[7]

$$\frac{d_t}{y_t - g} \leq \kappa. \tag{10}$$

By equation (3),

$$\frac{d_t}{m^d_{t+1}} = 1 - \frac{m^d_t \pi^{-1}_t}{m^d_{t+1}} = 1 - \frac{c_t}{m^d_{t+1}} = 1 - \frac{y_t - g}{m^d_{t+1}} \leq 1 - \frac{1}{\kappa}\frac{d_t}{m^d_{t+1}},$$

that is,

$$\frac{d_t}{m^d_{t+1}} \leq \frac{\kappa}{1+\kappa} \equiv \lambda.$$

Notice that if, instead, the constraint is a deficit to (private) GDP constraint of the form

$$\frac{g + (R_t - 1)b_t - \tau n_t}{y_t - g} = \frac{d_t + (b_{t+1} - b_t)}{y_t - g} \leq \kappa,$$

then $d_t / m^d_{t+1} \leq \lambda$ as long as $(b_{t+1} - b_t) \geq 0$, as in a (targeted) steady-state budget. We abstract from the exact nature of the constraint, but we assume that *ex-post* policies satisfy

$$\hat{d}^{(P)}\left(m^d_t, m^d_{t+1}\right) = \min\left\{d^{(P)}\left(m^d_t\right), \lambda m^d_{t+1}\right\} \qquad \textbf{(11)}$$

for some policy parameter λ. In particular, we are interested in studying how the stability of inflation target policies is affected by such a fiscal constraint parameter. Notice, however, that such a constraint does not mitigate (and may actually worsen) the indeterminacy problem of REE. More specifically, with full commitment and rational expectations, there is no rationale for imposing constraints of this type [see Evans et al. (2000b)]. Of course, with limited commitment and rational expectations, there may be a stabilizing role for fiscal constraints [see, e.g., Giovannetti et al. (2000)]. As in Evans et al. (2000b), this paper shows that, with full commitment and adaptive expectations, there is also a stabilizing role for fiscal constraints.

2.2.1. Precautionary savings. Unfortunately, the λ constraint is not enough to avoid currency collapses (i.e., it guarantees $1/p_t \geq 0$ but not $1/p_t > 0$). One may consider policies explicitly aimed at avoiding such extreme events, however. As long as there is some minimum (residual) demand for money, currency collapses cannot occur. Here, as in Evans et al. (2000b), we assume the existence of an $\epsilon > 0$, such that the representative agent's demand for real balances satisfies $m^d(\pi^e) = \max\{b = \pi^e, \epsilon\}$. As we will see, such an assumption only plays a role in our global analysis in the sense that, without it, the rare event of a currency collapse cannot be dismissed.[8]

2.3. Introducing Adaptive Expectations

We consider that private agents predict inflation as a constant. In other words, we follow Cagan (1956) in considering a general class of learning rules in which agents condition data focusing on a *minimal state variable* (MSV) solution. In particular,

$$\pi^e_{t+1} = \pi^e_t + \alpha\left(\pi_{t-1} - \pi^e_t\right), \qquad \textbf{(12)}$$

where previous-period, and not-current period, inflation is used to update forecasts. This formulation is consistent with the underlying informational structure of the model and with agents not overreacting to current events (i.e., having some behavioral inertia).[9] We also assume that the weight on realized inflation, α_t, is exogenous. Nevertheless, experimental evidence shows that the parameter α_t tends to increase when observed paths are nonstationary. In fact, in a nonstationary environment, to use a tracking procedure (i.e., keeping α_t constant) is a better learning rule than to use a stochastic approximation procedure (with $\alpha_t \searrow 0$), such as standard least-squares procedures. Since, on the one hand, the asymptotic analysis of the stochastic approximation case has been done by Evans et al. (2000) (only for the F policy) and, on the other hand, we want to allow for a wide range of tracking procedures, we should consider the whole class $\alpha_t = \alpha \in (0, 1)$.[10]

3. DYNAMIC MODEL WITH ADAPTIVE EXPECTATIONS

In this section we provide the main stability results. We start by considering some general properties of the adaptive expectations process under a general inflation target policy $d^{(P)}$. Given such a policy, substituting (12) into the intertemporal equilibrium condition (4), we obtain a second order difference equation in expected inflation rates:

$$\pi^e_{t+1} = (1-\alpha)\pi^e_t + \alpha\phi^{(P)}\left(\pi^e_{t-1}, \pi^e_t\right),$$

which, under our assumptions, takes the form

$$\pi^e_{t+1} = (1-\alpha)\pi^e_t + \alpha \frac{\max\left\{b - \pi^e_{t-1}, \epsilon\right\}}{\max\left\{b - \pi^e_t, \epsilon\right\} - \hat{d}^{(P)}\left(\pi^e_{t-1}, \pi^e_t\right)}. \tag{13}$$

As usual, a second order difference equation is more easily studied by writing it as an equivalent system of two first order difference equations. In order to do this, let $x_t = \pi^e_{t-1}$ and $y_t = \pi^e_t$. Then equation (13) can be written in the form $(x_{t+1}, y_{t+1}) = T^{(P)}(x_t, y_t)$, where $T^{(P)}$ is the two-dimensional map

$$T^{(P)}: \begin{cases} x_{t+1} = y_t \\ y_{t+1} = (1-\alpha)y_t + \alpha \dfrac{m(x_t)}{m(y_t) - \hat{d}^{(P)}(x_t, y_t)}. \end{cases} \tag{14}$$

with $m(z) = \max\{\epsilon, b - z\}$, $\hat{d}^{(P)}$ given by equation (11), $b \geq \pi^*$, $\alpha \in (0, 1)$, $\lambda \in [0, 1]$ and $\epsilon > 0$ is a small parameter.

3.1. Some General Properties of the Models

The map (14), the iteration of which defines the time evolution of the system in the space of expected inflation, is a nonlinear piecewise continuous map on $\mathbf{R}^2_+$. However, its behavior changes along the lines $x = b_\epsilon$ and $y = b_\epsilon$, where $b_\epsilon \equiv b - \epsilon$.

Correspondingly, we can subdivide $\mathbf{R}_+^2$ into the following four regions:

$$R(\mathrm{I}) = \{(x, y) \mid 0 \le x < b_\epsilon, 0 \le y < b_\epsilon\},$$

$$R(\mathrm{II}) = \{(x, y) \mid x \ge b_\epsilon, 0 \le y < b_\epsilon\},$$

$$R(\mathrm{III}) = \{(x, y) \mid x > b_\epsilon, y > b_\epsilon\},$$

$$R(\mathrm{IV}) = \{(x, y) \mid 0 \le x < b_\epsilon, y > b_\epsilon\}.$$

Notice that, by assumption, $E \equiv (\pi^*, \pi^*)$ is in region $R(\mathrm{I})$ and outside this region there is only a residual, ϵ, demand for real balances [i.e., for m_{t-1}^d in $R(\mathrm{II})$ or m_t^d in $R(\mathrm{III})$ or both m_{t-1}^d and m_t^d in $R(\mathrm{IV})$]. Therefore, we are particularly interested in the behavior of (14) in $R(\mathrm{I})$. The following result shows that, provided the fiscal constraint is not too loose, the regions, $R(\mathrm{II})$, $R(\mathrm{III})$, and $R(\mathrm{IV})$ are transition regions.

LEMMA 1. *Assume* $\lambda < 1 - 1/b_\epsilon$. *Then, for any initial condition* $(x_0, y_0) \in \mathbf{R}_+^2$, *a process* $\{x_t, y_t\}$ *generated by* (14) *visits* $R(\mathrm{I})$ *infinitely often. In particular, either* $R(\mathrm{I})$ *is an absorbing region for* $\{x_t, y_t\}$ *or, eventually,* $\{x_t, y_t\}$ *follows a path through the regions* $R(\mathrm{I}) \to R(\mathrm{IV}) \to R(\mathrm{III}) \to R(\mathrm{II}) \to R(\mathrm{I})$.

Proof. See Appendix A.

Notice that, when the fiscal constraint is binding, the map (14) reduces to the sub map

$$T_\lambda\colon \begin{cases} x_{t+1} = y_t, \\ y_{t+1} = (1-\alpha) y_t + \dfrac{\alpha}{1-\lambda} \dfrac{m(x_t)}{m(y_t)} \end{cases} \tag{15}$$

which has a unique fixed point at $E_\lambda \equiv [1/(1-\lambda), 1/(1-\lambda)]$. The assumption of Lemma 1 implies that E_λ is in region $R(\mathrm{I})$; that is, the condition $\lambda < 1 - 1/b_\epsilon$ guarantees that the map T_λ, active when there is only a residual demand for real balances, does not allow the process to be absorbed outside region (I). It does not guarantee, however, that the process eventually remains in region (I) because, there may be cycling behavior along the four regions. In fact, Lemma 1 allows the existence of cyclic dynamics, which may be periodic or not, that move "clockwise" visiting the four regions in the order $R(\mathrm{I}) \to R(\mathrm{IV}) \to R(\mathrm{III}) \to R(\mathrm{II}) \to R(\mathrm{I})$, with fast transitions (just one time period) for $R(\mathrm{II}) \to R(\mathrm{I})$ and $R(\mathrm{IV}) \to R(\mathrm{III})$ or $R(\mathrm{II})$, and with slower transitions for $R(\mathrm{III}) \to R(\mathrm{II})$ and $R(\mathrm{I}) \to R(\mathrm{IV})$. The existence of this type of large-amplitude oscillation is strictly related to the value of the parameter ϵ, in the sense that the amplitude of the oscillations is inversely proportional to ϵ. We return to this issue when we analyze the global dynamics of the models.

3.2. Local Stability of π^*

We first study the asymptotic stability (i.e., whether $\pi_t^e \to \pi^*$) of paths with initial conditions in a neighborhood of the target (i.e., $\|(\pi_0^e, \pi_1^e) - E\| < \rho$ for some $\rho > 0$. Such local stability analysis of (14) around π^* is relatively straightforward. It requires the characterization of the map (14) in region (I), possibly establishing conditions guaranteeing that the fiscal constraint is not binding for expectations close to the target and, finally, studying the eigenvalues of the corresponding Jacobian. We first briefly discuss the three policies and then compare them in terms of their local stability properties. For all of the policies, in the subregion of R(I) where the fiscal constraint is not binding, the map (14) reduces to the sub map $T_*^{(P)}$ whose fixed points are the same than those of the rational expectations map (5). For convenience, policies are discussed in reverse order with respect to their appearance; that is, I, F, and O. In what follows, given an inflation target π^*, we let $\Omega \equiv \{(b, \alpha) \mid b > \pi^*, \alpha \in (0, 1)\}$.

3.2.1. Policy I. The restriction of $T^{(I)}$ to region R(I) is given by

$$T^{(I)}|_{R(\mathrm{I})}\colon \begin{cases} x' = y \\ y' = (1-\alpha)y + \alpha \dfrac{m(x)}{m(y) - \min\left\{\dfrac{\pi^* - 1}{\pi^*} m(x), \lambda m(y)\right\}}. \end{cases} \tag{16}$$

The line s of equation $y = s(x) \equiv [(\pi^* - 1)/\lambda\pi^*]x + [1 - (\pi^* - 1)/\lambda\pi^*]b_\epsilon$ separates the R(I) into two subregions:

$$R(\mathrm{I}_A) = \{(x, y) \in R(\mathrm{I}) \mid y < s(x)\} \quad \text{and} \quad R(I_B) = \{(x, y) \in R(\mathrm{I}) \mid y > s(x)\}.$$

The map $T^{(I)}|_{R(I)}$ can be written in the equivalent form

$$T^{(I)}|_{R(I)}\colon \begin{cases} T^{(I)}|_{R(\mathrm{I}_B)} = T_\lambda & \text{if} \quad (x_t, y_t) \in R(\mathrm{I}_B) \\ T^{(I)}|_{R(\mathrm{I}_A)} = T_*^{(I)}\colon \begin{cases} x_{t+1} = y_t \\ y_{t+1} = (1-\alpha)y + \alpha\pi^* \dfrac{m(x_t)}{\pi^* m(y_t) - (\pi^* - 1)m(x_t)} \end{cases} & \text{if} \quad (x_t, y_t) \in R(\mathrm{I}_A) \end{cases}$$

The map $T_*^{(\mathrm{I})}$ has the unique fixed point $E^* = (\pi^*, \pi^*)$, which is also a fixed point of $T^{(\mathrm{I})}$ provided that $E^* \in (\mathrm{I}_A)$, that is, if the condition $1 - 1/\pi^* < \lambda \leq 1$ holds. In other words, the target equilibrium is a steady state of the model if the fiscal constraint on seignorage is not too tight. With such a condition, the fixed point of the map T_λ, $E_\lambda = [1/(1-\lambda), 1/(1-\lambda)]$, is not a fixed point of $T^{(\mathrm{I})}$; that is, $T^{(\mathrm{I})}(E_\lambda) = T_*^{(\mathrm{I})}(E_\lambda) \neq E_\lambda$.

We will restrict the fiscal constraint to satisfy

$$\lambda \in \Delta^* \equiv \left(1 - 1/\pi^*, 1 - 1/b_\epsilon\right). \tag{17}$$

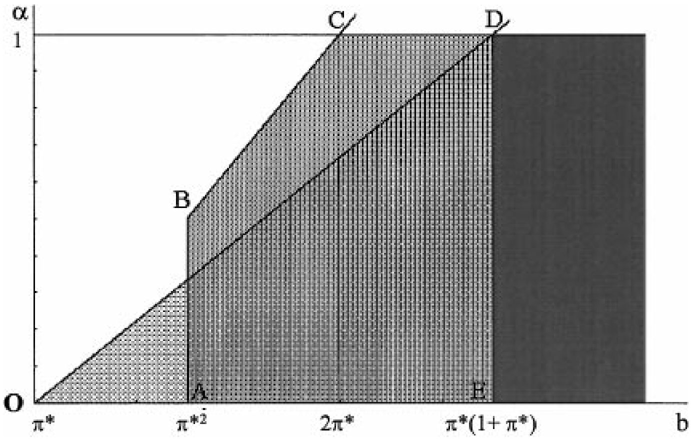

FIGURE 1. A superposition, in the parameter space Ω, of the regions of local stability of the target equilibrium E^* under the three different policies, with $\pi^* = 1.5$. The region with left boundary OD is the stability region of E^* under I, the one bounded by ABC refers to policy F, and the one with left boundary ED refers to policy O.

As long as condition (17) is satisfied, λ does not affect the local stability properties of E^*. Indeed, let $\Omega_s^{\mathrm{I}} = \{(b, \alpha) \in \Omega | b > \pi^*(1 + \alpha\pi^*)\}$ (see the region below the line OD in Figure 1). The following result is proved in Appendix B.1.

LEMMA 2. *Assume* $\lambda \in \Delta^*$ *[i.e., condition* (17)*]. If* $(b, \alpha) \in \Omega_s^1$*, then* E^* *is locally stable with policy I.*

In the complementary region, $\Omega_u^{\mathrm{I}} = \{(b, \alpha) \mid b - \pi^{*^2}\alpha - \pi^* < 0\} E^*$ is unstable. In particular, following the arguments given in the Appendix B.1, if the point (b, α) crosses the line

$$b = b_h^{(\mathrm{I})}(\alpha) = (1 + \alpha\pi^*)\pi^*, \tag{18}$$

passing from Ω_u^{I} to Ω_s^{I}, a *subcritical Neimark–Hopf bifurcation* occurs which, at least for $(b, \alpha) \in \Omega_s^{\mathrm{I}}$ close to the bifurcation curve (18), creates a repelling closed invariant curve Γ around the stable fixed point E^*, which constitutes the boundary of the basin of attraction $\mathcal{B}(E^*)$ of E^*. More precisely, for $b > b_h^{(\mathrm{I})}(\alpha)$, a range of values of b exists such that E^* is locally asymptotically stable (a stable focus), with a basin of attraction bounded by a closed curve whose radius is proportional to $\sqrt{b - b_h^{(\mathrm{I})}(\alpha)}$. Analogously, for a fixed value of the parameter $b \in (\pi^*, \pi^*(1 + \pi^*))$, the subcritical Neimark–Hopf bifurcation occurs at

$$\alpha = \alpha_h^{(\mathrm{I})}(b) = \frac{b - \pi^*}{\pi^{*^2}} \tag{19}$$

and E^* is stable for $\alpha < \alpha_h^{(\mathrm{I})}$ with the basin of attraction bounded, at least for values of α close to $\alpha_h^{(\mathrm{I})}$, by a closed curve whose radius increases proportionally to $\sqrt{\alpha_h^{(\mathrm{I})}(b) - \alpha}$.

3.2.2. Policy F. The restriction of $T^{(F)}$ to the region (I) is given by

$$T^{(F)}|_{R(\mathrm{I})}\colon \begin{cases} x_{t+1} = y_t \\ y_{t+1} = (1-\alpha)y_t + \alpha \dfrac{m(x_t)}{m(y_t) - \min\left\{\dfrac{\pi^*-1}{\pi^*}m(\pi^*),\, \lambda m(y_t)\right\}}. \end{cases} \qquad \mathbf{(20)}$$

The horizontal line q of equation $y = q(x) \equiv b_\epsilon - [(\pi^*-1)/\lambda\pi^*](b_\epsilon - \pi^*)$ separates the region $R(\mathrm{I})$ into two subregions,

$$R(\mathrm{I}_A) = \{(x,y) \in R(\mathrm{I}) \mid y < q(x)\} \quad \text{and} \quad R(\mathrm{I}_B) = \{(x,y) \in R(\mathrm{I}) \mid y > q(x)\},$$

such that the map $T^{(F)}|_{(\mathrm{I})}$ can be written in the equivalent form

$$T^{(F)}|_{R(\mathrm{I})}\colon \begin{cases} T^{(F)}|_{(\mathrm{I}_B)} = T_\lambda & \text{if} \quad (x_t,y_t) \in R(\mathrm{I}_B) \\ T^{(F)}|_{(\mathrm{I}_A)} = T_*^{(F)}\colon \begin{cases} x_{t+1} = y_t \\ y_{t+1} = (1-\alpha)y_t + \alpha\pi^* \dfrac{m(x_t)}{\pi^* m(y_t) - (\pi^*-1)m(\pi^*)}. \\ \qquad \text{if} \quad (x_t,y_t) \in R(\mathrm{I}_A). \end{cases} \end{cases}$$

As in the corresponding REE map (5), the map $T_*^{(F)}$ has two fixed points: the target $E^* = (\pi^*, \pi^*)$ and $B^* = (b/\pi^*, b/\pi^*)$. These points also are fixed points for $T^{(F)}$ provided that they belong to the region (I_A) where the dynamics of $T^{(F)}$ are governed by the restriction $T_*^{(F)}$. It is easy to see that $E^* \in (\mathrm{I}_A)$ if, $\lambda > 1 - 1/\pi^*$, which is satisfied if condition (17) holds, and $B^* \in (\mathrm{I}_A)$ if $\lambda > 1 - \pi^*/b$. (Notice that $1 - 1/\pi^* < 1 - \pi^*/b$ if $b > {\pi^*}^2$). As with policy I, we assume that condition (17) is satisfied.

On the basis of the analysis of the eigenvalues given in Appendix B.2, the target fixed point E^* is stable in the region $\Omega_s^F = \{(b,\alpha) \in \Omega \mid b > {\pi^*}^2 \text{ and } b > \pi^*(1+\alpha)\}$ (see the shaded region bounded by the lines AB and BC in Figure 1) and, for $(b,\alpha) \in \Omega_s^F$, B^* is a saddle point. The two fixed points of $T_*^{(F)}$ exchange stability via a *transcritical bifurcation* at $b = {\pi^*}^2$ at which $E^* = B^*$, so that the fixed point characterized by lower inflation is the stable one.

The unique fixed point E_λ of the map T_λ is also a fixed point for $T^{(F)}$, provided it belongs to the region (I_B); that is,

$$b_\epsilon - \frac{\pi^*-1}{\lambda\pi^*}\left(b_\epsilon - \pi^*\right) < \frac{1}{1-\lambda} < b_\epsilon.$$

Furthermore, E_λ is locally stable provided $b > b_h^\lambda(\alpha) = (\alpha+1)/(1-\lambda)$ (see Appendix B.4). From these conditions for the existence and stability of the fixed points, we obtain the following result:

LEMMA 3. *Assume that* $\lambda \in \Delta^*$[*i.e., condition* (17)] *and let* $(b,\alpha) \in \Omega$:

(i) *If* $b < {\pi^*}^2$, *then the map* $T^{(F)}$ *has three fixed points*: E^*, B^*, *and* E_λ. *If* $\alpha < \pi^* - 1$, *then* E^* *is unstable and* B^* *is stable, while if* $\alpha < (1-\lambda)b - 1$ *then* E_λ *is locally stable.*

(ii) *The target E^* is locally stable provided that $(b, \alpha) \in \Omega_s^F$. Furthermore, if $\lambda < 1 - \pi^*/b_\epsilon$, then E^* is the only fixed point of $T^{(F)}$, whereas if $1 - \pi^*/b_\epsilon < \lambda < 1 - 1/b_\epsilon$, then the map $T^{(F)}$ has three fixed points, E^*, B^* and E_λ, where B^* is unstable and E_λ, is stable if $\alpha < (1 - \lambda)b - 1$.*

As with policy I, for $b > \pi^{*^2}$ and high values of α, the target equilibrium E^* is unstable [the only attractor being a big "cyclic" set $A(\epsilon)$]; then, E^* becomes stable for decreasing values of α through a *subcritical Neimark–Hopf bifurcation* at the line $b = b_h^{(F)}(\alpha) = \pi^*(\alpha + 1)$.

3.2.3. Policy O. The restriction of $T^{(O)}$ to region $R(\mathrm{I})$ is given by

$$T^{(O)}|_{R(\mathrm{I})}: \begin{cases} x_{t+1} = y_t \\ y_{t+1} = (1-\alpha)y_t + \alpha \dfrac{m(x_t)}{m(y_t) - \min\left\{m(\pi^*) - \frac{1}{\pi^*}m(x_t), \lambda m(y_t)\right\}}. \end{cases} \tag{21}$$

The line r of equation $y = r(x) \equiv -(1/\lambda\pi^*)x + \pi^*/\lambda + (b_e/\lambda)(1/\pi^* + \lambda - 1)$ separates the region $R(\mathrm{I})$ into two subregions,

$$R(\mathrm{I}_A) = \{(x, y) \in R(\mathrm{I}) \mid y < r(x)\} \quad \text{and} \quad R(\mathrm{I}_B) = \{(x, y) \in R(\mathrm{I}) \mid y > r(x)\},$$

such that the map $T^{(O)}|_{(\mathrm{I})}$ can be written in the equivalent form

$$T^{(O)}|_{R(\mathrm{I})}: \begin{cases} T^{(O)}|_{R(\mathrm{I}_B)} = T_\lambda & \text{if} \quad (x_t, y_t) \in R(\mathrm{I}_B) \\ T^{(O)}|_{R(\mathrm{I}_A)} = T_*^{(O)}: \begin{array}{l} x_{t+1} = y_t \\ y_{t+1} = (1-\alpha)y_t + \alpha\pi^* \dfrac{m(x_t)}{m(y_t) - \left(m(\pi^*) - \frac{1}{\pi^*}m(x_t)\right)}. \end{array} & \text{if} \quad (x_t, y_t) \in R(\mathrm{I}_A). \end{cases}$$

As in the corresponding REE map (5), $T_*^{(O)}$ has two fixed points: the target $E^* = (\pi^*, \pi^*)$ and $B^* = (b/(1+\pi^*), b/(1+\pi^*))$. These points are also fixed points for $T^{(O)}$, provided that they belong to the region $R(\mathrm{I}_A)$ where the dynamics of $T^{(O)}$ are governed by the sub map $T_*^{(O)}$. It is easy to see that, with condition (17), $E^* \in R(\mathrm{I}_A)$ and $B^* \in R(\mathrm{I}_A)$ if $\lambda > 1 - (1 + \pi^*)/b$. {Notice that $1 - 1/\pi^* < 1 - [(1 + \pi^*)/b]$ if $b > \pi^*(1 + \pi^*)$.} Therefore, the characterization is similar to that obtained for policy F. With policy O, the two fixed points of $T_*^{(O)}$ exchange stability via a *transcritical bifurcation* at $b = \pi^*(1 + \pi^*)b = \pi^{*^2}$ at which $E^* = B^*$. As with policy O, the fixed point characterized by lower inflation is locally stable under adaptive learning. However, in contrast, in this case the condition $b > \pi^*(1 + \pi^*)$ is the only condition for the stability of E^*; that is, $\Omega_s^O = \{(b, \alpha) \in \Omega \mid b > \pi^*(1 + \pi^*)\}$ (see the shaded region at the right of line ED in Figure 1). The unique fixed point

E_λ of the map T_λ is also a fixed point for $T^{(O)}$, provided it belongs to the region $R(\mathrm{I}_B)$; that is, if

$$-\frac{1}{\lambda\pi^*}\frac{1}{1-\lambda}+\frac{\pi^*}{\lambda}+\frac{b_\epsilon}{\lambda}\left(\frac{1}{\pi^*}+\lambda-1\right)<\frac{1}{1-\lambda}<b_\epsilon.$$

Furthermore, E_λ is locally stable, provided that $b>(\alpha+1)/(1-\lambda)$ (see Appendix B.4). In summary, we obtain local stability results that almost parallel those of policy F.

LEMMA 4. *Assume* $\lambda\in\Delta^*$ [*i.e., condition* (17)] *and let* $(b,\alpha)\in\Omega$:

(i) *If* $b<\pi^*(1+\pi^*)$, *then the map* $T^{(O)}$ *has three fixed points*: E^*, B^*, *and* E_λ. E^* *is locally unstable,* B^* *is locally stable, and* E_λ *is locally stable if* $\alpha<(1-\lambda)b-1$;

(ii) *The target* E^* *is locally stable, provided that* $(b,\alpha)\in\Omega_s^O$. *Furthermore, if* $\lambda<1-(1+\pi^*)/b_\epsilon$, *then* E^* *is the only fixed point of* $T^{(O)}$, *whereas if* $1-(1+\pi^*)/b_\epsilon<\lambda<1-1/b_\epsilon$, *then the map* $T^{(O)}$ *has three fixed points,* E^*, B^*, *and* E_λ, *where* B^* *is unstable and* E_λ *is stable if* $\alpha<(1-\lambda)b-1$.

3.2.4. Ranking policies according to their local stability properties. Lemmas 2–4 show how the local stability properties of the inflation target π^* differ across policies. In particular, assuming condition (17), the stability of the inflation target under the policies I, F, and O holds in the following domains of the parameters' space Ω:

$$\begin{aligned}\Omega_s^I &= \{(b,\alpha)\in\Omega \mid b>\pi^*(1+\alpha\pi^*)\}\\ \Omega_s^F &= \{(b,\alpha)\in\Omega \mid b>\pi^{*^2} \quad\text{and}\quad b>\pi^*(1+\alpha)\}\\ \Omega_s^O &= \{(b,\alpha)\in\Omega \mid b>\pi^*(1+\pi^*)\}.\end{aligned}$$

Therefore, we say that policy $\boldsymbol{P}$ dominates policy $\boldsymbol{P}'$, in terms of its local stability properties, if the inflation target equilibrium π^* is locally stable in a larger domain of the parameter space Ω, and denote such preference by $\boldsymbol{P}\succ_l\boldsymbol{P}'$. Then, as corollary to Lemmas 2–4 we have the following propositions.

PROPOSITION 1. *Assume that* $\lambda\in\Delta^*$ [*i.e., condition* (17)] *and let* $\pi^*>1$. *Then, policy O is dominated in terms of its local stability properties. In particular,* $F\succ_l O$ *and* $I\succ_l O$.

Figure 1 illustrates Proposition 1 for $2>\pi^*>1$. Notice that, as long as $b>\pi^{*2}$, the unconditional policy F dominates the other policies in terms of its local stability properties. This result is consistent with Friedman's views.

A local stability ranking is not uniquely determined by $\succ_l$. For example, provided that the target is locally stable, we may be interested in whether convergence is monotone, which can make it easier to "pattern recognize" the tendency for inflation to converge to the target. Alternatively, we may be interested in the speed of convergence to the target. As we show in Appendix C, provided that the target is locally stable, only with policy O is convergence always monotone, whereas for other policies, monotone convergence requires a small enough value of α. Nevertheless, in terms of speed of convergence, policy O also tends to be dominated.

More formally, let $\Omega_s^* = \Omega_s^I \cap \Omega_s^F \cap \Omega_s^O$; that is, $\Omega_s^* \subset \Omega$ denotes the region of parameters where the inflation target is locally stable under the three policies under consideration. We say that $\boldsymbol{P} \succ_s \boldsymbol{P}'$ on a subset $A \subseteq \Omega_s^*$ if, for any $(b, \alpha) \in A$, paths (starting in a neighborhood of π^*) converge faster under the policy $\boldsymbol{P}$ than under the policy $\boldsymbol{P}'$. The following proposition (proved in the Appendix C) provides the corresponding characterization.

PROPOSITION 2. *Assume that* $(b, \alpha) \in \Omega_s^*$ *and* $\pi^* > 1$. *Then,*

(i) *there exists an* $\bar{\alpha}$ *such that, for all* $\alpha \leq \bar{\alpha}$, *all three policies have a monotone path*;
(ii) *there exists an* $\alpha_1 \in (\bar{\alpha}, 1)$ *and an* $\alpha_2 \in (\alpha_1, 1)$ *such that, for all* $\alpha \leq \alpha_1, I \succ_s O$, *and, for all* $\alpha \leq \alpha_2, F \succ_s O$.

As we have shown, the local stability analysis already allows us to rank inflation target policies and, in particular, it suggests disregarding the optimal policy O in favor of alternative policies. On the other hand, differences based on the eigenvalues of the Jacobian of $T_*^{(P)}$ tend to be relatively small and, therefore, the rankings are not very sharp. We now turn in the next subsection to the more interesting and novel global analysis of the three policies.

3.3. Global Stability of π^*

As in the preceding subsection, we first briefly discuss global dynamics under the alternative policies and then we summarize the results comparing the three policies. As we will see, even if the local analysis also provides useful information concerning the global dynamics of the system, a more complete understanding is based on the study of the basins of attraction and, in particular, of some *global bifurcations* that cause qualitative changes in such basins, whose characterization requires the use of computer graphics. We focus our attention on the basin of attraction of π^*, $\mathcal{B}(E^*)$, defined as the set of points of the plane x, y that generate trajectories converging to E^*. Of particular interest is the role played by the fiscal constraint parameter λ and by the tracking parameter α in enlarging $\mathcal{B}(E^*)$. The global analysis becomes quite complex because of the possible coexistence of different attractors. As we will see, in all of these respects the three, apparently very similar, policies behave quite differently. Such differences could not be captured in a model in which only the asymptotic case $\alpha \searrow 0$ is analyzed [e.g., Evans et al. (2000)].

3.3.1. Policy I: The role of fiscal constraints. As we have seen in Lemma 1, even when the inflation target is locally stable, there may be cycling paths following a large cyclical movement across the four regions. Figure 2 illustrates such behavior for policy I. In particular, Figure 2a shows, in the phase space x,y, the coexistence

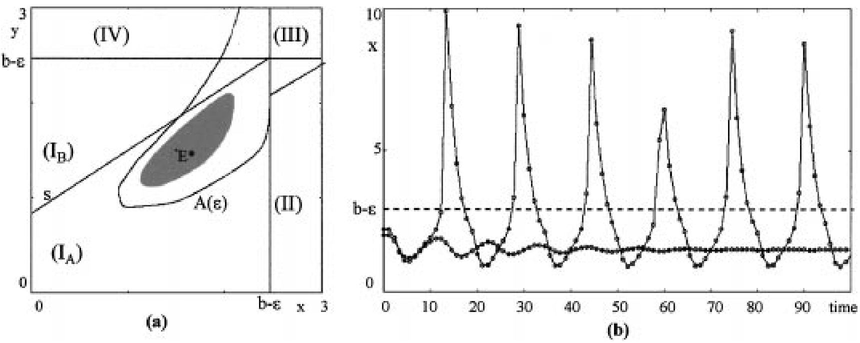

Figure 2. Numerical simulations of the model under policy I with $\pi^* = 1.5, \alpha = 0.4$, $\lambda = 0.5$, $b = 2.75$, and $\epsilon = 0.03$, that is, just after the subcritical Neimark–Hopf bifurcation at which the target inflation fixed point E^* becomes stable, occurring at $b_h^{(1)}(0.4) = 2.4$. (a) The basin of attraction of E^* is represented by the gray region, whereas the basin of the "cyclic" attractor $A(\epsilon)$ is represented by the white region (only partially visible in the figure). (b) Two sequences of expected inflation rates are represented relative to time, one generated by an initial condition taken in the gray region of (a) and the other one generated by an initial condition taken in the white region.

of a large "cyclic" attractor $A(\epsilon)$, whose basin is represented by the white region, with the SREE π^* whose basin $\mathcal{B}(E^*)$ is represented by the gray region. Figure 2b shows two paths, each of which starts from an initial expected inflation taken in a different basin of attraction.

In Figure 2a, $\mathcal{B}(E^*)$ is contained in the interior of subregion (I_A). This is a snapshot corresponding to fixed values of b, α, and λ. Nevertheless, changing these parameters also causes $\mathcal{B}(E^*)$ to change. In particular, numerical simulations show how the size of $\mathcal{B}(E^*)$ increases for decreasing values of α (or increasing values of b) until the basin boundary $\partial\mathcal{B}(E^*)$ has a contact with the big cyclic attractor $A(\epsilon)$. This contact causes the disappearance of $A(\epsilon)$ [Gumowski and Mira (1978, 1980)] and, consequently, E^* becomes a global attractor; that is, $\mathcal{B}(E^*)$ covers the whole phase space. Such a contact bifurcation is called *final bifurcation* in Mira et al. (1996) and Abraham et al. (1997) or *boundary crisis* in Grebogi et al. (1983). This bifurcation cannot be revealed by a local study, that is, based on the linear approximation of the dynamical system.

An interesting result is obtained if the influence of the parameter λ on the size and the shape of $\mathcal{B}(E^*)$ is considered. In fact, even if λ does not influence the local stability of E^* when condition (17) is assumed, it may influence the shape and the size of $\mathcal{B}(E^*)$. This is clearly shown in Figure 3, where we start with a situation similar to that of Figure 2a (see Figure 3a) and, keeping all of the other parameters fixed, we successively decrease λ, making the fiscal constraint tighter. In Figure 3b, $\mathcal{B}(E^*)$ intersects the subregion $R(\mathrm{I}_B)$ where dynamics are dominated by the sub map T_λ. The contact between the basin boundary $\partial\mathcal{B}(E^*)$ and the line s, which separates the subregions $R(\mathrm{I}_A)$ and $R(\mathrm{I}_B)$, causes a sudden enlargement of

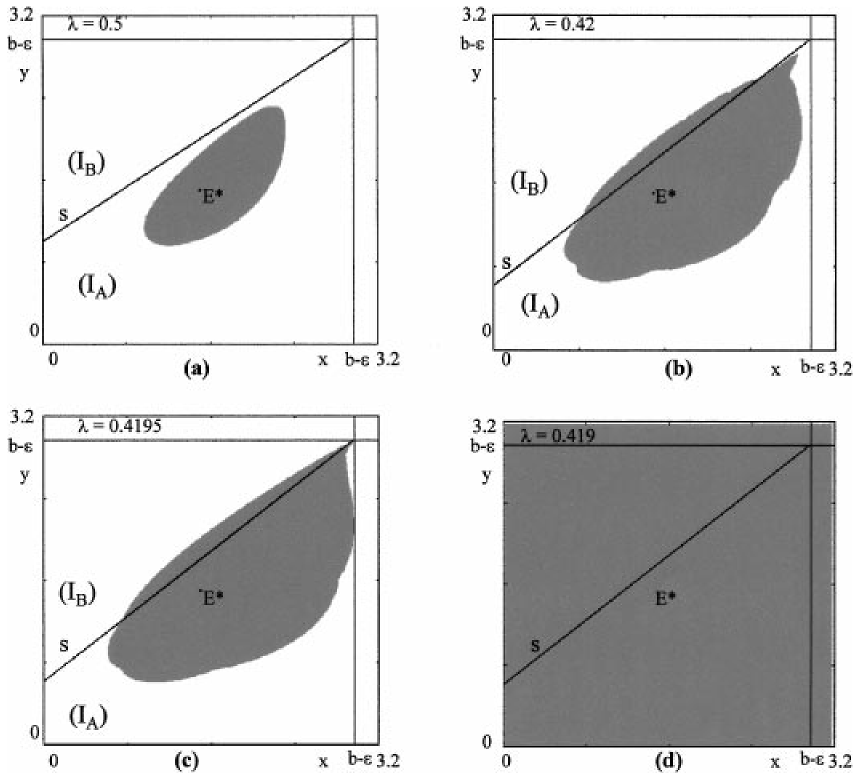

FIGURE 3. Numerical simulations of the model under policy I with $\pi^* = 1.5, \alpha = 0.6$, $b = 3$, $\epsilon = 0.03$, and four different values of λ, decreasing from (a) to (d). The gray region represents the basin of the target equilibrium E^*. In (a), $\lambda = 0.5$, and the basin is entirely included in the region (I_A). In (b), $\lambda = 0.42$, after the contact between the basin boundary and the line s. In (c), $\lambda = 0.4195$ at the contact between the basin boundary and the line $x = b - \epsilon$. In (d), $\lambda = 0.419$, after the contact between the basin boundary and the line $x = b - \epsilon$, the basin of E^* covers the whole plane; that is, E^* is globally stable.

the basin $\mathcal{B}(E^*)$. In fact, after such contact, if E_λ is stable for T_λ and $E_\lambda \in \mathcal{B}(E^*)$, then some trajectories starting from region $R(\mathrm{I}_B)$ may move toward E_λ and, consequently, enter the basin $\mathcal{B}(E^*)$. We may say that E_λ behaves as a *catalyst* because it attracts trajectories coming from the subregion $R(\mathrm{I}_B)$ and then it conveys them toward E^* because $E_\lambda \in \mathcal{B}(E^*)$. Moreover, a small reduction of λ causes $\mathcal{B}(E^*)$ to increase to the point where the basin boundary $\partial\mathcal{B}(E^*)$ *contacts* the line b_ϵ (see Figure 3c), producing a *global* (or *contact*) *bifurcation*. As Figure 3d shows, as a result of such a global bifurcation, $\mathcal{B}(E^*)$ coversthe entire phase space under consideration, so that global stability is achieved.

In summary, Figure 3 shows how fiscal constraints can enhance the global stability properties of an inflation target policy (such as I) even when the constraints have no effect on local stability properties of the inflation policy.

It is important to remark that, since the equations of the curves that form $\partial\mathcal{B}(E^*)$ are not known, an analytical computation of the parameters values at which the contacts between $\partial\mathcal{B}(E^*)$ and the lines s and $b = b_\epsilon$ occur is not possible. Hence, these parameters can only be revealed numerically, by a graphical analysis. Indeed, computational methods are a standard tool in the global study of dynamical systems of dimension greater than one [see, e.g., Mira et al. (1996), Brock and Hommes (1997)].

3.3.2. Policy F: The coexistence of two attracting fixed points. As Lemma 3(ii) shows, the fixed points E^* and E_λ may coexist, both being locally stable. In this case of two coexisting attractors, the initial condition is crucial in order to forecast the long-run behavior of the system; it is therefore important to study the boundaries of the respective basins of attraction. As with policy I, when E^* is the only attractor, decreasing α or λ, or increasing b, enhances the stability of π^*, and $\mathcal{B}(E^*)$ expands. However, when both E^* and E_λ are attractors, these changes of parameters tend to enhance the stability properties of both attractors and it may well be that the effect is stronger for E_λ, in which case $\mathcal{B}(E_\lambda)$ will enlarge while $\mathcal{B}(E^*)$ will contract. This is shown in Figure 4, where we start in a situation in which both attractors coexist, but just after the subcritical Neimark–Hopf bifurcation at which E_λ becomes stable and, therefore, $\mathcal{B}(E^*)$ encompasses almost all of the phase space [see Figure 4a; notice that the Neimark–Hopf bifurcation at which E_λ becomes stable occurs at $\alpha = (1-\lambda)b - 1 = 0.25$]. In Figures 4b–d, we successively reduce the tracking parameter α while keeping all other parameters constant. As α is decreased, $\mathcal{B}(E_\lambda)$ enlarges and its boundary has a contact with the line q. After this contact, a sudden change of $\mathcal{B}(E_\lambda)$ is observed, as shown in Figure 4b. Now the boundary of the basin $\mathcal{B}(E_\lambda)$ includes the saddle point $\mathcal{B}^*$ and, consequently, points that are very close to E^* belong to $\mathcal{B}(E_\lambda)$. Furthermore, if α is further decreased, $\mathcal{B}(E_\lambda)$ continues to enlarge until a contact with the line $x = b_\epsilon$ occurs (see Figure 4c), which marks another evident qualitative change, as Figure 4d shows.

In summary, Figure 4 shows how the presence of coexisting attractors (as may occur under policy F) can induce counterintuitive effects on the stability properties of the inflation target π^* when parameters are changed.

3.3.3. Policy O: The coexistence of two attracting fixed points and a chaotic attractor. Lemma 4(ii) shows that, with the policy O, the fixed points E^* and E_λ can coexist as attractors. However, as Figure 5 shows, the situation may be more complex: In particular, Figure 5b shows the existence of a chaotic attractor around E_λ. In this figure the dark-gray and the light-gray regions represent the basins of E^* and E_λ, respectively, whereas the points of the white region converge to the chaotic attractor. Notice that the basin $\mathcal{B}(E_\lambda)$ is formed by two disjoint portions. However, as the parameter α is decreased, the chaotic attractor disappears after a *contact* with its basin boundary, a typical *final bifurcation* (or *boundary crisis*); see Figure 5b.

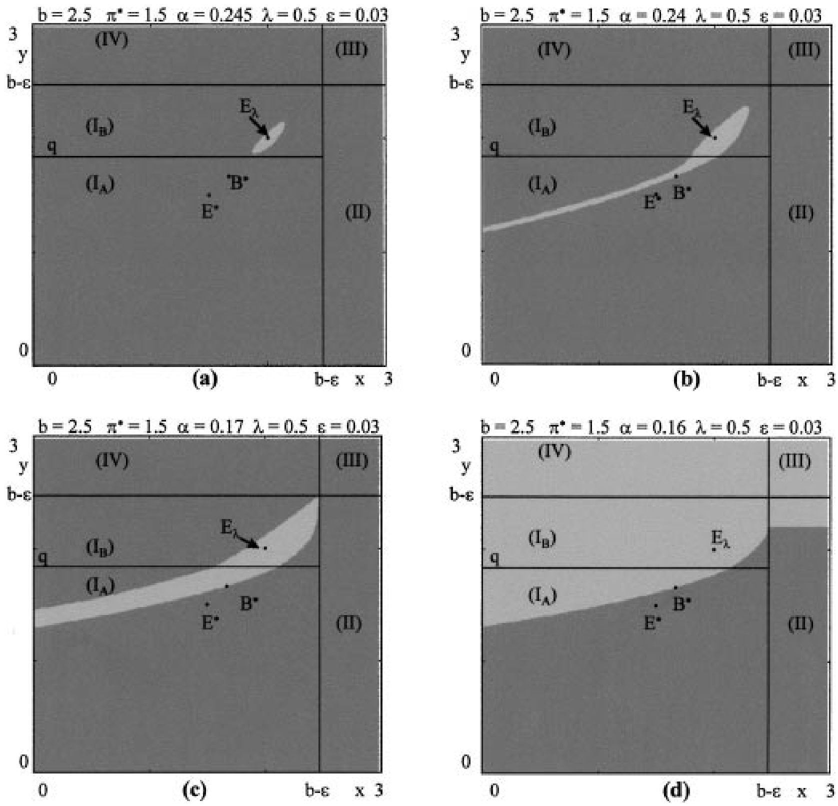

FIGURE 4. Numerical simulations of the model with policy F, with parameters $\pi^* = 1.5$, $\lambda = 0.5$, $b = 2.5$, $\epsilon = 0.03$, and four different values of α, decreasing from (a) to (d), such that the two stable equilibria E^* and E_λ coexist. The dark-gray region represents the basin $\mathcal{B}(E^*)$ of the target equilibrium E^*; the light-gray region represents the basin $\mathcal{B}(E_\lambda)$ of the higher-inflation equilibrium E_λ.

In summary, Figure 5 shows that the global dynamics can be quite complex. However, decreasing α (or increasing b) tends to simplify the dynamics of the model in favor of the attracting fixed points. As in Figure 4, however, stability may be enhanced more for E_λ than for E^*

3.3.4. Comparing policies according to their global stability properties with the help of fiscal constraints. The results on global dynamics given above are interesting but do not lead to a clear ranking of policies according to their global stability properties. To provide such a comparison, we restrict our attention to values of $\lambda \in \Delta^* \equiv (1 - 1/\pi^*, 1 - 1/b_\epsilon)$ [i.e., where condition (17) is satisfied] and check, by numerical computation, which values of λ, b, and α produce

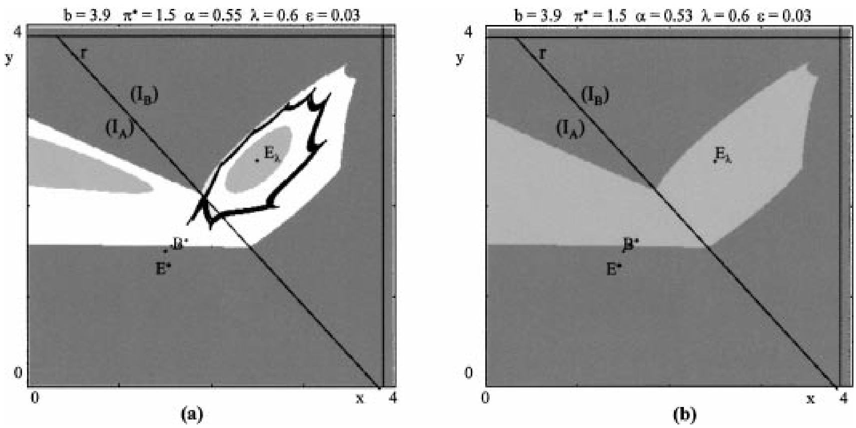

FIGURE 5. Numerical simulations of the model with policy O, with parameters $\pi^* = 1.5$, $\lambda = 0.6$, $b = 3.9$, $\epsilon = 0.03$, and two different values of α, such that the two stable equilibria E^* and E_λ coexist. The dark-gray region represents the basin $\mathcal{B}(E^*)$ of the target equilibrium E^*; the light-gray region represents the basin $\mathcal{B}(E_\lambda)$ of the higher-inflation equilibrium E_λ. (a) For $\alpha = 0.55$, a chaotic attractor also exists around E_λ, whose basin is represented by the white region. The basin $\mathcal{B}(E_\lambda)$ is formed by two disjoint portions. (b) For $\alpha = 0.53$, the chaotic attractor no longer exists.

"global convergence." More precisely, given a set of parameters (α, b, λ), we numerically generate paths from all initial conditions (x_0, y_0) taken within a fine grid in a wide portion of the (x,y) plane, and we count how many of such paths converge to the target. Figure 6 shows the results of these computations, made for many values of (b, λ), whose values are represented on the axes, and two different values of α. From Lemmas 3(ii) and 4(ii), for values (λ, b) between the curves $\lambda^*(b) = 1 - 1/b_\epsilon, \lambda^F(b) = 1 - \pi^*/b_\epsilon$, and $\lambda^O(b) = 1 - (1 + \pi^*)/b_\epsilon$, respectively, the attractor E^* may coexist with the attractor E_λ, whereas for values of (λ, b) below $\lambda^F(b)$ and $\lambda^O(b)$, E^* is the unique attractor. In contrast, for policy I, there is a unique fixed point that can be an attractor [E_λ is in subregion $R(\mathrm{I}_A)$], which results in a better performance of this policy in terms of global stability for relatively low values of α. For relatively high values of α, however, the target may cease to be stable and policy F may dominate policy I in terms of global stability.

In summary, Figure 6[11] reinforces the local stability ranking of policies. In particular, the global stability results are consistent with Propositions 1 and 2 in showing that the so-called optimal policy O tends to be outperformed, as a stabilization policy, by either the unconditional Friedman policy F or the adaptive inertia policy I when private agents form their expectations adaptively.

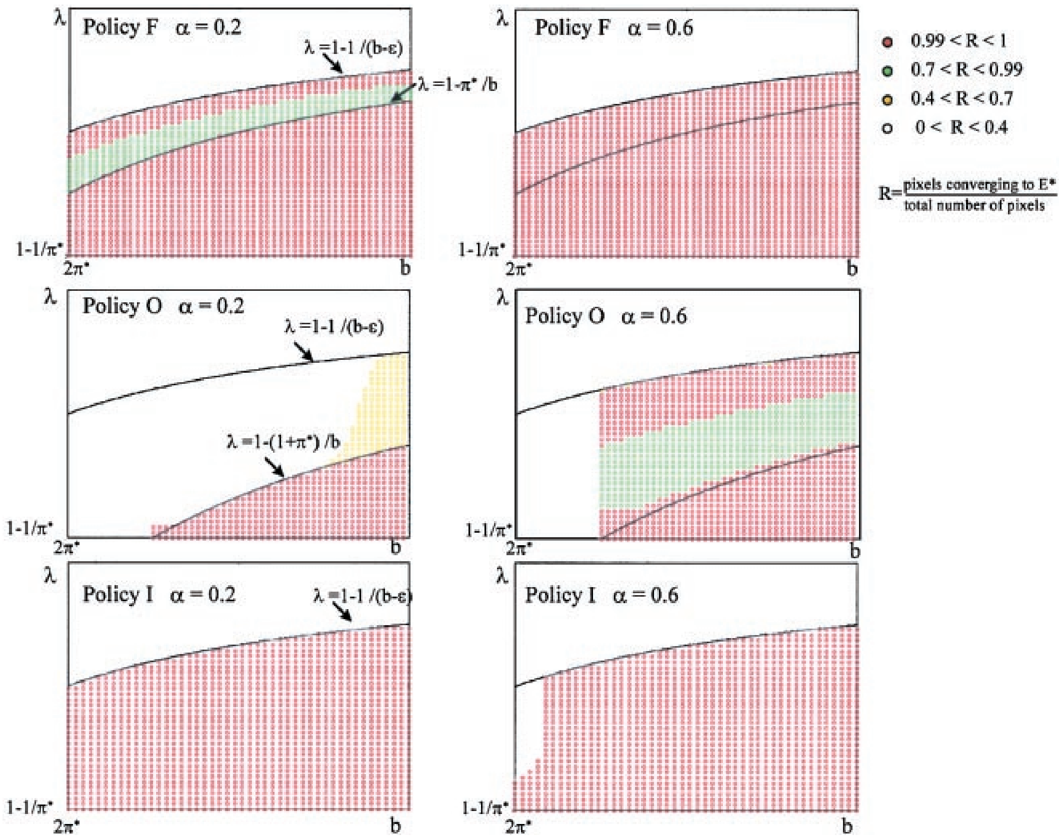

Figure 6. Numerical computations of the extension of the basin of the target equilibrium E^*. All of the figures are obtained with $\pi^* = 1.5$, $\epsilon = 0.03$, and $\alpha = 0.2$ (figures on the left) or $\alpha = 0.6$ (figures on the right). The different colors represents different values of the fraction R of initial conditions that generate trajectories converging to the target equilibrium E^*, according to the legend in the figure.

4. CONCLUSIONS

Stabilization policies must be judged by their stability properties. Within rational expectations equilibria, such a statement is not even meaningful. It is meaningful, however, when we consider that agents may form their expectations adaptively. Experimental evidence [see, e.g., Marimon and Sunder (1993, 1994, 1995)] supports this adaptive view and can provide an empirical ground for our stability results.[12] The fact that our local and global stability rankings are consistent is encouraging. In particular, our results reinforce Friedman's caution against "overly reactive" rules. Friedman had an intuition about policy lags that could apply to any model. In contrast, we provide a careful stability analysis of a relatively simple model without policy lags. Even so, some lessons emerge that are likely to apply to other models. First, and foremost, the *misspecification* that private agents have rational expectations when, in fact, they do not, may lead to a wrong policy design, in the sense that alternative designs of stabilization policies may outperform the rules designed under the rational expectations assumption. Second, even leaving aside time-consistency considerations or "fiscal theory of money" considerations [see, e.g., Woodford (1996)], fiscal constraints, in particular, seignorage constraints, may play an important role in helping stabilization policies to achieve their goals.[13] Third, even if monetary authorities follow—with full commitment—their announced inflation target rules, inflation may differ substantially from the target. Whereas, for example, inflationary episodes above the target are usually associated with loose monetary policy or weak monetary authorities, in our economies such instability may well correspond to the fact that, due to the existence of money substitutes, the inflation target may not be too far from the level of inflation in which there is a currency collapse. Furthermore, our global analysis also provides a good reason to study the point of *currency collapse*: It is the point where a global-contact bifurcation occurs, resulting in a qualitative improvement of the stability properties of the policy.

There is room for further research in several directions: studying other misspecified models, introducing stochastic learning, and so on. In such extensions, it would be interesting to see if the relatively good performance (as a stabilization policy) of Friedman's-constant-money-growth rule persists. We find it a remarkable result that may generalize to other environments.

NOTES

1. See Sargent (1999) for a discussion of adaptive models with misspecified beliefs.

2. For a detailed account of E-stability theory, see, for example, Evans and Honkapohja (2000a).

3. See, for example, Mira et al. (1996), Abraham et al. (1997), and Bischi et al. (1998) for an introduction to the these results on *contact bifurcations.*

4. Implicitly we assume that, within equivalent policies resulting in the same *d* policy, there is (local) Ricardian equivalence; that is, present-value considerations do not discriminate among these equivalent policies.

5. In a stochastic model, the question is what should the government do when, at some confidence level, it infers that the predictions of private agent are not consistent with rational expectations, given the government policy.

6. See, for example, Marimon and McGrattan (1994) and Fudenberg and Levine (1998).

7. For example, in the EMU, seignorage of the ECB is restricted; furthermore, the Growth and Stability Pact constrains deficits and, in the United States, balanced-budget proposals are recurrently being considered.

8. Notice that, for notational convenience, we also denote by $m^d(\pi^e)$ the demand for real balances with precautionary savings.

9. Lettau and van Zandt (1999), in contrast with Marcet and Sargent (1989a,b), show that if agents react to *current* prices and do not focus on MSV solutions, the stability properties of the adaptive learning process change. Recently, however, Adam (2000) has shown that, if Cagan's hyperinflationary model is properly developed to meaningfully allow for conditioning on current prices, most of the Marcet and Sargent results prevail.

10. Notice that one could also consider that agents give some weight to the announced target, such as $\pi^e_{t+1} = (1-\gamma_t)[(1-\alpha_t)\pi^e_t + \alpha_t\pi_{t-1}] + \gamma_t\pi^*$. However, although in such a rule tends to help the stability properties of the target, it complicates the analysis without providing new insights.

11. Similar computations, not reported here, are available on request.

12. In fact, Evans et al. (2000b) provide some experimental results showing the stabilization power of fiscal constraints.

13. Notice, however, that if fiscal constraints are too tight, the target may not be a stationary equilibrium.

14. The rigorous proof of the subcritical nature of the Hopf bifurcation requires the evaluation of some long expressions involving derivatives of the map up to order 3. In this case, we claim numerical evidence.

REFERENCES

Abraham, R., L. Gardini & C. Mira (1997) *Chaos in Discrete Dynamical Systems (a Visual Introduction in Two Dimensions*). New York: Springer–Verlag.

Adams, K. (2000) *Learning Behavior: Microeconomic and Macroeconomic Implications*. Unpublished Ph.D. Dissertation, European University Institute, Florence, Italy.

Barucci, E., G.I. Bischi & R. Marimon (1998) The Stability of Inflation Target Policies. Manuscript, European University Institute.

Benhabib, J., S. Schmitt-Grohé & M. Uribe (1999) The Perils of Taylor Rules. Mimeo, New York University.

Bernake, B.S. & I. Mihov (1997) What does the Bundesbank target?" *European Economic Review* 41, 1025–1053.

Bischi, G.I., L. Gardini & C. Mira (1998) Maps with denominator: Some generic properties. *International Journal of Bifurcations and Chaos* 9, 119–153.

Brock, W.A. & C. Hommes (1997) A rational route to randomness. *Econometrica* 65, 1059–1095.

Bullard, J. & K. Mitra (1999) Learning About Monetary Policy Rules. Mimeo, Federal Reserve Bank of St Louis.

Cagan, P. (1956) The monetary dynamics of hyerinflation. In M. Friedman (ed.), *Studies in the Quantity Theory of Money*. Chicago: University of Chicago Press.

Christiano, L.J. & C.J. Gust (1999) Taylor Rules in a Limited Participation Model. NBER working paper 7017.

Clarida, R. & M. Gertler (1996) How Does the Bundesbank Conduct Monetary Policy?" NBER working paper 5581.

Clarida, R., J. Galí & M. Gertler (1997a) Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory. Mimeo, New York University.

Clarida, R., J. Galí & M. Gertler (1997b) Monetary Policy Rules in Practice: Some International Evidence, Mimeo, New York University.

Evans, G.W. & S. Honkapohja (2000a) *Adaptive Learning and Macroeconomic Dynamics*. Princeton, NJ: Princeton University.

Evans, G.W., S. Honkapohja & R. Marimon (2001) Convergence in monetary inflation models with heterogeneous learning rules. *Macroeconomic Dynamics* 5, 1–31.

Farmer, R. (1999) *The Economics of Self-Fulfilling Prophecies*. Cambridge, MA: MIT Press.

Federal Reserve Bank of Kansas City (1996) *Achieving Price Stability*. Symposium proceedings. Jackson Hole, WY: FRB Kansas City.

Friedman, M. (1948) A monetary and fiscal framework for economic stability. *American Economic Review* 38, 245–264.

Friedman, M. (1960) *A Program for Monetary Stability*. New York: Fordham University Press.

Fudenberg, D. & D. Levine (1998) *The Theory of Learning in Games*. Cambridge, MA: M.I.T. Press.

Giovannetti, G., J. Diaz, R. Marimon & P. Teles (2000) Nominal Debt as a Burden to Monetary Policy. Mimeo, EUI.

Grebogi, C., E. Ott & J.A. Yorke (1983) Crises, sudden changes in chaotic attractors, and transient chaos. *Physica 7D*, 181–200.

Gumowski, I. & C. Mira (1978) Bifurcation déstabilisant une solution chaotique d'un endomorphisme du 2nd ordre. *Comptes Rendus de l'Academic des Sciences Paris, Série A*, 286, 427–431.

Gumowski, I. & C. Mira (1980) *Dynamique Chaotique*. Toulouse, France: Cepadues Editions.

Guckenheimer, J. & P. Holmes (1983) *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. New York: Springer-Verlag.

Lettau, T. & T. van Zandt (1999) Robustness of Adaptive Expectations as an Equilibrium Selection Device. CENTER Discussion paper 9598, Tilburg University.

Lorenz, H.W. (1993) *Nonlinear Dynamical Economics and Chaotic Motion*, 2nd ed. New York: Springer-Verlag.

Marcet A. & T.J. Sargent (1989a) Least squares learning and the dynamics of hyperinflation. In W.A. Barnett, J. Geweke, & K. Shell (eds.), *Economic Complexity: Chaos, Sunspots, Bubbles, and Nonlinearity*. Cambridge, UK: Cambridge University Press.

Marcet A. & T.J. Sargent (1989b) Convergence of least-squares learning in environments with hidden state variables and private information. *Journal of Political Economy* 97, 1306–1322.

Marimon, R. (1997) Learning from learning in economics. In D.M. Kreps & K.F. Wallis (eds.), *Advances in Economics and Econometrics: Theory and Applications*, Vol. I. Cambridge, UK: Cambridge University Press.

Marimon, R. & E. McGrattan (1994) On adaptive learning in strategic games. In A. Kirman & M. Salmon (eds.), *Learning and Rationality in Economics*. Oxford: Blackwell Publ.

Marimon, R. & S. Sunder (1993) Indeterminacy of equilibria in a hyperinflation world: experimental evidence. *Econometrica* 61, 1073–1107.

Marimon, R. & S. Sunder (1994) Expectations and learning under alternative monetary regimes: An experimental approach. *Economic Theory* 4, 131–162.

Marimon, R. & S. Sunder (1995) Does a constant money growth rule help stabilize inflation? Experimental evidence. *Carnegie-Rochester Conference Series on Public Policy* 43, 111–156.

McCallum, B.T. (1997) Issues in the Design of Monetary Policy Rules. NBER working paper 6016.

Mira, C., L. Gardini, A. Barugola & J.-C. Cathala (1996) *Chaotic Dynamics in Two-Dimensional Noninvertible Maps*. Singapore: World Scientific.

Mishkin, F.S. & A.S. Posen (1997) Inflation Targeting: Lessons from Four Countries. NBER working paper 6126.

Sargent, T.S. (1999) *The Conquest of American Inflation*. Princeton, NJ: Princeton University Press.

Sargent, T.S. & N. Wallace (1987) Inflation and the government budget constraint. In A. Razin & E. Sadka (eds.), *Economic Policy in Theory and Practice*. New York: Macmillan.

Svensson, L.E.O. (1997) Inflation forecast targeting: Implementing and monitoring inflation. *European Economic Review* 41, 1111–1146.

Taylor, J.B. (1993) Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series on Public Policy* 39, 195–214.

Taylor, J.B. (ed.) (1999) *Monetary Policy Rules*. Chicago: University of Chicago Press.
Woodford, M. (1994) Nonstandard indicators of moneatry policy: Can their usefulness be judged from forecasting regressions? In G.N. Mankiw (ed.), *Monetary Policy*. Chicago: University of Chicago Press.
Woodford, M. (1996) Control of the Public Debt: A Requirement for Price Stability? NBER working paper 5684.

APPENDIX A: PROOF OF LEMMA 1

We first prove that all the trajectories starting out of region $R(\mathrm{I})$ enter region $R(\mathrm{I})$ after a finite number of steps. In fact,

(a) If $(x_t, y_t) \in R(\mathrm{II})$, then $(x_{t+1}, y_{t+1}) \in R(\mathrm{I})$, because in the map (14) $y_t < b_\epsilon$ implies $x_{t+1} < b_\epsilon$ and $y_{t+1} = (1-\alpha)y_t < b_\epsilon$.

(b) If $(x_t, y_t) \in R(\mathrm{III})$ and $\lambda < 1 - 1/b_\epsilon$, then $(x_{t+k}, y_{t+k}) \in R(\mathrm{II})$ for a finite $k > 0$. Infact, in region $R(\mathrm{III})$, we have $m(x) = \epsilon$ and $m(y) = \epsilon$; hence, the map $T^{(P)}$ becomes

$$T^{(P)}|_{R(\mathrm{III})}\colon \begin{cases} x_{t+1} = y_t \\ y_{t+1} = (1-\alpha)y_t + \alpha \dfrac{1}{1-\lambda}. \end{cases}$$

This is a linear map with a triangular structure, the second component only being dependent on the second variable, and it is immediate to see that y_t converges to $1/(1-\lambda)$ at a speed $(1-\alpha)^t$, hence the entrance inside the region $R(\mathrm{II})$ after a finite number of steps is ensured if $1/(1-\lambda) < b_\epsilon$; that is, $\lambda < 1 - 1/b_\epsilon$.

(c) If $(x_t, y_t) \in R(\mathrm{IV})$, then $(x_{t+1}, y_{t+1}) \in R(\mathrm{III})$ or $(x_{t+1}, y_{t+1}) \in R(\mathrm{II})$, because $y_t > b_\epsilon$ implies $x_{t+1} = y_t > b_\epsilon$.

To complete the proof, we now show that a trajectory may transit from region $R(\mathrm{I})$ to region $R(\mathrm{IV})$, so that $R(\mathrm{I})$ is not trapping. In fact, in region $R(\mathrm{I})$, we have

$$T^{(P)}|_{R(\mathrm{I})}\colon \begin{cases} x_{t+1} = y_t \\ y_{t+1} = (1-\alpha)y_t + \alpha \dfrac{b - x_t}{b - y_t - \min\left\{d^{(P)}(x_t), \lambda(b - y_t)\right\}} \end{cases}$$

from which it is evident that a movement from region (I) to region (II) is impossible, because $y_t < b_\epsilon \Rightarrow x_{t+1} = y_t < b_\epsilon$, where as we may have $(x_t, y_t) \in R(I)$ and $(x_{t+1}, y_{t+1}) \in R(\mathrm{IV})$ whenever y_t is sufficiently close to b and x_t is sufficiently small. ■

APPENDIX B: LOCAL STABILITY ANALYSIS

In this Appendix, we analyze the local stability of the fixed points of the maps $T_*^{(I)}$, $T_*^{(F)}$, $T_*^{(O)}$, and T_λ. Such analysis is obtained by the standard study of the eigenvalues, that is, the solutions of the characteristic equation

$$P(z) = z^2 - \mathrm{Tr} \cdot z + \mathrm{Det} = 0, \qquad \textbf{(B.1)}$$

where Tr and Det are the trace and the determinant of the Jacobian matrix computed at the fixed point. A sufficient condition for the stability is expressed by the following system of inequalities

$$P(1) = 1 - \mathrm{Tr} + \mathrm{Det} > 0; \quad P(-1) = 1 + \mathrm{Tr} + \mathrm{Det} > 0; \quad 1 - \mathrm{Det} > 0 \qquad \textbf{(B.2)}$$

that give necessary and sufficient conditions for the two eigenvalues of (B.1) be inside the unit circle of the complex plane [see, e.g., Gumowski and Mira (1980 p. 159)].

B.1 MAP $T_*^{(I)}$

The Jacobian matrix of the map $T_*^{(I)}$ evaluated at the unique fixed point E^* is

$$DT_*^{(I)}(\pi^*, \pi^*) = \begin{bmatrix} 0 & 1 \\ -\dfrac{\alpha\pi^{*2}}{b-\pi^*} & 1 - \alpha + \dfrac{\alpha\pi^{*^2}}{b-\pi^*} \end{bmatrix}. \qquad \textbf{(B.3)}$$

The characteristic equation (B.1) has coefficients $\mathrm{Tr} = \mathrm{Tr}^{(I)} = 1 - \alpha + \alpha\pi^{*2}/(b - \pi^*)$ and $\mathrm{Det} = \mathrm{Det}^{(I)} = \alpha\pi^{*2}/(b - \pi^*)$. The conditions $P(1) > 0$ and $P(-1) > 0$ are always satisfied, and the only condition for the stability of E^* is $1 - \mathrm{Det} > 0$; that is,

$$\frac{\alpha\pi^{*2} - b + \pi^*}{b - \pi^*} < 0.$$

Since $b > \pi^*$ in the parameter space Ω, a sufficient condition for the stability of E^* is

$$\alpha\pi^{*^2} - b + \pi^* < 0. \qquad \textbf{(B.4)}$$

The vanishing of the left-hand side of (25) gives a line, in the parameter space b, α, such that if (b, α) crosses that line from left to right a pair of complex conjugate eigenvalues enters the unit circle and a subcritical Neimark–Hopf bifurcation occurs at which the fixed point E^* is changed from unstable focus to stable focus, and a repelling closed invariant orbit is created around it [14] [see, e.g., Guckenheimer and Holmes (1983, p. 162)]. Just after its creation, such a closed curve is smooth and approximately of circular shape, with radius proportional to the square root of the distance of the point (b, α) from the bifurcation line, at least for values of (b, α) close to the bifurcation curve [see, e.g., Guckenheimer and Holmes (1983, p. 305)].

B.2 MAP $T_*^{(F)}$

The Jacobian matrix at the fixed point E^*, is

$$\mathrm{DT}^{(F)}(\pi^*, \pi^*) = \begin{bmatrix} 0 & 1 \\ -\dfrac{\alpha\pi^*}{b-\pi^*} & 1-\alpha+\dfrac{\alpha\pi^{*2}}{b-\pi^*} \end{bmatrix}. \qquad \textbf{(B.5)}$$

Hence the characteristic equation (B.1) has coefficients $\mathrm{Tr} = \mathrm{Tr}^{(F)} = 1-\alpha+\alpha\pi^{*2}/(b-\pi^*)$ and $\mathrm{Det} = \mathrm{Det}^{(F)} = \alpha\pi^*/(b-\pi^*)$. In this case, we have $P(1) = \alpha[(b-\pi^{*2})/(b-\pi^*)] > 0$ if $b > \pi^{*^2}$ (being $b > \pi^*$ in the parameter space Ω). At $b = \pi^{*^2}$ the fixed point E^* merges with the other fixed point B^* and one eigenvalue is equal to 1. This situation corresponds to a *transcritical* (or *stability exchange*) bifurcation. The other two conditions, $P(-1) > 0$ and $1-\mathrm{Det} > 0$, become, respectively,

$$\frac{b(2-\alpha)+\pi^{*^2}[(\pi^*+2)\alpha-2]}{b-\pi^*} > 0 \quad \text{and} \quad \frac{\pi^*(\alpha+1)-b}{b-\pi^*} < 0. \qquad \textbf{(B.6)}$$

The former is always satisfied for $(b, \alpha) \in \Omega$, whereas the vanishing of the numerator of the latter gives a bifurcation curve at which a subcritical Neimark–Hopf bifurcation occurs.

The Jacobian matrix of the map $T_*^{(F)}$, evaluated in the other fixed point B^*, becomes

$$\mathrm{DT}^{(F)}\left(\frac{b}{\pi^*}, \frac{b}{\pi^*}\right) = \begin{bmatrix} 0 & 1 \\ -\dfrac{\alpha}{\pi^*-1} & 1-\alpha+\dfrac{\alpha b}{\pi^*(\pi^*-1)} \end{bmatrix}. \qquad \textbf{(B.7)}$$

In this case,

$$P(1) = \alpha\frac{\pi^{*2}-b}{\pi^*(\pi^*-1)} > 0 \quad \text{if} \quad b < \pi^{*^2}.$$

This confirms that the stability properties of E^* and B^* are exchanged at $b = \pi^{*^2}$, when the two fixed points merge. The other conditions $P(-1) > 0$ and $1-\mathrm{Det} > 0$, become, respectively,

$$\frac{\pi^{*^2}(2-\alpha)+2(\alpha-1)\pi^*+\alpha b}{\pi^*(\pi^*-1)} > 0 \quad \text{and} \quad \frac{\alpha-\pi^*+1}{\pi^*-1} < 0. \qquad \textbf{(B.8)}$$

For $\pi^* > 1$, the first condition is satisfied for each $\alpha \in (0, 1)$, whereas the second condition is satisfied for $\alpha < \pi^* - 1$. Hence, if $1 < \pi^* < 2$ and $b < \pi^{*^2}$, then the equation $\alpha = \pi^* - 1$ defines a bifurcation curve at which a subcritical Hopf bifurcation occurs, the fixed point B^* being a stable focus for $\alpha < \pi^* - 1$. If $b > \pi^{*^2}$, then B^* is a saddle-point, with eigenvalues $0 < z_1 < 1$ and $z_2 > 1$, a straightforward consequence of the inequalities $P(-1) > 0$, $P(1) < 0$ and $P(0) > 0$. These arguments allow us to give the following classification of the stability properties as the parameters π^*, b, and α vary: If $\pi^* > 1$, then E^* is a locally stable fixed point if

$$b > \pi^{*^2} \quad \text{and} \quad b > b_h^{(F)}(\alpha), \quad \text{with} \quad b_h^{(F)}(\alpha) = \pi^*(\alpha+1); \qquad \textbf{(B.9)}$$

B^* is locally stable if $b < \pi^{*^2}$ and $0 < \alpha < \pi^* - 1$.

B.3. MAP $T_*^{(O)}$

The Jacobian matrix at the fixed point E^*, is

$$\mathrm{DT}_*^{(O)}(\pi^*, \pi^*) = \begin{bmatrix} 0 & 1 \\ 0 & 1-\alpha+\dfrac{\alpha\pi^{*2}}{b-\pi^*} \end{bmatrix}, \tag{B.10}$$

and so, the eigenvalues are always real, $z_1 = 0, z_2 = 1-\alpha+[\alpha\pi^{*2}/(b-\pi^*)]$, and E^* is stable if $b > \pi^*(1+\pi^*)$.

At B^*, we have

$$\mathrm{DT}^{(O)}\left(\frac{b}{1+\pi^*}, \frac{b}{1+\pi^*}\right) = \begin{bmatrix} 0 & 1 \\ \alpha\dfrac{b-\pi^*(1+\pi^*)}{\pi^{*2}(1+\pi^*)} & 1-\alpha+\dfrac{\alpha b}{\pi^*(1+\pi^*)} \end{bmatrix} \tag{B.11}$$

so $P(1) > 0$ for $b < \pi^*(1+\pi^*)$, thus confirming that at $b = \pi^*(1+\pi^*)$ the two fixed points exchange their stability, and the conditions, that is, $P(-1) > 0$ and $1 - \mathrm{Det} > 0$, are always satisfied, provided that $\pi^* > 1$. If $b > \pi^*(1+\pi^*)$, then the fixed point B^* is a saddlepoint, with $-1 < z_1 < 0$ and $z_2 > 1$, a straightforward consequence of the inequalities $P(-1) > 0$, $P(1) < 0$ and $P(0) > 0$. The local stability properties of the two fixed points, for if $\pi^* > 1$, can be summarized as follows: for $b > \pi^*(1+\pi^*)$, E^* is stable and B^* is unstable; for $\pi^* < b < \pi^*(1+\pi^*)$, E^* is unstable and B^* is stable.

B.4. MAP T_λ

The Jacobian matrix of the map (15) evaluated at the unique fixed point E_λ is

$$\mathrm{DT}_\lambda\left(\frac{1}{1-\lambda}, \frac{1}{1-\lambda}\right) = \begin{bmatrix} 0 & 1 \\ -\dfrac{\alpha}{b(1-\lambda)-1} & 1-\alpha+\dfrac{\alpha}{b(1-\lambda)-1} \end{bmatrix}.$$

the characteristic equation (B.1) has coefficients

$$\mathrm{Tr} = 1-\alpha+\frac{\alpha}{b(1-\lambda)-1} \quad \text{and} \quad \mathrm{Det} = \frac{\alpha}{b(1-\lambda)-1}.$$

The condition $P(1) > 0$ is always satisfied, hence the stability conditions reduce to

$$2-\alpha+\frac{2\alpha}{b(1-\lambda)-1} > 0 \quad \text{and} \quad \frac{\alpha-b(1-\lambda)+1}{b(1-\lambda)-1} < 0 \tag{B.12}$$

which are both satisfied in the set (see Figure 7).

$$\Omega_s^\lambda = \left\{(b,\alpha)\in\Omega \;\middle|\; \left(b < \frac{1}{1-\lambda} \text{ and } b < b_f^\lambda(\alpha)\right) \quad \text{or} \quad \left(b > \frac{1}{1-\lambda} \text{ and } b > b_h^\lambda(\alpha)\right)\right\}.$$

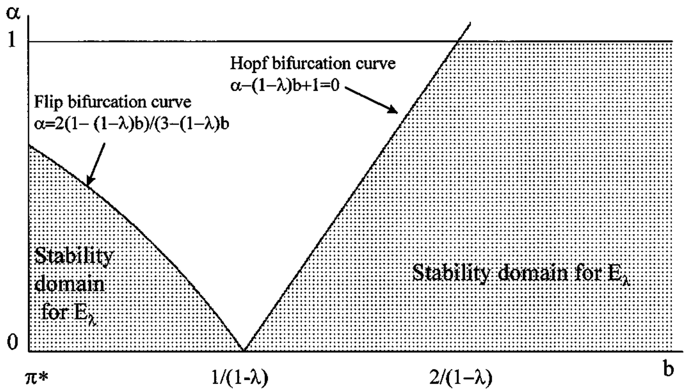

FIGURE 7. Stability regions for the fixed point E_λ of the map T_λ. The gray-shaded area represents the regions of local stability of E_λ in the parameters space (b, α).

In particular, the equation

$$b = b_h^\lambda(\alpha) = \frac{\alpha + 1}{1 - \lambda} \qquad \textbf{(B.13)}$$

gives a bifurcation curve at which a subcritical Neimark–Hopf bifurcation occurs.

APPENDIX C: PROOF OF PROPOSITION 2

Proposition 2 is a straightforward consequence of the following basic properties of linear two-dimensional discrete dynamical systems [see, e.g., Lorenz (1993, p. 255)]:

- If the eigenvalues z_1 and z_2 of the Jacobian matrix computed at the fixed point E^* are complex conjugate with modulus $|z_1| = |z_2| = \sqrt{\text{Det}} < 1$, where Det is the Jacobian determinant, then the convergence to the fixed point is oscillatory and the distance $\|(x_t, y_t) - E^*\|$ reduces at a rate proportional to $(\sqrt{\text{Det}})^t$.
- If the eigenvalues are real and both inside the unit circle, say $0 < |z_1| < |z_2| < 1$, then the distance $\|(x_t, y_t) - E^*\|$ reduces at a rate proportional to $|z_2|^t$, and if z_2 is positive, then the convergence is monotone in the long run, because the *dominant eigenvalue*, that is, the eigenvalue with largest modulus, determines the qualitative behavior of the linear system as $t \to \infty$.

Of course, the first case occurs if the discriminant $\Delta = \text{Tr}^2 - 4\,\text{Det} < 0$, and the second if the opposite (weak) inequality holds. In our case, let $B = \alpha\pi^*/(b - \pi^*)$. Then, $\text{Tr}^{(I)} = \text{Tr}^{(F)}(\alpha) = \text{Tr}^{(O)} = 1 - \alpha + B\pi^*$, $\text{Det}^{(F)} = B$, $\text{Det}^{(I)} = B\pi^*$, and $\text{Det}^{(O)} = 0$. Since $b > \pi^*$ and $\alpha \in (0, 1)$, then for all policies considered we have $\text{Tr}^{(P)} > 0$. Hence, in the case of real eigenvalues, the dominant eigenvalue is positive, given by $z_2^{(P)} = 0.5(\text{Tr}^{(P)} + \sqrt{\Delta^{(P)}}) > 0$. This means that whenever $\Delta^{(P)} > 0$ we have monotone convergence in the long run.

However, from the above equalities, it follows that

$$\Delta^{(I)} = \Delta^{(F)} - 4(\pi^* - 1)B; \quad \Delta^{(I)} = \Delta^{(O)} - 4B\pi^*; \quad \Delta^{(F)} = \Delta^{(O)} - 4B$$

and Proposition 2(i) follows from the fact that $B \searrow 0$ as $\alpha \searrow 0$.

The binary relations of Proposition 2(ii) can be obtained easily from the previous equalities, recalling that, when convergence is monotone, the speed of convergence is given by 0.5 $[\mathrm{Tr}^{(P)} + \sqrt{\Delta^{(P)}}]$, and when it is oscillatory by $\sqrt{\mathrm{Det}^{(P)}}$. For example, to see that $F \succ_s O$, notice that $0.5\,(\mathrm{Tr}^{(O)} + \sqrt{\Delta^{(O)}}) = \mathrm{Tr}^{(O)} = 1 - \alpha + B\pi^*$, while $\sqrt{\mathrm{Det}^{(F)}} = \sqrt{B}$. ■