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Global Analysis of a Nonlinear Model with Learning

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In this paper we consider a nonlinear model with learning, motivated by a paper by Dimitri (1988) dealing with a quadratic generalization of the linear model of Bray (1983), and we give a theoretical explanation of the numerical simulations shown in Dimitri's paper, where the critical role of the initial conditions is stressed. A weighted average with exponentially decreasing weights (fading memory), more general than the one proposed by Bray, is considered, and results on the global dynamics of such a learning process are obtained through the reduction to an equivalent two-dimensional map. We show that even if the map governing the long run behavior of the model with fading memory is the same as that of a standard adaptive rule, in the case of multiplicity of attractors the basins of attraction are different, that is, starting from the same initial condition different asymptotic behaviors may be obtained for the two kinds of learning. The main results of this paper are an exact delimitation of the basins of attraction of Dimitri's model, and the study of the bifurcations through which the structure of the basins becomes rather complex. The procedure outlined for the delimitation of the basins of attraction is quite general and it is suitable to be applied to other nonlinear models with the same kind of learning.

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Introduction

The concept of rational expectations equilibrium (REE) is often considered as the standard outcome for the long run behavior of an economic system, to which the system converges through a learning

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process. The fact that, differently from physical or biological systems, the evolution of economic and social systems is influenced by the presence of agents who, in making their decisions, use their memories and learn from the past experience, had already been stressed by Lindhal (1939) and Hicks (1939).

Many authors have proposed dynamic models, endowed with some learning rules which describe how the economic agents make forecasts by using the information gained from the data observed in the past, to prove convergence, or non convergence, to a REE (see e.g. Bray, 1983; Lucas, 1986; Marcet and Sargent, 1989; Bullard, 1994; Balasko and Royer, 1996). These models are often obtained by the introduction of additional dynamic variables, called expected values, that describe the evolution of agents' forecasts over time. In particular Bray (1983) examined a linear model where agents use a simple arithmetic mean of the prices observed in the past to obtain a "reasonable" forecast of the next price

$$z_t = {}_{t+1}p_t^{(e)} = \frac{1}{t}\sum_{k=1}^{t} p_k \qquad (1)$$

where, following Bray's notation, ${}_{t+1}p_t^{(e)}$ represents the price expected at period $(t+1)$, the expectation being held at the previous period t (that is, on the basis of the observations available in period t). In the following we shall use the more concise notation z_t instead of ${}_{t+1}p_t^{(e)}$. The example studied in Bray (1983) can be written in the form:

$$p_{t+1} = a + bz_t + \varepsilon_{t+1} \qquad (2)$$

where ε_t is a sequence of independently and identically distributed random variables with zero mean and finite variance. If z_t is computed as in (1) Bray shows that the sequence $\{z_t\}$ of the expected prices converges, with probability one, to the REE $\bar{p} = \dfrac{a}{1-b}$, provided that $b < 1$.

Holmes and Manning (1988) used the learning rule (1) in a nonlinear cobweb model, with supply and demand functions characterized by constant elasticities, and proved that such a type of learning has a stabilizing effect on the long run dynamics. However they remark that, in their particular nonlinear model, the short and intermediate run dynamics can be rather complex and of considerable interest.

In 1988 Dimitri proposed the following modification of Bray's model, obtained by the introduction of a quadratic nonlinearity:

$$p_{t+1} = \beta z_t^2 + \delta + \varepsilon_{t+1} \qquad (3)$$

with the aim of studying the effect of such a nonlinearity on Bray's result. The model (3) has two rational expectations equilibria when $\beta\delta < \frac{1}{4}$, given by

$$q^* = \frac{1+\sqrt{1-4\beta\delta}}{2\beta}; \quad p^* = \frac{1-\sqrt{1-4\beta\delta}}{2\beta} \qquad (4)$$

For model (3) with learning rule (1) Dimitri gives a necessary condition for the local stability of a REE α^*, expressed by

$$2\beta\alpha^* < 1 \qquad (5)$$

This condition gives a selection rule among the two different equilibria (4) since only the REE p^* satisfies (5), which is very similar to the condition $b < 1$ for Bray's model since both express the fact that the multiplier at the equilibrium must be less than one, but, as repeatedly stressed in Dimitri's paper, differing from the linear case the stability condition (5) is only local, so that some questions related to the global behavior of the model should be investigated. In fact Dimitri writes "... the evolution of the model is indeed very much dependent upon the starting position..." and presents numerical simulations by which he clearly shows that, with the same set of parameters, the convergence toward the locally stable equilibrium is not ensured for initial conditions far from the equilibrium point, or, even starting near the equilibrium, divergent price sequences are obtained if random shocks are applied.

However, the problem of the global analysis of the nonlinear model (3) is left open in Dimitri's paper and, to our knowledge, no serious steps have been made, in the economic literature, toward this direction. Indeed, many studies are devoted to the local stability analysis of the steady states of models with learning, whereas the question of the delimitation of the basins of attraction, essential in the understanding of the global dynamical properties of nonlinear models with coexisting attractors, has been somewhat neglected (the importance of the delimitation of the basins of attraction in nonlinear dynamic economic models has been recently emphasized by Brock and Hommes, 1997).

In the model (3) there are two possible kinds of asymptotic behavior: convergence to the fixed point p^* or divergence. Thus we can say that there are two coexisting attractors: the REE p^* and infinity. The delimitation of the boundary that separates the set of initial conditions that generate trajectories converging to the REE (i.e. the basin of attraction of p^*) from the set of initial conditions that generate unbounded trajectories (i.e. the basin of attraction of infinity) for the model (3) with learning rule (1) is the main goal of this paper. In order to obtain this result, we consider a more general class of models, with expectations on the current variable, of the form

$$p_{t+1} = f(z_t) \tag{6}$$

with initial condition (i.c.) p_1, where the expected price z_t is expressed as a weighted arithmetic mean

$$z_t = \sum_{k=1}^{t} a_{tk} p_k, \quad \text{with} \quad a_{tk} \geq 0,\ k = 1, \ldots, t \quad \text{and} \quad \sum_{k=1}^{t} a_{tk} = 1 \tag{7}$$

with weights of earlier observations distributed as the terms of a nonincreasing geometric progression of ratio $\rho \in [0, 1]$, that is

$$a_{tk} = \frac{\rho^{t-k}}{W_t}, \quad \text{with} \quad W_t = \sum_{k=1}^{t} \rho^{t-k} = \begin{cases} \frac{1-\rho^t}{1-\rho} & \text{if } 0 \leq \rho < 1 \\ t & \text{if } \rho = 1 \end{cases} \tag{8}$$

It can be noticed that this learning rule is a generalization of that proposed by Bray, since, for $\rho = 1$, (8) gives a uniform distribution of weights, $a_{tk} = \frac{1}{t}$ for each $1 \leq k \leq t$, so that (7) reduces to (1). For lower values of ρ we obtain the more realistic situation of a *fading memory*, in which earlier observations receive less weight than recent ones (see Friedman, 1979 and Radner's comment to Bray, 1983). In the other limiting case $\rho = 0$ (7) reduces to static expectations, that is $z_t = {}_{t+1}p_t^{(e)} = p_t$, which means that agents believe that present price will remain also in the next period. Thus, by using ρ as a varying parameter in the interval [0, 1], we can explore the effect of a learning rule with more or less rapidly fading memory, and in particular of Bray's learning (1) in the limiting case $\rho = 1$.

In this paper a general method for the analysis of the global properties of the class of models (6) with learning given by (7) and (8) is applied to Dimitri's model (3). The method is based on the reduction of the model with learning to an equivalent two dimensional map. We show that the difference equation governing the asymptotic behavior of a model with fading memory is the same as that obtained by a standard adaptive rule, but the basins of attraction are different, and may be rather complex.

In Section 1 we show how the reduction of the model (6) with learning to a two-dimensional map is obtained, and we discuss the relation between the attractors of such a map and those of the model (6) endowed with a standard adaptive learning. The main result given in this section is related to the method for the delimitation of the basins of attraction of the model (6) with fading memory.

In Section 2 these results are applied to the study of the basins of attraction of (3) with fading memory, and the limiting case $\rho \to 1$ is used to explain the structure of the basins, and their bifurcations, for the model (3) with Bray's learning (1), so that a theoretical explanation of the numerical results presented in Dimitri (1988) is given.

1. *General Results*

The model (6), endowed with learning rule (7) and geometric weights (8), i.e.

$$z_t = {}_{t+1}p_t^{(e)} = \sum_{k=1}^{t} \frac{\rho^{t-k}}{W_t} p_k \tag{9}$$

can be written as a first order non-autonomous recurrence in the expected prices:

$$z_{t+1} = \sum_{k=1}^{t+1} \frac{\rho^{t+1-k}}{W_{t+1}} p_k = \frac{\rho W_t}{W_{t+1}} \sum_{k=1}^{t} \frac{\rho^{t-k}}{W_t} p_k + \frac{1}{W_{t+1}} p_{t+1} = \tag{10}$$

$$= \frac{\rho W_t}{W_{t+1}} z_t + \frac{1}{W_{t+1}} f(z_t),\ t = 1, 2, \ldots$$

where W_t, defined in (8), is the t^{th} partial sum of the geometric series of ratio ρ, so that it can be obtained recursively as

$$W_{t+1} = 1 + \rho W_t, \quad W_1 = 1 \tag{11}$$

The recurrence (10), with $\rho W_t = W_{t+1} - 1$, can be written as an adaptive rule

$$z_{t+1} = z_t + \alpha_t (p_{t+1} - z_t) \tag{12}$$

with a time-dependent speed of adjustment

$$\alpha_t = \frac{1}{W_{t+1}}$$

From the definition (8) of W_t it follows that α_t is a decreasing sequence, with $\alpha_t \in (0, 1)$ for each t and $\alpha_t \to (1-\rho)$ as $t \to +\infty$. Hence the non-autonomous recurrence (10) tends to the *limiting form*

(13) $$z_{t+1} = g(z_t) = \rho z_t + (1-\rho) f(z_t)$$

This means that in the long run the model with learning (9) behaves like a model with a standard adaptive rule (see Appendix A for more details on adaptive learning and its relations with (9)). However, as suggested by the numerical results of Dimitri and the remarks of Holmes and Manning, the equivalence only holds for the limit sets, whereas the short run behavior may be very different, and may have a strong influence on the structure of the basins of attraction. In other words, the transient dynamics occurring during the early iterates of (10), when it is different from its limiting form (13), may be crucial for the asymptotic behavior of the trajectories when more than one coexisting attractor is present. That is, given an initial condition p_1 the trajectory (and the long run behavior) of (10) may be very different from the one obtained by iterating (13) with the same initial condition.

A global characterization of the dynamical properties of (10) can be obtained by writing (10) as a two-dimensional autonomous map. In fact, if W_t is taken as a dynamical variable, recursively defined by (11), the following two-dimensional autonomous map is readily obtained from (10):

(14) $$T: \begin{cases} z_{t+1} = \dfrac{\rho W_t}{1+\rho W_t} z_t + \dfrac{1}{1+\rho W_t} f(z_t) \\ W_{t+1} = 1 + \rho W_t \end{cases}$$

This map is equivalent to (6) with learning rule (9) in the sense that the projection on the z-axis of any trajectory of (14) starting from the point

(15) $$(z_1, W_1) = (p_1, 1)$$

where $W_1 = 1$ is obtained from (8) with $t = 1$, represents a sequence of expected prices $\{z_t\}$ from which the corresponding sequence $\{p_t\}$ of actual prices, starting from the initial price p_1, can be obtained as the images under the function f:

(16) $$p_{t+1} = f(z_t) \qquad t = 1, 2, \ldots$$

In other words, if $\{(z_1, W_1), (z_2, W_2), \ldots, (z_t, W_t), \ldots\}$ is the sequence generated by the map T starting from the initial condition $(z_1, W_1) = (p_1, 1)$, then $\{z_1, z_2, \ldots, z_t, \ldots\}$ is the sequence of expected prices starting from $z_1 = p_1$, and $\{p_1, p_2 = f(z_1), \ldots, p_t = f(z_{t-1}), \ldots\}$ is the corresponding sequence of actual prices. Thus the study of the general model (6) with an infinite-horizon learning rule (9) is reduced to that of a two-dimensional map with initial conditions constrained on the line $W = 1$ (*line of initial conditions*). The dynamics with Bray's expectations (1) are obtained for $\rho = 1^-$. This is the basic idea from which the results of this section follow.

Since the initial conditions are to be taken on the line $W = 1$, the trajectories are confined in the half-plane $W > 1$. In fact this half-plane is mapped into itself by T because the second difference equation in (14), which gives the dynamics of the variable W, is independent of z and gives a monotonically increasing sequence (the partial sums of a geometric series of ratio ρ). Furthermore, if $0 \le \rho < 1$, the sequence $\{W_t\}$ converges to the sum of the geometric series

(17) $$W^* = \frac{1}{1-\rho}$$

For $0 \le \rho < 1$ the line $W = W^*$ is an invariant and globally attracting line for the map T, on which the ω-limit sets of all its trajectories must be located. For this reason we shall call this line *line of ω-limit sets*. The restriction of T to this line is given by the one-dimensional map

(18) $$g_\rho(z) = \rho z + (1-\rho) f(z)$$

already obtained in (13) as the limiting form of the non-autonomous recurrence (10). The map (18) will be called *limiting map*, since it governs the asymptotic behavior of the map T. This implies, as proved in Bischi *et al.* (1996), that any k-cycle $A = \{z_1^*, \ldots, z_k^*\}$ of the map $g_\rho(z)$ is in one-to-one correspondence with a k-cycle $\mathcal{A} = A \times \{W^*\} = \{(z_1^*, W^*), \ldots, (z_k^*, W^*)\}$ of the map T, located on the line of ω-limit sets. Some properties of the map (18) are given in the Appendix A. The stability of the attractors of the model (6) with learning (9), and their basins of attraction, can be studied on the basis of the following proposition, which is a summary of the main results given in Bischi *et al.* (1996):

Proposition 1. Let A be a k-cycle, $k \ge 1$, of the map $g_\rho(z)$, $0 \le \rho < 1$. Then

(*i*) if A is attracting for the limiting map $g_\rho(z)$, then the set $\mathcal{A}=A\times\{W^*\}$ is an attracting cycle of the map T, and $f(A)$ is an attracting cycle of the model (6) with learning (9);
(*ii*) the basin of attraction D_1 of the attractor $f(A)$ of the model (6) with learning (9) is given by the intersection of the two-dimensional basin $\mathcal{B}$ of the cycle $\mathcal{A}$ of the map T with the line of initial conditions $W=1$.

A sketch of the proof of this proposition is given in Appendix B.

We recall that the case $k=1$ corresponds to a fixed point z^* of $g_\rho(z)$, and $f(A)=f(z^*)=z^*$ is a REE, since the fixed points of $g_\rho(z)$ are also fixed points of $f(z)$ (see Appendix A).

Part (i) of Proposition 1 confirms that the asymptotic behavior, i.e. the kinds of attractors and their stability properties, are the same as those of a standard adaptive learning rule with adaptive coefficient $\alpha=1-\rho$. For example, a sufficient condition for the attractivity of a REE z^*, under learning (9) with $\rho<1$, is given by $|g'_\rho(z^*)|<1$, that is,

$$-\frac{\rho+1}{1-\rho}<f'(z^*)<1 \tag{19}$$

This condition is always satisfied as $\rho\to 1^-$ provided that $f'(z^*)<1$. In other words, for the general model (6) with Bray's learning, the steady states z^* characterized by $f'(z^*)<1$ are locally attracting equilibria, whereas those with $f'(z^*)>1$ are repelling. This confirms, and extends, the stability results obtained, for particular models, by Bray (1983) and Dimitri (1988).

However, the most important implications of Proposition 1 are due to part (ii), since it suggests a general procedure to obtain the boundaries of the basins of attraction when two or more coexisting attractors are present, as often occurs in the case of nonlinear models. In these cases the knowledge of the exact structure of the basins of attraction is crucial, as suggested by the numerical simulations of Dimitri, especially when a random component is considered as in (2) and (3). Such knowledge cannot be obtained from the limiting map g_ρ, because the initial conditions are to be taken on the line $W=1$, whereas g_ρ only governs the dynamics near the line of ω-limit sets $W=W^*$. This means that only a global knowledge of the two-dimensional map T allows one to follow the whole trajectory from the line of the initial conditions to that of the ω-limit sets, thus taking into account the role of the short-run behavior, during which the dynamics is not governed by the limiting map g_ρ.

In the limiting case $\rho\to 1^-$, even if the line of the ω-limit sets moves infinitely far from that of initial conditions, the z variable of the map T always converges to a fixed point z^* of $f(z)$ (i.e. a REE), so the basin of attraction of $D_1(z^*)$ can be obtained from the knowledge of the two-dimensional basin of the points that generate trajectories of the map T that indefinitely approach the line $z=z^*$.

We observe that even if the trajectories of map T starting from the line of initial conditions $W=1$ are entirely included in the region of the phase plane (z, W) with $W\in(1, W^*)$, we shall study the properties of the map T in the larger region with $W\in(-1/\rho, W^*)$. In fact map (14) is a rational map which is not defined in whole plane, because the denominator of the first component vanishes on the points of the line

$$W=-\frac{1}{\rho} \tag{20}$$

which will be called a *singular line* below. Bischi and Gardini (1997) show that the presence of this line, and in particular the existence, on it, of points in which the first component of T assumes the form 0/0, has important consequences on the structure of the basins and their global bifurcations. From such points, called *focal points* in Bischi and Gardini (1997), fans of basins boundaries arise giving peculiar finger-shaped structures, called *lobes*.

It is easy to see that the focal points of map (14) are related to the existence of REE, since in a point (z^*, W^*), where z^* is a fixed point of $f(z)$, the first component of T has the form 0/0.

The existence of lobes, issuing from the focal points, has important consequences on the structure of the basins of attraction of the model with learning (9) whenever they intersect the line of initial conditions $W=1$. This occurrence causes the creation of basins with a complicated topological structure, such as basins formed by many disjointed intervals, as will be shown in the next section.

2. *Structure of the Basins of Attraction*

In this section we consider model (3) with learning rule (9) written in the form of the equivalent two-dimensional recurrence (14):

$$T_\varepsilon:\begin{cases} z_{t+1}=\dfrac{\rho W_t}{1+\rho W_t}z_t+\dfrac{1}{1+\rho W_t}(\beta z_t^2+\delta+\varepsilon_{t-1}) \\ W_{t+1}=1+\rho W_t \end{cases} \tag{21}$$

with initial condition (15). As explained in Section 1, the sequence $\{p_t\}$ of real prices can be obtained from that of expected prices $\{z_t\}$ as

$$p_{t+1} = \beta z_t^2 + \delta + \varepsilon_{t+1} \tag{22}$$

In order to study the global stability properties of the process with learning we first consider the deterministic part of the recurrence (21), i.e. the map T obtained by setting $\varepsilon_t = 0$ for each t, then we consider the effect of the stochastic component as a sequence of shocks which cause random variations of the z variable, represented by horizontal displacements of the phase point (z, W) of the map T. These displacements may, or may not, cause a crossing of the boundaries separating different basins of attraction.

In order to better explain this point we give and comment the results for model (3) with learning (1), obtained, in the limit case $\rho \to 1^-$, following the procedure that will be described below, in the Subsections 2.1 and 2.2.

If $\beta\delta < 1/4$, and the memory ratio ρ is sufficiently close to 1, then the only bounded attractor of the limiting map $g_\rho(z)$ is the REE p^*, in addition to an attractor at infinity (i.e. associated with unbounded sequences). Hence, under the above assumptions, there are only two different asymptotic behaviors of the map (21): convergence to the locally attracting fixed point $\mathcal{P} = (p^*, W^*)$ or divergence. Let $\mathcal{F}$ denote the boundary (or frontier) that, in the strip $W \in (-1/\rho, W^*)$, separates the basin $\mathcal{B}(\mathcal{P})$ from the basin $\mathcal{B}(\infty)$ of points generating unbounded sequences. As we shall prove in the Subsections 2.1, 2.2 and 2.3, the following results hold:

If $\rho \to 1^-$ then

- for $-2-\sqrt{2} < \beta\delta < 1/4$ the boundary $\mathcal{F}$ is formed by the vertical segment ω, given by the portion of the line of equation

$$z = q^* \tag{23}$$

with $W \in (-1/\rho, W^*)$, the segment ω_{-1}, given by the portion of the line of equation

$$\beta z + W + \beta q^* = 0 \tag{24}$$

with $W \in (-1/\rho, W^*)$, and the arc ω_{-2} of the ellipse of equation

$$\beta^2 z^2 + \beta z W + W^2 + (2 + \beta q^*) W + \beta(q^* + \delta) + 1 = 0 \tag{25}$$

located above the singular line $W = -1/\rho$. These three curves cross the singular line (20) in the focal points

$$F_q = (q^*, -1/\rho) \quad \text{and} \quad F_q = (p^*, -1/\rho) \tag{26}$$

- for $\beta\delta < -2-\sqrt{2}$ the boundary $\mathcal{F}$ also includes higher rank preimages ω_{-k}, $k > 2$, of the line ω, which have loops that bound lobes of $\mathcal{B}(\infty)$, issuing from the focal points F_q and F_p, as shown Figures 1, 2 and 3.

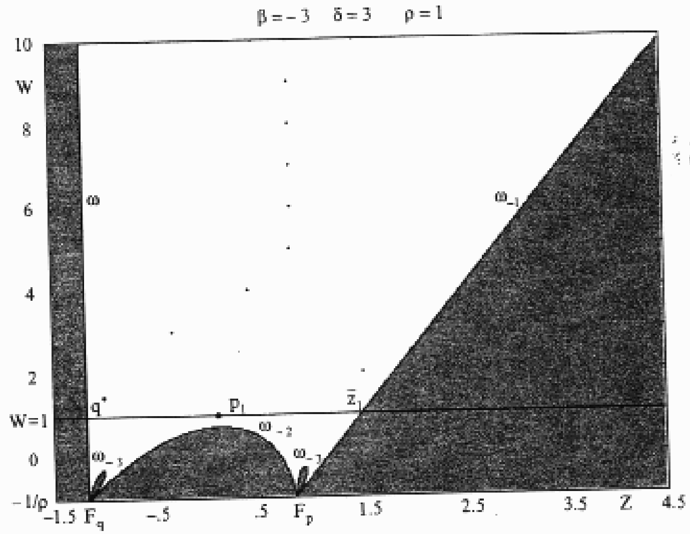

Figure 1

Phase space (z, W) of the map T with $\beta = -3$, $\delta = 3$, $\rho = 1$ (the limiting case of Bray's learning). The line $W=1$ represents the line of initial conditions. The white area represents the set of points that generate trajectories $\{z_t, W_t\}$ of T with $z_t \to p^*$, the grey-shaded area represents the set of points that give diverging trajectories. The portion $(q^*, \bar{z}_1)$ of the line $W = 1$ included inside the white region represents the one-dimensional basin of attraction $D_1(p^*)$ of the REE p^* for the model with learning. For this set of parameters there are two lobes, bounded by the curves ω_{-3}, issuing from the focal points F_q and F_p. The arc ω_{-2} has not reached the line of initial conditions, hence $D_1(p^*)$ is a unique interval. The dots represent the first 10 points of a trajectory, converging to the REE p^*, that starts from the initial condition p_1 belonging to $D_1(p^*)$.

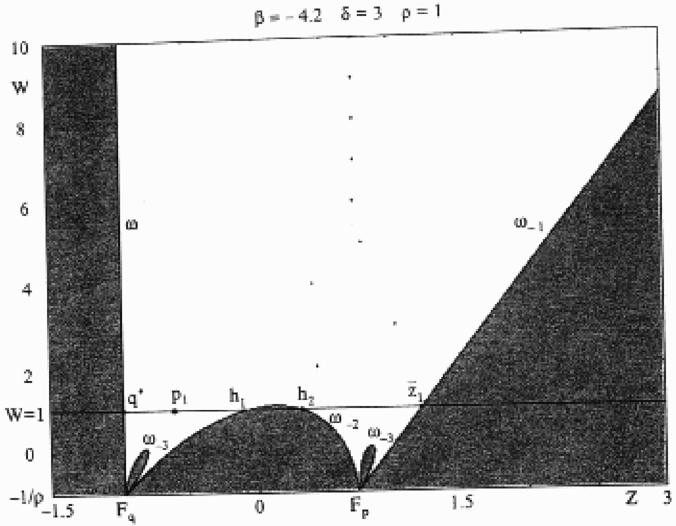

Figure 2

With the same values of the parameters δ and ρ as in Figure 1 the basins are represented for $\beta = -4.2$. The basin of attraction $D_1(p^*)$ of the REE $p^* = 0.734$ is formed by the union of two disjointed intervals, separated by a "hole" (h_1, h_2) of $\mathcal{B}(\infty)$. The dots represent the first 10 points of a trajectory, converging to the REE p^*, that starts from the initial condition p_1 belonging to $D_1(p^*)$.

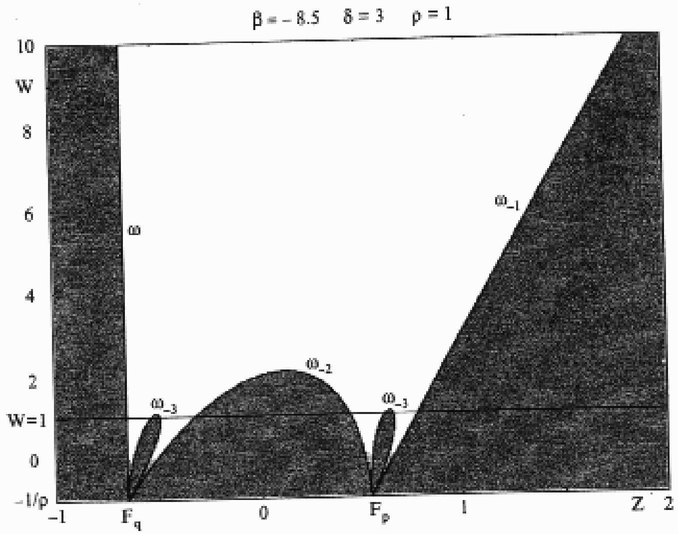

Figure 3

With $\beta = -4$, and the parameters δ and ρ as in the previous figures, the basin $D_1(p^*)$ is split into 4 disjointed intervals, separated by three "holes" of $\mathcal{B}(\infty)$.

Notice that, in the figures, $\beta > 0$ and $\delta > 0$ have been used, as in the numerical simulations proposed by Dimitri (1988), even if, due to the meaning of the parameters of the model, $\beta > 0$ and $\delta < 0$ should be considered. The only difference is that with $\beta > 0$ we have $q^* > p^*$ whereas in our figures, obtained with $\beta < 0$, $q^* < p^*$. This fact does not cause any substantial change in the geometrical arguments that will be described below, and the main results are not affected by this different choice since, as we shall see, the bifurcations of the basins of attraction, as well as the local bifurcations of the map g_p, are only influenced by the aggregate parameter $\sigma = -\beta\delta$. Hence, in the following we shall use the same parameters as in Dimitri's paper, in order to compare the results of our analysis with his simulations.

On the basis of Proposition 1, the knowledge of $\mathcal{F}$ allows us to obtain an exact delimitation of the one-dimensional basin of attraction $D_1(p^*)$ of the REE of the model (3) with learning (1), together with its bifurcations, occurring as $\sigma = -\beta\delta$ is increased (we recall that β represents the "weight" of the nonlinear term in (3)):

Proposition 2. Let us consider the model (3) with learning (1). Then
(*i*) for $\beta\delta > -27/4 - 2\sqrt{6}$ the basin of attraction of the REE p^* is a unique interval

$$D_1(p^*) = \mathcal{B}(\mathcal{P}) \cap \{W = 1\} = (q^*, -q^* - 1/\beta)$$

bounded by the intersections of the line of initial conditions $W = 1$ with the two lines ω and ω_{-1}, of equations (23) and (24) respectively;

(*ii*) at $\beta\delta = -27/4 - 2\sqrt{6}$ a bifurcation occurs at which the basin $D_1(p^*)$ is transformed from a unique interval into the union of two disjointed intervals, due to the contact of the ellipse ω_{-2} with the line $W = 1$ at the point $z_H = \frac{1}{2\beta}$. Just after the bifurcation, for $\beta\delta < -27/4 - 2\sqrt{6}$, the two intervals are separated by a "hole", whose points belong to the basin of infinity, with middle point z_H and extrema given by the z coordinates h_1 and h_2 of the two intersections of ω_{-2} with the line $W = 1$;
(*iii*) as $\beta\delta$ is further decreased other bifurcations occur, due to contacts between lobes $\omega_{-k} \in T^{-k}(\omega)$, with $k > 2$, and the line of initial conditions $W = 1$. Just after the contact of ω_{-k}, 2^{k-2} new "holes" of the basin of infinity are created, so that $D_1(p^*)$ becomes the union of 2^{k-1} disjoint intervals.

Before giving the proof of these results, we illustrate their meaning by some numerical simulations, shown in Figures 1, 2 and 3.

Figure 1 is obtained with the same set of parameters as those used in Table 5 of Dimitri's paper, i.e. $\beta = -3$ and $\delta = 3$. The basins $\mathcal{B}(\mathcal{P})$ and $\mathcal{B}(\infty)$ are represented by different colors, white and grey respectively. Since $\beta\delta = -9 > -27/4 - 2\sqrt{6}$ the basin of the REE $p^* = 0.847$ is given by the unique interval $D_1(p^*) = (q^*, -q^* - 1/\beta) = (-1.18, 1.51)$, according to Proposition 2. For example, starting from the initial price $p_1 = 0.1 \in D_1(p^*)$ we obtain the sequence of expected prices $z_2 = 1.535$, $z_3 = -0.333$, $z_4 = 0.417$, z_5 0.829, $z_6 = 0.847$, ... converging to p^* (the corresponding trajectory of the map T is shown in Figure 1), whereas starting from $p_1 = 1.6 \notin D_1(p^*)$, we obtain the diverging sequence $z_2 = -1.54$, $z_3 = -2.39$, $z_4 = -5.36$, $z_5 = -20.94$, $z_6 = -236.26$, ...(not represented in the figure).

The global analysis of the basins of the deterministic part of model (3) allows us to obtain information about the maximum spread of the stochastic variable which ensures convergence to the REE p^*. In fact, the line ω_{-1}, which forms the right part of the boundary $\mathcal{F}$, intersects the line $W = W_t = t$, on which is located the phase point after t time steps, in the point of z-coordinate

$$\bar{z}_t = -q^* - \frac{t}{\beta} \tag{27}$$

Hence, in the situation shown in Figure 1, a stochastic disturbance ε_t, applied at time t, does not cause divergence if the phase point of the map T remains inside the interval

$$D_t(p^*) = (q^*, \bar{z}_t) \tag{28}$$

whose width increases as t increases. For example, with the parameters used in Figure 1, we obtain $\bar{z}_1 \simeq 1.51$, $\bar{z}_5 = 2.85$, $\bar{z}_{10} \simeq 4.51$. This suggests that the maximum spread of the random variable which does not cause divergence increases with t, that is, the stable equilibrium is more robust as economic agents acquire more knowledge. However, as noticed by Dimitri on the basis of his numerical simulations, at each time step a random perturbation with a relatively high variability can give a strictly positive probability of non-convergence. From our analysis such qualitative considerations can be changed into quantitative ones, in the sense that at each time step (28) gives us the exact distance of the boundaries of $\mathcal{B}(\infty)$ from p^*.

Figure 2 is obtained with the same value of the parameter δ as in Figure 1, and $\beta = -4.2$, so that we are just after the bifurcation described in part (ii) of Proposition 2. In fact, in the situation shown in Figure 2 we have $D_1(p^*) = (q^*, h_1) \cup (h_2, -q^* - 1/\beta) = (-0.97, -0.078) \cup (0.317, 1.21)$. Starting from the initial price $p_1 = -0.6$, we obtain a sequence $\{z_t\}$ converging to p^* (shown in Figure 2) whereas starting from $p_1 = 0.1$ we obtain the sequence $z_2 = 1.52$, $z_3 = -1.25$, $z_4 = -1.84$, $z_5 = -3.71$, $z_6 = -12.27$, $z_7 = -100.46$, ...diverging to $-\infty$.

Thus, for the set of parameters used to obtain Figure 2 we have a non-connected basin of the REE, formed by two disjointed intervals of which only one contains the point p^*, called the *immediate basin* of the attracting point p^*. In such a situation, a price sequence starting from an initial price near p^*, i.e. in its immediate basin, may give diverging dynamics if a small random shock brings it into the basin $\mathcal{B}(\infty)$, whereas a larger shock may bring the initial price inside the other part of the basin of p^*, beyond the hole, so that the future evolution of the process with learning will remain convergent to p^*. This is a counterintuitive result arising from the global analysis of the basin of attraction of the stable REE. From Figure 2 it can be seen that ω_{-2} does not intersect the lines $W = W_t$ for $t > 1$. This implies that as t increases an exogenous perturbation, which causes a horizontal displacement of the phase point, cannot bring it inside the hole for sufficiently high values of t. Furthermore, as remarked above, the size, in the z direction, of $\mathcal{B}(\mathcal{P})$, increases for higher values of W, so that stronger shocks are necessary to bring the phase point inside $\mathcal{B}(\infty)$. These two features of the global structure of the basins of the map T imply that the system is less vulnerable, with respect to exogenous perturbations, as time goes on.

If $\sigma = -\beta\delta$ is further increased new holes are created, as shown in Figure 3, obtained with $\delta = 3$ and $\beta = -8.5$. In this case a trajectory converging to $p^* = 0.538$ is obtained, for example, starting from $p_1 = -0.4$, whereas starting from $p_1 = 0.62$, which is closer to the REE, a diverging trajectory is obtained.

2.1. *A General Procedure for the Delimitation of the Basin Boundaries*

We now describe the procedure followed to obtain the boundary $\mathcal{F}$ in the model with an exponentially fading memory, described by the weighted average (9) with $0 < \rho < 1$. For each $0 \leq \rho < 1$ and $\beta\delta < 1/4$ the map T has two fixed points: $\mathcal{Q} = (q^*, W^*)$ and $\mathcal{P} = (p^*, W^*)$. The fixed point $\mathcal{Q}$ is a saddle point, with unstable manifold along the line $W = W^*$ and local stable manifold along the invariant line $z = q^*$ (see Appendix B).

According to Proposition 1, the attractors of T are obtained from those of the limiting map $g_\rho(z)$, and, as explained in Appendix A, for any $\lambda \in (-1/4, 2)$ the map $g_\rho(z)$ has a unique bounded attractor. This attractor is the fixed point p^* if the stability condition (A6) holds true. As remarked in Appendix A, such a condition is always satisfied if the memory ration ρ is sufficiently close to 1, whereas for lower values of ρ the bounded attractor of the map $g_\rho(z)$ may be a cycle or even a chaotic interval. In any case, in the following we shall denote by A the bounded attractor of the limiting map g_ρ, and by $\mathcal{A}$ the corresponding bounded attractor of T located on the line $W = W^*$. Let $\mathcal{B}(\mathcal{A})$ denote the basin of attraction of $\mathcal{A}$, defined as the open set of points of the phase plane of T whose trajectories converge to $\mathcal{A}$.

Since $\mathcal{A}$ is the only bounded attractor of the map T, the complementary set of the closure of $\mathcal{B}(\mathcal{A})$ is $\mathcal{B}(\infty)$, i.e. the set of points having divergent trajectories. The goal of this section is the determination of the frontier $\mathcal{F} = \partial\mathcal{B}(\mathcal{A}) = \partial\mathcal{B}(\infty)$ that separates the two basins. Such a frontier behaves as a repelling set for the points near it, since it acts as a watershed for the trajectories of the map T. Points belonging to $\mathcal{F}$ are mapped into $\mathcal{F}$ both under forward and backward iteration of T: more exactly $T(\mathcal{F}) \subseteq \mathcal{F}$, $T^{-1}(\mathcal{F}) = \mathcal{F}$ (see Mira *et al.*, 1994; Mira *et al.*, 1996, ch. 5). This implies that if a saddle-point $\mathcal{Q}$ belongs to $\mathcal{F}$, then $\mathcal{F}$ must also contain the whole stable manifold (see Gumowski and Mira, 1980; Mira *et al.*, 1996). We consider the local stable manifold of the saddle point $\mathcal{Q}$, located on the line $z = q^*$ (see Appendix B). A trajectory of T starting on the left of the line $z = q^*$ is diverging, and this line behaves as a repelling line, because the unstable manifold of the saddle $\mathcal{Q}$, along the line $W = W^*$, has a branch pointing toward the bounded attractor $\mathcal{A}$, and the opposite branch going to infinity (see Figure 4). The other parts of $\mathcal{F}$ can be obtained by taking all the pre-images of the local stable set (see Gumowski and Mira, 1980; or Mira *et al.*, 1996)

$$\mathcal{F} \subset \cup_{n \geq 0} T^{-n}(\{z = q^*\}) \tag{29}$$

where $T^{-n}(z, W)$ represents the set of all the rank-n pre-images of a given point (z, W).

The map T, given by (21) with $\varepsilon_t = 0$, is a noninvertible map. This means that even if a point (z_t, W_t) has a unique image under the application of T, $(z_{t+1}, W_{t+1}) = T(z_t, W_t)$, the backward iteration of T is not uniquely defined, since given a point (z_{t+1}, W_{t+1}) its pre-images (z_t, W_t) are obtained by solving a second degree algebraic system, that has two real solutions, given by

$$T^{-1}: \begin{cases} z_t = \dfrac{(1 - W_{t+1}) \pm \sqrt{(1 - W_{t+1})^2 + 4\beta(z_{t+1}W_{t+1} - \delta)}}{2\beta} \\ W_t = \dfrac{W_{t+1} - 1}{\rho} \end{cases} \tag{30}$$

if $\Delta = (1 - W_{t+1})^2 + 4\beta(z_{t+1}W_{t+1} - \delta) > 0$, or no real solutions if $\Delta < 0$. In the former case we say that the point (z_{t+1}, W_{t+1}) has two pre-images, given by (30), in the latter case we say that the point (z_{t+1}, W_{t+1}) has no pre-images. Following the terminology of Mira *et al.* (1996) we say that the plane is divided into two regions, called Z_2 and Z_0, whose points have two or no pre-images respectively. These two regions are separated by the curve defined by the equation

$$\Delta(z, W) = (1 - W)^2 + 4\beta z W - 4\beta\delta = 0 \tag{31}$$

called critical curve LC (from the French "Ligne Critique"). The points belonging to LC have two coincident pre-images located on the line LC_{-1} given by

$$\rho W + 2\beta z = 0 \tag{32}$$

obtained from the first of (30) with $\Delta = 0$ and $W_{t+1} = \rho W_t + 1$. The curve LC_{-1} can also be obtained as the locus of points at which the determinant of the Jacobian matrix of T vanishes, and $LC = T(LC_{-1})$ (see Gumowski and Mira, 1980; or Mira *et al.*, 1996 for a review of the method of critical curves in two-dimensional noninvertible maps). The curves LC and LC_{-1} for the map (21) are represented in Figure 4.

The knowledge of the curves LC and LC_{-1} is important in the computation of the pre-images of the local stable set of $\mathcal{Q}$ from which $\mathcal{F}$ is obtained according to (29). The segment ω of the line $z = q^*$ entirely lies inside the region Z_2. In fact, under the assumption $4\beta\delta < 1$, which ensures the existence of the fixed points, we have $\Delta(q^*, W) > 0$ since $4\beta q^* W > 2W$ from the expression of q^* given in (4). From (30) with $z_{t+1} = q^*$ we get the two preimages of ω: one belongs to the same invariant line $z = q^*$ and the other one is on the line of equation

$$\beta z + \rho W + \beta q^* = 0 \tag{33}$$

This line intersects the line of initial conditions $W = 1$ in the point of z-coordinate

$$\bar{z}_1 = -q^* - \frac{\rho}{\beta} \tag{34}$$

According to (29), also line (33) belongs to $\mathcal{F}$. The portion of this line located below the critical curve LC belongs to the region Z_2, hence it has two pre-images, say ω^1_{-2} and ω^1_{-2}, whose equations can be obtained from

(30) with $z_{t+1} = -\rho W_{t+1} - \beta q^*$. These two pre-images are located at opposite sides with respect to the line LC_{-1} and merge in the point H, given by the merging pre-images of the point $H_1 = \omega_{-1} \cap LC$ (see Figure 4). After some algebraic manipulation it is possible to see that such pre-images belong to the curve of equation

$$\beta^2 z^2 + \beta\rho z W + \rho^3 W^2 + \rho(2\rho + \beta q^*)W + \beta(q^* + \delta) + \rho = 0 \tag{35}$$

This curve is an ellipse if $\rho > 1/4$, a parabola if $\rho = 1/4$, a branch of hyperbola if $0 < \rho < 1/4$. The curve (35) crosses the singular line (20) at the focal points F_q and F_p defined in (26). According to (29) also the curve

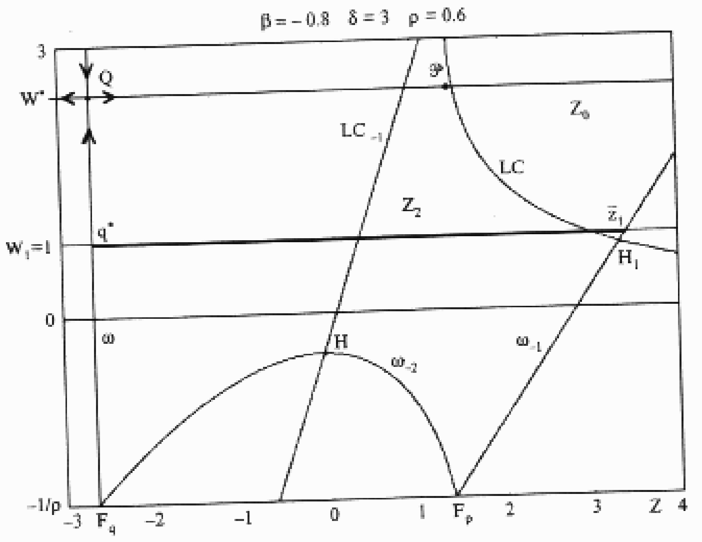

Figure 4

The frontier $\mathcal{F}$ that separates $\mathcal{B}(\mathcal{P})$ from $\mathcal{B}(\infty)$, given by the stable set of the saddle point $\mathcal{Q} = (q^*, W^*)$, is represented for memory ratio $\rho = 0.6$ and parameters $\delta = 3$, $\beta = -0.8$. The line $W = W^*$ is the line of ω-limit sets, the thicker part of the line of initial conditions $W = 1$ represents the one-dimensional basin of attraction $D_1(p^*)$ under the process with learning. The curves denoted by LC and LC_{-1} are portions the critical curve (31) and its preimage (32). For this set of parameters the point $H_1 = LC \cap \omega_{-1}$ is below the line $W = 1$, hence its preimage H is below the line $W = 0$. This implies that there are no lobes issuing from the focal points F_q and F_p.

(35) belongs to the frontier $\mathcal{F}$, but we are only interested in its portion above the singular line, denoted by ω_{-2} in Figure 4. As long as the point of intersection H_1 between LC and the line ω_{-1} is below the line $W = W_1 = 1$, the whole curve ω_{-2} lies below the z axis, so that the pre-images of ω_{-2} are located below the singular line, as can be easily deduced from the second component of (30).

As $\sigma = -\beta\delta$ increases the critical curve LC moves upwards, and when it reaches the line $W = W_1 = 1$ the curve ω_{-2} reaches the z axis, so that its pre-images ω_{-3} appear, issuing from the two focal points F_q and F_p (as proved in Bischi and Gardini, 1997). For example, in Figure 5 the point H_1 is above the line $W = 1$, and consequently its pre-image H, which is on the top of the arc ω_{-2}, is above the line $W = 0$. The two pre-images of the portion of ω_{-2} above the z axis are the lobes issuing from the focal points.

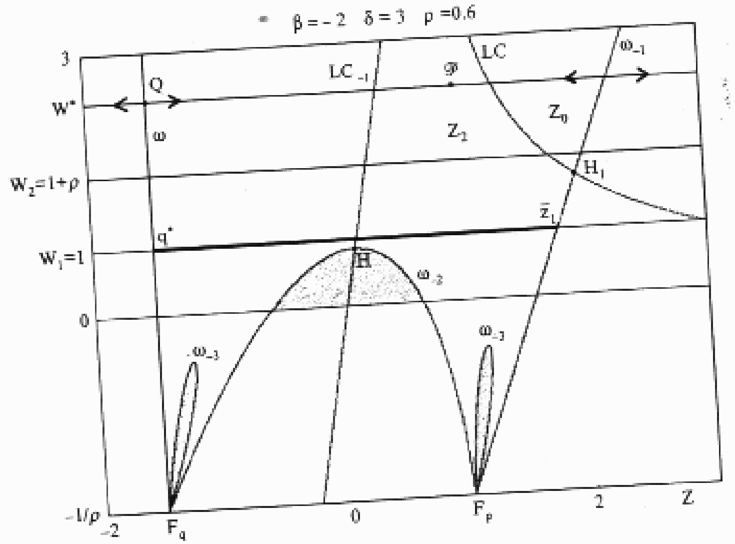

Figure 5

With $\beta = -2$, and the parameters δ and ρ as in the fig. 4, the point $H_1 = LC \cap \omega_{-1}$ is above the line $W = W_1 = 1$, but below the line $W = W_2 = 1 + \rho$. Hence its preimage H is between the lines $W = 0$ and $W = W_1 = 1$. This implies that the two preimages of the grey shaded portion of $\mathcal{B}(\infty)$ are given by two lobes, denoted by ω_{-3}, issuing from the focal points F_q and F_p. The basin of attraction $D_1(p^*)$, represented by the thicker part of the line of initial conditions $W = 1$, is given by the interval $(q^*, \bar{z}_1)$.

2.2. *Bifurcation from Connected to Non Connected Basin*

For the set of parameters used in Figure 5 the arc ω_{-2} of $\mathcal{F}$ does not intersect the line of initial conditions $W = 1$, thus it does not affect the basin of attraction $D_1(A)$ of the bounded sequences of the process with learning. This is due to the fact that the point $H_1 = \omega_{-1} \cap LC$ is located below the line $W = W_2 = 1 + \rho$, and this implies that its pre-image H is located below the line of initial condition $W = W_1 = 1$. In fact, due to the particular structure of the second equation of the map L, we have

(36) $$W_t = \sum_{k=0}^{t-1} \rho^k$$

and this in turn implies that the pre-images of any point of a line $W = W_t$ are located on the line $W = W_{t-1}$, as can be easily computed by the second equation of (30).

As $\sigma = -\beta\delta$ is increased, a value will be reached, say $\sigma = \sigma_1^*$, at which the point H_1 is on the line $W = W_2 = 1 + \rho$ and, as a consequence, the curve ω_{-2} becomes tangent to the line of initial conditions $W = W_t = 1$, the tangency point being the intersection between the line LC_{-1} and the line $W = 1$, say $(z_H, 1)$ where

(37) $$z_H = -\frac{\rho}{2\beta}$$

The value σ_1^*, which can be easily computed from the tangency condition between the curve of equation (35) and the line $W = 1$, represents a bifurcation of the basin $D_1(A)$ of initial conditions which generate bounded price sequences. In fact for $\sigma < \sigma_1^*$, the basin $D_1(A)$ is the interval $(q^*, \bar{z}_1)$, whereas for $\sigma > \sigma_1^*$, a *hole* is created around z_H, whose points belong to $\mathcal{B}(\infty)$, bounded by the two intersections $(h_1, 1)$ and $(h_2, 1)$ between the curve (35) and the line $W = 1$ (see Figure 6).

Notice that for the set of parameters used in Figure 6, i.e. $\delta = 3$, $\beta = -3$ and $\rho = 0.6$, the fixed point p^* is no longer attracting, since the stability condition (43) is not satisfied, and the attractor $A = \{z_1^*, z_2^*\}$ is a cycle of period 2, represented in Figure 6 by the two small circles on the line $W = W^*$. A remarkable fact is that the periodic point z_1^* falls inside the "hole", so that the trajectory starting from it does not converge to the cycle (it diverges).

The situation becomes even more complex as σ is further increased. A value $\sigma = \sigma_2^*$ is reached at which the point H_1 is on the line

$W = W_3 = 1 + \rho + \rho^2$: at this value of σ two lobes of $\mathcal{B}(\infty)$, bounded by ω_{-3}, reach the line of initial conditions, the tangency points being the two pre-images H^1_{-1} and H^2_{-1} of the point H (see Figure 6). This gives a second bifurcation of the basin $D_1(A)$, at which two new holes are created around the tangency points.

Other similar bifurcations occur at $\sigma = \sigma_k^*$, at which ω_{-k-1}, belonging to the set $T^{-k-1}(\omega)$, become tangent to the line of initial conditions. This implies that 2^k new holes are created. The result of this sequence of bifurcations is that the basin $D_1(A)$ becomes smaller and smaller, with the structure which is typical of a Cantor set.

In Figure 7, obtained for $\rho = 0.6$, $\delta = 3$ and $\beta = -4$, the attractor A is a chaotic interval, and its basin of attraction if given by the union of 16 disjoint intervals, separated by holes of $\mathcal{B}(\infty)$.

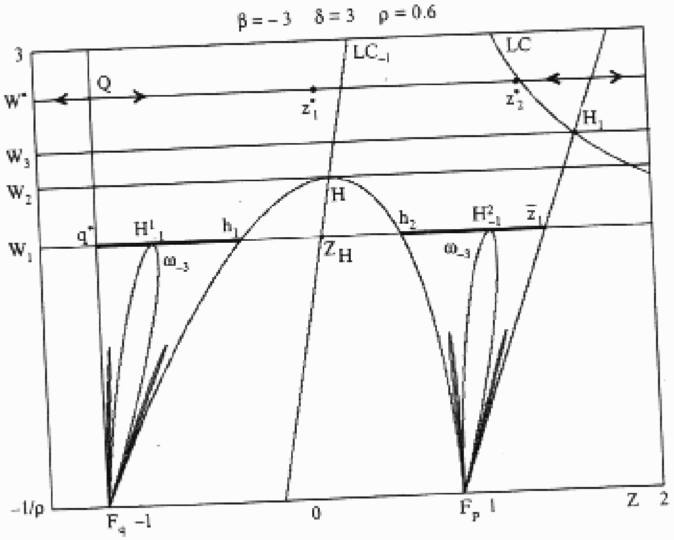

Figure 6

With $\beta = -3$, and the parameters δ and ρ as in the fig. 4, the bounded attractor of the process with learning is a cycle of period 2, $A = \{z_1^*, z_2^*\}$. The basin $D_1(A) = (q^*, h_1) \cup (h_2, \bar{z}_1)$ is formed by the union of two disjoint intervals. The point $H_1 = LC \cap \omega_{-1}$ belongs to the line $W = W_3 = 1 + \rho + \rho^2$. Hence its preimage H belongs to the line $W = W_1 = 1$. This is the bifurcation occurring at $\sigma = \sigma_2^*$ at which the lobes ω_{-3} are tangent to the line of initial conditions.

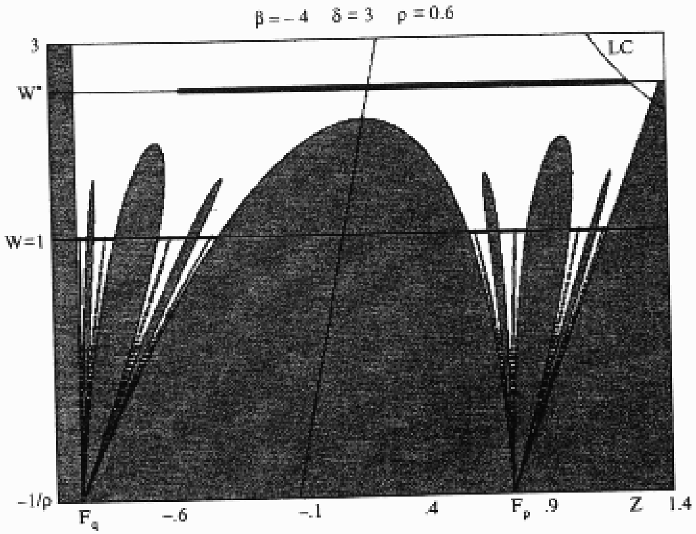

Figure 7

With $\beta = -4$, and the parameters δ and ρ as in the fig. 4, the bounded attractor A of the process with learning is a chaotic interval. The basin $D_1(A)$ is formed by the union of 16 disjoint intervals.

2.3. *The Limiting Case of Bray's Learning*

The global analysis of the basin boundaries described in the previous subsection holds for any value of the memory ratio ρ belonging to the interval (0, 1). In particular it holds in the limiting case $\rho \to 1^-$. In this case the singular line (20), where the focal points F_q and F_p (from which the lobes of $\mathcal{B}(\infty)$ arise) are located, has equation $W = -1$. The equations of the curves (24) and (25) are obtained, respectively, from (33) and (35) with $\rho = 1$. Furthermore, the simpler form of (25), with respect to (35), allows us an easy computation of the tangency condition between the arc ω_{-2} and the z axis, and the value $\sigma = 2 + \sqrt{2}$, at which the lobes issuing from the focal points appear, is obtained. Also the tangency condition between the arc ω_{-2} and the line of initial conditions $W = 1$ can be easily obtained, which gives the bifurcation value $\sigma_1^* = 27/4 + 2\sqrt{6}$ appearing in Proposition 2.

This completes the proof of Proposition 2.

Concluding Remarks

In this paper the global dynamical properties of a nonlinear model with expectations on the current variable, proposed by Dimitri (1988) as a quadratic generalization of the linear model of Bray (1983), are studied. In order to give a theoretical explanation of the remarks on the role of the initial conditions, given by Dimitri on the basis of numerical simulations, we have proposed the learning rule (9), where the expected price is computed as a weighted average of the values of the past, from which the simple average proposed by Bray, and used by Dimitri, can be obtained as a limiting case.

The study of the global dynamical properties of a model (6) with learning (9) is performed through the reduction to an equivalent two-dimensional map. This allowed us to define a general procedure to obtain the exact delimitation of the basins of attraction of the attracting sets. Our results show that the difference equation that governs the long-run dynamics of the model with fading memory (9) is the same as that of a standard adaptive learning rule, but, given an initial condition, the fate of the trajectory may be different if the nonlinear model is characterized by multiplicity of equilibria.

The global analysis of the basins given in this paper allowed us to obtain information about the maximum spread of a stochastic variable so that convergence to the REE is ensured. Our results suggest that the maximum spread of the random variable which does not cause divergence increases with time, i.e. a REE is more robust as economic agents acquire more knowledge.

In other words, in a nonlinear model with multiple equilibria, endowed with a learning rule of the form (9), the study of the limiting equation (13) is not sufficient to obtain a forecasting on the asymptotic behavior of the trajectories. In fact, only a global study of the two-dimensional map (14) can give a complete understanding of the fate of the trajectories of the model with learning. Furthermore, the delimitation of the basins of attraction of the two-dimensional map (14) is necessary to understand the possible effects of stochastic disturbances, even during the short and intermediate run dynamics. This fact is in agreement with the analysis given in Vercelli (1994) where, in the context of a general discussion on the concept of equilibrium, it is stressed that the robustness of an equilibrium situation in an economic model can be only established through a global analysis of the endogenous dynamic behavior of the model.

In this paper we have applied our results to Dimitri's model, following his claim for a global analysis of his nonlinear model. However, the methods used in this paper can be applied to the study of the global properties of other nonlinear models endowed with a learning rule (9), or its limiting case (1). For example, very similar structures of the basins are obtained for other models represented by unimodal maps, like the cobweb model proposed by Jensen and Urban (1984), or the one proposed by Artstein (1983), endowed with learning (9). An application to a cobweb model in which the limiting map is bimodal, with coexisting attracting cycles, is given in Bischi and Naimzada (1995).

Applications of the procedure exposed in this paper to fictitious plays (see e.g. Shapley, 1964, and references therein) also give interesting results, both for the kinds of attractors and for the structure of the basins. Such applications will be the object of further studies.

APPENDIX A

Adaptive Learning, Infinite Memory and Limiting Map

If at each time period the expected price $z_t = {}_{t+1}p_t^{(e)}$ is computed according to the standard adaptive rule

(A1) $$z_{t+1} = z_t + \alpha (p_{t+1} - z_t)$$

where $\alpha \in [0, 1]$ is the constant speed of adjustment, the time evolution of the expected prices is governed by the difference equation

(A2) $$z_{t+1} = g_\alpha(z_t) = (1 - \alpha)\, z_t + \alpha f(z_t)$$

obtained by inserting the law of motion (6) into (A1). The properties of the map (A2) are well known. It is a convex combination of the map f and the identity map, so its graph is included inside the region between the graph of f and the diagonal. This implies that the map $g_\alpha(z)$ and the map $f(z)$ have the same fixed points, which are REE. Instead, the cycles of g_α are in general different from those of f, because the two maps have different graphs for $\alpha < 1$. For example, even if the map f has cycles of arbitrary period, like in the case of a quadratic map, the map $g_\alpha(z)$ cannot have cycles at all for sufficiently low values of α. This can be intuitively understood since the map $g_\alpha(z)$ approaches the diagonal as $\alpha \to 0$, hence a value $\bar{\alpha} \in (0, 1)$ exists such that $g_\alpha(z)$ is an increasing function for any $\alpha \in (0, \bar{\alpha})$, and an increasing map cannot have cycles of period $k > 1$ (see e.g. Devaney, 1986 or Mira, 1987).

Notice that (A2) has the same form as the limiting map (13) with $\rho = 1 - \alpha$. Hence the limiting form of (10), which is also the limiting map governing the asymptotic behavior of the two-dimensional map T, has the same properties as (A2). This means, for example, that any more complex asymptotic behavior so that the convergence to a REE is not possible if the memory ratio ρ is sufficiently close to 1, is always true in the limit

$\rho \to 1$, i.e. in the case of Bray's learning (1). For the model (6) with $f(z) = (\beta z^2 + \delta)$, i.e. the function proposed by Dimitri, we have

$$g_\rho(z) = \rho z + (1-\rho)(\beta z^2 + \delta) \tag{A3}$$

which is a quadratic map conjugate, by the linear transformation $z = \frac{1}{\beta(1-\rho)}x - \frac{\rho}{\beta(1-\rho)}$, to Myrberg's map

$$x_{t+1} = x_t^2 - \lambda \tag{A4}$$

with parameter

$$\lambda = \frac{\rho^2 - 2\rho - 4\beta\delta(1-\rho)^2}{4} \tag{A5}$$

This is a well known map (see e.g. Mira, 1987, or Gumowski and Mira, 1980), which is also conjugate to the logistic map $y_{t+1} = \mu y_t(1-y_t)$ through the linear change of variable $y = -\frac{x}{\mu} + \frac{1}{2}$, from which the relation between the parameters is $\lambda = \frac{\mu^2}{4} - \frac{\mu}{2}$. Hence, like Myrberg and the logistic map, the map (A3) can have attracting cycles of any period and can also exhibit chaotic behavior (see e.g. Devaney, 1986; Baumol and Benhabib, 1989).

For any set of parameters β, δ and ρ the dynamical properties of the map $g_\rho(z)$ can be obtained from those of Myrberg's map with parameter λ given by (A5): the fixed point q^* is always repelling, whereas p^* is attracting for $1/4 < \lambda < 3/4$, corresponding to the condition

$$\frac{\rho^2 - 2\rho - 3}{4(1-\rho)^2} < \beta\delta < \frac{1}{4} \tag{A6}$$

For each $\lambda \in (3/4, 2)$ a unique bounded attractor exists, which can be a cycle of period $k > 1$ or a chaotic attractor. For $\lambda > 2$ no bounded attractors exist, i.e. the generic trajectory of Myrberg's map is diverging (see Mira, 1987).

It is evident that the sufficient condition (A6) for the attractivity of p^* is always satisfied for ρ sufficiently close to 1. This contrasts with the very rich dynamical behavior shown with low values of ρ (see Mira, 1987 for a description of the complex behavior of Myrberg's map or Devaney, 1986 for the logistic map). This suggests that high values of the memory ratio ρ, as well as low values of the speed of adjustment α in the standard adaptive rule, have a stabilizing effect on the asymptotic dynamics.

However, while the basin of attraction if the REE p^* under the standard adaptive learning (A1) is simply obtained from a study of the one-dimensional map $g_\alpha(z)$, the basin of the same REE for the model with learning (9) is different, since it must be obtained by the procedure indicated by part (ii) of Proposition 1.

This difference can be also understood by writing the adaptive learning rule as an infinite weighted average (see e.g. Nerlove, 1958 or Gandolfo, 1981 page 7):

$$z_t = \alpha \sum_{k=-\infty}^{t} (1-\alpha)^{t-k} x_k \tag{A7}$$

A comparison between (A7) and (9), with $\rho = 1 - \alpha$, shows a substantial difference between the two weighted averages, since the average (9) has a finite number of terms for each finite time t (the terms of such average tend to become infinitely many only in the limit $t \to +\infty$), whereas the average (A7) is always formed by infinitely many terms, for each t. Hence, even if the two averages become more and more similar as t increases, they are different at each finite time, especially for values of t close to the "starting point" of Bray's average (or its generalization (9)).

APPENDIX B

Sketch of the Proof of Proposition 1

The proof is based on the fact that the line $W = W^* = \dfrac{1}{1-\rho}$ is *invariant*, because $W_t = W^*$ implies $W_{t+1} = W^*$ for each t, and *globally attracting*, because the second component of (14) is such that the sequence $\{W_t\}$ is always monotonically convergent to W^*. This implies that the cycles of the two-dimensional map (14) and those of its restriction to the line $W = W^*$, given by the map $g_\rho(z)$ defined in (18), are in one-to-one correspondence. Moreover, if A is an attracting cycle of g_ρ then $\mathcal{A}$ is also attracting for T, and if A is a repelling cycle of g_ρ then $\mathcal{A}$ is a saddle cycle for T, with unstable manifold along the invariant line $W = W^*$, and stable manifold transverse to it. In fact, the Jacobian matrix of the map (14) is a triangular matrix that, computed in the points of the line $W = W^*$, becomes

(B1) $$DT(z, W^*) = \begin{bmatrix} g'_\rho(z) & \rho(1-\rho)^2[z - f(z)] \\ 0 & \rho \end{bmatrix}$$

At any fixed point z^* of $f(z)$, $DT(z^*, W^*)$ becomes diagonal, with eigenvalues

(B2) $$\lambda_1 = g'(z^*) = \rho + (1-\rho) f'(z^*) \quad \text{and} \quad \lambda_2 = \rho$$

and corresponding eigenvectors parallel to the z axis and the W axis respectively. It follows that a fixed point (z^*, W^*) of the map T is an attracting node if and only if $|g'_\rho(z^*)| < 1$, i.e. z^* is an attracting fixed point for the limiting map $g_\rho(z)$, whereas it is a saddle, with unstable manifold along the line $W = W^*$ and stable manifold along the vertical line $z = z^*$, if and only if z^* is a repelling fixed point for the map g. A similar reasoning holds for a k-cycle, with $k > 1$. In fact, if $A = \{z^*_1, z^*_2, \ldots, z^*_k\}$ is a k-cycle of the map $g_\rho(z)$ then any point (z^*_i, W^*)

is a fixed point of T^k and the Jacobian matrix $DT^k(z^*_i, W^*) = \prod_{i=1}^{k} DT(z^*_i, W^*)$ is a triangular matrix with eigenvalues

$$\lambda_1^{(k)} = \prod_{i=1}^{k} g'_\rho(z^*_i) \quad \text{and} \quad \lambda_2^{(k)} = \rho^k$$

and corresponding eigendirections along the line $W = W^*$ and transverse to it respectively. The eigenvalue $\lambda_1^{(k)}$ is the multiplier of the k-cycle A of the map $g_\rho(z)$, hence, as in the case of a fixed point, the corresponding cycle of T is an attracting node (saddle) when A is attracting (repelling) for g_ρ. From the definition of local stability it follows that when $\mathcal{A}$ is stable for T an open neighborhood around each periodic point (z^*_i, W^*) exists whose points generate trajectories of T converging to $\mathcal{A}$, and the union of these neighborhoods is called the local stable set, $\mathcal{S}_{loc}(\mathcal{A})$, of $\mathcal{A}$.

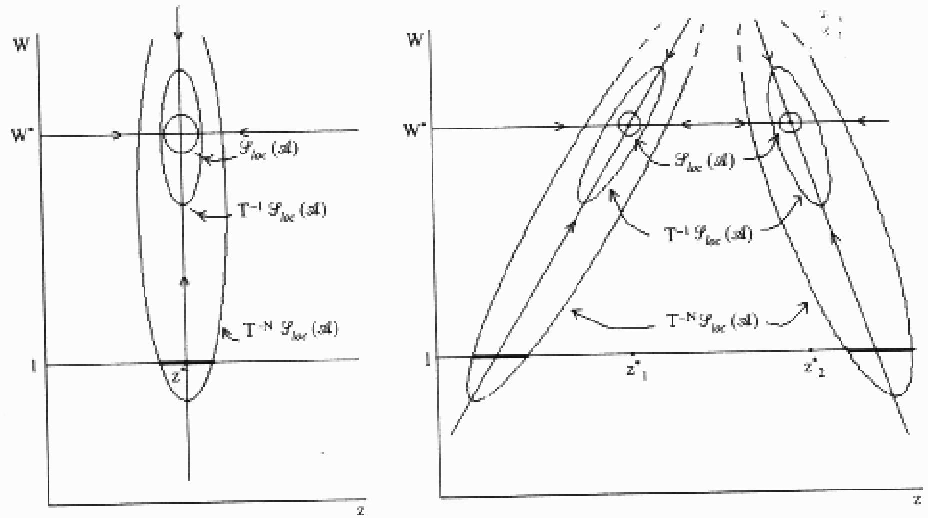

Figure 8

Schematic representation of the geometric arguments produced to prove Proposition 1. The line $W = W^*$ is the line of the ω-limit sets of the two-dimensional map T, the line $W = 1$ is the line of initial conditions. The projection on the z axis of a trajectory of T starting from the line $W = 1$ is a trajectory of the model (6) with learning expressed by (9). (a) Case of an attracting fixed point. (b) Case of an attracting cycle.

The total basin of $\mathcal{A}$ can be constructed by taking the union of all the pre-images of $\mathcal{S}_{loc}(\mathcal{A})$, i.e. $\mathcal{B}(\mathcal{A}) = \bigcup_{n=0}^{\infty} T^{-n}(\mathcal{S}_{loc}(\mathcal{A}))$. Because of the particular structure of the second equation of T, which is a linear contraction in the W direction, we have that each inverse of T is an expansive map in the W direction. This implies that a finite value N exists such that $\bigcup_{n=0}^{N} T^{-n}(\mathcal{S}_{loc}(\mathcal{A}))$ intersects the line of initial conditions $W=1$. Hence a subset of positive one-dimensional measure of the line $W=1$ exists whose points generate trajectories converging to $\mathcal{A}$ (see Figure 8). This proves that A is an attractor for (10), and $f(A)$ is the corresponding attractor for the sequences of actual prices. The total basin of attraction $D_1(A)$ for the process with learning is given by the intersection the two-dimensional basin $\mathcal{B}(\mathcal{A})$, with the line of initial conditions $W=1$

$$D_1(A) = \mathcal{B}(\mathcal{A}) \cap \{W=1\}$$

In Figure 8 the geometric idea behind the proof is illustrated. An attracting fixed point z^* always belongs to $D_1(z^*)$. In fact the line $z = z^*$ is an invariant line, because $z_t = z^*$ implies $z_{t+1} = z^*$ for each t, and this implies that a set of preimages of $\mathcal{S}_{loc}(\mathcal{A})$ expands along that line (see Figure 8a). Instead, for an attracting cycle $A = \{z_1^*, z_2^*, ..., z_k^*\}$ it may happen that some periodic point $z_1^* \notin D_1(A)$, i.e. the trajectory starting from z_1^* does not converge to $\mathcal{A}$ (see the point z_1^* of Figure 6).

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