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Basins of attraction in an evolutionary model of boundedly rational consumers

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Abstract. We consider a two-dimensional discrete dynamical system that describes an evolutionary approach to the economic problem of consumers' demand under the assumption of bounded rationality. The adaptive process is obtained by the iteration of a nonlinear map of the plane characterized by the presence of a denominator that can vanish. Recent results on the global bifurcations of these kinds of maps are used to explain the structure of the basins of attraction. The equilibria of the dynamic model represent rational choices of the consumers, i.e. solutions of utility maximization problems under budget constraints. We derive conditions under which multiple steady states emerge, and the related question of the study of the basins of attraction and their global bifurcations is approached by using geometric and numerical methods.

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1 Introduction

The traditional approach to the economic theory of consumer demand is based on the assumption that consumers are *rational agents*, that is, they make their choices according to the maximization of a given function of the quantities of goods consumed, called a utility function, subject to budget constraints. This requires each consumer to have a complete knowledge of the utility function, is able to exploit fully all the available information on the economic parameters involved (e.g. prices and budget) and has got the computational skill required to solve the optimization problem. Many authors argue that these assumptions are too strong, and economic models should take into account human limited ability to solve maximization problems. This leads to the weaker concept of *bounded rationality*, which assumes that agents make repeated choices following a trial and error (or adaptive) method, and at each time they correct the previous choices on the basis of the observation of their effects. Sometimes this repeated adaptive process converges to an equilibrium point that corresponds to the same choice of a rational agent, i.e. the agents learn to behave rationally in the long run. This possibility may be seen as an "evolutionary explanation" of the assumption of rational behavior. In a seminal paper by Alchian (1950) this "evolutionary approach" is described to explain how non fully rational economic agents (firms in that case) follow a "Darwinian" evolution, characterized by adaptive trial and error methods, that may lead them to converge to a rational behavior in the long run.

In this paper we consider a dynamic model, recently proposed by D'Orlando and Rodano (2005), that describes iterated consumer choices. They assume that at each time period the consumers update their consumption choices on the basis of the observed discrepancy between expected utility and realized utility. Moreover, the utility function is updated according to the consumption choice of the previous period. The discrete time evolution of this adaptive process is represented by the iteration of a nonlinear two-dimensional map, whose steady states represent local maximum points of the utility function, i.e. the choices of a rational consumers. We prove that a range of parameters exist such that three rational equilibria are present. In this case the evolutionary model may act as an equilibrium selection device, i.e. some equilibria may be more likely to be reached than others when an evolutionary dynamic is introduced (some equilibria may be not reached at all if they are unstable under the chosen adaptive process). Of course, only those rational equilibria which are stable under an adaptive process can be "learned" by the agents. Moreover, when several coexisting stable equilibria are present, a situation denoted as *multistability*, the evolutionary process becomes *path dependent*, that is, the final outcome depends on the initial conditions. In these situations a study of the basins of attraction has a crucial importance. In general, a complete study of the boundaries that separate the different basins of attraction and their qualitative changes as the parameters of the model vary is not an easy task in dynamical systems of dimensions greater than one. In fact, this requires a global analysis of the dynamical system, i.e. a study which is not based on linear approximations. The discrete dynamical system proposed by D'Orlando and Rodano is represented by the iteration of a nonlinear two-dimensional map characterized by the presence of a denominator that vanishes along a curve, and one component of the map assumes the form 0/0 at one point. This gives us the opportunity to apply some of the methods recently introduced in Bischi et al. (1999, 2003) for the study of two-dimensional map with a denominator. There the concepts of *focal point* and *prefocal curve* have been defined to explain the creation of particular structures of the basins called *lobes*. Roughly speaking, a *prefocal curve* is a set of points which are mapped (or "focalized") into a single point, called *focal point,* by the inverse function (if the map is invertible) or by at least one of the inverses (if the map is noninvertible). In this paper we detect the typical lobe structure by a numerical computation of the basins, and we explain the global (or contact) bifurcations that lead to their creation, by using the concepts of focal point and prefocal curve.

The paper is organized as follows. Section 2 gives a short description of the economic dynamic model is given. Section 3 studies the existence of equilibrium points, and the presence of a range of parameters leading to coexistence of three equilibria is proved. Section 4 recalls some definitions related to the basins, and their bifurcations, for two-dimensional maps characterized by the presence of a vanishing denominator, on the basis of some recent papers on these topics. Section 5 applies these definitions and results to the study of the global bifurcations of the basins of the economic dynamic model considered in this paper. Section 6 concludes and outlines further studies to be done on the same economic model.

2 The model

Following D'Orlando and Rodano (2005), let us consider a utility function $U(x, \overline{x})$, where x is the quantity of a given good and $\overline{x}$ represents the aggregated quantity of all the other goods that the consumer can buy. As usual in economics, the utility function represents the satisfaction obtained by the consumer as a consequence of the consumption of the goods considered. If p is the unit price of good x and $\overline{p} = 1$ is a reference price of the other goods, the budget constraint becomes

$$px + \overline{x} = m, \tag{1}$$

where m is the amount of money that the consumer can use to buy goods. The rational choice of the consumer is the solution $(x^*, \overline{x}^*)$ of the problem of maximization of U under the budget constraint (1). If we exclude corner solutions, the rational choice is a solution of the system

$$\frac{\partial U/\partial x}{\partial U/\partial \overline{x}} = p; \qquad px + \overline{x} = m. \tag{2}$$

The economic intuition behind this system is the following: The price of the good balances the relative gain of utility the consumer expects from the consumption of a unit of the good considered.

D'Orlando and Rodano (2005) assume that the consumer is not able to compute the solutions of this problem, and they consider the following discrete time[1] adjustment process

$$x_t = x_{t-1} + \mu \left[S(x_{t-1}) - p \right], \tag{3}$$

where $S = \frac{\partial U/\partial x}{\partial U/\partial \overline{x}}$ and μ represents an adjustment speed. This adaptive process is based on the assumption that at any time period t the quantity x_t that the consumer buys is obtained as a correction of the quantity chosen in the previous period, x_{t-1}, according to the discrepancy between the price and the observed relative utility gain $S(x_{t-1})$. It is plain that a steady state of this dynamic process is a rational choice, i.e. a solution of (2).

Following again D'Orlando and Rodano, we consider the famous Cobb–Douglas utility function:

$$U(x, \overline{x}) = x^{\alpha} \overline{x}^{1-\alpha} \tag{4}$$

in which the exponents measure the respective marginal utility (if we multiply the quantity of the given good by a factor t, utility is multiplied by a factor t^{α}).

[1] A discrete time setting is assumed because after a consumer buys a good a certain time interval is to be waited in order to have money to buy the good again.

From the utility function we get

$$S(x)=\frac{\alpha}{1-\alpha}\frac{\overline{x}}{x}=\frac{\alpha}{1-\alpha}\frac{m-px}{x}$$

and the adjustment process (3) becomes

$$x_t=x_{t-1}+\mu\left(\frac{\alpha}{1-\alpha}\frac{m}{x_{t-1}}-\frac{1}{1-\alpha}p\right). \tag{5}$$

D'Orlando and Rodano also assume that the consumer's preferences are influenced by past choices, in the sense that a consumer may increasingly prefer the good that was consumed in the past periods (due to habits of skillness gained) or may decreasingly prefer the goods consumed in the past because she/he becomes tired of it. These situations may be modelled by assuming that the exponent α of the utility function (4) changes over time according to

$$\alpha_t=\alpha\left(x_{t-1}\right). \tag{6}$$

Among the many possible choices of the function $\alpha\left(x\right)$ D'Orlando and Rodano propose the following increasing function, characterized by a sigmoid shape

$$\alpha_t=\frac{1}{k_1+k_2\cdot k_3^{x_{t-1}}}, \tag{7}$$

where $k_1>1$, $k_2>0$ and $0<k_3<1$, so that $0<\alpha_t<1$. In fact, $\alpha_{\min}=\alpha\left(0\right)=\dfrac{1}{k_1+k_2}$ and $\lim_{x\to\infty}\alpha\left(x\right)=\dfrac{1}{k_1}=\alpha_{\max}$, with $\alpha\left(0\right)<\alpha_{\max}<1$. Putting all things together, the dynamical system that describes the evolutionary process is represented by the iteration of the following two-dimensional map

$$\begin{cases} x'=x+\mu\left(\dfrac{\alpha}{1-\alpha}\dfrac{m}{x}-\dfrac{p}{1-\alpha}\right) \\ \alpha'=\dfrac{1}{k_1+k_2k_3^x} \end{cases}, \tag{8}$$

where $'$ denotes the unit-time advancement operator. That is, if the right hand side variables represent the dynamic variables at period t then the left hand side ones represent the state variables at period $(t+1)$.

In the following we shall study the existence of steady states of the adaptive process described above, and in the cases of coexistence of equilibria we shall focus our attention on the study of their basins of attraction.

3 Existence of rational equilibria

The steady states of the adaptive model described in the previous section are the fixed points of the map (8), i.e. the solutions of the system

$$\begin{cases} \alpha=\dfrac{p}{m}x \\ \alpha=\dfrac{1}{k_1+k_2k_3^x} \end{cases} \tag{9}$$

obtained by setting $x' = x$ and $\alpha' = \alpha$ in (8). The solutions of this system cannot be analytically computed. However, they can be graphically represented as the intersections between a line through the origin of angular coefficient p/m and a sigmoid curve, as shown in Fig. 1, where three sets of parameters are considered such that three, one and two solutions are obtained respectively.

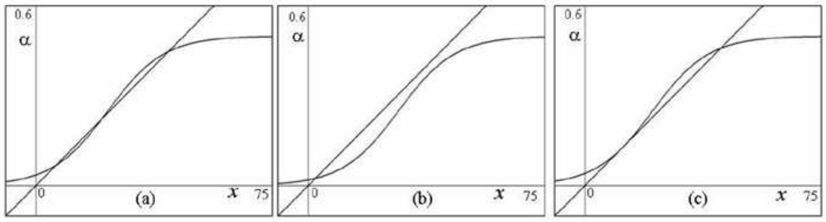

Figure 1:

The following proposition, proved in the Appendix, gives the ranges of the parameters for which these different cases occur.

PROPOSITION 1 *At least one positive fixed point of* (8) *exists for each set of parameters such that* $k_1 > 1$, $k_2 > 0$ *and* $0 < k_3 < 1$. *Three positive steady states exist provided that*

$$\frac{m}{p} > -4\frac{k_1}{\ln k_3} \quad and \quad k_2^{\min} < k_2 < k_2^{\max}, \tag{10}$$

where

$$k_2^{\min} = \frac{\frac{2mpk_1}{m-\sqrt{m^2+\frac{4mpk_1}{\ln k_3}}} - k_1}{k_3^{\frac{m-\sqrt{m^2+\frac{4mpk_1}{\ln k_3}}}{2pk_1}}}; \qquad k_2^{\max} = \frac{\frac{2mpk_1}{m+\sqrt{m^2+\frac{4mpk_1}{\ln k_3}}} - k_1}{k_3^{\frac{m+\sqrt{m^2+\frac{4mpk_1}{\ln k_3}}}{2pk_1}}}. \tag{11}$$

As argued in the introduction, interesting situations for economists are obtained when locally stable equilibria exist, because in these cases the trajectories of the adaptive process that converge to a stable equilibrium represent situations of consumers that *learn to be rational* along the evolutionary process. However, even if a unique stable equilibrium exists, it is important to have an estimate of the extent of its basin of attraction, that is, how far the initial condition can be taken so that the convergence to the rational equilibrium is guaranteed. To better explain this point, let us consider a set of parameters such that a unique positive equilibrium exists, namely $p = 10$, $m = 500$, $\mu = 0.537$, $k_1 = 10$, $k_3 = 0.432$, $k_2 = 57$ (for this set of parameters $-4k_1/\ln k_3 = 47.657$, $k_2^{\min} = 80.36$ and $k_2^{\max} = 82.7$). A numerical computation of the basin of attraction of the positive steady state, say A, is given by the white region shown in Fig. 2a, where the dark grey region represents the set of initial conditions leading to negative values of the dynamical variables, i.e. economically unfeasible trajectories (in the following we shall denoted this set as the *unfeasible set*). As we can see in Fig. 2a, there is a quite complicated structure of the boundary that separated the basin of the rational equilibrium from the unfeasible set, especially in the region around the point $O = (0, 0)$.

The situation becomes even more involved if we consider a set of parameters such that three positive equilibria exist, as shown in Fig. 2b and 2c, obtained with the set of parameters $p = 10$, $m = 1677$, $\mu = 0.917$, $k_1 = 2.4$, $k_3 = 0.9$, $k_2 = 78$ (for this set of parameters $-4k_1/\ln k_3 = 91.12$, $k_2^{\min} = 40.92$ and $k_2^{\max} = 221.73$). In this case the steady state denoted by A is a saddle point that generated through a flip bifurcation a stable cycle of period two denoted by $A_2 = \{A_2^{(1)}, A_2^{(2)}\}$, the steady state denoted by C is a stable node, and the one in the middle, denoted by B, is a saddle point, whose stable set forms the boundary $\mathcal{F}$ that separates the basins of attraction $\mathcal{B}(A_2)$ and $\mathcal{B}(C)$ of the stable cycle A_2 and the stable equilibrium C respectively. In Fig. 2b and 2c the white region is, again, the basin $\mathcal{B}(A_2)$, the pale grey region is $\mathcal{B}(C)$ and the dark grey region represents, again, the unfeasible set. Also in this case the structure of the basins reveals some peculiarities in the region around the point O, where the three basins are conveyed through the point O. A remarkable feature is that there are initial conditions that generate trajectories that converge at the upper stable equilibrium C even if they are much closer to the lower stable 2 period cycle A_2. If one considers evolutive processes starting from different initial conditions that are gradually changed, for example along a path from point H to point K in Fig. 2c, unfeasible trajectories are obtained first (dark grey portion of the path), then trajectories that converge at the upper stable rational equilibrium, then trajectories that converge at the cycle, then trajectories converging to the upper equilibrium again and so on. In other words, it is not easy to forecast what rational behavior (if any) will be learned by the boundedly rational consumer.

In the following we shall focus our attention on the problem of understanding the global dynamical properties of the map (8) that are responsible for the creation of such peculiar structures of the basins, formed by lobes issuing from the point $O = (0, 0)$.

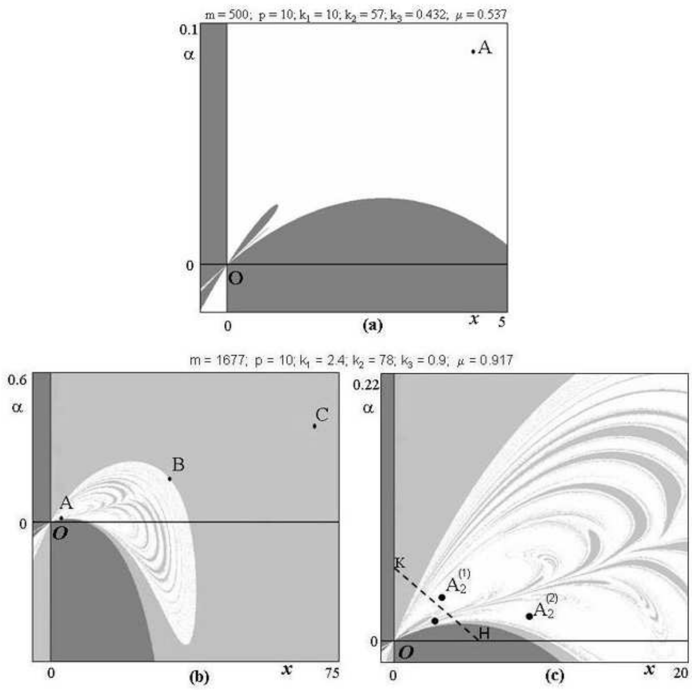

Figure 2:

4 Basins and their bifurcations

We first recall some basic definitions and properties[2] of the basins of attraction for a discrete dynamical system defined by the iteration of a two-dimensional map $T : (x, y) \to (x', y')$ defined by $(x', y') = T(x, y) = (f(x, y), g(x, y))$, where $(x, y) \in \mathbb{R}^2$ and f, g are assumed to be real valued functions. The point $(x', y') \in \mathbb{R}^2$ is called a rank-1 image of the point (x, y) under T, and (x, y) is called a rank-1 preimage of (x', y'). A set $A \subset \mathbb{R}^2$ is *trapping* if it is mapped into itself, $T(A) \subseteq A$, i.e. if $(x, y) \in A$ then also $T(x, y) \in A$. A trapping set is *invariant* if it is mapped onto itself: $T(A) = A$, i.e. all the points of A are images of points of A. A closed invariant set A is an attractor if it is *asymptotically stable*, i.e. if a neighborhood U of A exists such that $T(U) \subseteq U$ and $T^t(x, y) \to A$ as $t \to +\infty$ for each $(x, y) \in U$.

The *Basin of an attractor* A is the set of all points that generate trajectories converging to A

$$\mathcal{B}(A) = \left\{ (x, y) \mid T^t(x, y) \to A \quad \text{as } t \to +\infty \right\}. \tag{12}$$

[2]For a more detailed and rigorous treatment, see e.g. Mira et al., 1996.

Starting from the definition of stability, let U be a neighborhood of an attractor A whose points converge to A. Of course $U \subseteq \mathcal{B}(A)$, but also the points of the phase space which are mapped inside U after a finite number of iterations belong to $\mathcal{B}(A)$. Hence, the *total basin of* A (or briefly the basin of A) is given by the open set $\mathcal{B}(A) = \bigcup_{n\geq 0} T^{-n}(U)$, where $T^0(x,y) = (x,y)$ and $T^{-n}(x,y)$ represents the set of rank-n preimages of (x,y), i.e. the set of points that are mapped into (x,y) after n iterations of the map T. The basin $\mathcal{B}(A)$ is trapping under T and invariant under T^{-1}, i.e.

$$T^{-1}(\mathcal{B}(A)) = \mathcal{B}(A), \qquad T(\mathcal{B}(A)) \subseteq \mathcal{B}(A).$$

The boundary $\mathcal{F} = \partial\mathcal{B}(A)$ behaves as a repelling set for the points near it, since it acts as a watershed for the trajectories of the map T. Points belonging to $\mathcal{F}$ are mapped into $\mathcal{F}$ both under forward and backward iteration of T. More exactly

$$T^{-1}(\partial\mathcal{B}(A)) = \partial\mathcal{B}(A), \qquad T(\partial\mathcal{B}(A)) \subseteq \partial\mathcal{B}(A).$$

We remark that $T^{-1}(\partial\mathcal{B}(A)) = \partial\mathcal{B}(A)$ implies that if a curve segment belongs to $\partial\mathcal{B}(A)$ then also all its preimages must belong to $\partial\mathcal{B}(A)$. In particular, $\partial\mathcal{B}(A)$ includes the stable set of any fixed point (or cycle) of T belonging to $\partial\mathcal{B}(A)$.

So, in order to study the structure of the boundaries of a basin, the properties of the inverse (or inverses if a map is noninvertible, see Mira et al., 1996) must be considered.

It is worth noticing that the map (8) is not defined in the whole plane because the denominator of its first component

$$x' = \frac{(1-\alpha)x^2 + \alpha\mu m - x\mu p}{x(1-\alpha)} \tag{13}$$

vanishes along the two lines $x = 0$ and $\alpha = 1$, and assumes the form $0/0$ in the point $O = (0,0)$. Following the terminology introduced in Bischi et al., 1999, the two lines constitute the *set of non-definition* δ_s, and the point O may be a *focal point* (see also Bischi et al., 2003, 2005). Bischi et al., 1999, 2003 show that the structure of the basins of a map with a denominator that can vanish is strongly influenced by the presence of a *focal point*, where the map assumes the form $0/0$. As we shall see, this is also true in the case of the map (8). Before proving this statement we briefly recall the main definitions and properties related to focal points.

4.1 Definition of *focal point* and *prefocal set*

Let us consider a two-dimensional map, defined by $(x', y') = (F(x,y), G(x,y))$, with at least one of the components F or G containing a denominator that can vanish. This implies that the map is not defined in the whole plane. This characteristic is the source of some particular dynamical behaviors, as recently evidenced in Bischi et al., 1999, 2003, where new singularities, called *focal point* and *prefocal curve*, have been defined to characterize some new kinds of global bifurcations. Roughly speaking, a *prefocal curve* is a set of points for which at least one inverse exists which maps (or "focalizes") the whole set into a single point, called *focal point*.

In order to simplify the exposition, let us assume that, like in the case of the map (8) only one of the two functions defining the map T has a denominator which can vanish

$$T : \begin{cases} x' = F(x, y) = N(x, y)/D(x, y) \\ y' = G(x, y) \end{cases}, \tag{14}$$

where x and y are real variables, $G(x, y)$, $N(x, y)$ and $D(x, y)$ are continuously differentiable functions defined in the whole plane $\mathbb{R}^2$. Hence, the *set of nondefinition* of the map T, defined as the set of points where at least one denominator vanishes, reduces to

$$\delta_s = \{(x, y) \in \mathbb{R}^2 \mid D(x, y) = 0\}. \tag{15}$$

We assume that δ_s is given by the union of smooth curves of the plane. The two-dimensional recurrence obtained by the successive iterations of T is well defined provided that the initial condition belongs to the set E given by $E = \mathbb{R}^2 \setminus \bigcup_{k=0}^{\infty} T^{-k}(\delta_s)$, where $T^{-k}(\delta_s)$ denotes the set of the rank-k preimages of δ_s, i.e. the set of points which are mapped into δ_s after k applications of T ($T^0(\delta_s) \equiv \delta_s$). Indeed, the points of δ_s, as well as all their preimages of any rank that constitute a subset of $\mathbb{R}^2$ of zero Lebesgue measure, must be excluded from the set of initial conditions that generate non interrupted sequences by the iteration of the map T, so that $T : E \to E$.

Now consider a bounded and smooth simple arc γ, parameterized as $\gamma(\tau)$, transverse to δ_s (like γ_a and γ_c in Fig. 3), such that $\gamma(0) = (x_0, y_0)$ and $\gamma \cap \delta_s = \{(x_0, y_0)\}$. We are interested in its image $T(\gamma)$. As $(x_0, y_0) \in \delta_s$ we have, according to the definition of δ_s, $D(x_0, y_0) = 0$, but in general $N(x_0, y_0) \neq 0$. Hence

$$\lim_{\tau \to 0\pm} T(\gamma(\tau)) = (\infty, G(x_0, y_0)), \tag{16}$$

where ∞ means either $+\infty$ or $-\infty$. This means that the image $T(\gamma)$ is made up of two disjoint unbounded arcs asymptotic to the line of equation $y = G(x_0, y_0)$ (see Fig. 3).

A different situation may occur if the point $Q(x_0, y_0) \in \delta_s$ is such that not only the denominator but also the numerator vanishes in it, i.e. $D(x_0, y_0) = N(x_0, y_0) = 0$. In this case, the first component of the limit (16) takes the form $0/0$. This implies that this limit may give rise to a finite value, so that the image ($T(\gamma_b)$ in Fig. 3) is a bounded arc crossing the line $y = G(x_0, y_0)$ in the point $(x, G(x_0, y_0))$, where

$$x = \lim_{\tau \to 0} F(x(\tau), y(\tau)). \tag{17}$$

It is clear that the value x in (17) depends on the arc γ. Furthermore it may have a finite value along some arcs and be infinite along other ones. This leads

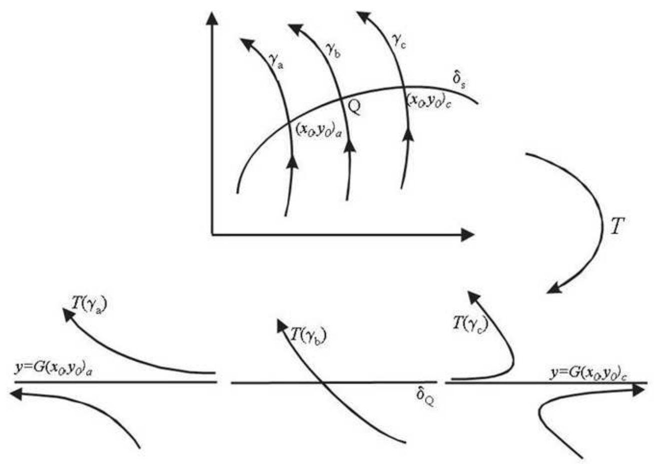

Figure 3:

us to the following definition of the singular sets of *focal point* and *prefocal curve* (Bischi et al., 1999):

Definition. Consider the map T in (14). A point $Q = (x_0, y_0)$ is a focal point if at least one component of T takes the form $0/0$ in Q and there exist smooth simple arcs $\gamma(\tau)$, with $\gamma(0) = Q$, such that $\lim_{\tau\to 0} T(\gamma(\tau))$ is finite. The set of all such finite values, obtained by taking different arcs $\gamma(\tau)$ through Q, is the prefocal set δ_Q, the equation of which is $y = G(Q)$.

Let us consider a simple focal point, i.e. a simple root of the algebraic system

$$N(x, y) = 0, \qquad D(x, y) = 0.$$

We recall that a focal point $Q = (x_0, y_0)$ is simple if $\overline{N}_x\overline{D}_y - \overline{N}_y\overline{D}_x \neq 0$, where $\overline{N}_x = \dfrac{\partial N}{\partial x}(x_0, y_0)$ and analogously for the other partial derivatives. In this case a one-to-one correspondence is defined between the point $(x, G(Q))$, in which $T(\gamma)$ crosses δ_Q, and the slope m of γ in Q:

$$m \to (x(m), G(Q)), \quad \text{with} \quad x(m) = (\overline{N_x} + m\overline{N_y})/(\overline{D_x} + m\overline{D_y}) \tag{18}$$

and

$$(x, G(Q)) \to m(x) \quad \text{with} \quad m(x) = (\overline{D_x}x - \overline{N_x})/(\overline{N_y} - \overline{D_y}x). \tag{19}$$

These relations can be obtained by using a method either based on a series expansion of the functions $N(x, y)$ and $D(x, y)$ in a neighborhood of $Q = (x_0, y_0)$, or by considering the Jacobian determinant of the inverse map T^{-1} (or one of the inverses if the map is noninvertible). In fact, from the definition of the prefocal curve, it follows that the Jacobian det $\left(DT^{-1}\right)$ must necessarily vanish in the points of δ_Q. Indeed, if the map T^{-1} is defined in δ_Q, then all the points of the line δ_Q are mapped by T^{-1} into the focal point Q. This means that T^{-1} is not locally invertible in the points of δ_Q, being a many-to-one map, and this implies that its Jacobian cannot be different from zero in the points of δ_Q.

From the relations (18), (19) it results that different arcs γ_j, passing through a focal point Q with different slopes m_j, are mapped by T into bounded arcs $T(\gamma_j)$ crossing δ_Q in different points $(x(m_j),G(Q))$ (see the qualitative picture in Fig. 4a)

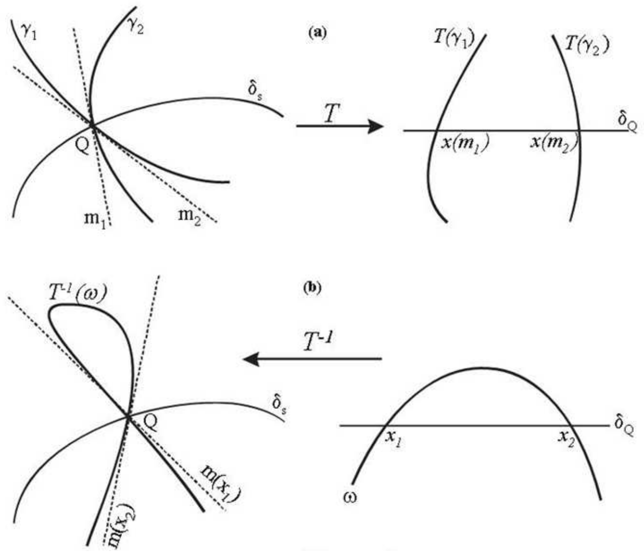

Figure 4:

Interesting properties are obtained if the inverse of T (or the inverses, if T is a noninvertible map) is (are) applied to a curve that crosses a prefocal curve. Let δ_Q be a prefocal curve whose corresponding focal point is Q. Then each point sufficiently close to δ_Q has its rank-1 preimage in a neighborhood of the focal point Q. If the inverse T^{-1} is continuous along δ_Q then all the points of δ_Q are mapped by T^{-1} in the focal point Q. Roughly speaking, we can say that the prefocal curve δ_Q is "*focalized*" by T^{-1} in the focal point Q, or, more concisely, that $T^{-1}(\delta_Q) = Q$. We note that the map T is not defined in Q, thus T^{-1} cannot to be strictly considered as an inverse of T in the points of δ_Q, even if T^{-1} is defined in δ_Q. In fact, if an arc ω crosses δ_Q in two distinct points, say $(x_1, G(Q))$ and $(x_2, G(Q))$ then its preimage $T^{-1}(\omega)$ must include a loop with double point in Q, as shown in Fig. 4b.

When the presence of a vanishing denominator induces the existence of focal points, important effects on the geometrical and dynamical properties of the map T can be observed. Indeed, a contact of an arc ω with a prefocal curve δ_Q, gives rise to important qualitative changes in the shape of the preimages $T_j^{-1}(\omega)$. When the arcs ω are portions of phase curves of the map T, such as *invariant closed curves or stable sets* of saddles that form *basin boundaries*, one has that contacts between singularities of different nature generally induce important qualitative changes, which constitute *new types of global bifurcations* that change the structure of the basins. In fact, let us consider a smooth curve segment ω that moves towards a prefocal curve δ_Q until it crosses through δ_Q (see Fig. 5) so that only a focal point $Q = T^{-1}(\delta_Q)$ is associated with δ_Q. The prefocal set δ_Q belongs to the line of equation $y = G(Q)$, and the one-to-one correspondence defined by (18) and (19) holds. When ω moves toward δ_Q, its preimage $\omega_{-1} = T^{-1}(\omega)$ moves towards Q. If ω becomes tangent to δ_Q in a point $C = (x_c, G(Q))$, then ω_{-1} has a cusp point at Q. The slope of the common tangent to the two arcs, that join at Q, is given by $m(x_c)$, according to (19). If the curve segment ω moves further, so that it crosses δ_Q at two points $(x_1, G(Q),)$ and $(x_2, G(Q))$, then ω_{-1} forms a *loop with a double point* at the focal point Q. Indeed, the two portions of ω that intersect δ_Q are both mapped by T^{-1} into arcs through Q, and the tangents to these two arcs of ω_{-1}, issuing from the focal point, have different slopes, $m(x_1)$ and $m(x_2)$ respectively, according to (19).

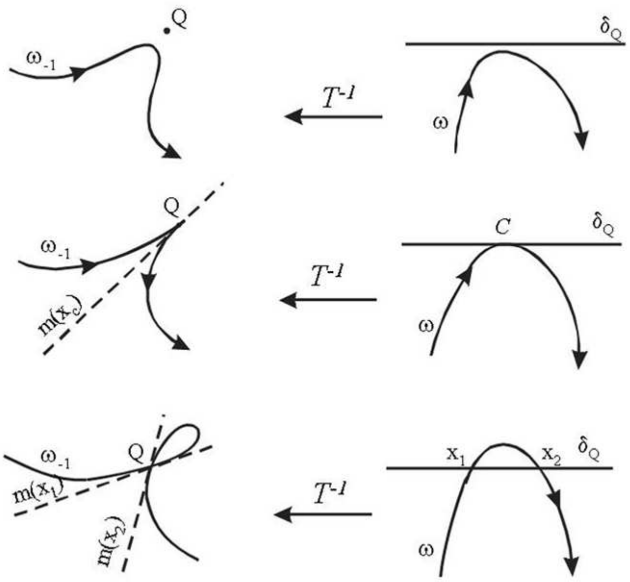

Figure 5:

When ω is an arc belonging to a basin boundary $\mathcal{F}$, the qualitative modifications of the preimages $T_j^{-1}(\omega)$ of ω, due to a tangential contact of ω with the prefocal curve, can be particularly important for the global dynamical properties of the map T. As recalled above, a basin boundary $\mathcal{F}$ is backward invariant, i.e. $T^{-1}(\mathcal{F}) = \mathcal{F}$, so if ω is an arc belonging to $\mathcal{F}$, then all its preimages of any rank must belong to $\mathcal{F}$. This implies that if a portion ω of $\mathcal{F}$ has a tangential contact with a prefocal curve δ_Q, then a cusp points, located in the focal point Q, is included in the boundary $\mathcal{F}$. It results that if the basin boundary $\mathcal{F}$ was smooth before the contact with the prefocal curve δ_Q, such a contact gives rise to points of non smoothness. When $\mathcal{F}$ crosses through δ_Q in two points, after the contact $\mathcal{F}$ must contain a loop with double points in Q. So, *a contact of a basin boundary with a prefocal curve gives rise to a new type of basin bifurcation that causes the creation of cusp points and, after the crossing, of loops (called "lobes"), along the basin boundary.*

As we shall see in the next section, this is the basic mechanism leading to the particular structure of the basins shown in Fig. 2.

5 Basins' bifurcations for the adaptive process

The first component of the map (8) assumes the form

$$x' = F(x, \alpha) = \frac{N(x, \alpha)}{D(x, \alpha)} = \frac{(1-\alpha)x^2 + \alpha\mu m - x\mu p}{x(1-\alpha)} \tag{20}$$

hence its denominator vanishes along the lines

$$\delta_s = \{(x, \alpha) \mid x = 0 \text{ or } \alpha = 1\} \tag{21}$$

and it assumes the form $0/0$ in the point $Q = (0, 0)$. Indeed, this is a focal point for the map T, and the corresponding prefocal curve is

$$\delta_Q = \left\{ (x, \alpha) \mid \alpha = \frac{1}{k_1 + k_2} \right\}$$

obtained by inserting the coordinates of Q in the second component of (8).

We now show that inverse T^{-1} focalizes the line δ_Q into the unique point Q. In fact, it is easy to compute the inverse of T by solving the system (8) with respect to x and α. In fact, from the second equation we obtain

$$x = \frac{\ln\left(\dfrac{1}{\alpha' k_2} - \dfrac{k_1}{k_2}\right)}{\ln k_3} \tag{22}$$

and from the first equation

$$\alpha = \frac{x^2 - xx' - \mu p x}{x^2 - xx' - \mu m}, \tag{23}$$

where each x in the right hand side can be replaced by using (22), so that the inverse map T^{-1} is obtained. The first component (22) of the inverse is defined in the range of the map T, given by

$$0 < \alpha' < \frac{1}{k_1} \tag{24}$$

and the second component, (23), is defined in $E' = \mathbb{R}^2 \setminus \bigcup_{k=0}^{\infty} T^k(\delta'_s)$, where δ'_s is the locus of points where the denominator of (23) vanishes. It is immediate to check that $T^{-1}(\delta_Q) = Q = (0,0)$. In fact, if we insert $\alpha' = 1/(k_1 + k_2)$ in (22) we get $x = 0$, and from (23) with $x = 0$ we get $\alpha = 0$.

This allows us to explain the structure of the basins observed in Fig. 2 on the light of the results recalled in the previous section. Let us consider first the situation shown in Fig. 6a, obtained with parameters $k_1 = 10$; $k_2 = 20$; $k_3 = 0.432$; $m = 500$; $p = 10$; $\mu = 0.537$. In this case we have only an equilibrium point, a stable node, whose basin boundary is separated from the unfeasible set by the line δ_s of equation $x = 0$ and its preimages. The rank-1 preimage of δ_s, say $\delta_s^{-1} = T^{-1}(\delta_s)$ (see Fig. 6a), is close to the prefocal line δ_Q, and the rank-2 preimage, denoted by $\delta_s^{-2} = T^{-1}\left(\delta_s^{-1}\right)$, is also shown. As the parameter k_2 is increased, the curve δ_s^{-1} moves upwards, until it has a tangential contact with the prefocal line (Fig. 6b). This implies that portion of the basin boundary formed by δ_s^{-2} has a cusp point in Q, and a further increase of k_2 leads to a crossing between δ_s^{-1} and δ_Q. This causes the creation of a lobe issuing from the focal point Q, formed by the rank one preimage of the portion of the unfeasible set above δ_Q (Fig. 6c). When this lobe will grow up for increasing values of k_2 it will reach the line δ_Q and this will mark the creation of a new lobe and so on, thus leading to the situation shown in Fig. 2a.

Let us consider, now, a situation of bistability, like the one shown in Fig. 2b, where a "rational equilibrium" and a stable cycle coexist, each with its own basin of attraction. The boundary that separates these two basins is formed by the stable set of the saddle point. Also in this case, any contact of a basin boundary with the prefocal curve will cause the creation of a lobe of the same basin issuing from the focal point. Let us start from the situation shown in Fig. 7a, obtained with parameters $k_1 = 2.4$; $k_2 = 44$; $k_3 = 0.9$; $m = 1677$; $p = 10$; $\mu = 0.917$. As the parameter k_2 is increased, the basin boundary of the stable equilibrium crosses the prefocal line δ_Q. This causes, again, the creation of a lobe issuing from the focal point Q, formed by the rank one preimage of the basin of attraction of the stable equilibrium (Fig. 7b). For increasing values of k_2 also the preimages of rank two, three, four and so on will cross the line δ_Q creating an high number of lobes.

This sequence of global bifurcations is at the basis of the formation of basins' structures like those shown in Fig. 2b and 2c.

As a final remark, we can stress that attractors which are different from stable equilibria can be observed among the dynamic scenarios of the dynamical system generated by the iteration of the map (8), such as attracting periodic cycles, like the one as shown in Fig. 2c, or chaotic attractors. In these cases the asymptotic behavior of the adaptive process will never converge at a rational behavior, i.e. the consumers never learn to behave rationally. These kinds of

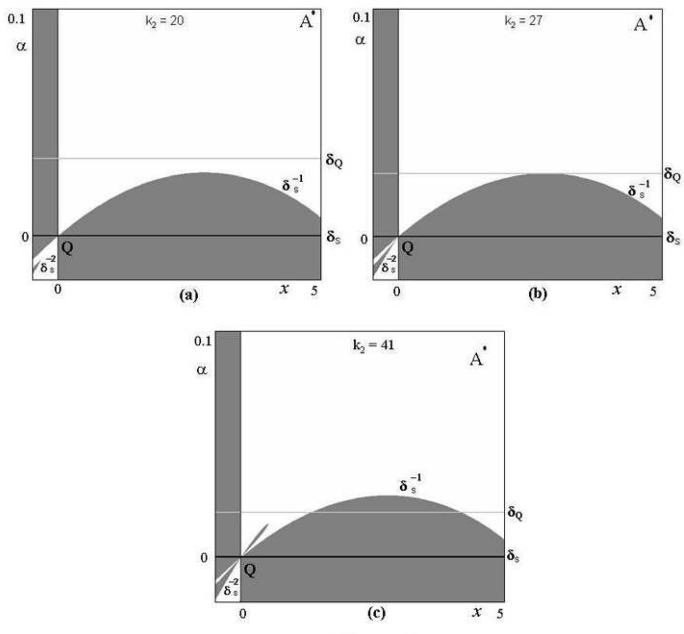

Figure 6:

disequilibrium dynamics may exhibit interesting features, both from a mathematical and an economic point of view, and will be studied elsewhere. However, it is worth noticing here that other properties of the maps with denominator, already stressed in Bischi et al., 1999, can be applied in this context. In fact, also the inverse map T^{-1}, defined by (22) and (23), has a vanishing denominator, and its second component assumes the form 0/0 in the point $\left(\frac{m}{p} - \mu p, \frac{1}{k_1 + k_2 k_3^{\frac{m}{p}}}\right)$. This is a focal point of the inverse map, and the associated prefocal line has equation $x = \frac{m}{p}$. As indicated in Bischi et al., 1999, a simple way to detect this is based on the study of the determinant of the Jacobian matrix of the map T, given by

$$DT = \begin{bmatrix} 1 - \dfrac{\alpha\mu m}{x^2(1-\alpha)} & \dfrac{\mu m - \mu p x}{x(1-\alpha)^2} \\ -\dfrac{k_2 k_3^x \ln k_3}{(k_1 + k_2 k_3^x)^2} & 0 \end{bmatrix} \tag{25}$$

hence

$$\det DT = 0 \;\Leftrightarrow\; x = \frac{m}{p}. \tag{26}$$

Then it is easy to see that the image by T of the line at which the Jacobian determinant vanishes is a single point

$$T\left(x = \frac{m}{p}, y\right) = \left(\frac{m}{p} - \mu p, \frac{1}{k_1 + k_2 k_3^{\frac{m}{p}}}\right)$$

i.e. that line is "focalized" by T, thus confirming that the line $x = m/p$ is the prefocal line of T^{-1}. If a chaotic attractor exists that crosses the prefocal curve of the inverse, then it will include a so called "knot" (following the terminology introduced by Bischi et al., 1999).

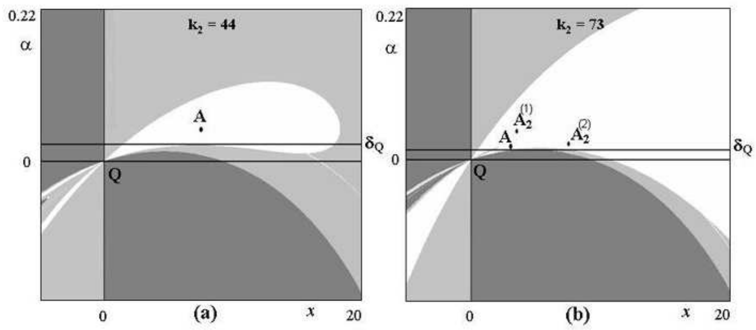

Figure 7:

6 Conclusions

In this paper the theory of focal points and prefocal curves (see Bischi et al., 1999, 2003, 2005) has been used to explain some peculiar structures of the basins of attraction of a discrete time dynamical system proposed by D'Orlando and Rodano (2005) to model the behavior of boundedly rational consumers. In fact, this dynamical system is obtained by the iteration of a map characterized by the presence of a denominator that can vanish, and assumes the form 0/0 in a point of the phase plane.

Maps with focal points and prefocal sets naturally arise in discrete dynamical systems of the plane found in several applications, such as economic modeling (see Bischi and Naimzada, 1997, Bischi et al., 2000) or numerical iterative methods (see Billings and Curry, 1996, Gardini et al., 1999). These papers exhibit basins with structures which are peculiar of maps with a vanishing denominator, called *lobes* and *crescents*, that can be explained in terms of contacts between basin boundaries and prefocal curves. For the dynamical system described in this paper a study of the basins is particularly important, because convergence to a stable equilibrium is interpreted as the possibility that non rational consumers can learn to behave rationally in the long run. So, on one side the assumption of rationality seems to be too strong an assumption, a kind of limiting case, if compared with real economic systems where agents have a limited ability to compute, on the other side the convergence of the evolutionary process analyzed in this paper can be seen as a justification of the assumption of rationality. This was Nash's concern: we can attain a rational outcome as the asymptotic outcome of an evolutionary process (see e.g. Marimon, 1997). Moreover, in the case with multiple rational equilibria, the evolutionary model may act as a selection device, and the kind of rational choice reached by the evolutionary dynamics becomes path dependent, i.e. crucially depends on the initial conditions. This is the reason why we focused our attention on the study of the basins of attraction.

Even more interesting situations of equilibrium selection arise when there are attractors of the adaptive dynamics which are not rational equilibria. This leads to situations of coexistence, for the bounded rationality dynamics, of rational attractors with non-rational attractors, so that long-run behavior is characterized by agents which continue to make choices different from the rational ones. Parameter constellations that give rise to the presence of chaotic attractors can be easily evidenced in the numerical explorations of the model considered in this paper. Their properties, which be investigated in the future, may have some interesting topological features due to the existence of focal points of the inverse map that can also cause the creation of particular kinds of chaotic attractors. Indeed a focal point, generated by the inverse map, may behave like a "*knot*", where infinitely many invariant curves of an attracting set shrink into a set of isolated points, as shown in Bischi et al., 1999.

Appendix

Proof of Proposition 1. First of all we notice that a solution of (9) always exists, because the difference between the two functions at right hand sides

$$g(x) = s(x) - r(x) = \frac{1}{k_1 + k_2 k_3^x} - \frac{p}{m}x$$

is such that $g(0) = s(0) = 1/(k_1 + k_2) > 0$ and $\lim_{x\to+\infty} g(x) = \frac{1}{k_1} - \lim_{x\to+\infty} \frac{p}{m}x = -\infty$.

In order to find necessary and sufficient condition for the existence of three solutions of (9) let us write the equation

$$\frac{p}{m}x = \frac{1}{k_1 + k_2 k_3^x} \tag{27}$$

in the equivalent form

$$\frac{m}{pk_2 x} - \frac{k_1}{k_2} = k_3^x \tag{28}$$

from which

$$\frac{\ln\left(\frac{m}{pk_2 x} - \frac{k_1}{k_2}\right)}{\ln k_3} = x. \tag{29}$$

We now consider the equivalent system:

$$\begin{cases} z = x \\ z = \dfrac{\ln\left(\frac{m}{pk_2 x} - \frac{k_1}{k_2}\right)}{\ln k_3} \end{cases} \tag{30}$$

and we want that the two right hand sides have the same derivatives in two points (a necessary condition for having three solutions) and then we ask for which values the points of equal derivative are true tangency points between the two curves represented by the two equations (30). The two derivatives coincide at the solutions of the equation

$$-\frac{m}{\ln k_3 \left(mx - pk_1 x^2\right)} = 1 \tag{31}$$

i.e.

$$pk_1 \ln k_3 x^2 - m \ln k_3 x - m = 0. \tag{32}$$

Two solutions

$$x_1 = \frac{m - \sqrt{m^2 + \frac{4mpk_1}{\ln k_3}}}{2pk_1}; \qquad x_2 = \frac{m + \sqrt{m^2 + \frac{4mpk_1}{\ln k_3}}}{2pk_1} \tag{33}$$

exist provided that

$$m^2 + \frac{4mpk_1}{\ln k_3} > 0 \tag{34}$$

i.e.

$$\frac{m}{p} > \frac{-4k_1}{\ln k_3}. \tag{35}$$

The upper point of tangency, at $x = x_2$, is characterized by the condition:

$$\frac{p}{m}\left(\frac{m + \sqrt{m^2 + \frac{4mpk_1}{\ln k_3}}}{2pk_1}\right) = \frac{1}{k_1 + k_2 k_3^{\frac{m+\sqrt{m^2+\frac{4mpk_1}{\ln k_3}}}{2pk_1}}} \tag{36}$$

i.e

$$\frac{2mpk_1}{m + \sqrt{m^2 + \frac{4mpk_1}{\ln k_3}}} = k_1 + k_2 k_3^{\frac{m+\sqrt{m^2+\frac{4mpk_1}{\ln k_3}}}{2pk_1}} \tag{37}$$

from which $k_2^{\max}$ is get. Analogously, the tangency condition for $x = x_1$ gives $k_2^{\min}$. This completes the proof. □

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