

Chapter 14

KNOWLEDGE ACCUMULATION IN AN R&D NETWORK

*G.I. Bischi and F. Lamantia**

DEMQ, Dipartimento di Economia e Metodi Quantitativi
Università di Urbino "Carlo Bo", Via Saffi 42, Urbino, Italy
Dipartimento di Scienze Aziendali, Università della Calabria,
Via P. Bucci 3C, Rende (CS), Italy

Abstract

In this chapter we propose an oligopoly model where firms invest in cost reducing R&D for producing homogeneous goods that are sold in a market. In particular, we assume that R&D efforts build up a "stock of knowledge", which, in turn, reduces marginal costs of production. Moreover, though all firms are competitors in the marketplace, they can decide to collaborate through cooperation agreements in a network for sharing their R&D results.

14.1. Introduction

In traditional models of dynamic oligopolistic competition, it is often accepted that a single decision variable (quantity or price) can summarize all strategic decisions of firms (see e.g. [1–4]). This often leads to models that are analytical tractable but too simplified for realistic applications. However, when dealing with the production of technological goods, other dynamic variables become of paramount importance, such as R&D investments and knowledge (see e.g. [5–9]). Here we propose a general framework for describing an industry where firms strategically interact and choose several decision variables with interrelated effects at discrete times. In particular, we consider an oligopoly where n competitors produce

*E-mail addresses: gian.bischi@uniurb.it, lamantia@unical.it

homogeneous goods and invest in R&D. These R&D efforts are (irreversible) investments that firms decide for increasing their overall knowledge level, otherwise subject to obsolescence (see [10–12]). Knowledge has a positive impact on unit production costs, through a so-called R&D production function, see [9]. Moreover, firms can be organized in a R&D network, i.e., they can cooperate and share R&D results, even if they remain competitors on the marketplace, as proposed in [7]. However, R&D investments are not entirely private, as a fraction of them can spill over for free to competitors (see also [13–15] on this point). Hence, in our model, at each time period, each firm has two different choices to make: the level of R&D effort to exert, and then the quantity to produce. However, for reducing the complexity of the problem, we assume that at each time period firms are able to “solve” the problem of optimal production choice, according to the R&D efforts, by backward induction. In this way, the model becomes a repeated two stage game, so that the choice of an R&D effort strategy is always followed by an “optimal” subsequent choice of quantities. In order to keep the setting analytically tractable, we introduce some simplifying assumptions, which allow us to set up the model in terms of a representative firm. First of all, we assume that firms are homogeneous, so that also total knowledge is a homogeneous quantity within the industry. In addition, the network structure is fixed, so that firms have to decide the level of their R&D efforts and the quantity to produce within a given network structure. We discuss briefly the effect of a change in the level of collaboration activity. With this respect, firms’ agreements for sharing R&D results are more alliances (long-term instances) than coalitions (short-lived instances), as it is often the case when agreements originate from Joint Ownership relationship (see [16]). After describing the general framework of the model, we give a specific functional form for the R&D production function. So we propose a dynamic formulation of the model, in terms of gradient dynamic (see [17–20]). As a consequence of the assumption on agents’ homogeneity, this dynamic model is bidimensional, with dynamic variables given by R&D effort and knowledge. In particular, the equilibria of the dynamic model are also equilibria for the corresponding static model. In this way the proposed dynamic model is useful for two different purposes: first, it describes the “out of equilibrium” decisions of a representative firm that engages the competition over time. Second, in case of convergence to an equilibrium point, the dynamic model is a numerical tool for finding these equilibria, that, in general, can not be found algebraically. In fact, we derive analytical results on existence, uniqueness and stability of an R&D effort equilibrium with a strict concave R&D production function, i.e., when the marginal productivity of knowledge is decreasing. When the R&D production function obeys the “law of variable proportions”, some insights can be given by numerical methods. The chapter is organized as follows. In Section 14.2 the static model is derived, both in general terms and for a specific R&D production function. For the latter a dynamic version is developed in Section 14.3, where also some specific questions on existence, uniqueness and stability of equilibria are addressed. Section 14.4 concludes.

14.2. The Static Model

14.2.1. General Formulation

We consider a homogenous-product oligopoly where n quantity setting firms operate in a market characterized by a linear inverse demand function $p = a - bQ$, $a, b > 0$, where Q is the total output in the market. These n firms are assumed to form a symmetric network¹ of degree k , with $0 \leq k \leq n - 1$, inside which firms can form bilateral agreements for sharing R&D results. So every firm has the same number of collaborative ties k , a parameter that represents the level of collaborative attitude of the network (level of collaboration activity). In this model we assume that the network structure is given; however efforts and quantities are dynamically chosen by firms at each (discrete) time period. Every euro spent by firms concurs to form the knowledge gain of firm i at time t (also called effective R&D level) as

$$E_i(t) = x_i(t) + kx_i(t) + \beta_i x_i(t) [(n-1) - k], \quad (14.1)$$

where $x_i(t)$ is the R&D effort by firm i at time t and $\beta_i \in [0, 1]$ is the absorptive capacity related to the ability of firm i to gain knowledge for free from non-connected nodes in the network (i.e. spillovers, see [9]). The second term in the right hand side of (14.1) represents the total effort exerted by linked firms, whereas the third term represents the efforts by non-linked ones. As present investments in R&D can produce effects for subsequent periods, with a suitable discount factor, we model the time t total (or accumulated) knowledge of firm i as

$$z_i(t) \stackrel{\text{def}}{=} \sum_{k=0}^t \rho^{t-k} E_i(k) = E_i(t) + \rho \sum_{k=0}^{t-1} \rho^{t-1-k} E_i(k) = E_i(t) + \rho z_i(t-1), \quad (14.2)$$

where $\rho \in [0, 1]$ gives a measure of how rapidly information becomes obsolete: values close to 1 represent a system where even the results of very old R&D efforts contribute to current knowledge, whereas values close to 0 imply that only very recent efforts give significant contributions to the total knowledge z_i . For similar formulations of knowledge accumulation see [6, 9, 10]. An increment of total knowledge can reduce individual production costs; for this respect, we assume that firm i has a marginal cost function at time t of the form

$$c_i(t) = c_0 - c_0 f_i(z_i(t)), \quad (14.3)$$

where c_0 is the marginal cost without R&D efforts (equal for all firms) and $f_i(z_i)$ is the R&D production function that we assume (see also [9]):

- (i) $0 \leq f_i < 1$ with $f_i(0) = 0$ and $\lim_{z \rightarrow +\infty} \frac{\partial f_i(z)}{\partial z_i} = 0$;
- (ii) $f_i = f_i(z_i)$ twice continuously differentiable with $\frac{\partial f_i}{\partial z_i} > 0$; and
- (iii) $\exists \bar{z} \geq 0$, such that $\frac{\partial^2 f_i(z)}{\partial z_i^2} \geq 0 \forall z < \bar{z}$ and $\frac{\partial^2 f_i(z)}{\partial z_i^2} \leq 0, \forall z > \bar{z}$.

¹We recall that in a symmetric network all nodes (firms) have the same number of links.

Notice that, by assumption (ii), firms' knowledge has a benefit impact on its unit production cost. Moreover, by (iii), for $\bar{z} = 0$, the R&D production function is always concave and it obeys the so-called law of diminishing marginal productivity; on the other hand, for $\bar{z} > 0$ (and a nonlinear $f_i(\cdot)$) the R&D production function obeys the "law of variable proportions". The latter can be easily justified in many economic situations. For instance, let us suppose that the firms are developing a new computer software and R&D effort are represented by programmers working on the project. If too few programmers are involved, they have to solve all possible issues. By hiring more programmers, one can assign them to deal with specific parts of the project, up to a point where they indeed are too many and diminishing marginal productivity is reached. In the following we assume that $a > c_0$, i.e., a minimum level of profitability exists to attract firms in the market. Note that taking the identity $f_i(x) = x$ as R&D production function and without knowledge accumulation ($\rho = 0$) the cost function simply reduces to the one proposed in [8]². In any case, all firms are rivals in the market place, and they calculate their optimal quantity by solving a profit maximization problem. Then, given optimal quantities as functions of efforts, they can assess how R&D efforts increase their individual profits; due to the R&D network of collaboration and spillovers, each firm calculates these cost-reducing efforts taking into account not only the network structure it belongs to (number of firms in the network and number of linked partners) but also the average cost structure of other firms. Following [8], we assume that each oligopolist i in the network maximizes its own profit function

$$\pi_i = \left\{ a - b \left[q_i + \sum_{p \neq i} q_p \right] - c_i \right\} q_i - \gamma x_i^2,$$

where q_i is the quantity produced (and sold) by agent i and γx_i^2 , $\gamma > 0$, is the cost of R&D effort (see [7, 21]).

The optimal quantity of firm i is

$$q_i = \frac{a - nc_i + \sum_{p \neq i} c_p}{b(1+n)} \quad (14.4)$$

with corresponding optimal expected profit (see the Appendix for details)

$$\pi_i^{\text{ex}}(x_i, x_{-i}) = \left[\frac{a - nc_i(x_i, x_{-i}) + \sum_{p \neq i} c_p(x_p, x_{-p})}{\sqrt{b}(1+n)} \right]^2 - \gamma x_i^2. \quad (14.5)$$

Given this setting, each firm tries to maximize the optimal individual profit with respect to its own R&D effort x_i . Substituting the cost functions (14.3) of representative agents in the network, we can reformulate (14.5) as $\pi_i(x_i, x_{-i})$, which is a function of R&D efforts only.

²From the point of view of spillovers, we could also assume that the absorptive capacity β_i in (14.1) depends on accumulated knowledge, with properties similar to the ones of the R&D production function (see [9]). For the sake of simplicity, here we assume that spillover parameters are constant.

14.2.2. An Example with a Specific R&D Production Function

From now on we consider the following specific functional form of the R&D production function:

$$f_i(z) = \frac{z^r}{1+z^r},$$

where the parameter $r > 0$ models the effectiveness of R&D investments. This function satisfies the assumptions (i)–(iii) listed in the previous section. We observe that $f_i(z_i)$ is strictly concave for $0 < r \leq 1$, i.e., marginal productivity of knowledge is always decreasing according to the "law of diminishing marginal productivity"; for values $r > 1$, $f_i(z_i)$ is convex for low values of knowledge and then concave with an inflection point at $\bar{z} = \sqrt[r]{\frac{r-1}{r+1}}$ (strictly increasing in $r > 1$) that corresponds to the level of maximum marginal productivity. In the second case the R&D production function obeys the "law of variable proportions".

With this specific functional form, we can rewrite (14.3) as

$$c_i(t) = \frac{c_0}{1 + (x_i(t) + kx_i(t) + \beta_i(n-k-1)x_i(t) + \rho z(t-1))^r} \quad (14.6)$$

By the assumption of symmetry of the network, we can write the effective R&D level for a firm i , $i \neq l$ that is linked to i in the network as

$$E_{i_l}(t) = kx_i(t) + x_i + \beta_{i_l} x_i(t) [(n-1) - k],$$

and analogously the effort for a firm l , not linked to i as

$$E_{l_{-i}}(t) = (k+1)x_l(t) + \beta_{l_{-i}} [x_l(t)(n-2-k) + x_i]$$

with cost functions c_{i_l} and $c_{l_{-i}}$ derived from (14.6).

Now, due to the symmetry of the network, we can impose that the R&D effort by a generic firm (different from i) is simply x_l . So, for the firm i , the terms that represent the total cost of production for the rest of the industry, to be included in (14.5), are given by

$$\sum_{p \neq i} c_p(t) = kc_{i_l}(t) + (n-1-k)c_{l_{-i}}(t).$$

Substituting all these cost functions into (14.5) and assuming the same spillover parameters across the industry ($\beta_i = \beta$, $i = 1, \dots, n$) also total knowledge is homogeneous, thus we simply write z instead of z_i and z_l . All in all, the expected profit function for firm i can be written as

$$\pi_i^{\text{ex}}(x_i, x_l) = \frac{1}{b(1+n)^2} \left(a - \frac{c_0(n-k)}{1 + (x_i + x_l(k + \beta(n-k-1)) + \rho z)^r} \right) \quad (14.7)$$

$$+ \frac{c_0(n-k-1)}{1 + (x_l + kx_l + \beta(x_l + x_l(n-k-2)) + \rho z)^r} - \gamma x_i^2, \quad (14.8)$$

where x_i and x_l here represent the R&D effort expected for next time period. Note that (14.7) also depends on z , the current stock of knowledge. With naive expectations à la Cournot, i.e. $x_i = x_i(t)$, we can express the expected profit for time $t+1$ in function of state variables at time t , i.e., R&D effort $x_i(t)$ and total knowledge $z(t)$.

Assuming interior optimum, it is possible to write the F.O.C. for maximum profit of firm i , i.e., $\frac{\partial \pi_i^{\text{ex}}(x_i, x_l)}{\partial x_i} = 0$ ³. By the assumption of symmetry of the network, each agent trivially

³Second order conditions are satisfied for sufficiently high costs of R&D efforts γ .

solves the same problem and so she exerts the same effort (i.e. after taking the derivative w.r.t. x_i we can simplify the expression by imposing $x_i = x_i = x$). The expression for the F.O.C. of a representative firm is

$$\frac{\partial \pi_i^{\text{ex}}}{\partial x_i} \Big|_{x_i=x_i} = \frac{2c_0(-k + \beta(1+k-n) + n)rA^{r-1}(a - c_0 + aA^r)}{b(1+n)^2(1+A^r)^3} - 2\gamma x_i = 0 \quad (14.9)$$

where $A = (1 - \beta)x_i(1 + k) + \beta x_i n + \rho z$

In particular, a change of R&D efforts influences:

- the current cost of effort;
- the unitary cost reduction through the R&D production function for the firm and for linked competitors;
- the unitary cost reduction for competitors through spillovers.

Expression (14.9) represents the tool to model how R&D efforts exerted by the representative firm changes over time according to the gradient dynamic, as explained in the next section. As we already noticed, the static model is a two stage game, so the choice of an R&D effort strategy induces an "optimal" subsequent choice of quantities, according to (14.4). It is interesting to observe that in the homogeneous case the ex-post marginal cost is equal overall the industry, and by (14.4) the realized quantity is always strictly positive.

14.3. Dynamic Adjustment of R&D Efforts and Knowledge

We assume that, due to the network structure of R&D collaborations and spillover externalities, agents are not able to coordinate themselves and optimize their profits with respect to R&D effort levels in one shot; this assumption is fully justified by the fact that the equation specifying a Nash equilibrium is nonlinear and not analytically solvable in general cases. So, it is reasonable to assume that even a skilled player is unable to select at once an optimal effort level. Consequently, firms are assumed to behave myopically, i.e., they adaptively adjust their efforts over time towards an optimal strategy by following the direction of the local estimate of expected profits gradient, according to the so called "gradient dynamics" (see [17–20]). In order to keep low the dimension of the dynamic system, we assume that firms are homogeneous, so a single equation can model the dynamic choices of effort over time for a representative firm, and the other equation keeps track of the total knowledge of this agent, so that the dynamic model assumes the form⁴

$$\begin{cases} x' = \max \left[0, x + \alpha(x) \frac{\partial \pi^{\text{ex}}}{\partial x}(x, z) \right] \\ z' = x \{ 1 + k + \beta [(n-1) - k] \} + \rho z \end{cases} \quad (14.10)$$

where ' is the unit time advancement operator, x represents the R&D effort at time period t of the representative firm (we henceforth suppress the subscript i) and $\alpha(\cdot)$ is a positive

⁴Note that we assume irreversible R&D efforts, i.e., $x \geq 0$. In principle one can formulate the model with reversible investments, but then a constraints on positivity of knowledge must be introduced.

function that represents the speed of adjustment. Note that, given the initial effort x_0 , due to (14.2), the initial condition on total knowledge is forced to be

$$z(0) = x(0) \{ 1 + k + \beta [(n-1) - k] \}. \quad (14.11)$$

In other words, the dynamic model is represented by the iterated 2-dimensional map (14.10) with initial condition (i.c. henceforth) taken on a 1-dimensional submanifold of the phase space. At each time period the representative firm decides the R&D effort to be exerted in the next period, consequently changing the accumulated knowledge as well. Notice that a choice in the R&D effort also implies a choice of optimal quantity to produce, according to (14.4). To simplify the model we assume a constant speed of adjustment $\alpha(x) = \alpha$, i.e., the absolute increments or decrements of R&D efforts are directly proportional to marginal profits.

14.3.1. Fixed Point Analysis

From the first equation in (14.10), it is immediate to notice that an equilibrium of the dynamical system is a point where $\frac{\partial \pi^{\text{ex}}}{\partial x}$ vanishes, indicating that it corresponds to a Nash equilibrium level of R&D effort. Moreover, if the effectiveness of R&D vanishes, $r \rightarrow 0$, we also get that optimal effort $x^* \rightarrow 0$ and the model reduces to a standard Cournot Oligopoly with marginal cost given by $\frac{c_0}{2}$. The case $r = 1$, where knowledge capital follows the law of diminishing marginal productivity, is particularly interesting, because the more involved cases with $0 < r < 1$ share similar qualitative properties, e.g., the same properties of existence and uniqueness of a positive equilibrium. For this particular case, we show the following

Proposition 14.1 *Let us consider the model (14.10) when $r = 1$. For all economic meaningful parameters, the system (14.10) has exactly one equilibrium $E = (x^*, z^*)$ with strictly positive coordinates.*

Proof. From the second equation in (14.10) we get the stationary level of total knowledge as

$$z = \eta x = \frac{\{ 1 + k + \beta [(n-1) - k] \}}{1 - \rho} x, \quad (14.12)$$

which, substituted back in $\frac{\partial \pi^{\text{ex}}}{\partial x} \Big|_{x_i=x_i}$, identifies a positive equilibrium point of (14.10) as a zero of the following function (well defined for $r > 0$):

$$g(x; r) = \frac{2c_0(n - k + \beta(1 + k - n))r(1 - \rho)(x\eta)^r(a - c_0 + a(x\eta)^r)}{b(1 + n)^2(1 + k + \beta(n - 1 - k))x(1 + (x\eta)^r)^3} - 2\gamma x. \quad (14.13)$$

For the case $r = 1$, existence of a strictly positive equilibrium then follows, by noticing that:

- $g(x; r)$ is continuous in $(0, +\infty)$;
- $\lim_{x \rightarrow 0^+} g(x; 1) = \frac{2(a - c_0)c_0((n - k)(1 - \beta) + \beta)}{b(1 + n)^2} > 0$;
- $\lim_{x \rightarrow +\infty} g(x; r) = -\infty$.

Moreover, it is possible to show (by some easy but quite cumbersome algebraic manipulations) that $g(x; 1) = 0$ is equivalent to the research of the roots of a concave fourth degree polynomial in $(0, +\infty)$, thus ruling out the case of three positive equilibria. ■

When $r > 1$, more scenarios are possible, as we can have, depending on the parameters, cases without any equilibrium, with one positive equilibrium or with coexistence of multiple (e.g., two) equilibria. In particular, the function $g(x; r)$ in (14.13) is continuous, with $\lim_{x \rightarrow 0^+} g(x; r) = 0$ and $\lim_{x \rightarrow +\infty} g(x; r) = -\infty$; moreover for $r \geq 3$, we can prove analytically that $\lim_{x \rightarrow 0^+} \frac{dg(x; r)}{dx} = -2\gamma < 0$, so (except some non generic singular cases) either zero or an even number of equilibria exist. These various situations are shown in Figure 14.1, obtained for parameters $a = 5$; $b = 1$; $n = 20$; $k = 10$; $\rho = 0.55$; $\beta = 0$; $c_0 = 2$; $\gamma = 5$. In Figure 14.1a, a typical case with $r = 1$ is depicted, where existence and uniqueness of the positive equilibrium is analytically shown in Proposition 14.1. For $r > 1$ all possible configurations are shown in Figure 14.1: one positive equilibrium with $r = 1.5$ in Fig. 14.1b; no positive equilibrium with $r = 3$ in Fig. 14.1c; two positive equilibria with $r = 5$ in Fig. 14.1d.

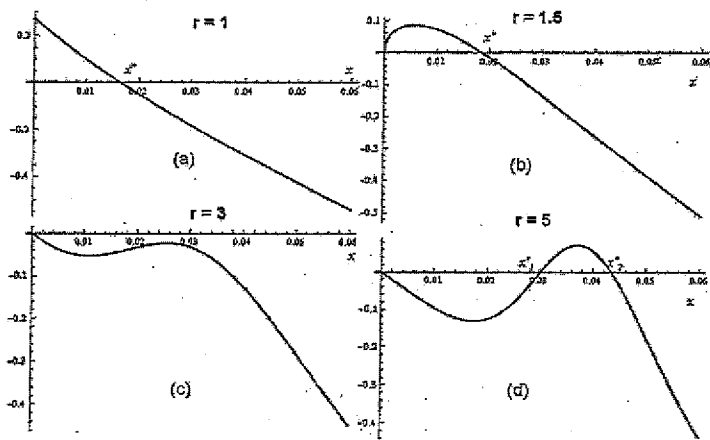


Figure 14.1. Representation of (14.13), whose zeroes are positive R&D effort equilibria, with $a = 5$; $b = 1$; $n = 20$; $k = 10$; $\rho = 0.55$; $\beta = 0$; $c_0 = 2$; $\gamma = 5$. (a) Uniqueness whenever $r = 1$; (b) with these fixed parameters, a unique equilibrium exists for $r = 1.5$; (c) no positive equilibrium for $r = 3$; (d) multiple equilibria for $r = 5$.

14.3.2. Stability of Fixed Points

Given the computational difficulty of the model, an equilibrium value of effort x^* (with corresponding knowledge z^*) is unlikely to be chosen by firms at the onset of the game. For this reason it is also important to get some insights on the stability of fixed points.

By Proposition 14.1, in the benchmark case $r = 1$ (whose importance has been stressed in the previous section) there is a unique fixed point $E = (x^*, z^*)$.

In general, we recall that the stability conditions for an equilibrium of a bidimensional map can be expressed by

$$\begin{cases} 1 - Tr + Det > 0 \text{ (Divergence boundary)} \\ 1 + Tr + Det > 0 \text{ (Flip boundary)} \\ 1 - Det > 0 \text{ (Flutter boundary)} \end{cases}$$

where Tr and Det denote the trace and the determinant of the Jacobian matrix of (14.10) at the unique fixed point $E = (x^*, z^*)$, see, e.g., citeML01. Under the constraints on parameter values, these conditions can be written respectively as:

$$\begin{cases} \gamma > -\frac{A(x^*)}{1-\rho} \\ \gamma < \frac{A(x^*)}{(1+\rho)} + \frac{1}{\alpha} \\ 2\gamma + \frac{1-\rho}{\rho\alpha} > 0 \end{cases}$$

where the quantity $A(x^*) = \frac{c_0(1+k)(n-k)(3c_0-2\alpha(1+x^*+x^*k+\rho z^*))}{b(1+n)^2(1+x^*+x^*k+\rho z^*)^2}$. In any case, the third condition is always satisfied, and so Neimark-Sacker bifurcations are ruled out for all parameter values. Condition $A(x^*) \geq 0$ implies that also the first condition for stability is satisfied, and so a destabilization of the fixed point $E = (x^*, z^*)$ could take place only through flip bifurcations for sufficiently high values of γ . Otherwise, for $A(x^*) < 0$ also transcritical or pitchfork bifurcations are, in principle, possible for low values of γ . More detailed analytical conditions can be obtained in particular sub-cases. For instance when $\beta = 0$ (no spillovers), by (14.12), we have that condition $A(x^*) > 0$ is always satisfied whenever $x^* > \frac{1-\rho}{2(1+k)}$; otherwise, for $0 < x^* \leq \frac{1-\rho}{2(1+k)}$, that condition is equivalent to $2\alpha > \frac{3c_0(1-\rho)}{(1-\rho)+x^*(1+k)}$. Under these circumstances, only flip bifurcations are possible.

Now we examine the cases with $r > 1$. Analytical conditions for stability are not easy in the general case, so we mainly rely on numerical simulations. However, we remark that all cases described below will exemplify scenarios which are observable for broader ranges of parameters.

Let us inspect, in detail, the situation of Figure 14.1d, where, besides the null equilibrium $x_0^* = 0$, two positive equilibria values of effort are present, labeled $x_1^* \approx 0.03006$ and $x_2^* \approx 0.04338$. Numerically we have that $E_2 = (x_2^*, \eta x_2^*)$, see (14.12), is a stable fixed point for all $\alpha \in [0, 0.15032]$, and for higher speed of adjustment, E_2 loses stability through a flip bifurcation; on the other hand, $E_1 = (x_1^*, \eta x_1^*)$ is an unstable fixed point for all values of α . Now we fix a speed of adjustment for which E_1 is stable, namely $\alpha = 0.1$, and we change the spillover parameter β . As β is increased, the equilibrium values of effort and knowledge of $E_1(\beta)$ and $E_2(\beta)$ increase and decrease, respectively. At $\tilde{\beta} \approx 0.4192$ a saddle-node bifurcation occurs, at which the two positive equilibria merge and disappear (see Fig. 14.2a). Consequently, the only attractor of the system for higher spillover values is the zero level, where no R&D efforts are exerted. The asymptotic values of R&D efforts are shown as thick lines in the bifurcation diagram of Figure 14.2b, where $\beta \in [0, 1]$. Here the dashed curve represents the unstable equilibrium level of R&D efforts as a function of β .

From an economic point of view, this can be easily justified: in fact, β measures the fraction of R&D efforts spilling over from free to non-linked competitors, so it is not surprising that the steady level R&D investments are decreasing in it. More interestingly, when $r > 1$, so that the R&D production function follows the "law of variable proportions", a spillover threshold level could exist ($\tilde{\beta}$ in our example), such that long run equilibria investments are discontinuous in spillovers, i.e., the representative firm stops investing in R&D. Note that the consequence of this bifurcation is irreversible. In fact, the reduction of β below $\tilde{\beta}$ (for instance by a regulator imposing a stricter law on intellectual properties protection), does not restore the previous R&D equilibrium level that remains zero. This "path-dependence" property of R&D effort originates a *hysteresis effect* in the model. It is clear that spillovers introduces a free-riding problem: as a consequence of absence of R&D investments, knowledge also reduces to zero, due to its obsolescence rate; and marginal costs, according to (14.3), are maximized, so that at the end all firms are worst off.

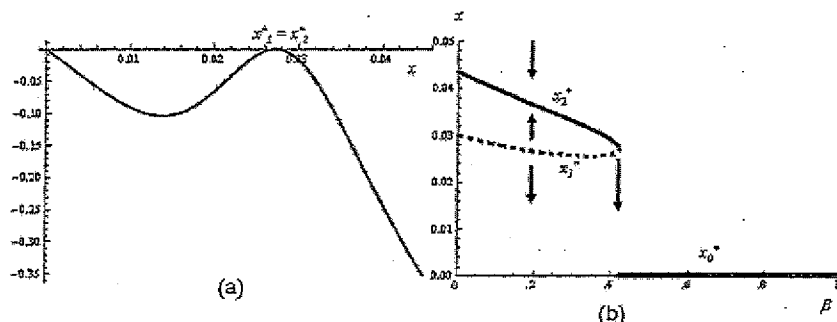


Figure 14.2. Parameters as in Figure 14.1d, but $\tilde{\beta} \approx 0.4192$. (a) Graph of (14.13) at the bifurcation point, showing the merging of equilibria x_1^* and x_2^* ; (b) bifurcation diagram with $\beta \in [0, 1]$. The thick lines represent stable equilibria, the dashed ones unstable equilibria.

Next, we consider the influence of the level of collaboration activity k on the system. In general, our numerical experiments (valid for both concave and convex-concave production functions) confirm that the long run level of R&D effort x is decreasing in the number of collaboration ties k , as shown for a particular case in Figure 14.3a, obtained with parameters $a = 10; b = 1; n = 20; r = 4; \rho = 0.9; \beta = 0.15; c_0 = 1.5; \gamma = 4.5, \alpha = 0.15, k \in [0, 19] \cap \mathbb{Z}$, with initial condition $x(0) = 0.4$ and, according to (14.11), $z(0) = 3.52$.

This is in accordance with proposition 4 in [8], as the result of two counterbalancing effects: In fact, when a firm has more collaborators, an increase in its own R&D efforts not only lowers its production costs (beneficial effect), but also it reduces the costs for collaborators that, in turn, become tougher competitors (disadvantageous effect). If we inspect the corresponding long run level of knowledge, we notice that it is maximized for an intermediate level of collaboration activity. In our numerical example, the maximum level of knowledge is reached for $k = 8$, as shown in the bifurcation diagram for z in Figure 14.3b.

Note that by (14.3) and the monotonicity of $f(z)$, the maximum level of knowledge corresponds to the minimum level of firms' marginal cost. This is also in line with proposition 5 in [8], stating that "cost reduction is maximum when each firm is linked with roughly half of the other firms".

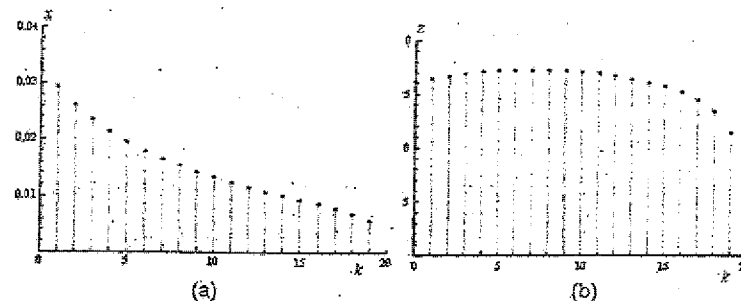


Figure 14.3. Sequence of asymptotic equilibria varying $k \in [0, 19] \cap \mathbb{Z}$. Remaining parameters are given by $a = 10; b = 1; n = 20; r = 4; \rho = 0.9; \beta = 0.15; c_0 = 1.5; \gamma = 4.5, \alpha = 0.15$. (a) R&D effort equilibria x^* ; (b) knowledge equilibria z^* .

We end our analysis by discussing the long run dynamic behavior of the system as the parameter r varies, i.e., regulating the inflection point of the R&D production function. For low level of r (inflection of R&D production function for low levels of knowledge), we mainly observe convergence to the stable fixed point, whereas for high levels of r (inflection of R&D production function for high levels of knowledge), the dynamics are characterized by high period cycles or chaotic motion. In Figure 14.4, obtained with parameters $a = 10; b = \frac{1}{2}; n = 20; k = 5; \rho = 0.9; \beta = 0.2; c_0 = 1.5; \gamma = 2.5, \alpha = 0.18$ and $r \in (0, 20]$, with initial condition $(x(0), z(0)) = (0.4, 3.52)$, a bifurcation diagram is depicted with varying R&D efforts, showing the well-known route to chaos through a period doubling cascade of bifurcations.

14.4. Conclusion

In this chapter we have introduced a stylized model of an R&D network, with knowledge that accumulates over time and can spill over to competitors for free. In particular, we have specified an R&D production function that regulates how total knowledge influences production costs. Moreover, depending on a parameter, we consider both the case of a concave R&D production function and a convex-concave one. For the first case, we proved that a unique equilibrium for R&D efforts exists with a corresponding stationary level of total knowledge of the industry. For this benchmark case we also discussed some analytical conditions for asymptotic stability of the unique equilibrium. Other interesting results are obtained with a convex-concave production function. In this case it is possible to observe a discontinuous transition from a positive equilibrium to absence of investments as spillovers

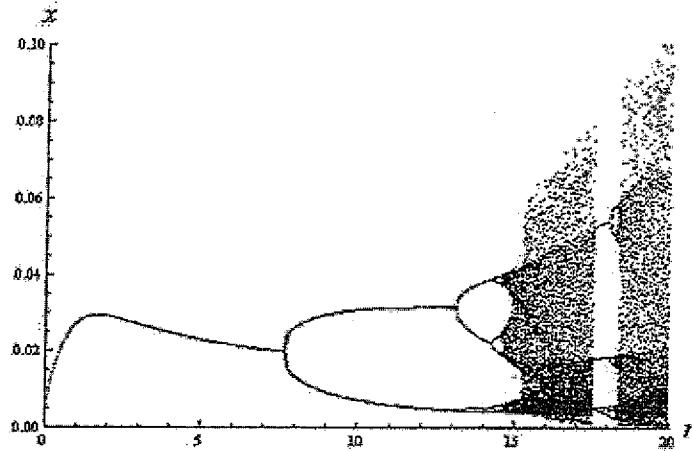


Figure 14.4. Period doubling route to chaos with $r \in (0, 20]$ and $a = 10$; $b = \frac{1}{2}$; $n = 20$; $k = 5$; $\rho = 0.9$; $\beta = 0.2$; $c_0 = 1.5$; $\gamma = 2.5$, $\alpha = 0.18$.

are increased; this phenomenon shows hysteresis effects, so it can even be irreversible as spillovers are reduced back to the previous level. This suggests that it is important to protect intellectual properties to limit the disincentive to invest in R&D as a consequence of free rider behaviors. Moreover, our numerical experiments suggest that, both in the case of strictly concave or convex-concave production function, R&D efforts are decreasing as the number of links in the network is increased. However, total knowledge appears to be always maximized for intermediate levels of collaboration activity. These effects are in total accordance with analogous results of the literature on R&D networks, reported in [8]. To end our discussion on the model, we observe that complex dynamic behavior can arise when the R&D production function is characterized by an inflection point for very high values of knowledge. Possible improvements of the model include spillovers dependence on accumulated knowledge (or absorptive capacity, see [9]) and relaxing the assumption of myopic behavior, like in [23], where a differential game with infinite time horizon is analyzed.

A. Alternative Derivation of Profit Function

Profit function for i -th oligopolist

$$\pi_i(q_i) = (a - bQ)q_i - c_i(q_i), \quad (\text{A.14})$$

where Q is the total industry output, $a, b > 0$ and cost function $c_i(q_i)$.

From F.O.C. we get

$$\frac{\partial \pi_i}{\partial q_i} = a - bQ - bq_i - c'_i(q_i) = 0, \quad (\text{A.15})$$

i.e.,

$$bq_i = a - bQ - c'_i(q_i), \quad i = 1, \dots, n. \quad (\text{A.16})$$

Summing up the n relations in (A.16), it results

$$b \sum_{i=1}^n q_i = bQ = na - bnQ - \sum_{j=1}^n c'_j(q_j),$$

from which we get

$$bQ = \frac{na - \sum_{j=1}^n c'_j(q_j)}{n+1}. \quad (\text{A.17})$$

Substituting (A.17) into (A.16) we get

$$\begin{aligned} q_i^* &= \frac{1}{b} \left[a - \frac{na - \sum_{j=1}^n c'_j(q_j)}{n+1} - c'_i(q_i) \right] \\ &= \frac{a - nc'_i(q_i) + \sum_{j \neq i} c'_j(q_j)}{b(n+1)}. \end{aligned} \quad (\text{A.18})$$

That is the Cournot-Nash equilibrium quantity for oligopolist i . Now let us consider the linear cost function of the form $c_i(q_i) = c_i q_i$ for all agents with marginal costs $c'_i(q_i) = c_i$. In this case the profit function (A.14) at the equilibrium, using (A.17) and (A.18), becomes

$$\begin{aligned} \pi_i(q_i^*) &= (a - bQ)q_i^* - c_i(q_i^*) \\ &= \left(\frac{a + c_i + \sum_{j \neq i} c_j}{n+1} \right) \left(\frac{a - nc_i + \sum_{j \neq i} c_j}{b(n+1)} \right) - c_i \frac{a - nc_i + \sum_{j \neq i} c_j}{b(n+1)} \\ &= \frac{a^2 + 2a \sum_{j \neq i} c_j + (\sum_{j \neq i} c_j)^2 - 2anc_i + 2c_i \sum_{j \neq i} c_j + n^2 c_i^2}{b(n+1)^2} \\ &= \left[\frac{a - nc_i + \sum_{j \neq i} c_j}{\sqrt{b(n+1)}} \right]^2, \end{aligned}$$

that coincides with the first part of (14.5).

References

- [1] Okuguchi, K. (1976). *Expectations and stability in oligopoly models*. Springer-Verlag.
- [2] Vives, X. (1999). *Oligopoly pricing : Old ideas and new tools*. Cambridge, MA: The MIT Press, .
- [3] Puu, T., & Sushko, I. (2002), *Oligopoly dynamics, Models and tools*. New York: Springer-Verlag.
- [4] Bischi, G.I., Chiarella, C., Kopel, M. & Szidarovszky, F. (2009). *Nonlinear oligopolies: Stability and bifurcations*. New York: Springer-Verlag.
- [5] Nelson, R. R. (1982). The role of knowledge in R&D efficiency. *The Quarterly Journal of Economics*, **97** (3), 453–470.
- [6] Spence, M. (1984). Cost reduction, competition and industry performance. *Econometrica*, **52**, 101–121.
- [7] D'Aspremont, C., & Jacquemin, A. (1988). Cooperative and noncooperative R&D duopoly with spillovers. *The American Economic Review*, **78** (5), 1133–1137.
- [8] Goyal, S., & Moraga-Gonzales, J. L. (2001). R&D networks, *RAND Journal of Economics*, **32** (4), 686–707.
- [9] Confessore, G., & Mancuso, P. (2002). A dynamic model of R&D competition. *Research in Economics*, **56**, 365–380.
- [10] Bischi, G. I., & Lamantia, F. (2004). A competition game with knowledge accumulation and spillovers. *International Game Theory Review*, **6**, 323–342.
- [11] Petit, M. L., & Tolwinski, B. (1999). R&D cooperation or competition? *European Economic Review*, **43**, 185–208.
- [12] Petit, M.L., Sanna-Randaccio, F., & Tolwinski, B. (2000). Innovation and foreign investment in a dynamic oligopoly, *International Game Theory Review*, **2** (1), 1–28.
- [13] Audretsch, D.B., & Feldman, M.P. (1996). R & D Spillovers and the geography of innovation and production. *American Economic Review*, **86**, 630–640.
- [14] Bischi, G.I., & Lamantia, F. (2002). Nonlinear duopoly games with positive cost externalities due to spillover effects. *Chaos, Solitons & Fractals*, **13**, 805–822.

- [15] Bischi, G.I., Dawid, H., & Kopel, M. (2003). Gaining the competitive edge using internal and external spillovers: A dynamic analysis. *Journal of Economic Dynamics and Control*, **27**, 2171–2193.
- [16] Gauvin, S. (1995). Network of Innovators: Evidence from Canadian patents. *Group Decision and Negotiation*, **4**, 411–428.
- [17] Flam, S.D. (1993). Oligopolistic competition: From stability to chaos. In F. Gori, L. Geronazzo & M. Galeotti (Eds.), *Nonlinear dynamics in economics and social sciences. Lecture Notes in Economics and Mathematical Systems*. (pp. 232–399). Springer.
- [18] Furth, D. (1986). Stability and instability in oligopoly. *Journal of Economic Theory*, **40**, 197–228.
- [19] Bischi, G.I., & Naimzada, A. (1999). Global analysis of a duopoly game with bounded rationality. In *Advances in dynamic games and applications: Vol. 5* (pp. 361–385). Birkhauser.
- [20] Szabó, G., & Fáth, G. (2007). Evolutionary games on graphs. *Physics reports*, **446**, 97–216.
- [21] Qiu, L. D. (1997). On the dynamic efficiency of Bertrand equilibria. *Journal of Economic Theory*, **75**, 213–229.
- [22] Medio, A., & Lines, M. (2001). *Nonlinear dynamics*, Cambridge University Press.
- [23] Cellini, R., & Lambertini, L. (2009). Dynamic R&D with spillovers: Competition vs cooperation. *Journal of Economic Dynamics & Control*, **33**, 568–582.

Chapter 15

ON MARKET INERTIA AND STABLE COMPETITION

*F. Szidarovszky and S. Yousefi**

Systems and Industrial Engineering Department, The University of
 Arizona, Tucson, Arizona 85721-0020, USA
 Centre for Advanced Research in Nature and Society, P. Box 20,
 DK-5884, Denmark

Abstract

In this chapter we look into the asymptotic stability of dynamic competition in markets operating under inertia. We derive stability conditions that are general and apply them to classical models of duopolies. We also obtain conditions that are established through an explicit discretization from continuous dynamics to discrete dynamics. Such an approach is suited to be incorporated in numerical procedures and provides comparison between different stabilization regimes.

15.1. Introduction

Issues related to the stability of the equilibrium and to the existence and uniqueness of the equilibrium are regarded as central to the oligopoly literature (see [1, 2] for detailed treatment of the topic). Although the literature on dynamic oligopolies deals with both discrete and continuous systems, the issues related to the comparison of the different stability regimes with the simultaneous study of asymptotical properties of these systems are rather neglected (see [2, 3] for extensive treatment on the history as well as on recent studies in dynamic oligopolies). In this chapter we intend to address and fill up this gap.

*E-mail addresses: szidar@sie.arizona.edu, sy@carinas.org